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N-Δ axial transition form factors ⋆

B. Golli[a,b], L. Amoreira[c,e], M. Fiolhais[d,e], and S. Širca[f,b]

[a] Faculty of Education, University of Ljubljana, 1000 Ljubljana, Slovenia
[b] J. Stefan Institute, 1000 Ljubljana, Slovenia
[c] Department of Physics, University of Beira Interior, 6201-001 Covilhã, Portugal
[d] Department of Physics, University of Coimbra, 3004-516 Coimbra, Portugal
[e] Centre for Computational Physics, University of Coimbra, 3004-516 Coimbra, Portugal
[f] Faculty of Mathematics and Physics, University of Ljubljana, 1000 Ljubljana, Slovenia

Abstract. We review some basic properties of the N-Δ transition axial amplitudes and relate them to the strong πNΔ form-factor. In models with the pion cloud we derive a set of constraints on the pion wave function which guaranty the correct behaviour of the amplitudes in the vicinity of the pion pole. Corrections due to the spurious center-of-mass motion are calculated to the leading order in the inverse baryon mass. We give explicit expressions for the amplitudes in the Cloudy Bag Model and show that they rather strongly underestimate the experimental values.

1 Introduction

The weak N-Δ transition amplitudes yield important information about the structure of the nucleon and the Δ, and in particular about the role of chiral mesons since they explicitly enter in the expression for the axial part of the weak current. There exist only very few calculations in quark models [1,2] yet none of them includes the mesonic degrees of freedom. This can be traced back to the difficulty of incorporating consistently the pion field which is necessary to describe the correct low-Q^2 behaviour of the amplitudes. Obviously, this can be done only in the models that properly incorporate the chiral symmetry.

The aim of this work is to study the axial amplitudes of the N-Δ transition in models with quarks and chiral mesons. In Sec. 2 we introduce expressions for the axial helicity amplitudes and relate them to the experimentally measured quantities, $C_i^A, i = 3, 6$, the so called Adler form-factors. We derive the analog of the Goldberger-Treiman relation that relates the leading axial form factor, C_5^A, to the strong πNΔ coupling constant. In Sec. 3 we calculate the amplitudes in a simple isobar model that includes the pion. In Sec. 4 we study some general properties of the axial amplitudes in quark models that include the pion and possibly also its chiral partner, the σ-meson. We derive a set of constraints on the pion field and show that in models that satisfy these constraints the pion pole appears only in the C_6^A form-factor. Furthermore, if the meson self-interaction is absent in the model, i.e. if the pion interacts only with quarks, the pion contributes solely to

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the C_6^A form-factor while the C_4^A andC_5^A form-factors pick up only the contribution from quarks. In most quark models the nucleon and the Δ are calculated as localized states while the expressions for the amplitudes require states with good linear momenta. In Sec. 5 we use the wave packet formalism to derive corrections to the amplitudes calculated between localized states and show that the approximations are valid for momenta that are small compared to typical baryon masses. In Sec. 6 we give explicit expressions for the axial as well as the strong form-factors in the Cloudy Bag Model (CBM) and make a simple estimate of their strengths.

The calculation of the form-factors in the CBM as well as in the linear σ-model that includes besides the pion also the σ-meson is presented and compared to the experimentally measure form-factors in [3,4] and in the contribution of Simon Širca [5] to these Proceedings.

2 Same basic properties of transition amplitudes

2.1 Definition of the helicity amplitudes

The weak transition amplitudes are defined as the matrix elements of the weak interaction Hamiltonian

$$M = \langle \Delta | H | N, W \rangle = W^{(-)}_{a\mu} \langle \Delta | V^{a\mu} - A^{a\mu} | N \rangle \tag{1}$$

where a is the isospin index. For simplicity we shall assume $a = 0$ and will not write it explicitly. For the axial part alone we have:

$$M^A = \sqrt{\frac{4\pi\alpha_W}{2K_0}} \sum_\lambda e_{\mu\lambda} \langle \Delta | A^\mu | N \rangle = \sqrt{\frac{4\pi\alpha_W}{2K_0}} \left[\langle \Delta | A^0 | N \rangle - \sum_\lambda \varepsilon_\lambda \cdot \langle \Delta | \mathbf{A} | N \rangle \right] , \tag{2}$$

where

$$K_0 = \frac{M_\Delta^2 - M_N^2}{2M_\Delta} \quad \text{and} \quad 4\pi\alpha_W = \frac{4\pi\alpha}{\sin^2\theta_W} \approx 0.443 . \tag{3}$$

The 4-momentum of the incident weak boson (W) is

$$k^\mu = (k_0, 0, 0, k) , \quad k_0 = \frac{M_\Delta^2 - M_N^2 - Q^2}{2M_\Delta} , \quad k = \sqrt{k_0^2 + Q^2} . \tag{4}$$

The helicity amplitudes are defined as

$$\tilde{S}^A = -\langle \Delta^+(p'), s_\Delta = \tfrac{1}{2} | A_0^0(0) | N^+(p) s_N = \tfrac{1}{2} \rangle , \tag{5}$$

$$\tilde{A}^A_{\frac{3}{2}} = -\langle \Delta^+(p'), s_\Delta = \tfrac{3}{2} | \varepsilon_+ \cdot \mathbf{A}(0) | N^+(p) s_N = \tfrac{1}{2} \rangle , \tag{6}$$

$$\tilde{A}^A_{\frac{1}{2}} = -\langle \Delta^+(p'), s_\Delta = \tfrac{1}{2} | \varepsilon_+ \cdot \mathbf{A}(0) | N^+(p) s_N = -\tfrac{1}{2} \rangle , \tag{7}$$

$$\tilde{L}^A = -\langle \Delta^+(p'), s_\Delta = \tfrac{1}{2} | \varepsilon_0 \cdot \mathbf{A}(0) | N^+(p) s_N = \tfrac{1}{2} \rangle . \tag{8}$$

2.2 The Adler form-factors

Experimentalists measure the so called Adler form-factors defined as [6]:

$$\langle\Delta^+(p')|A_{\alpha(a=0)}|N^+(p)\rangle = \bar{u}_{\Delta\alpha}\,\frac{C_4^A(Q^2)}{M_N^2}\,p'_\mu q^\mu u_N - \bar{u}_{\Delta\mu}\,\frac{C_4^A(Q^2)}{M_N^2}\,p'_\alpha q^\mu u_N$$
$$+\bar{u}_{\Delta\alpha}\,C_5^A(Q^2)\,u_N + \bar{u}_{\Delta\mu}\,\frac{C_6^A(Q^2)}{M_N^2}\,q^\mu q_\alpha u_N + \bar{u}_{\Delta\alpha}\,\frac{C_3^A(Q^2)}{M_N}\,\gamma_\mu q^\mu u_N\ , \quad (9)$$

where $p'_\mu = (M_\Delta; 0, 0, 0)$ and $q^\mu = (\omega; 0, 0, k)$, and $u_{\Delta\alpha}$ is the Rarita-Schwinger spinor:

$$u_\alpha(p, s_\Delta) = \sum_{\lambda', s} C^{\frac{3}{2}s_\Delta}_{1\lambda'\frac{1}{2}s}\, e_{\alpha\lambda'}(p)u(p, s)\,. \quad (10)$$

Here

$$e^\mu_\lambda(p) = \left[\frac{\boldsymbol{\varepsilon}_\lambda\cdot\mathbf{p}}{M_\Delta}\ ,\ \boldsymbol{\varepsilon}_\lambda + \frac{\mathbf{p}(\boldsymbol{\varepsilon}_\lambda\cdot\mathbf{p})}{M_\Delta(p_0+M_\Delta)}\right]\ , \quad (11)$$

and $u(p, s)$ is the usual bispinor for a spin $\frac{1}{2}$ particle. For the Δ at rest it has a simple form (e.g. [7], 414):

$$e^\mu_\lambda = (0\ ,\ \boldsymbol{\varepsilon}_\lambda)\ , \qquad u(p, s) = \begin{pmatrix}1\\0\end{pmatrix}\chi_{\frac{1}{2}s}\ , \quad (12)$$

where $\boldsymbol{\varepsilon}_\lambda$ are the polarization vectors. The form-factor C_3^A is small; in models with s-wave quarks and p-wave pions it is even identically 0; we shall therefore assume $C_3^A = 0$ in the further derivations.

The helicity amplitudes can now be easily related to the form factors. For $\alpha = 0$ the evaluation is straightforward, while for $\alpha \neq 0$ we multiply (9) by e^α_λ and use the following relations:

$$e^\alpha_\lambda \bar{u}_\alpha(p, s_\Delta)u_N = \boldsymbol{\varepsilon}_\lambda \sum_{\lambda', s} C^{\frac{3}{2}s_\Delta}_{1\lambda'\frac{1}{2}s}(-\boldsymbol{\varepsilon}^*_{\lambda'})\bar{u}(p, s)u_N = -C^{\frac{3}{2}s_\Delta}_{1\lambda\frac{1}{2}s_N}\ , \quad (13)$$

$$e^\alpha_\lambda q_\alpha = -k\delta_{\lambda,0}\ , \qquad \bar{u}_\mu(p, s_\Delta)q^\mu u_N = -kC^{\frac{3}{2}s_\Delta}_{1\lambda\frac{1}{2}s_N}\ . \quad (14)$$

We obtain

$$\tilde{S}^A = -\left[k\,\frac{C_4^A}{M_N^2}\,M_\Delta - \omega k\frac{C_6^A}{M_N^2}\right]\sqrt{\frac{2}{3}}\,, \quad (15)$$

$$\tilde{A}^A_{\frac{3}{2}} = -\left[\frac{C_4^A}{M_N^2}\,\omega M_\Delta + C_5^A\right] = \sqrt{3}\,\tilde{A}^A_{\frac{1}{2}}\ , \quad (16)$$

$$\tilde{L}^A = -\left[\frac{C_4^A}{M_N^2}\,\omega M_\Delta + C_5^A - \frac{k^2}{M_N^2}C_6^A\right]\sqrt{\frac{2}{3}}\,. \quad (17)$$

The Adler form-factors read

$$C_6^A = \frac{M_N^2}{k^2}\left[-\tilde{A}^A_{\frac{3}{2}} + \sqrt{\frac{3}{2}}\tilde{L}^A\right]\ , \quad (18)$$

$$C_5^A = -\sqrt{\frac{3}{2}}\left(\tilde{L}^A - \frac{k_0}{k}\,\tilde{S}^A\right) - \frac{k_0^2 - k^2}{M_N^2}\,C_6^A\,, \tag{19}$$

$$C_4^A = \frac{M_N^2}{kM_\Delta}\left[-\sqrt{\frac{3}{2}}\,\tilde{S}^A + \frac{k_0 k}{M_N^2}\,C_6^A\right]. \tag{20}$$

2.3 The off-diagonal Goldberger-Treiman relation

Let us compute the divergence of the axial current between the Δ and N (9). Using (14) we get ($q^2 \equiv -Q^2$):

$$\langle \Delta^+(P)|\partial^\alpha A_\alpha|N^+(p)\rangle = \mathrm{i}k\left[C_5^A(q^2) + \frac{C_6^A(q^2)}{M_N^2}\,q^2\right] C^{\frac{3}{2}\frac{1}{2}}_{10\frac{1}{2}\frac{1}{2}}\,. \tag{21}$$

In the chiral limit the divergence has to vanish. From the above expression we would conclude that $C_5^A(q^2) = 0$ which is experimentally not the case. Hence $C_6^A(q^2)$ should have a pole at $q^2 = 0$ such that

$$C_6^A(q^2) = -\frac{M_N^2 C_5^A(q^2)}{q^2}\,. \tag{22}$$

As in the nucleon case, we relate this term to the term in the axial current that is responsible for the pion decay: $A^\alpha_{a\,\mathrm{pole}}(x) = f_\pi \partial^\alpha \pi_a(x)$. We can therefore identify the C_6^A-term in (9) with:

$$\bar{u}_{\Delta\mu}\,\frac{C_6^A(q^2)}{M_N^2}\,q^\mu q_\alpha u_N = \mathrm{i}q_\alpha f_\pi \langle \Delta^+(P)|\pi_0(0)|N^+(p)\rangle\,. \tag{23}$$

Indeed, the pion propagator behaves as q^{-2} in the chiral limit.

In the real world the pion mass is finite and we write the pion field as

$$\langle \Delta^+(P)|\pi_0(0)|N^+(p)\rangle = \mathrm{i}\frac{G_{\pi N\Delta}(q^2)}{2M_N}\,\frac{\bar{u}_{\Delta\mu}\,q^\mu u_N}{-q^2 + m_\pi^2}\,\sqrt{\frac{2}{3}}\,. \tag{24}$$

while the vanishing of (21) is replaced by PCAC:

$$\langle \Delta^+(P)|\partial^\alpha A_{\alpha\,a}|N^+(p)\rangle = -m_\pi^2\, f_\pi \langle \Delta^+(P)|\pi_a(0)|N^+(p)\rangle\,. \tag{25}$$

Replacing the LHS of (25) by (21) and using (23) and (24) we find

$$\mathrm{i}q^\alpha \bar{u}_{\Delta\alpha} u_N\left[C_5^A(q^2) + f_\pi \frac{G_{\pi N\Delta}(q^2)}{2M_N}\,\frac{q^2}{-q^2 + m_\pi^2}\,\sqrt{\frac{2}{3}}\right] = \mathrm{i}q^\alpha \bar{u}_{\Delta\alpha} u_N\,\frac{G_{\pi N\Delta}(q^2)}{2M_N}\,\frac{m_\pi^2\, f_\pi}{-q^2 + m_\pi^2}\,\sqrt{\frac{2}{3}}\,. \tag{26}$$

We finally obtain

$$C_5^A(q^2) = f_\pi\,\frac{G_{\pi N\Delta}(q^2)}{2M_N}\,\sqrt{\frac{2}{3}}\,, \tag{27}$$

the *off-diagonal Goldberger-Treiman relation*, which – strictly speaking – holds only in the limit $q^2 \to m_\pi^2$. Assuming a smooth behaviour of the amplitudes for q^2 in the vicinity of m_π^2 we can expect (27) to remain valid for sufficiently small q^2 in the experimentally accessible range.

3 The axial current in a simple isobar model with pions

The aim of this section is to derive the amplitudes in a simple model in order to study the contribution of pions to the amplitudes and to analyze the qualitative behaviour of the amplitudes. The derivation in this section is based on the standard derivation of the diagonal Goldberger-Treiman relation and PCAC (see e.g. [7]).

We investigate the axial hadronic current in a model with two structureless fermion fields, the nucleon and the Δ, and the pion field. Since we are interested here only in the nucleon-Δ transition we shall write down explicitly only the pertinent parts of the Lagrangian and of the hadron current. The nucleon and the Δ (at rest) satisfy the Dirac equation

$$(i\gamma_\mu \partial^\mu - E_N)\psi_N = 0\ , \qquad (i\gamma_\mu \partial^\mu - M_\Delta)\psi_\Delta = 0\ . \tag{28}$$

We assume the following form of the $\pi N\Delta$ interaction

$$\mathcal{L}_{\pi N\Delta} = -iG_{\pi N\Delta}\bar{\psi}_\Delta \gamma_5 T_a \psi_N \pi_a\ , \tag{29}$$

where we introduce the transition operator $\vec{T}$ (and $\boldsymbol{\Sigma}$) by

$$\langle \tfrac{3}{2} t_\Delta | T_a | \tfrac{1}{2} t_N \rangle = C^{\frac{3}{2} t_\Delta}_{1 a \frac{1}{2} t_N}\ , \qquad \langle \tfrac{3}{2} s_\Delta | \Sigma_\lambda | \tfrac{1}{2} s_N \rangle = C^{\frac{3}{2} s_\Delta}_{1 \lambda \frac{1}{2} s_N}\ . \tag{30}$$

(Note that γ^μ has a more complicated structure:

$$\boldsymbol{\gamma} = \begin{vmatrix} 0 & \mathbf{S} \\ -\mathbf{S} & 0 \end{vmatrix}\ , \tag{31}$$

where the generalized Pauli matrices $\mathbf{S}$ act in the space spanned by the $S = \frac{1}{2}$ and $S = \frac{3}{2}$ subspaces:

$$\mathbf{S} = \begin{vmatrix} \boldsymbol{\sigma} & \boldsymbol{\Sigma} \\ \boldsymbol{\Sigma}^\dagger & \boldsymbol{\sigma}_{\Delta\Delta} \end{vmatrix}\ . \tag{32}$$

The generalized isospin is introduce in the same way.)

The nucleon bispinor can be written as

$$u_N(p) = \sqrt{\frac{E_N + M_N}{2M_N}} \begin{pmatrix} 1 \\ \frac{\boldsymbol{\Sigma}\cdot\mathbf{p}}{E_N + M_N} \end{pmatrix} \chi_{\frac{1}{2}s_N} \xi_{\frac{1}{2}t_N} \approx \begin{pmatrix} 1 \\ \frac{\boldsymbol{\Sigma}\cdot\mathbf{p}}{2M_N} \end{pmatrix} \chi_{\frac{1}{2}s_N} \xi_{\frac{1}{2}t_N}\ , \tag{33}$$

with χ and ξ describing respectively the spin and isospin part of the bispinor, and

$$p^\mu = (E_N, \mathbf{p})\ , \qquad E_N = \sqrt{M_N^2 + \mathbf{p}^2} \approx M_N\ . \tag{34}$$

We assume that Δ is at rest, $p'^\mu = (M_\Delta; 0, 0, 0)$, hence

$$u_\Delta(p') = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \chi_{\frac{3}{2}s_\Delta} \xi_{\frac{3}{2}t_\Delta}\ . \tag{35}$$

In the model, the transition part of the axial current takes the form:

$$A^\mu_a = g^\Delta_A \bar{\psi}_\Delta \gamma^\mu \gamma_5\, \tfrac{1}{2} T_a \psi_N + f_\pi \partial^\mu \pi_a\ . \tag{36}$$

Using the Dirac equations (28) and the Klein-Gordon equation for the (pertinent part of the) pion field:

$$\left(\partial_\mu \partial^\mu + m_\pi^2\right) \pi_a = -\mathrm{i} G_{\pi N \Delta} \bar{\psi}_\Delta \gamma_5 T_a \psi_N \tag{37}$$

we immediately obtain

$$\partial_\mu A_a^\mu = \mathrm{i} g_A^\Delta \, \tfrac{1}{2}(M_\Delta + M_N) \bar{\psi}_\Delta \gamma_5 \, T_a \psi_N - \mathrm{i} f_\pi G_{\pi N \Delta} \bar{\psi}_\Delta \gamma_5 T_a \psi_N - f_\pi m_\pi^2 \, \pi_a \, . \tag{38}$$

In the limit $m_\pi \to 0$ the current is conserved provided

$$\tfrac{1}{2}(M_\Delta + M_N) g_A^\Delta = f_\pi G_{\pi N \Delta} \tag{39}$$

which is the *off-diagonal Goldberger-Treiman relation* (27). The constant g_A^Δ is related to the experimentally measured $C_5^A(0)$ by

$$g_A^\Delta = \frac{2M_N}{M_\Delta + M_N} \sqrt{6}\, C_5^A(0) \, , \qquad C_5^A(0) = 1.22 \pm 0.06 \, . \tag{40}$$

We now evaluate the matrix elements of the transition axial current. In this case the solution of (37) is

$$\langle \Delta(p')|\pi_a(\omega, \mathbf{k})|N(p)\rangle = -\mathrm{i} \, \frac{G_{\pi N \Delta}}{2M_N} \, \frac{\langle \Delta| \, (-\boldsymbol{\Sigma} \cdot \mathbf{k}) \; T_a |N\rangle}{(-\omega^2 + \mathbf{k}^2 + m_\pi^2)} \tag{41}$$

with $\omega = M_\Delta - M_N$, $\mathbf{k} = -\mathbf{p}$. For the time-like component of the current we get

$$\begin{aligned}\langle \Delta(p')|A_a^0(0)|N(p)\rangle &= -k \, \frac{g_A^\Delta}{2M_N} \langle \Delta|\Sigma_0 \, \tfrac{1}{2} T_a|N\rangle + \mathrm{i}\omega f_\pi \langle \Delta(p')|\pi_a|N(p)\rangle \\ &= -\left[\frac{g_A^\Delta \, k}{4M_N} + \frac{f_\pi G_{\pi N \Delta}}{2M_N} \, \frac{\omega k}{(-q^2 + m_\pi^2)}\right] \langle \Delta|\Sigma_0 \, T_a|N\rangle \, . \end{aligned} \tag{42}$$

The spatial part is

$$\begin{aligned}\langle \Delta(p')|\mathbf{A}_a(0)|N(p)\rangle &= g_A^\Delta \langle \Delta|\boldsymbol{\Sigma} \, \tfrac{1}{2} T_a|N\rangle + \mathrm{i}\mathbf{k} f_\pi \langle \Delta(p')|\pi_a|N(p)\rangle \\ &= \tfrac{1}{2} \, g_A^\Delta \langle \Delta|\boldsymbol{\Sigma} \, T_a|N\rangle - \frac{f_\pi G_{\pi N \Delta}}{2M_N} \, \frac{\mathbf{k}}{(-q^2 + m_\pi^2)} \, \langle \Delta|(\boldsymbol{\Sigma} \cdot \mathbf{k}) \, T_a|N\rangle \, . \end{aligned} \tag{43}$$

The helicity amplitudes introduced in the first section (for 4-vector momentum transfer $q^\mu = p'^\mu - p^\mu = (\omega; 0, 0, k)$) are now expressed as

$$\tilde{S}^A = \left[k \, \frac{g_A^\Delta}{4M_N} + k \, \frac{f_\pi G_{\pi N \Delta}}{2M_N} \, \frac{\omega}{(-q^2 + m_\pi^2)}\right] \sqrt{\frac{2}{3}}^{\,2} \, , \tag{44}$$

$$\tilde{A}^A_{\frac{3}{2}} = -\tfrac{1}{2} \, g_A^\Delta \sqrt{\frac{2}{3}} = \sqrt{3} \tilde{A}^A_{\frac{1}{2}} \, , \tag{45}$$

$$\tilde{L}^A = \left[-\tfrac{1}{2} \, g_A^\Delta + \frac{f_\pi G_{\pi N \Delta}}{2M_N} \, \frac{k^2}{(-q^2 + m_\pi^2)}\right] \sqrt{\frac{2}{3}}^{\,2} \, . \tag{46}$$

Using (39) we are now able to explicitly check that PCAC holds in the model:

$$\begin{aligned}\langle\Delta^+(p')|\partial_\mu A^\mu_{a=0}|N^+(p)\rangle &= -i\left(\omega\tilde{S}^A - k\tilde{L}^A\right)\\ &= -m_\pi^2 f_\pi\langle\Delta^+(p')|\pi_0|N^+(p)\rangle\ . \qquad (47)\end{aligned}$$

In this model we can express the Adler form-factors solely in terms of either g_A^Δ or $G_{\pi N\Delta}$:

$$C_6^A = \frac{1}{\sqrt{6}} f_\pi M_N \frac{G_{\pi N\Delta}}{-q^2 + m_\pi^2}\,, \qquad (48)$$

$$C_5^A = \frac{1}{\sqrt{6}} \frac{M_\Delta + M_N}{2M_N}\, g_A^\Delta = \sqrt{\frac{2}{3}}\, \frac{f_\pi G_{\pi N\Delta}}{2M_N}\,, \qquad (49)$$

$$C_4^A = -\frac{1}{\sqrt{6}} \frac{M_N}{2M_\Delta}\, g_A^\Delta = -\frac{M_N^2}{M_\Delta(M_\Delta + M_N)}\, C_5^A \approx -0.33\, C_5^A\ . \qquad (50)$$

The relations derived above show that only C_6^A exhibits the pole behavior while in the other two amplitudes the pole behavior cancels out and the result is the same as if we used only the fermion part of the axial current. In the next section we shall see that this property holds in a vast class of models that fulfill certain virial relations.

4 Helicity amplitudes in models with the pion cloud

We investigate quark models that include the pion and possibly also its chiral partner, the σ-meson. The part of the Hamiltonian that involves pions can be written in the following form:

$$H_\pi = \int d\mathbf{r} \left\{ \tfrac{1}{2}\left[\vec{P}_\pi^2 + (\nabla^2 + m_\pi^2)\vec{\pi}^2\right] + U(\sigma, \vec{\pi}) + \sum_t j_t \pi_t \right\}\ . \qquad (51)$$

Here j_t represents the quark pseudoscalar-isovector source term, t is the third component of the isospin, and $U(\sigma, \vec{\pi})$ a possible meson self-interaction term (such as the Mexican hat potential of the linear σ-model). Let $|N\rangle$ and $|\Delta\rangle$ be the ground state and the excited state describing the Δ with $H|N\rangle = E_N|N\rangle$ and $H|\Delta\rangle = E_\Delta|\Delta\rangle$, then we can write the following virial theorems (relations):

$$\langle N|[H, \vec{P}_\pi]|N\rangle = \langle N|H\vec{P}_\pi - \vec{P}_\pi H|N\rangle = 0\,, \qquad (52)$$

$$\langle \Delta|[H, \vec{P}_\pi]|\Delta\rangle = 0\,, \qquad (53)$$

$$\langle \Delta|[H, \vec{P}_\pi]|N\rangle = (E_\Delta - E_N)\langle\Delta|\vec{P}_\pi|N\rangle = i(E_\Delta - E_N)^2\langle\Delta|\vec{\pi}|N\rangle\,. \qquad (54)$$

We have used $\vec{P}_\pi = i[H, \vec{\pi}]$ in the last line. We call (54) the *off-diagonal virial relation (theorem)*. (Note that there is no off-diagonal relation of this type for the σ-field because it is scalar-isoscalar and the matrix elements vanish identically.)

We now evaluate the commutators on the LHS using (51):

$$(-\Delta + m_\pi^2)\langle N|\pi_t(\mathbf{r})|N\rangle = -(-1)^t\langle N|J_{-t}(\mathbf{r})|N\rangle\,, \qquad (55)$$

$$(-\Delta + m_\pi^2)\langle \Delta|\pi_t(\mathbf{r})|\Delta\rangle = -(-1)^t\langle \Delta|J_{-t}(\mathbf{r})|\Delta\rangle\,, \qquad (56)$$

$$(-\Delta + m_\pi^2 - \omega_*^2)\langle \Delta|\pi_t(\mathbf{r})|N\rangle = -(-1)^t\langle \Delta|J_{-t}(\mathbf{r})|N\rangle\,. \qquad (57)$$

We have defined $\omega_* = (E_\Delta - E_N)$ and

$$J_t(\mathbf{r}) = j_t(\mathbf{r}) + (-1)^t \,\frac{\partial U(\sigma,\vec{\pi})}{\partial \pi_{-t}(\mathbf{r})}\,, \tag{58}$$

and used

$$[\pi_{t'}(\mathbf{r}'), P_{\pi,t}(\mathbf{r})] = i(-1)^t \delta_{t,-t'}\delta(\mathbf{r}' - \mathbf{r})\,. \tag{59}$$

These relations hold for the exact solutions; in an approximate computational scheme we can use these relations as constraints on the approximate states.

We now show an important property of the axial transition amplitudes which holds for the states that satisfy the above virial relations. Let us split the axial current into two parts:

$$\vec{A}^\alpha = \vec{A}^\alpha_{np} + \vec{A}^\alpha_{pole}\,, \tag{60}$$

$$\vec{A}^\alpha_{np} = \bar{\psi}\gamma^\alpha\gamma_5\,\tfrac{1}{2}\vec{\tau}\psi + (\sigma - f_\pi)\partial^\alpha\vec{\pi} - \vec{\pi}\partial^\alpha\sigma\,, \tag{61}$$

$$\vec{A}^\alpha_{pole} = f_\pi\partial^\alpha\vec{\pi}\,. \tag{62}$$

We can now relate the non-pole contribution (61) to the first term in (36) and (obviously) the pole contribution to the second term in (36). Since the off-diagonal virial relation (57) coincides with (41), the evaluation is similar to the derivation presented in the previous section. The pole term (62) contributes only to the longitudinal and the scalar amplitude, hence:

$$C^A_{6\,(pole)} = -if_\pi\,\frac{M_N^2}{k}\,\sqrt{\frac{3}{2}}\,\langle\Delta^+_{s_\Delta=\frac{1}{2}}|\pi_0(0)|N^+_{s_n=\frac{1}{2}}\rangle\,, \tag{63}$$

$$C^A_{5\,(pole)} = 0\,,$$

$$C^A_{4\,(pole)} = 0\,.$$

5 Calculation of form-factors between localized states

The amplitudes (5)-(8) are defined between states with good 4-momenta p' and p respectively while in the model calculations localized states are used. We can use such states in our calculation of amplitudes by interpreting them as wave packets of states with good linear momenta:

$$|B(\mathbf{r})\rangle = \int d\mathbf{p}\;\varphi(\mathbf{p})\,e^{i\mathbf{p}\cdot\mathbf{r}}|B(\mathbf{p})\rangle\,. \tag{64}$$

The spin-momentum dependence of $|B(\mathbf{p})\rangle$ is expressed by the bispinor

$$u_B(\mathbf{p}) = \sqrt{\frac{E+M}{2M}}\begin{pmatrix}1\\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+M}\end{pmatrix}\chi_{spin}\,. \tag{65}$$

Requiring (65) is normalized, $\langle B(\mathbf{p})|B(\mathbf{p})\rangle = 1$, we have

$$\int d\mathbf{r}\langle B(\mathbf{r})|B(\mathbf{r})\rangle = (2\pi)^3\int d\mathbf{p}\,|\varphi(\mathbf{p})|^2 = 1\,. \tag{66}$$

We now relate matrix elements between localized states to matrix elements between states with good momenta. We start by a matrix element between localized states:

$$\int d\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}} \langle \Delta | M(\mathbf{r}) | N \rangle = \int d\mathbf{r} \int d\mathbf{p}' \int d\mathbf{p}\, e^{i(\mathbf{k}-\mathbf{p}'+\mathbf{p})\cdot\mathbf{r}} \langle \Delta(\mathbf{p}') | M(\mathbf{r}) | N(\mathbf{p}) \rangle \times \varphi^*_\Delta(\mathbf{p}') \varphi_N(\mathbf{p}) . \tag{67}$$

Since the matrix element $\langle \Delta(\mathbf{p}') | M(\mathbf{r}) | N(\mathbf{p}) \rangle$ does not depend on $\mathbf{r}$ (all $\mathbf{r}$-dependence is contained in the exponential) we can substitute it by its value at $\mathbf{r} = 0$. We then carry out the $\mathbf{r}$ integration yielding $\delta(\mathbf{p} - \mathbf{p}' + \mathbf{k})$, and the above matrix element reads:

$$\int d\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}} \langle \Delta | M(\mathbf{r}) | N \rangle = (2\pi)^3 \int d\mathbf{p}\, \langle \Delta(\mathbf{p}+\mathbf{k}) | M(0) | N(\mathbf{p}) \rangle\ \varphi^*_\Delta(\mathbf{p}+\mathbf{k}) \varphi_N(\mathbf{p}) \ . \tag{68}$$

From the parameterization of the axial current (9) we can read off the p' and p dependence and plug it into (68). We neglect terms of the order p^2/M^2, e.g. the last term in the expression (11) for $e^\mu_\lambda(p)$. We find:

$$\bar{u}_\alpha(p', s_\Delta = \tfrac{1}{2}) q^\alpha u_N(s = \tfrac{1}{2}) = \left[\frac{M_\Delta - M_N}{M_\Delta} p'_3 - k \right] \sqrt{\frac{2}{3}} \tag{69}$$

and

$$\bar{u}_0(p', s_\Delta = \tfrac{1}{2}) u_N(s = \tfrac{1}{2}) = \frac{p'_3}{M_\Delta} \sqrt{\frac{2}{3}} \ . \tag{70}$$

We can carry out the integration over $\mathbf{p}$ since $C_i(q^2)$ do not depend on $\mathbf{p}$. We assume $\varphi_\Delta(\mathbf{p}) \approx \varphi_N(\mathbf{p}) \equiv \Pi_{i=1}^3 \varphi(p_i)$. A typical integral gives:

$$\begin{aligned} (2\pi)^3 \int d\mathbf{p}\, p_3\, \varphi(\mathbf{p}+\mathbf{k}) \varphi(\mathbf{p}) &= 2\pi \int dp_3\, p_3 \varphi(p_3 + k) \varphi(p_3) \\ &= 2\pi \int dq\, (q - \tfrac{1}{2}k)\, \varphi(q + \tfrac{1}{2}k) \varphi(q - \tfrac{1}{2}k) \\ &= -\tfrac{1}{2}k \left[1 - \tfrac{1}{2}k^2 \int dq\, \varphi'(q)^2 + \ldots \right] \\ &\approx -\tfrac{1}{2}k \left[1 - \tfrac{1}{2}k^2 \langle z^2_{\text{c.m.}} \rangle \right] \ , \end{aligned} \tag{71}$$

where we have taken into account that φ are normalized and used the relation ($\tilde{\varphi}(z)$ is the Fourier transform of $\varphi(q)$):

$$\int dq\, \varphi'(q)^2 = \int dz\, z^2 \tilde{\varphi}(z)^2 = \langle z^2 \rangle \ . \tag{72}$$

(Integrating p'_3 we would get $\frac{1}{2}k$.) Here $\langle z^2_{\text{c.m.}} \rangle = \frac{1}{3} \langle r^2_{\text{c.m.}} \rangle$ is a typical spread of the wave packet describing the center-of-mass motion of the localized state and is of the order of the inverse baryon mass. Clearly, in this approximation it is not meaningful to calculate the form-factor to very high k. We finally obtain (neglecting terms of the order k^2/M^2):

$$\tilde{S}^A = -\left[k\,\frac{M_\Delta}{M_N^2}\,C_4^A + \frac{k}{2M_\Delta}\,C_5^A - \frac{\omega k}{M_N^2}\,\frac{M_\Delta + M_N}{2M_\Delta}\,C_6^A\right]\sqrt{\frac{2}{3}}\,, \tag{73}$$

$$\tilde{A}^A_{\frac{3}{2}} = = -\left[\omega\,\frac{M_\Delta}{M_N^2}\,C_4^A + C_5^A\right] = \sqrt{3}\,\tilde{A}^A_{\frac{1}{2}}\,, \tag{74}$$

$$\tilde{L}^A = = -\left[\omega\,\frac{M_\Delta}{M_N^2}\,C_4^A + C_5^A - \frac{k^2}{M_N^2}\,\frac{M_\Delta + M_N}{2M_\Delta}\,C_6^A\right]\sqrt{\frac{2}{3}}\,. \tag{75}$$

We now express the experimental amplitudes in terms of the helicity amplitudes as

$$C_6^A = \frac{M_N^2}{k^2}\left[-\tilde{A}^A_{\frac{3}{2}} + \sqrt{\frac{3}{2}}\tilde{L}^A\right]\frac{2M_\Delta}{M_\Delta + M_N}\,, \tag{76}$$

$$C_5^A = -\sqrt{\frac{3}{2}}\left(\tilde{L}^A - \frac{k_0}{k}\,\tilde{S}^A\right)\frac{2M_\Delta}{M_\Delta + M_N} - \frac{k_0^2 - k^2}{M_N^2}\,C_6^A\,, \tag{77}$$

$$C_4^A = \frac{M_N^2}{kM_\Delta}\left[-\sqrt{\frac{3}{2}}\,\tilde{S}^A + \frac{k_0 k}{M_N^2}\,\frac{M_\Delta + M_N}{2M_\Delta}\,C_6^A\right] - \frac{M_N^2}{2M_\Delta^2}\,C_5^A\,. \tag{78}$$

The strong form-factor can be treated in the same way. The general coupling of the pion field to the baryon is written in the form

$$H_{B-\pi} = \int d\mathbf{r}\,J_a^\pi(\mathbf{r})\pi_a(\mathbf{r})\,, \tag{79}$$

where $J_a^\pi(\mathbf{r})$ is the baryon strong pseudoscalar- isovector current. The N-Δ transition matrix element is parameterized as

$$\langle\Delta^+(p')|J_a^\pi(0)|N^+(p)\rangle = -i\bar{u}_{\Delta\mu}\,\frac{G_{\pi N\Delta}(q^2)}{2M_N}\,q^\mu u_N\,, \tag{80}$$

where $q = p' - p$. Using (69) we find

$$\langle\Delta^+(p')|J_a^\pi(0)|N^+(p)\rangle = -i\frac{G_{\pi N\Delta}(q^2)}{2M_N}\left[\frac{M_\Delta - M_N}{M_\Delta}p'_3 - k\right]C^{\frac{3}{2}\frac{1}{2}}_{10\frac{1}{2}\frac{1}{2}}\,. \tag{81}$$

We now use of relation (68) as well as (71) to obtain

$$\frac{G_{\pi N\Delta}(q^2)}{2M_N}\,\frac{M_\Delta + M_N}{2M_\Delta} = \frac{1}{ik}\langle\Delta||\int d\mathbf{r}e^{i\mathbf{k}\cdot\mathbf{r}}J(\mathbf{r})||N\rangle\,. \tag{82}$$

6 Helicity amplitudes in the Cloudy Bag Model

The Cloudy Bag Model (CBM) is the simplest example of a quark model with the pion cloud that fulfills the virial constraints (52)-(54) provided we take the usual perturbative form for the pion profiles [8,9]. We also take the N-Δ splitting equal to the experimental value, $\omega \equiv M_\Delta - E_N$. Since the pion contribution to the axial current has the form of the pole term in (62), only the quarks contribute to the C_5^A and C_4^A amplitudes.

The helicity amplitudes and the Adler form-factors simplify further if we make the usual assumption of the same quark profiles for the nucleon and the Δ. In this case the scalar amplitude picks up only the pion contribution while the quark term is identically zero. The transverse amplitude $\tilde{A}^A_{\frac{3}{2}} = \sqrt{3}\,\tilde{A}^A_{\frac{1}{2}}$ has only the quark contribution while the longitudinal amplitude has both:

$$\tilde{A}^A_{\frac{3}{2}}(Q^2) = -\frac{1}{\sqrt{6}}\int dr\, r^2 \left[j_0(kr)\left(u^2 - \frac{1}{3}v^2\right) + \frac{2}{3}\, j_2(kr)v^2\right] \langle\Delta|| \sum \sigma\tau ||N\rangle, \quad (83)$$

$$\tilde{L}^A(Q^2) = -\frac{2}{3}\left\{\frac{1}{2}\int dr\, r^2 \left[j_0(kr)\left(u^2 - \frac{1}{3}v^2\right) - \frac{4}{3}\, j_2(kr)v^2\right] - \frac{\omega_{MIT}}{\omega_{MIT}-1}\,\frac{m_\pi}{2f_\pi}\,\frac{j_1(kR)}{kR}\,\frac{k^2}{(Q^2+m_\pi^2)}\right\} \langle\Delta|| \sum \sigma\tau ||N\rangle\,, \quad (84)$$

$$\tilde{S}^A(Q^2) = \frac{2}{3}\,\frac{\omega_{MIT}}{\omega_{MIT}-1}\,\frac{m_\pi}{2f_\pi}\,\frac{j_1(kR)}{kR}\,\frac{\omega k}{(Q^2+m_\pi^2)} \langle\Delta|| \sum \sigma\tau ||N\rangle\,. \quad (85)$$

Here k and $Q^2 \equiv -q^2$ are related through (4), $\omega_{MIT} = 2.04$, and

$$\langle\Delta|| \sum \sigma\tau ||N\rangle = \sqrt{Z_N Z_\Delta}\left\{ 2\sqrt{2} + \frac{\sqrt{2}}{27\pi}\,\mathcal{P}\int_0^\infty dk\, k^2\, \rho^2(k)\left[\frac{25}{\omega_k^2(\omega_k-\omega)} + \frac{2}{\omega_k(\omega_k^2-\omega^2)}\right] + \frac{25\sqrt{2}}{27\pi}\int_0^\infty dk\, k^2\, \rho^2(k)\left[\frac{5}{4\omega_k^3} + \frac{1}{\omega_k^2(\omega_k+\omega)}\right]\right\}, \quad (86)$$

where

$$\rho(k) = \frac{\omega_{MIT}}{\omega_{MIT}-1}\,\frac{j_1(kR)}{\sqrt{2\pi}\, f_\pi\, R^3}\,. \quad (87)$$

and Z_N and Z_Δ are the usual wave-function-renormalization constants [8].

The strong transition form-factor $G_{\pi N\Delta}(Q^2)$ is:

$$\frac{G_{\pi N\Delta}(Q^2)}{2M_N} = \frac{\omega_{MIT}}{\omega_{MIT}-1}\,\frac{1}{2f_\pi}\,\frac{j_1(kR)}{kR}\,\langle\Delta|| \sum \sigma\tau ||N\rangle\,\frac{2M_\Delta}{M_\Delta+M_N}\,. \quad (88)$$

Similarly as in (47) we can now explicitly show that PCAC is fulfilled provided the off-diagonal GT relation holds in the model. Since the Lagrangian is invariant under the chiral transformation both relation should hold for the exact solution, but this is of course not obvious for the approximate solution. In the model it is straightforward to evaluate the pertinent quantities at $Q^2 = -m_\pi^2$. We prefer to give here the expressions at $Q^2 = 0$ which take much simpler forms, e.g.:

$$C_5^A(0) = \frac{1}{\sqrt{6}}\int dr\, r^2 \left[j_0(kr)\left(u^2 - \frac{1}{3}v^2\right) - \frac{4}{3}\, j_2(kr)v^2\right] \langle\Delta|| \sum \sigma\tau ||N\rangle\,\frac{2M_\Delta}{M_\Delta+M_N}\,, \quad (89)$$

with $k = K_0$ (see (3).

The constant (89) can be easily evaluated for the degenerate N and Δ and neglecting pion corrections to $\langle\Delta\| \sum \sigma\tau \|N\rangle$:

$$C_5^A(0) = \frac{1}{\sqrt{6}} \frac{3g_A^\circ}{5} 2\sqrt{2} = 0.755 \;, \tag{90}$$

where $g_A^\circ = 1.09$ is the value of the nucleon g_A in the MIT bag model. Clearly, (90) strongly underestimates the experimental value (40). In the same limit, the strong coupling is

$$g_{\pi N\Delta} \equiv G_{\pi N\Delta}(0) \frac{m_\pi}{2M_N} = \sqrt{\frac{72}{25}}\, g_{\pi NN}^\circ = 1.39 \;. \tag{91}$$

Here $g_{\pi NN}^\circ = 0.82$ is the CBM value without pion correction. Again, (91) strongly underestimates the experimental value of 2.2, though the off-diagonal Goldberger-Treiman relation is exactly fulfilled in this approximation.

In [3] we show that the pion corrections improve the results in particular the ratio of the strong $g_{\pi N\Delta}$ and $g_{\pi NN}$ coupling constants but the value of $C_5(0)$ remains far below the experimental value. A possible solution, described and discussed in [3–5] is to include the contribution of the σ-meson which enters the expression for the axial current (61) and considerably increases the value of C_5^A.

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