ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 22 (2022) #P2.09 / 317–326
https://doi.org/10.26493/1855-3974.2577.25d
(Also available at http://amc-journal.eu)

Linkedness of Cartesian products of complete graphs*

Leif K. Jørgensen

Department of Mathematical Sciences, Aalborg University, Denmark

Guillermo Pineda-Villavicencio †, Julien Ugon ‡

Federation University, Ballarat, Australia and
School of Information Technology, Deakin University, Geelong, Australia

Received 9 March 2021, accepted 26 August 2021, published online 27 May 2022

Abstract

This paper is concerned with the linkedness of Cartesian products of complete graphs. A graph with at least $2k$ vertices is k*-linked* if, for every set of $2k$ distinct vertices organised in arbitrary k pairs of vertices, there are k vertex-disjoint paths joining the vertices in the pairs.

We show that the Cartesian product $K^{d_1+1} \times K^{d_2+1}$ of complete graphs K^{d_1+1} and K^{d_2+1} is $\lfloor (d_1+d_2)/2 \rfloor$-linked for $d_1, d_2 \geq 2$, and this is best possible.

This result is connected to graphs of simple polytopes. The Cartesian product $K^{d_1+1} \times K^{d_2+1}$ is the graph of the Cartesian product $T(d_1) \times T(d_2)$ of a d_1-dimensional simplex $T(d_1)$ and a d_2-dimensional simplex $T(d_2)$. And the polytope $T(d_1) \times T(d_2)$ is a *simple polytope*, a (d_1+d_2)-dimensional polytope in which every vertex is incident to exactly d_1+d_2 edges.

While not every d-polytope is $\lfloor d/2 \rfloor$-linked, it may be conjectured that every simple d-polytope is. Our result implies the veracity of the revised conjecture for Cartesian products of two simplices.

Keywords: k-linked, cyclic polytope, connectivity, dual polytope, linkedness, Cartesian product.

Math. Subj. Class. (2020): 05C40, 52B05

*The authors want to thank the referee for his/her comments, which have certainly helped to improve the presentation of the paper.

†Corresponding author. Guillermo would like to thank the hospitality of Leif Jørgensen, and the Department of Mathematical Sciences at Aalborg University, where this research started.

‡Julien Ugon's research was supported by the ARC discovery project DP180100602.

E-mail addresses: leifkjorgensen@gmail.com (Leif K. Jørgensen), work@guillermo.com.au (Guillermo Pineda-Villavicencio), julien.ugon@deakin.edu.au (Julien Ugon)

1 Introduction

Denote by $V(X)$ the vertex set of a graph. Given sets A, B of vertices in a graph, a path from A to B, called an $A-B$ *path*, is a (vertex-edge) path $L := u_0 \dots u_n$ in the graph such that $V(L) \cap A = \{u_0\}$ and $V(L) \cap B = \{u_n\}$. We write $a - B$ path instead of $\{a\} - B$ path, and likewise, write $A - b$ path instead of $A - \{b\}$.

Let G be a graph and X a subset of $2k$ distinct vertices of G. The elements of X are called *terminals*. Let $Y := \{\{s_1, t_1\}, \dots, \{s_k, t_k\}\}$ be an arbitrary labelling and (unordered) pairing of all the vertices in X. We say that Y is *linked* in G if we can find disjoint $s_i - t_i$ paths for $i \in [1, k]$, the interval $1, \dots, k$. The set X is *linked* in G if every such pairing of its vertices is linked in G. Throughout this paper, by a set of disjoint paths, we mean a set of vertex-disjoint paths. If G has at least $2k$ vertices and every set of exactly $2k$ vertices is linked in G, we say that G is *k-linked*.

This paper studies the linkedness of Cartesian products of complete graphs. Linkedness of Cartesian products has been studied in the past [4]. The *Cartesian product* $G_1 \times G_2$ of two graphs G_1 and G_2 is the graph defined on the pairs (v_1, v_2) with $v_i \in G_i$ and with two pairs (u_1, u_2) and (v_1, v_2) being adjacent if, for some $\ell \in \{1, 2\}$, $u_\ell v_\ell \in E(G_\ell)$ and $u_i = v_i$ for $i \neq \ell$. We prove that the Cartesian product $K^{d_1+1} \times K^{d_2+1}$ of complete graphs K^{d_1+1} and K^{d_2+1} is $\lfloor (d_1 + d_2)/2 \rfloor$-linked for $d_1, d_2 \geq 0$, and that there are products that are not $\lfloor (d_1 + d_2 + 1)/2 \rfloor$-linked; hence this result is best possible. Here K^t denotes the complete graph on t vertices.

Our result is connected to questions on the linkedness of a polytope. A (convex) polytope is the convex hull of a finite set X of points in $\mathbb{R}^d$; the *convex hull* of X is the smallest convex set containing X. The *dimension* of a polytope in $\mathbb{R}^d$ is one less than the maximum number of affinely independent points in the polytope; a set of points $\vec{p}_1, \dots, \vec{p}_k$ in $\mathbb{R}^d$ is *affinely independent* if the $k - 1$ vectors $\vec{p}_1 - \vec{p}_k, \dots, \vec{p}_{k-1} - \vec{p}_k$ are linearly independent. A polytope of dimension d is referred to as a *d-polytope*.

The *Cartesian product* $P \times P'$ of a d-polytope $P \subset \mathbb{R}^d$ and a d'-polytope $P' \subset \mathbb{R}^{d'}$ is the Cartesian product of the sets P and P':

$$P \times P' = \left\{ \begin{pmatrix} p \\ p' \end{pmatrix} \in \mathbb{R}^{d+d'} \,\middle|\, p \in P,\, p' \in P \right\}.$$

The resulting polytope is $(d + d')$-dimensional. The *graph* $G(P)$ of a polytope P is the undirected graph formed by the vertices and edges of the polytope. It follows that the graph $G(P \times P')$ of the Cartesian product $P \times P'$ is the Cartesian product $G(P) \times G(P')$ of the graphs $G(P)$ and $G(P')$.

A d-simplex $T(d)$ is the convex hull of $d + 1$ affinely independent points in $\mathbb{R}^d$. The graph of $T(d)$ is the complete graph K^{d+1}. As a consequence, our result implies that the graph of the Cartesian product $T(d_1) \times T(d_2)$ is $\lfloor (d_1 + d_2)/2 \rfloor$-linked for $d_1, d_2 \geq 0$. Henceforth, if the graph of a polytope is k-linked we say that the polytope is also *k-linked*.

The first edition of the Handbook of Discrete and Computational Geometry [3, Problem 17.2.6] posed the question of whether or not every d-polytope is $\lfloor d/2 \rfloor$-linked. This question was answered in the negative by [2]. None of the known counterexamples are *simple d-polytopes*, d-polytopes in which every vertex is incident to exactly d edges. Hence, it may be hypothesised that the conjecture holds for such polytopes.

Conjecture 1.1. *Every simple d-polytope is $\lfloor d/2 \rfloor$-linked for $d \geq 2$.*

Cartesian products of simplices are simple polytopes, and so our result supports this revised conjecture. Furthermore, Cartesian products of simplices and duals of cyclic polytopes are related; the dual of a cyclic d-polytope with $d+2$ vertices is the Cartesian product of a $\lfloor d/2 \rfloor$-simplex and a $\lceil d/2 \rceil$-simplex [6, Example 0.6]. Hence we obtain that the dual of a cyclic d-polytope on $d+2$ vertices is also $\lfloor d/2 \rfloor$-linked for $d \geq 2$.

Unless otherwise stated, the graph theoretical notation and terminology follows from [1] and the polytope theoretical notation and terminology from [6]. Moreover, when referring to graph-theoretical properties of a polytope such as linkedness and connectivity, we mean properties of its graph.

2 Linkedness of Cartesian products of complex graphs

The contribution of this section is a sharp theorem (Theorem 2.1) that tells the story of the linkedness of Cartesian product of two complete graphs.

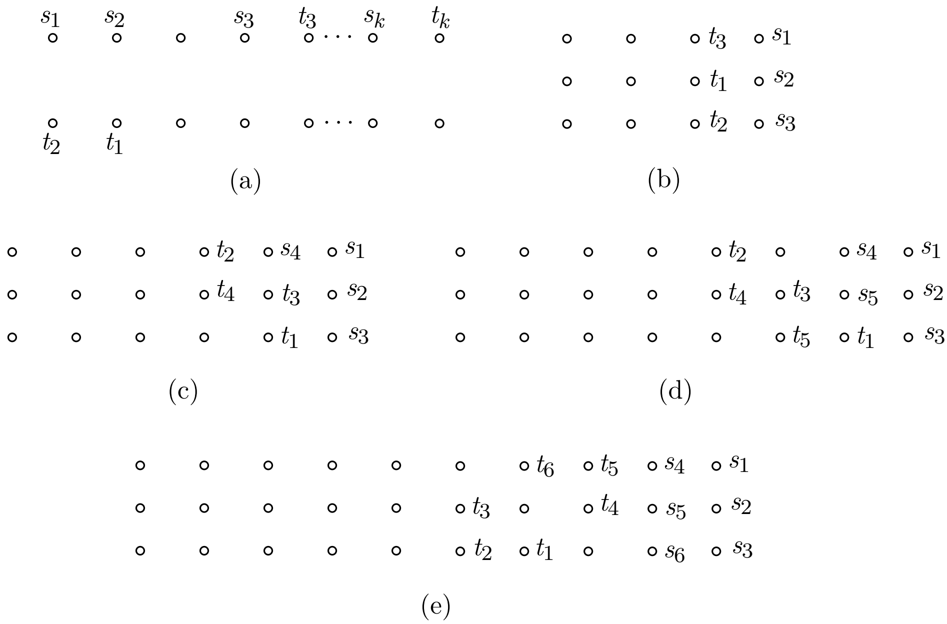

Figure 1: No feasible linkage problems for $K^{d_1+1} \times K^{d_2+1}$, $k = \lfloor (d_1+d_2+1)/2 \rfloor$, $d_1 \leq 2$ and $d_2 > d_1$. (a) The case $d_1 = 1$ and even d_2 with $d_2 > d_1$. (b) The case $d_1 = 2$ and $d_2 = 3$. (c) The case $d_1 = 2$ and $d_2 = 5$. (d) The case $d_1 = 2$ and $d_2 = 7$. (e) The case $d_1 = 2$ and $d_2 = 9$. Each row of each part (a)-(e) is a complete graph whose edges have not been drawn.

Theorem 2.1. *The Cartesian product of two complete graphs K^{d_1+1} and K^{d_2+1} is $\lfloor (d_1+d_2)/2 \rfloor$-linked for every $d_1, d_2 \geq 0$.*

Remark 2.2. Theorem 2.1 is best possible. There are products $K^{d_1+1} \times K^{d_2+1}$ that are not $\lfloor (d_1+d_2+1)/2 \rfloor$-linked:

1. $K^2 \times K^{d_2+1}$ for even $d_2 \geq 1$, and

2. $K^3 \times K^{d_2+1}$ for $d_2 = 1, 3, 5, 7, 9$.

For each of these cases, Figure 1 provides a pairing of terminals that cannot be $\lfloor (d_1 + d_2 + 1)/2 \rfloor$-linked. We conjecture these are the only such cases.

An immediate corollary of Theorem 2.1 is the following.

Corollary 2.3. *The Cartesian product of two simplices $T(d_1)$ and $T(d_2)$ is $\lfloor (d_1 + d_2)/2 \rfloor$-linked for every $d_1, d_2 \geq 0$.*

The notions of linkage, linkage problem, and valid path will simplify our arguments. A *linkage* in a graph is a subgraph in which every component is a path. Let X be a set of vertices in a graph and let $Y := \{\{s_1, t_1\}, \dots, \{s_k, t_k\}\}$ be a pairing of all the vertices of X. A *Y-linkage* $\{L_1, \dots, L_k\}$ is a set of disjoint paths with the path L_i joining the pair $\{s_i, t_i\}$ for $i = 1, \dots, k$. We may also say that Y represents our *linkage problem*, and if Y is linked in G then our linkage problem is *feasible* and *infeasible* otherwise. A path in the graph is called *X-valid* if no inner vertex of the path is in X. Let X be a set of vertices in a graph G. Denote by $G[X]$ the subgraph of G induced by X, the subgraph of G that contains all the edges of G with vertices in X. Write $G - X$ for $G[V(G) \setminus X]$.

Consider a linkage problem $Y := \{\{s_1, t_1\}, \dots, \{s_k, t_k\}\}$ on a set X of $2k$ vertices in a graph G. Consider a linkage $\mathcal{L}$ from a subset Z of X to some set Z' disjoint from X and label the vertices of Z' such that the path in $\mathcal{L}$ with end $z_i \in Z$ has its other end $z_i' \in Z'$. Then the linkage $\mathcal{L}$ in G *induces* a linkage problem Y' in $(G - V(\mathcal{L})) \cup Z'$ where the vertices of $X \setminus Z$ remain and the vertices of Z have been replaced by the vertices of Z'. Slightly abusing terminology, we also call terminals the vertices of Z'. If the problem Y' is feasible in $(G - V(\mathcal{L})) \cup Z'$, so is the problem Y in G.

Since we make heavy use of Menger's theorem [1, Theorem. 3.3.1], we next remind the reader of one of its consequences.

Theorem 2.4 (Menger's theorem)**.** *Let G be a k-connected graph, and let A and B be two subsets of its vertices, each of cardinality at least k. Then there are k disjoint $A - B$ paths in G.*

We fix some notation and terminology for the remaining of the section. Let G denote the graph $K^{d_1+1} \times K^{d_2+1}$. We think of $G = K^{d_1+1} \times K^{d_2+1}$ as a grid with $d_1 + 1$ rows and $d_2 + 1$ columns. In this way, the entry in Row i and Column j can be referred to as $G[i, j]$.

When we write about a row r of subgraph G' of G, we think of r as a subgraph of G' and as the number r so that we can write about the rth row of G' or G; this ambiguity should cause no confusion. An entry in the grid $K^{d_1+1} \times K^{d_2+1}$ with no terminal is said to be *free*, as is a row or a column of a subgraph of G with no terminal. A row or a column of a subgraph of G with every entry being occupied by a terminal is said to be *full*.

We need the following induced subgraphs of G:

$C_{ab\dots z}$, the subgraph formed by the union of Columns $a, b, \dots, z$;

$\bar{C}_{ab\dots z}$, the subgraph obtained by removing Columns $a, b, \dots, z$;

$R_{ab\dots z}$, the subgraph formed by the union of Rows $a, b, \dots, z$;

$\bar{R}_{ab\dots z}$, the subgraph obtained by removing Rows $a, b, \dots, z$;

A_α, the induced subgraph of $\bar{C}_{12}$ obtained by removing its first α rows; and

B_α, the subgraph of C_{12} obtained by removing its first α rows.

For instance, $\bar{C}_1$ denotes the subgraph of G obtained by removing the first column, C_{12} the subgraph formed by the first two columns of G, and $\bar{C}_{12}$ denotes the subgraph obtained by removing the first two columns of G; observe $\bar{C}_{12}$ is isomorphic to $K^{d_1+1} \times K^{d_2-1}$. Figure 2 depicts some of the aforementioned subgraphs of $K^{d_1+1} \times K^{d_2+1}$.

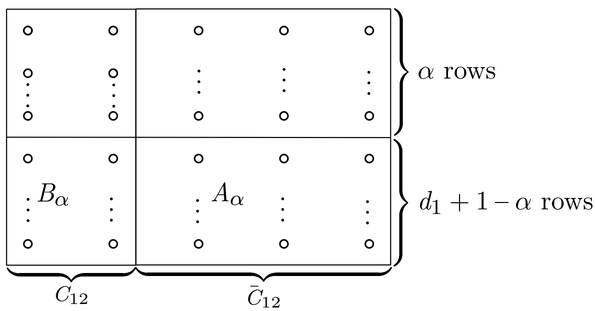

Figure 2: Depiction of the subgraphs B_α, A_α, C_{12}, and $\bar{C}_{12}$ of $K^{d_1+1} \times K^{d_2+1}$.

The connectivity of $K^{d_1+1} \times K^{d_2+1}$ is stated below.

Lemma 2.5 (Špacapan [5, Theorem 1]). *The (vertex)connectivity of $K^{d_1+1} \times K^{d_2+1}$ is precisely $d_1 + d_2$.*

We continue fixing further notation. Henceforth let $k := \lfloor (d_1 + d_2)/2 \rfloor$. And let X be a subset of $2k$ vertices of G and let $Y := \{\{s_1, t_1\}, \ldots, \{s_k, t_k\}\}$ be a pairing of all the vertices in X.

We first settle the simple cases of $(0, d_2)$ and $(1, d_2)$ for $d_2 \geq 0$.

Proposition 2.6 (Base cases). *For $d_2 \geq 0$ the Cartesian products $K^1 \times K^{d_2+1}$ and $K^2 \times K^{d_2+1}$ are both $\lfloor (1 + d_2)/2 \rfloor$-linked. This statement is best possible.*

Proof. The lemma is true for the pair $(0, d_2)$ for each $d_2 \geq 0$, since $K^1 \times K^{d_2+1} = K^{d_2+1}$ and K^{d_2+1} is $\lfloor (1 + d_2)/2 \rfloor$-linked. This is best possible.

The graph $K^2 \times K^{d_2+1}$ is $(1 + d_2)$-connected by Lemma 2.5. Use Menger's theorem (Theorem 2.4) to bring the $1 + d_2$ terminals to the subgraph $\bar{R}_1$ through a linkage $\{S_1, \ldots, S_k, T_1, \ldots, T_k\}$ with $S_i := s_i - \bar{R}_1$ and $T_i := t_i - \bar{R}_1$ for $i \in [1, k]$. Letting $\{\bar{s}_i\} := V(S_i) \cap V(\bar{R}_1)$ and $\{\bar{t}_i\} := V(T_i) \cap V(\bar{R}_1)$, we produce a new linkage problem $Y' := \{\{\bar{s}_1, \bar{t}_1\}, \ldots, \{\bar{s}_k, \bar{t}_k\}\}$ in $\bar{R}_1$ whose feasibility implies that of Y in G. To solve Y' link the pairs of Y' in the subgraph $\bar{R}_1$, which is isomorphic to K^{d_2+1}, using the $\lfloor (1 + d_2)/2 \rfloor$-linkedness of K^{d_2+1}. For even even d_2, Figure 1(a) shows an infeasible linkage problem with $\lfloor (2 + d_2)/2 \rfloor$ pairs in the graph $K^2 \times K^{d_2+1}$. $\square$

In what follows we aim to find a Y-linkage $\{L_1, \ldots, L_k\}$ in G with L_i joining the pair $\{s_i, t_i\}$ of Y for $i \in [1, k]$. Our proof is by induction on (d_1, d_2) with the base cases settled in Proposition 2.6. If there is a pair of Y, say $\{s_1, t_1\}$, lying in some column or row of G, say in Column 1, we send every terminal $s_i \in C_1$ that is different from s_1 and t_1 and that is not adjacent to t_i to the subgraph $\bar{C}_1$, and apply the induction hypothesis on $\bar{C}_1$. Otherwise, we may assume every pair of Y lies in two distinct columns or rows, say the pair $\{s_1, t_1\}$ lies in C_{12}; then we send every terminal $s_i \in C_{12}$ that is different from s_1 and

t_1 and that is not adjacent to t_i to the subgraph $\bar{C}_{12}$, and apply the induction hypothesis to $\bar{C}_{12}$. We develop these ideas below.

The definition of k-linkedness gives the following lemma at once; we will use it implicitly hereafter.

Lemma 2.7. *Let $\ell \leq k$. Let X be a set of 2ℓ distinct vertices of a k-linked graph K, let Y be a labelling and pairing of the vertices in X, and let Z be a set of $2k - 2\ell$ vertices in K such that $X \cap Z = \emptyset$. Then there exists a Y-linkage in K that avoids every vertex in Z.*

Besides, basic algebraic manipulation yields the following inequality.

Lemma 2.8. *If $x \geq 2$ and $y \geq 2$ then $x(y-1) > x + y - 3$.*

Proof. The inequality simplifies to $(x-1)(y-2) > -1$. □

We are now ready to put together all the elements of the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $k := \lfloor (d_1 + d_2)/2 \rfloor$. Then $d_1 + d_2 \geq 2k$.

Proposition 2.6 gives the result for the pairs $(d_1, 0)$, $(0, d_2)$, $(d_1, 1)$, and $(1, d_2)$ for each $d_1, d_2 \geq 0$. Hence, our bidimensional induction on (d_1, d_2) can start with the assumption of $d_1, d_2 \geq 2$.

We first deal with the case where a pair in Y, say $\{s_1, t_1\}$, lies in some column or some row of G, say in Column 1.

Case 1. A pair in Y, say $\{s_1, t_1\}$, lies in Column 1.

The induction hypothesis ensures that the subgraph $\bar{C}_1$ is $(k-1)$-linked. Hence it suffices to show that all the terminals in C_1 other than s_1, t_1 can be moved to $\bar{C}_1$ via a linkage; Menger's theorem (Theorem 2.4) guarantees this.

Let U be the set of terminals in C_1 other than s_1 and t_1, and let W be the set of terminals in $\bar{C}_1$. Then $|U| + |W| \leq d_1 + d_2 - 2$, as $|U| + |W| = 2k - 2$ and $2k \leq d_1 + d_2$. Besides, the subgraph $G - (W \cup \{s_1, t_1\})$ is $|U|$-connected, as G is $(d_1 + d_2)$-connected (Lemma 2.5). In the case of $d_1, d_2 \geq 2$, Lemma 2.8 yields that $\bar{C}_1$ has more than $|U \cup W|$ vertices:

$$|\bar{C}_1| = (d_1 + 1)d_2 > d_1 + 1 + d_2 + 1 - 3 > d_1 + d_2 - 2 = |U| + |W|.$$

Use Menger's theorem (Theorem 2.4) to bring the $|U|$ terminals in C_1 to the subgraph $\bar{C}_1$ through a linkage Y_U. For every path L in Y_U, if $s_i \in L$, let $\{\bar{s}_i\} := V(L) \cap V(\bar{C}_1)$ and if $t_i \in L$ let $\{\bar{t}_i\} := V(L) \cap V(\bar{C}_1)$. For $s_i \in W$ (respectively $t_i \in W$) let $\bar{s}_i = s_i$ (respectively $\bar{t}_i = t_i$). This produces a new linkage problem $Y' := \{\{\bar{s}_2, \bar{t}_2\}, \ldots, \{\bar{s}_k, \bar{t}_k\}\}$ in $\bar{C}_1$ whose feasibility implies that of Y in G, since s_1 and t_1 are adjacent in C_1. The $(k-1)$-linkedness of $\bar{C}_1$ now settles the case.

By symmetry, we can assume that every pair $\{s_i, t_i\}$ in Y lies in two different columns or rows and that s_i, t_i are not adjacent. Without loss of generality, assume that

$$s_1 \text{ is in Column 1 and } t_1 \text{ is in Column 2 of } C_{12}. \tag{$*$}$$

The induction hypothesis also ensures that both $\bar{C}_{12}$ and $\bar{R}_{12}$ are $(k-1)$-linked. We consider two further cases based on the number of terminals in C_{12} or R_{12}.

Case 2. The subgraph C_{12} contains precisely $d_1 + 2 - \alpha$ terminals, including $\{s_1, t_1\}$, where $0 \leq \alpha \leq d_1$.

Excluding $\{s_1, t_1\}$, there are at most d_1 terminals in C_{12}, and there are d_1+1 internally-disjoint $s_1 - t_1$ paths in C_{12} of length at most three: two length-two paths and $d_1 - 1$ length-three paths. One of these $s_1 - t_1$ paths, say L_1, avoids every other terminal in C_{12}.

Without loss of generality, assume that Row 1 in C_{12} is part of the path L_1; that is,

$$\{G[1,1], G[1,2]\} \subseteq V(L_1). \tag{$**$}$$

In the subcase $\alpha = d_1$, every pair in $Y \setminus \{s_1, t_1\}$ is in $\bar{C}_{12}$, and the induction hypothesis on $\bar{C}_{12}$ settles the subcase.

Suppose that $\alpha = d_1 - 1$, say C_{12} contains $\{s_1, t_1, s_2\}$. Then $s_2 \in B_1$ and $t_2 \in \bar{C}_{12}$. We may assume s_1, s_2 are in Column 1 and t_1 is in Column 2. We show there is an X-valid $s_2 - A_1$ path L_2' such that the vertex $\bar{s_2} \in V(L_2') \cap V(A_1)$ is either t_2 or a nonterminal.

Through each entry of Column 1 of B_1, there are $d_2 - 1$ paths form s_2 to A_1 of length at most two (one for each column in A_1). Moreover, there are at least $d_1 - 1$ free entries in Column 1 of B_1. Therefore, to ensure the existence of L_2', we need to show that at least one of these $(d_1 - 1)(d_2 - 1)$ paths from s_2 to A_1 either contains t_2 or a nonterminal in A_1. Indeed, according to Lemma 2.8, the inequality

$$(d_1 - 1)(d_2 - 1) > d_1 - 1 + d_2 - 3 \geq |X \setminus \{s_1, t_1, s_2, t_2\}|$$

holds for $d_1, d_2 \geq 2$. Hence we get the existence of L_2'. As a result, the solution of the new problem $Y' := \{\{\bar{s_2}, t_2\}, \{s_3, t_3\}, \dots, \{s_k, t_k\}\}$ in $\bar{C}_{12}$ induces a solution of the problem Y in G. And the solution of Y' follows from the $(k-1)$-linkedness of $\bar{C}_{12}$.

Henceforth assume that $\alpha \leq d_1 - 2$. To finalise Case 2, we require a couple of claims.

Claim 2.9. *Suppose that there are at most $d_1 + 2 - \alpha$ terminals in $B_{\alpha+1} = K^{d_1-\alpha} \times K^2$. Then there is an injection from the set of rows of $B_{\alpha+1}$ that contain two terminals x_1, x_2 such that $\{x_1, x_2\} \cap \{s_1, t_1\} = \emptyset$ to the set of rows of $B_{\alpha+1}$ that contain no terminal other than possibly s_1 and t_1.*

Proof. This follows from a simple counting argument. The number of rows in $B_{\alpha+1}$ is $d_1 - \alpha$. Let m denote the number of rows of $B_{\alpha+1}$ that contain two terminals x_1, x_2 such that $\{x_1, x_2\} \cap \{s_1, t_1\} = \emptyset$ and let $n := |(X \cap V(B_{\alpha+1})) \setminus \{s_1, t_1\}|$; that is, n counts the total number of terminals in $B_{\alpha+1}$ other than s_1 and t_1. It follows that the number of rows of $B_{\alpha+1}$ that contain precisely one terminal $x \notin \{s_1, t_1\}$ is $n - 2m$; either s_1 or t_1 may be in these rows. As a result, the number of rows of $B_{\alpha+1}$ that contain no terminal other than $\{s_1, t_1\}$ is $d_1 - \alpha - m - (n - 2m)$. Combining $n \leq d_1 - \alpha$ with all these numbers, we get that

$$d_1 - \alpha - m - (n - 2m) = d_1 - \alpha - n + m \geq d_1 - \alpha - (d_1 - \alpha) + m = m.$$

The claim is proved. $\square$

Claim 2.10. *Suppose that there are at most $d_1 + 2 - \alpha$ terminals in $B_{\alpha+1} = K^{d_1-\alpha} \times K^2$. If every row in the subgraph $A_{\alpha+1} = K^{d_1-\alpha} \times K^{d_2-1}$ of $\bar{C}_{12}$ has a free entry, then, for every terminal $x \notin \{s_1, t_1\}$ in $B_{\alpha+1}$, there is an X-valid $x - A_{\alpha+1}$ path L to a free entry in $A_{\alpha+1}$; and all these X-valid paths are disjoint.*

Proof. If a row of $B_{\alpha+1}$ contains exactly one terminal $x \notin \{s_1, t_1\}$, then send x to a free entry in the same row of $A_{\alpha+1}$. Let x_1 and x_2 be two terminals in $B_{\alpha+1}$ that satisfy

$\{x_1, x_2\} \cap \{s_1, t_1\} = \emptyset$ and occupy a row r_f of $B_{\alpha+1}$. From Claim 2.9 ensues the existence of a row r_e of $B_{\alpha+1}$ that contain no terminal other than possibly s_1 and t_1; in short, there is at least a free entry in r_e.

Consider a pair (r_f, r_e) of rows granted by Claim 2.9. Send either x_1 or x_2, say x_1, to the free entry in the row r_e of $A_{\alpha+1}$ passing through the corresponding free entry in the row r_e of $B_{\alpha+1}$, and send x_2 to a free entry in the row r_f of $A_{\alpha+1}$. The proof of the claim is now complete. □

Now suppose that $\alpha = 0$ or $2 \leq \alpha \leq d_1 - 2$. In this subcase, the subgraph $\bar{C}_{12}$ contains at most α full rows: if $\alpha + 1$ rows were full in $\bar{C}_{12}$ then there would be at least $(\alpha+1)(d_2-1)$ terminals in $\bar{C}_{12}$ but $(\alpha+1)(d_2-1) > d_2 - 2 + \alpha$ (Lemma 2.8). Even when the path L_1 uses the first row of C_{12} by $(**)$, there is no loss of generality by assuming that the full rows of $\bar{C}_{12}$ are among the *first* $\alpha + 1$ *rows* of $\bar{C}_{12}$. It follows that every row of $A_{\alpha+1}$ has a free entry.

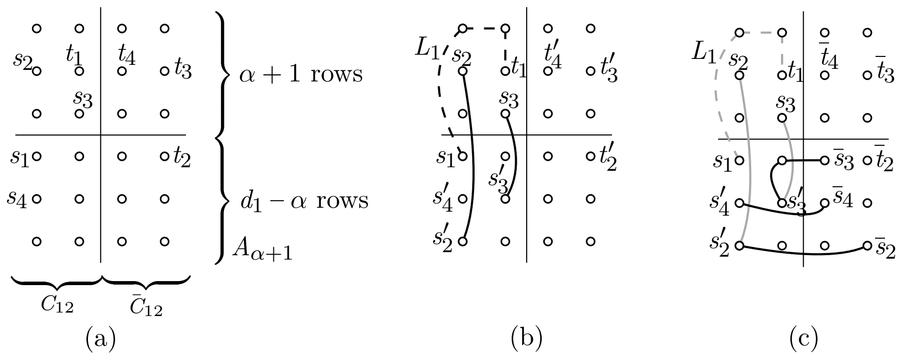

Figure 3: Auxiliary figure for Case 2 (a) This shows a scenario where $d_1 = 5$, $d_2 = 3$, and $\alpha = 2$. (b) The path $L_1 = s_1 - t_1$ in dashed line, the paths that send the terminals in $B_1 \setminus B_3$ other than s_1 and t_1 to B_3, and the resulting new linkage $Y' = \{\{s'_2, t'_2\}, \{s'_3, t'_3\}, \{s'_4, t'_4\}\}$ in $\bar{C}_{12} \cup B_{\alpha+1}$. (c) The paths that send the terminals in B_3 to A_3, and the resulting new linkage $Y'' = \{\{\bar{s}_2, \bar{t}_2\}, \{\bar{s}_3, \bar{t}_3\}, \{\bar{s}_4, \bar{t}_4\}\}$ in $\bar{C}_{12}$.

Next we show how to send to $B_{\alpha+1}$ the terminals other than s_1 and t_1 that are in the rows 2 to $\alpha + 1$ of C_{12}; that is, the terminals other than s_1 and t_1 that are in $B_1 \setminus B_{\alpha+1}$. For $\alpha = 0$, $B_1 \setminus B_{\alpha+1} = \emptyset$ and there is nothing to do. We now focus on the subcase $2 \leq \alpha \leq d_1 - 2$. Let n_1 and n_2 denote the number of terminals in $B_1 \setminus B_{\alpha+1}$ and $B_{\alpha+1}$, respectively. Then the following inequalities hold

$$
\begin{aligned}
n_1 + n_2 &\leq d_1 + 2 - \alpha \leq d_1 \quad (\text{since } 2 \leq \alpha),\\
n_1 + n_2 &\leq d_1 + 2 - \alpha \leq 2d_1 - 2\alpha = |V(B_{\alpha+1})| \quad (\text{since } \alpha \leq d_1 - 2).
\end{aligned}
$$

From the second inequality, it follows that there are at least n_1 free vertices in $B_{\alpha+1}$. Since B_1 is d_1-connected by Lemma 2.5, Menger's theorem gives n_1 disjoint paths in B_1 from the terminals in $B_1 \setminus B_{\alpha+1}$ to n_1 free entries in $B_{\alpha+1}$, avoiding the n_2 terminals in $B_{\alpha+1}$. For a terminal s_i in $B_1 \setminus B_{\alpha+1}$, let L'_i be the path from s_i to $B_{\alpha+1}$ and let $s'_i := V(L'_i) \cap B_{\alpha+1}$. Define t'_i similarly for a terminal t_i in $B_1 \setminus B_{\alpha+1}$. Furthermore, for s_i (respectively, t_i) in

$B_{\alpha+1} \cup \bar{C}_{12}$, let $s'_i := s_i$ (respectively, $t'_i := t_i$). This produces a new linkage problem $Y' := \{\{s'_2, t'_2\}, \dots, \{s'_k, t'_k\}\}$ in $\bar{C}_{12} \cup B_{\alpha+1}$. See Figure 3(b).

There are at most $d_1 + 2 - \alpha$ terminals in $B_{\alpha+1} = K^{d_1-\alpha} \times K^2$, and every row in $A_{\alpha+1} = K^{d_1-\alpha} \times K^{d_2-1}$ has a free entry. Hence, Claim 2.10 applies, and there is a linkage formed by X-valid paths from the terminals in $B_{\alpha+1}$, other than s_1 and t_1, to free entries in $A_{\alpha+1}$. For every such path L''_i, if $s'_i \in V(L''_i) \cap V(B_{\alpha+1})$, let $\{\bar{s}_i\} := V(L''_i) \cap V(A_{\alpha+1})$, and if $t'_i \in V(L''_i) \cap V(B_{\alpha+1})$, let $\{\bar{t}_i\} := V(L''_i) \cap V(A_{\alpha+1})$. Besides, for $s'_i \in \bar{C}_{12}$ (respectively $t'_i \in \bar{C}_{12}$), let $\bar{s}_i = s'_i$ (respectively, $\bar{t}_i = t'_i$). This produces a new linkage problem $Y'' := \{\{\bar{s}_2, \bar{t}_2\}, \dots, \{\bar{s}_k, \bar{t}_k\}\}$ in $\bar{C}_{12}$ whose feasibility implies that of Y', and therefore that of Y in G, by completing each linkage problem with the path L_1. See Figure 3(c).

Now we have a new linkage problem Y'' in $\bar{C}_{12}$ with $(k-1)$ pairs. The solution of Y'' in $\bar{C}_{12}$ implies a solution of the linkage problem Y in G. To link the pairs of Y'' use the $(k-1)$-linkedness of $\bar{C}_{12}$.

Finally assume that $\alpha = 1$. Then there are exactly $d_1 + 1$ terminals in C_{12} and at most $d_2 - 1$ terminals in $\bar{C}_{12}$. In a first scenario suppose that either both entries in $B_1 \setminus B_2$ are nonterminals or each terminal other than s_1 and t_1 in $B_1 \setminus B_2$ is adjacent to a nonterminal in B_2. Then we can send these terminals in $B_1 \setminus B_2$ to B_2. In the second scenario, suppose that there is a terminal s_i ($i \neq 1$) in $B_1 \setminus B_2$ whose neighbours in B_2 are all terminals. Then the column of s_i in B_1 would contain exactly d_1 terminals, including s_i. We send s_i to a free entry in A_1, in the same row as s_i (the first row of A_1): if this free entry didn't exist, then s_i would be adjacent to the $d_2 - 1$ terminals in A_1 and the $d_1 - 1$ terminals in B_2. Since there are $d_1 + d_2$ terminals in total, it would follow that s_i is adjacent to t_i. This contradiction shows that we can send s_i to a free entry in A_1.

In both scenarios, it remains to send the terminals other than s_1 and t_1 in $B_2 = K^{d_1-1} \times K^2$ to $A_2 = K^{d_1-1} \times K^{d_2-1}$. To do so, we reason as in the subcase $2 \leq \alpha \leq d_1 - 2$. It follows that there are at most $d_1 + 2 - 1$ terminals in B_2, and that every row in A_2 has a free entry. Claim 2.10 applies again and gives a linkage formed by X-valid paths from the terminals in B_2, other than s_1, t_1, to free entries in A_2.

With all the terminals other than s_1 and t_1 in $\bar{C}_{12}$, therein we have a new linkage problem Y' with $k-1$ pairs whose solution in $\bar{C}_{12}$ implies a solution of the linkage problem Y in G. To solve Y' in $\bar{C}_{12}$ use the $(k-1)$-linkedness of $\bar{C}_{12}$.

By symmetry, we also have the result if there are at most $d_2 + 2$ terminals in R_{12}, including $\{s_1, t_1\}$.

Case 1. The subgraph C_{12} contains at least $d_1 + 3$ terminals, including $\{s_1, t_1\}$.

This case reduces to the previous case. If C_{12} contains at least $d_1 + 3$ terminals then R_{12} contains at most $d_2 - 3 + 4 = d_2 + 1$ terminals, since there are four entries shared by C_{12} and R_{12}. Because we make no distinction between columns and rows, this case is already covered. This completes the proof of the theorem. $\square$

3 Duals of cyclic polytopes

There is a close connection between duals of cyclic d-polytopes with $d+2$ vertices and Cartesian products of complete graphs.

The *moment curve* in $\mathbb{R}^d$ is defined by $x(t) := (t, t^2, \dots, t^d)$ for $t \in \mathbb{R}$, and the convex hull of any $n > d$ points on it gives a *cyclic polytope* $C(n, d)$. The *combinatorics* of a cyclic polytope, the face lattice of the polytope faces partially ordered by inclusion, is independent of the points chosen on the moment curve. Hence we talk of the cyclic d-polytope on n vertices [6, Example 0.6].

For a polytope P that contains the origin in its interior, the *dual polytope* P^* is defined as

$$P^* = \{y \in \mathbb{R}^d \mid x \cdot y \leq 1 \text{ for all } x \text{ in } P\}.$$

If P does not contain the origin, we translate the polytope so that it does. Translating the polytope P changes the geometry of P^* but not its face lattice. The face lattice of P^* is the inclusion reversed face lattice of P. In particular, the vertices of P^* correspond to the facets of P, and the edges of P^* correspond to the $(d-2)$-faces of P. The *dual graph* of a polytope P is the graph of the dual polytope, or equivalently, the graph on the set of facets of P where two facets are adjacent in the dual graph if they share a $(d-2)$-face.

Duals of cyclic d-polytopes are simple d-polytopes. It is also the case that the dual of a cyclic d-polytope with $d+2$ vertices can be expressed as $T(\lfloor d/2 \rfloor) \times T(\lceil d/2 \rceil)$ ([6, Example 0.6]). From this observation and Theorem 2.1 the next corollary follows at once.

Corollary 3.1. *Duals of cyclic polytopes with $d+2$ vertices are $\lfloor d/2 \rfloor$-linked for every $d \geq 2$.*

ORCID iDs

Leif K. Jørgensen https://orcid.org/0000-0003-4922-3937
Guillermo Pineda-Villavicencio https://orcid.org/0000-0002-2904-6657
Julien Ugon https://orcid.org/0000-0001-5290-8051

References

[1] R. Diestel, *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*, Springer-Verlag, Berlin, 5th edition, 2017, doi:10.1007/978-3-662-53622-3.

[2] S. Gallivan, Disjoint edge paths between given vertices of a convex polytope, *J. Comb. Theory Ser. A* **39** (1985), 112–115, doi:10.1016/0097-3165(85)90086-x.

[3] J. E. Goodman and J. O'Rourke (eds.), *Handbook of Discrete and Computational Geometry*, Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, 1997, `http://www.csun.edu/~ctoth/Handbook/HDCG3.html`.

[4] G. Mészáros, On linkedness in the Cartesian product of graphs, *Period. Math. Hungar.* **72** (2016), 130–138, doi:10.1007/s10998-016-0113-8.

[5] S. Špacapan, Connectivity of Cartesian products of graphs, *Appl. Math. Lett.* **21** (2008), 682–685, doi:10.1016/j.aml.2007.06.010.

[6] G. M. Ziegler, *Lectures on Polytopes*, volume 152 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1995, doi:10.1007/978-1-4613-8431-1.