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Improving upper bounds for the distinguishing index

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Abstract

The distinguishing index of a graph G, denoted by $D'(G)$, is the least number of colours in an edge colouring of G not preserved by any non-trivial automorphism. We characterize all connected graphs G with $D'(G) \geq \Delta(G)$. We show that $D'(G) \leq 2$ if G is a traceable graph of order at least seven, and $D'(G) \leq 3$ if G is either claw-free or 3-connected and planar. We also investigate the Nordhaus-Gaddum type relation: $2 \leq D'(G) + D'(\overline{G}) \leq \max\{\Delta(G), \Delta(\overline{G})\} + 2$ and we confirm it for some classes of graphs.

Keywords: Edge colouring, symmetry breaking in graph, distinguishing index, claw-free graph, planar graph.

Math. Subj. Class.: 05C05, 05C10, 05C15, 05C45

1 Introduction

We follow standard terminology and notation of graph theory (cf. [12]). In this paper, we consider general, i.e. not necessarily proper, edge colourings of graphs. Such a colouring f of a graph G *breaks an automorphism* $\varphi \in \mathrm{Aut}(G)$ if φ does not preserve colours of f. The *distinguishing index* $D'(G)$ of a graph G is the least number d such that G admits an edge colouring with d colours that breaks all non-trivial automorphisms (such a colouring is called a *distinguishing edge d-colouring*). Clearly, $D'(K_2)$ is not defined, so in this paper, a graph G is called *admissible* if neither G nor $\overline{G}$ contains K_2 as a connected component.

The definition of $D'(G)$ introduced by Kalinowski and Pilśniak in [17] was inspired by the *distinguishing number* $D(G)$ which was defined for general vertex colourings by Albertson and Collins [1]. Another concept is the *distinguishing chromatic number* $\chi_D(G)$

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introduced by Collins and Trenk [7] for proper vertex colourings. Both numbers, $D(G)$ and $\chi_D(G)$, have been intensively investigated by many authors in recent years [4, 5, 6, 9, 16].

Our investigation was motivated by the renowned result of Nordhaus-Gaddum [18] who proved in 1956 the following lower and upper bounds for the sum of the chromatic numbers of a graph and its complement (actually, the upper bound was first proved by Zykov [22] in 1949).

Theorem 1.1 ([18]). *If G is a graph of order n with the chromatic number $\chi(G)$, then*

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1.$$

Since then, Nordhaus-Gaddum type bounds were obtained for many graph invariants. An exhaustive survey is given in [2]. Here, we adduce only those closely related to the topic of our paper.

In 1964, Vizing [20] considered proper edge colourings and he proved Nordhaus-Gaddum type bounds for the chromatic index of a graph.

Theorem 1.2 ([20]). *If G is a graph of order n with the chromatic index $\chi'(G)$, then*

$$n - 1 \leq \chi'(G) + \chi'(\overline{G}) \leq 2(n-1).$$

In 2013, Collins and Trenk [8] proved Nordhaus-Gaddum type inequalities for the distinguishing chromatic number.

Theorem 1.3 ([8]). *For every graph of order n and distinguishing number $D(G)$ the following inequalities are satisfied*

$$2\sqrt{n} \leq \chi_D(G) + \chi_D(\overline{G}) \leq n + D(G).$$

Kalinowski and Pilśniak [17] also introduced a *distinguishing chromatic index* $\chi'_D(G)$ of a graph G as the least number of colours in a proper edge colouring that breaks all non-trivial automorphisms of G. They proved the following somewhat unexpected result.

Theorem 1.4 ([17]). *If G is a connected graph of order $n \geq 3$, then*

$$\chi'_D(G) \leq \Delta(G) + 1$$

unless $G \in \{C_4, K_4, C_6, K_{3,3}\}$ when $\chi'_D(G) \leq \Delta(G) + 2$.

The following Nordhaus-Gaddum type inequalities for the distinguishing chromatic index are the same as in Theorem 1.2 but we have to be more careful in the proof.

Theorem 1.5. *If G is an admissible graph of order $n \geq 3$, then*

$$n - 1 \leq \chi'_D(G) + \chi'_D(\overline{G}) \leq 2(n-1)$$

with the only exception $K_{1,4}$.

Proof. Without loss of generality we may assume that G is connected. It can be easily checked that the conclusion holds if $G \in \{K_4, C_6, \overline{C_6}, K_{3,3}\}$. Otherwise, $\chi'_D(G) \leq \Delta(G) + 1$. Suppose first that $\overline{G}$ is also connected. By Theorem 1.4,

$$\Delta(G) + \Delta(\overline{G}) \leq \chi'_D(G) + \chi'_D(\overline{G}) \leq \Delta(G) + \Delta(\overline{G}) + 2.$$

Clearly, $n-1 \leq \Delta(G)+\Delta(\overline{G}) \leq 2(n-2)$ since both G and $\overline{G}$ are connected.

Now, let $\overline{G}$ be disconnected (but admissible). If there are two nonisomorphic components of $\overline{G}$ of orders k_1 and k_2 such that $3 \leq k_1 \leq k_2$, then $\Delta(\overline{G}) \leq n-k_1-1 \leq n-4$, so $\chi'_D(\overline{G}) \leq n-2$. If $\overline{G}$ has $t \geq 2$ components isomorphic to a graph H of order at least three, then $\chi'_D(H) \leq \frac{n}{t}+1$ as $\Delta(H) \leq \frac{n}{t}-1$. Even if we wastefully add an extra colour for each additional copy of H, we get $\chi'_D(tH) \leq \frac{n}{t}+1+t-1 = \frac{n}{t}+t \leq n-2$ unless $G = K_{3,3}$ but this we already checked.

To complete the proof it is enough to settle the case when $\overline{G}$ has only one component H of order at least three and some isolated vertices. Hence, $\Delta(H) \leq n-2$. It is easy to check that $\chi'_D(G)+\chi'_D(\overline{G}) \leq 2(n-1)$ for $H \in \{K_4, C_6, \overline{C_6}, K_{3,3}\}$ except for $H = K_4$ when $G = K_{1,4}$. Otherwise, $\chi'_D(\overline{G}) \leq n-1$ and the conclusion holds unless $|G| = |H|+1$ and $\Delta(H) = n-2$. But then G has a unique vertex x of degree $n-1$ (hence, x is fixed by every automorphism of G) with a pendant edge. The graph $G-x$ has a distinguishing colouring with $n-1$ colours by Theorem 1.4 since $\Delta(G-x) \leq n-2$. It suffices to colour the pendant edge with a colour missing at x to see that $\chi'_D(G) \leq n-1$. □

Collins and Trenk observed in [8] that the Nordhaus-Gaddum type relation is trivial for the distinguishing number, as $D(G)+D(\overline{G}) = 2D(G)$ since $\mathrm{Aut}(\overline{G}) = \mathrm{Aut}(G)$ and every colouring of $V(G)$ breaking all non-trivial automorphisms of G also breaks those of $\overline{G}$.

In Section 4 we formulate and discuss the following conjecture.

Conjecture 1.6. *Let G be an admissible graph of order $n \geq 7$, and let $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$. Then*

$$2 \leq D'(G)+D'(\overline{G}) \leq \Delta+2.$$

In Section 2 we characterize graphs G which need exactly $\Delta(G)$ colours to break all non-trivial automorphisms. In Section 3 we give upper bounds for the distinguishing index of traceable graphs, claw-free graphs, planar graphs and 2-connected graphs.

2 Improved general upper bound

In the sequel, we make use of some facts proved in [17].

Proposition 2.1 ([17]). *$D'(P_n) = 2$ for every $n \geq 3$.*

Proposition 2.2 ([17]). *$D'(C_n) = 3$ for $n \leq 5$, and $D'(C_n) = 2$ for $n \geq 6$.*

Proposition 2.3 ([17]). *$D'(K_n) = 3$ if $3 \leq n \leq 5$, and $D'(K_n) = 2$ if $n \geq 6$.*

Proposition 2.4 ([17]). *$D'(K_{3,3}) = 3$, and $D'(K_{n,n}) = 2$ if $n \geq 4$.*

By the well-known theorem of Jordan (cf. [12]), every finite tree T has either a central vertex or a central edge, which is fixed by every automorphism of T. In the proof of Theorem 2.8, which is the main result of this section, we use Lemma 2.5, a simple generalization of the theorem of Jordan. Recall that the *eccentricity* of a vertex v in a connected graph G is the number

$$\varepsilon_G(v) = \max\{d(v,u) \ : \ u \in V(G)\}.$$

The *center* of a graph G is the set $Z(G)$ of vertices with minimum eccentricity. Clearly, the center of G is setwise fixed by every automorphism $\varphi \in \mathrm{Aut}(G)$, i.e. $\varphi(v) \in Z(G)$ if $v \in Z(G)$. A proper subgraph H of G is called *pendant* if it has only one vertex adjacent to vertices outside H.

Lemma 2.5. *Let G be a connected graph such that every cycle is contained in a clique. Then the center of G is either a single vertex or a maximal clique.*

Proof. The claim is true if G is a clique K_k of order $k \geq 1$. Otherwise, $\kappa(G) = 1$, and each block of G is a clique of order at least two. We then modify the standard proof of the theorem of Jordan for trees. Let G^- be a graph obtained from G by deleting $k-1$ vertices of degree $k-1$ in every pendant clique K_k with $k \geq 2$. Clearly, $\varepsilon_{G^-}(v) = \varepsilon_G(v) - 1$ for each $v \in V(G^-)$. Consequently, $Z(G^-) = Z(G)$. We continue this process until only one clique K_k is left for some $k \geq 1$. This clique is maximal whenever $k \geq 2$. □

A *symmetric tree*, denoted by $T_{h,d}$, is a tree with a central vertex v_0, all leaves at the same distance h from v_0 and all vertices that are not leaves of equal degree d. A *bisymmetric tree*, denoted by $T''_{h,d}$, is a tree with a central edge e_0, all leaves at the same distance h from the edge e_0 and all vertices which are not leaves of equal degree d.

Theorem 2.6 ([17])**.** *If T is a tree of order $n \geq 3$, then $D'(T) \leq \Delta(T)$. Moreover, equality is achieved if and only if T is either a symmetric or a bisymmetric tree.*

For connected graphs in general there is the following upper bound for $D'(G)$.

Theorem 2.7 ([17])**.** *If G is a connected graph of order $n \geq 3$, then*

$$D'(G) \leq \Delta(G)$$

unless G is C_3, C_4 or C_5.

It follows for connected graphs that $D'(G) > \Delta(G)$ if and only if $D'(G) = \Delta(G) + 1$ and G is a cycle of length at most 5. The equality $D'(G) = \Delta(G)$ holds for cycles of length at least 6, for K_4, $K_{3,3}$ and for all symmetric or bisymmetric trees. Now, we show that $D'(G) < \Delta(G)$ for all other connected graphs. A *palette* of a vertex is the multiset of colours of edges incident to it.

Theorem 2.8. *Let G be a connected graph that is neither a symmetric nor a bisymmetric tree. If the maximum degree of G is at least 3, then*

$$D'(G) \leq \Delta(G) - 1$$

unless G is K_4 or $K_{3,3}$.

Proof. Denote $\Delta = \Delta(G)$. The conclusion holds for trees due to Theorem 2.6. Then assume that G contains a cycle. The general idea of the proof is the following. If G does not contain a cycle of length greater than three, then we define G' as an empty graph. Otherwise, we consecutively delete pendant trees and pendant triangles until we obtain a subgraph G'. Then, we construct an edge colouring f with $\Delta - 1$ colours stabilizing all vertices of G' by every automorphism preserving f. Finally, we colour pendant subtrees and pendant triangles to complete a distinguishing colouring with $\Delta - 1$ colours of the whole graph G.

If $\Delta(G') = 2$, then G' is a cycle C_p having a distinguishing colouring with $\Delta - 1$ colours unless $p \in \{4, 5\}$ and $\Delta = 3$. In this case, it can be easily checked that the graph G'_+ induced by C_p and the independent edges of G incident to C_p can always be coloured with two colours such that the vertices of C_p are fixed by every colour preserving automorphism. So we can assume that $\Delta(G') \geq 3$. If $G' \in \{K_4, K_{3,3}\}$, then $G' \neq G$ due to the assumption, hence $\Delta \geq 4$, so we can stabilize K_4 or $K_{3,3}$ with three colours.

Let $N_i(v)$ denote the *i-th sphere in v*, i.e. the set of vertices of distance i from the vertex v. Let x be a vertex with maximum degree in G'. We colour with 1 all edges incident with x. In our edge colouring f of the graph G', the vertex x will be the unique vertex of maximum degree with the monochromatic palette $\{1, \ldots, 1\}$. Hence, x will be fixed by every automorphism φ preserving f. Consequently, φ maps each sphere $N_i(x)$ onto itself.

The first sphere $N_1(x)$ can be partitioned into subsets M_k, for $k = 0, \ldots, \Delta - 1$, defined as

$$M_k = \{v \in N_1(x) \; : \; |N_1(v) \cap N_2(x)| = k\}.$$

Denote $M_k = \{v_1, \ldots, v_{l_k}\}$. Thus, $l_0 + l_1 + \ldots + l_{\Delta-1} = \Delta$.

We want to find a colouring f of the edges of $G'[N_1(x) \cup N_2(x)]$ and, if necessary, of some subsequent spheres, such that each vertex of $N_1(x) \cup N_2(x)$ is fixed by every automorphism preserving this colouring. To do this, we proceed in a number of steps M_k, for $k = 0, \ldots, \Delta - 1$. In each step M_k, we find a colouring that fixes the vertices of M_k and their neighbours in $N_2(x)$.

Step $\mathbf{M_0}$. First we consider the case when the subgraph $G'[M_0]$ induced by the vertices of M_0 is connected. Observe that $\Delta(G'[M_0]) \leq \Delta - 1$ and, by Theorem 2.7, we can colour distinguishingly the edges of $G'[M_0]$ with $\Delta - 1$ colours, even if $G'[M_0]$ is a short cycle C_p with $3 \leq p \leq 5$. Indeed, if $G'[M_0] = C_3$ and $\Delta = 3$, then we would have $G = K_4$, but K_4 is excluded. Otherwise, $\Delta \geq 4$ and we can use a third colour in a short cycle C_p. It may happen that there exists a vertex $v \in M_0$ of degree Δ in G' (so $|M_0| = \Delta$) with a monochromatic palette $\{1, \ldots, 1\}$ in a colouring of $G'[M_0]$ given by Theorem 2.7. In this case, either G is a complete graph K_n with $n \geq 5$ so $D'(K_n) \leq \Delta - 1$ by Proposition 2.3, or it is not difficult to see that there exists a colour c such that there is no vertex with all incident edges coloured with c; whence we can exchange c and 1 in this colouring of $G'[M_0]$.

Now, let $G'[M_0]$ be disconnected. Let $z_1, \ldots, z_s$ be isolated vertices or end-vertices of isolated edges in $G'[M_0]$. Clearly, $s \leq \Delta - 1$ by the definition of G'. If $s = \Delta - 1$, then we colour with i every edge $z_i u$, where $u \in N_1(x) \setminus M_0$. Otherwise, we colour $z_i u$ with $i + 1$ for $i = 1, \ldots, s$. Thus, we avoid a monochromatic palette of $\{1, \ldots, 1\}$ at another vertex of maximum degree in G'.

We also have to distinguish all isomorphic components of $G'[M_0]$ of order greater than 2. Denote such a component by H and suppose that $G'[M_0]$ contains t components isomorphic to H, for some $t \geq 2$. Hence $t \leq \frac{\Delta}{3}$ and $\Delta(H) \leq \frac{\Delta}{t} - 1$. Therefore, we can choose distinct sets of $\frac{\Delta}{t}$ colours for every component since

$$\binom{\Delta - 1}{\frac{\Delta}{t}} \geq \binom{\Delta - 1}{3} \geq \frac{\Delta}{3} \geq t.$$

Thus each vertex of M_0 is fixed.

Step $\mathbf{M_1}$. For every $i = 1, \ldots, l_1$, we colour the edge $v_i u$, where $u \in N_2(x)$, with a distinct colour from $\{1, \ldots, \Delta - 1\}$. This is impossible only if $l_1 = \Delta$, when we have to have two vertices $a, b \in M_1$ with the same colour of edges aa' and bb', where a' and b' are neighbours of a and b in $N_2(x)$, respectively. If $G'[M_1]$ contains an edge e, then we colour it with 1, and all other edges of $G'[M_1]$ with 2. Then we choose exactly one of the vertices a, b incident to e. We proceed analogously when $G'[N_2(x)]$ contains an edge. Then all vertices of M_1 are fixed unless $l_1 = \Delta$ and neither $G'[N_1(x)]$ nor $G'[N_2(x)]$ contains an edge.

If $|N_2(x)| = 1$, then G' is isomorphic to $K_{2,\Delta}$. It is easy to see that $D'(K_{2,\Delta}) \leq \Delta - 1$ for $\Delta \geq 3$ (for $\Delta \geq 4$ this immediately follows from Lemma 3.1 and Corollary 3.8). If $2 \leq |N_2(x)| \leq \Delta - 1$, then choosing a and b such that a' has at least two neighbours in $N_1(x)$ and $b' \neq a'$ yields a colouring fixing $N_1(x) \cup N_2(x)$.

Suppose $|N_2(x)| = \Delta$. If there is a vertex $v \in N_2(x)$ with less than $\Delta - 1$ neighbours in $N_3(x)$, then we choose a such that $a' = v$, and it suffices to reserve a unique set of colours for the edges between a' and $N_3(x)$.

Hence, assume that every vertex of $N_2(x)$ has $\Delta - 1$ neighbours in $N_3(x)$. We select two vertices $a, b \in M_1$ and assume that the colours of the edges aa' and bb' are the same. Next, we implement the following Procedure SUBTREES (a, b), which we also use in subsequent steps.

Procedure SUBTREES (a, b)

We are given two vertices $a, b \in N_1(x)$ such that each their neighbour in $N_2(x)$ is adjacent to $\Delta - 1$ vertices of $N_3(x)$.

Let T_a be a maximal subtree of the graph $G'[\{a\} \cup \bigcup_{i \geq 2} N_i(x)]$, rooted at a, such that all leaves of T_a belong to the same sphere $N_{l-1}(x)$ and each vertex of $V(T_a) \cap N_{i-1}(x)$ has $\Delta - 1$ neighbours in $N_i(x)$ for $i = 3, \ldots, l$. Thus $l \geq 3$. Define a graph

$$\widetilde{T_a} = G'[\bigcup_{v \in V(T_a) \setminus \{a\}} N(v)],$$

i.e. $\widetilde{T}_a$ is a graph obtained from T_a by adding all edges incident with the leaves of T_a. Analogously, we define a tree T_b and a graph $\widetilde{T}_b$. Observe that the trees T_a and T_b are disjoint and non-empty.

The edges incident to the roots a and b are already coloured. For every other vertex of T_a and T_b, we colour its incident edges going to the next sphere with distinct colours from $\{1, \ldots, \Delta - 1\}$. Thus we obtain an edge colouring f. The only automorphism of T_a (as well as of T_b) preserving f is the identity. The vertex x will be fixed by every colour preserving automorphism φ. Consequently, φ maps $\widetilde{T}_a$ onto $\widetilde{T}_b$ whenever $\varphi(a) = b$. Thus, if $\widetilde{T}_a$ and $\widetilde{T}_b$ are not isomorphic, then f distinguishes all vertices in $V(T_a) \cup V(T_b)$. Hence, assume that the rooted graphs $\widetilde{T}_a$ and $\widetilde{T}_b$ are isomorphic. Observe that there exists exactly one non-trivial isomorphism $\psi_0 \colon V(T_a) \to V(T_b)$ preserving f since each vertex in T_a has a distinct coloured path from the root a.

Denote $W_l = (V(\widetilde{T}_a) \cup V(\widetilde{T}_b)) \cap N_l(x)$. By our choice of G', all vertices in W_l are of degree at least two in G'. It follows that one of the following three cases has to hold.

Case 1. There exist vertices in W_l adjacent to more than one vertex of W_{l-1}. Then we modify f by colouring again all edges between such vertices and W_{l-1} in order to break any possible permutation of W_l. A permutation of a set $L \subseteq W_l$ can be extended to an automorphism of G' that fixes all leaves of $\widetilde{T}_a \cup \widetilde{T}_b$ only if every vertex from L have the same set of neighbours $U = \{u_1, \ldots, u_d\}$ in W_{l-1}. Such a set L contains at most $\Delta - 1$ leaves since the number of edges joining U to W_l equals $d(\Delta - 1)$. Every permutation of L will be broken whenever for every vertex $w \in L$ the multiset of colours of the edges $wu_1, \ldots, wu_d$ will be distinct. Clearly, $d \leq \Delta$. There are $\binom{\Delta+d-2}{d}$ such possible multisets of $\Delta - 1$ colours. Clearly, $\binom{\Delta+d-2}{d} - 1 \geq \Delta - 1$ for $\Delta \geq 3$ and $d \geq 2$. We can exclude a

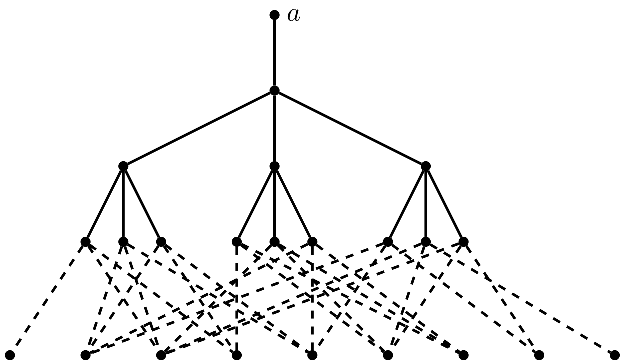

Figure 1: An example of the subgraph $\widetilde{T}_a$ for $\Delta = 4$ and $l = 4$. The edges of $\widetilde{T}_a$ between W_3 and W_4 that do not belong to the tree T_a are dashed.

rainbow multiset $P = \{1, \dots, d\}$ (or an almost rainbow multiset $P = \{1, \dots, \Delta-1, \Delta-1\}$ if $d = \Delta$) and we still have enough multisets to colour the edges incident with vertices of L. Moreover, for $d = \Delta$ we can also exclude a monochromatic palette $\{1, \dots, 1\}$ since $\binom{2\Delta-2}{\Delta} - 2 \geq \Delta - 1$ for $\Delta \geq 3$.

We partition the set W_l into maximal subsets L with the same set of neighbours and assign suitable multisets of colours to each set L. We thus obtain a colouring fixing all vertices from W_l unless ψ_0 can be extended to an isomorphism $\widetilde{\psi_0}$ of $\widetilde{T}_a$ onto $\widetilde{T}_b$ preserving this colouring. To break every such possible extension $\widetilde{\psi_0}$, it suffices to assign the excluded multiset P to one vertex of one set L.

Case 2. Every vertex in W_l has only one neighbour in W_{l-1} and the set of edges $F = E(G'[W_l])$ is non-empty. Then we colour one edge of F with 1, and all other edges in F with 2. This colouring fixes all vertices of $\widetilde{T}_a$ and $\widetilde{T}_a$ unless all edges in F are of the form $w\widetilde{\psi_0}(w)$, where $w\widetilde{\psi_0}(w)$ is one of possible extensions of ψ_0 to an isomorphism of $\widetilde{T}_a$ onto $\widetilde{T}_b$. In such a case, we choose one edge $ww' \in F$ and exchange colours of the edge wu, where $u \in W_{l-1}$, with another edge between u and W_l.

Case 3. Every vertex in W_l has only one neighbour in W_{l-1} and no neighbours in W_l. By the maximality of the trees T_a and T_b and the definition of G', each vertex in W_l has at least one neighbour in $N_{l+1}(x)$ and there exists a vertex $w_0 \in W_l$ with $s < \Delta - 1$ neighbours $y_1, \dots, y_s \in N_{l+1}(x)$. We colour each edge $w_0 y_j$ with colour $j+1$ for $j = 1, \dots, s$. Next, for every vertex $w \in W_l$, we colour the set of edges between w and $N_{l+1}(x)$ with a set of $\Delta - 1$ colours excluding the set $\{2, \dots, s+1\}$.

We thus obtained a colouring f of the edges of $G'[V(\widetilde{T}_a) \cup V(\widetilde{T}_b)]$, and the edges incident to W_l in Case 3, fixing all vertices of $\widetilde{T}_a$ and $\widetilde{T}_b$.

End of Procedure SUBTREES (a, b)

Step $\mathbf{M_2}$. For every $i = 1, \dots, l_2$, we colour the edges $v_i u_i^1$, $v_i u_i^2$ where $\{u_i^1, u_i^2\} \subseteq N_2(x)$, with distinct sets of colours from among $\binom{\Delta-1}{2}$ sets. This is impossible only in the following three cases (in each case, we can assume that neither $G'[N_1(x)]$ nor $G'[N_2(x)]$ contains an edge, otherwise we could construct a distinguishing colouring f of $G'[N_1(x) \cup N_2(x)]$ analogously as in step M_1):

a) $l_2 = \Delta = 4$. If there exist two vertices a and b in M_2 such that $N(a) \cap N(b) \cap N_2(x) \neq \emptyset$, then we colour with 2 both edges incident with b, and for the remaining vertices in M_2 we have distinct sets of colours from among $\binom{3}{2}$ sets. If for every two vertices $a, b \in M_2$, the set $N(a) \cap N(b) \cap N_2(x)$ is empty, then two vertices a and b are assign the same pair of distinct colours, and we can distinguish them in next spheres using the procedure SUBTREES (a, b).

b) $l_2 = \Delta - 1$ and $\Delta = 3$. Let $M_2 = \{a, b\}$. If $N(a) \cap N(b) \cap N_2(x) \neq \emptyset$, then we colour edges incident with a with colours 1 and 2, and both edges incident with b with 2. If the set $N(a) \cap N(b) \cap N_2(x)$ is empty, then a and b get the same pair of distinct colours and we can distinguish them in next spheres by the procedure SUBTREES (a, b).

c) $l_2 = \Delta = 3$. Let $M_2 = \{a, b, c\}$. If for two vertices of M_2, say a and b, the set $N(a) \cap N(b) \cap N_2(x)$ is non-empty, then we can colour with 2 both edges incident with b and we colour edges incident with the remaining vertices of M_2 with a couple $\{1, 2\}$. It is not difficult to verify that this way, for every configuration of neighbours of M_2, we can obtain colouring fixing the vertices of $N_1(x) \cup N_2(x)$ unless $|N(a) \cap N(b) \cap N(c) \cap N_2(x)| = 2$. But then $G' = G = K_{3,3}$, contrary to the assumption. If every vertex of $N_2(x)$ is adjacent only to one vertex of M_2, then the pairs of edges incident to a and b are assign the same pair of colours $\{1, 2\}$, and we distinguish them using the procedure SUBTREES (a, b). Both edges cu^1, cu^2 incident with c are coloured with 2, and to distinguish them, we split c into two vertices c^1 and c^2, each joined by an edge coloured with 2 to u^1 and u^2, respectively, and apply the procedure SUBTREES (c^1, c^2).

Step $\mathbf{M_k}$, for $\mathbf{k \geq 3}$. For every $i = 1, \ldots, l_k$, we colour the edges between v_i and $N_2(x)$ with distinct sets of k colours from among $\binom{\Delta-1}{k}$ sets. It is always possible whenever $\binom{\Delta-1}{k} \geq l_k$. This inequality does not hold only in two cases:

a) $k = \Delta - 2$ and $l_k = \Delta$. In this case we define a colouring with $\Delta - 1$ colours like in step M_2 a). Namely, if either a vertex of M_k or its neighbour in $N_2(x)$ is adjacent to a vertex in the same sphere, then we can define a colouring fixing all these vertices analogously as in step M_1 and step M_2. Also, if there are two vertices $a, b \in M_{\Delta-2}$ with a common neighbour in $N_2(x)$, we can assign the same palette to a and b as in the previous steps. Otherwise, two vertices $a, b \in M_{\Delta-2}$ are assign the same palette of $\Delta - 2$ colours and we distinguish them using Procedure SUBTREES (a, b).

b) $k = \Delta - 1$ and $l_k \geq 2$. Hence, $\Delta \geq 4$. For every $i = 1, \ldots, l_k$, the set of edges between $v_i \in M_{\Delta-1}$ and $N_2(x)$ will be assign a distinct multiset P^i of colours from the set $\{1, \ldots, \Delta - 1\}$, where only colour i appears twice. Moreover, one vertex can assign a rainbow palette $\{1, \ldots, \Delta - 1\}$. Thus every vertex of $M_{\Delta-1}$ will have a distinct palette, and hence will be stabilized. To stabilize the two vertices of $N_2(x)$ joined to v_i by edges of colour i, we examine the vertices $v_1, \ldots, v_{\Delta-1}$ of $M_{\Delta-1}$ in the following order.

   First, we consider each vertex v_i that have a neighbour $w_i \in N_2(x)$ with at least one but at most $\Delta - 2$ neighbours in $N_3(x)$. We choose another neighbour $w_i' \in N_2(x)$ of v_i and assign two distinct sets of colours for the edges going to $N_3(x)$ from w_i and w_i', respectively. We colour the edges $v_i w_i$ and $v_i w_i'$ with the same colour i. Thus all neighbours of v_i are stabilized.

In the next stage, we consider every vertex v_i with every neighbour in $N_2(x)$ adjacent to $\Delta - 1$ vertices of $N_3(x)$. We colour the set of edges between v_i and $N_2(x)$ with the palette P^i, where two edges v_iu_1, v_iu_2 are coloured with i. Then we delete v_i and introduce two vertices v_i^1, v_i^2 and edges $v_i^1u_1$ and $v_i^2u_2$ coloured with i. Then we use the procedure SUBTREES (v_i^1, v_i^2) to stabilize u_1 and u_2.

Further, we consider each vertex v_i with a neighbour $w_i \in N_2(x)$ incident to an edge w_iu, where $u \in N_2(x)$. First, we look for such an edge w_iu, which is already coloured. If there is no such edge, we take an uncoloured w_iu and colour it with colour 3. In both cases, we put colour i on the edge v_iw_i and another edge v_iw with $w \neq u$. After we examine each such vertex v_i, we colour with 2 all remaining edges contained in $N_2(x)$.

Finally, we are left with at most Δ vertices v_i such that every neighbour of v_i is adjacent only to (at least two) vertices of $N_1(x)$. We take a first such vertex v_i and assign colour i to two its incident edges v_iw_i and v_iw_i'. Thus all neighbours of v_i are stabilized unless common neighbours of w_i and w_i' were not considered yet. Then we take such a neighbour v_j and colour its incident edges with the palette P^j such that the edges v_jw_i and v_jw_i' have distinct colours. We repeat this procedure until only one vertex of $M_{\Delta-1}$ is left. We put a rainbow palette $\{1, \ldots, \Delta - 1\}$ on its incident edges.

After we accomplish steps $M_0, \ldots, M_{\Delta-1}$, we colour all uncoloured edges in subgraphs $G'[N_1(x)]$ and $G'[N_2(x)]$ with 2. Each vertex of $N_1(x) \cup N_2(x)$ is now fixed by every automorphism preserving our colouring f of edges of $G'[\{x\} \cup N_1(x) \cup N_2(x)]$, and of some edges between next spheres, if the procedure SUBTREES was used.

Then we recursively colour all yet uncoloured edges incident to consecutive spheres $N_i(x)$ as follows: for $v \in N_i(x)$, $i \geq 2$, we colour all edges vu, where $u \in N_{i+1}(x)$, with distinct colours from $\{1, \ldots, \Delta - 1\}$. This is always possible since every vertex of $N_i(x)$ has at most $\Delta - 1$ neighbours in $N_{i+1}(x)$. Finally, we colour all uncoloured edges with end-vertices in the same sphere with 2. Hence, all vertices of G' are fixed by any automorphism preserving our colouring f. It is also easily seen that the already coloured edges can save their colours. Moreover, it is not difficult to observe that x is the unique vertex of maximum degree with a monochromatic palette $\{1, \ldots, 1\}$. Thus, the whole subgraph G' (or G'_+) is fixed.

To end the proof, we colour pendant trees and triangles deleted from G at the beginning. First assume that G' is not empty. Let $N_i(G')$, for $i \geq 0$, be the set of vertices of distance i from G'. Then we recursively colour the edges incident to consecutive spheres $N_i(G')$ in the following way: for $v \in N_i(G')$, $i \geq 0$, we colour all edges vu, where $u \in N_{i+1}(G')$, with distinct colours from $\{1, \ldots, \Delta - 1\}$ and the remaining edges incident to v, contained in $N_i(x)$, with 2. Hence, all vertices of G will be fixed by any automorphism preserving our colouring f.

If G' is empty, then we start with the centre $Z(G)$ that is setwise fixed by every automorphism. It follows from Lemma 2.5 that $Z(G)$ either induces K_3, or K_2 (not contained in K_3), or K_1. Let first $Z(G)$ induce a triangle K_3. If $\Delta = 3$, then we stabilize $Z(G)$ by colouring with two colours all edges incident with vertices of $Z(G)$. When $\Delta \geq 4$, we can colour the edges of the triangle $Z(G)$ with three colours. Next, we recursively colour edges incident to subsequent spheres $N_i(Z(G))$ with $\Delta - 1$ colours.

If $Z(G)$ is an edge e, then $G - e$ has two components. We distinguish each of them by colouring subsequent spheres $N_i(Z(G))$ with $\Delta - 1$ colours. If the components are isomorphic, then by assumption, each of them has a triangle. We colour two edges of these triangles contained in a sphere $N_i(Z(G))$, for some $i \geq 2$, with two distinct colours.

Finally, let $Z(G)$ be a single vertex z. Hence, $G - z$ has $q \geq 2$ components, each joined to z by one or two edges. If $q < \Delta$, then we can easily colour distinguishingly the edges incident with subsequent spheres $N_i(z)$, $i \geq 0$, with $\Delta - 1$ colours. If $q = \Delta$, then we choose two components of $G - z$, at least one of them with a triangle, and colour their two edges incident with z with the same colour. Then we distinguish these two components by an edge of the triangle. □

3 Some classes of graphs

A graph G is called *asymmetric* if its automorphism group is trivial. Then obviously $D'(G) = 1$.

We say that a graph G is *almost spanned* by a subgraph H (not necessarily connected) if $G - v$ is spanned by H for some $v \in V(G)$. The following observation will play a crucial role in this section.

Lemma 3.1. *If a graph G is spanned or almost spanned by a subgraph H, then*

$$D'(G) \leq D'(H) + 1.$$

Proof. We colour the edges of H with colours $1, \ldots, D'(H)$, and all other edges of G with an additional colour 0. If φ is an automorphism of G preserving this colouring, then $\varphi(x) = x$, for each $x \in V(H)$. Moreover, if H is a spanning subgraph of $G - v$, then also $\varphi(v) = v$. Therefore, φ is the identity. □

3.1 Traceable graphs

Recall that a graph is *traceable* if it contains a Hamiltonian path.

Theorem 3.2. *If G is a traceable graph of order $n \geq 7$, then $D'(G) \leq 2$.*

Proof. Let $P_n = v_1 v_2 \ldots v_n$ be a Hamiltonian path of G. If $G = P_n$, then the conclusion follows from Proposition 2.1. If G is isomorphic to $P_n + v_1 v_3$, then we colour the edge $v_1 v_3$ with 1, and all other edges with 2 breaking all non-trivial automorphisms of G. So suppose that G contains an edge $v_i v_j$ distinct from $v_1 v_3$ and $v_{n-2} v_n$ with $i < j - 1$. Without loss of generality we may assume that $i - 1 \leq n - j$ (otherwise we reverse the labeling). It is easy to see that at least one of the graphs $P_n + v_i v_j - v_{j-1} v_j$, $P_n + v_i v_j - v_{j-1}$ or $P_n + v_i v_j - v_n$ is an asymmetric spanning or almost spanning subgraph of G for any $n \geq 7$. The conclusion follows from Lemma 3.1. □

The assumption $n \geq 7$ is substantial in Theorem 3.2 as $D'(K_{3,3}) = 3$.

3.2 Claw-free graphs

A $K_{1,3}$-free graph, called also *a claw-free graph*, is a graph containing no copy of $K_{1,3}$ as an induced subgraph. Claw-free graphs have numerous applications, e.g., in operations research and scheduling theory. For a survey of claw-free graphs and their applications consult [10].

A *k-tree* of a connected graph is its spanning tree with maximum degree at most k. Win [21] investigated spanning trees in 1-tough graphs and proved the following result.

Theorem 3.3 ([21])**.** *A* 2*-connected claw-free graph has a* 3*-tree.*

We use this result to give an upper bound for the distinguishing number of claw-free graphs.

Theorem 3.4. *If G is a connected claw-free graph, then $D'(G) \leq 3$.*

Proof. Assume first that G is 2-connected. By Theorem 3.3, G contains a 3-tree T. By Theorem 2.6, we have $D'(T) \leq 2$ if T is neither symmetric nor bisymmetric tree. In such a case, $D'(G) \leq 3$ by Lemma 3.1.

Let T be a symmetric tree $T_{h,3}$. Denote a central vertex of T by x and its neighbours by a, b, c. Since G is a claw-free graph, there exists in G at least one edge, say bc, in the neighbourhood of x in T. Define a subgraph $\widetilde{T} = T + bc$. We colour bc, xa and xb with 1, and xc with 2. Thus all vertices a, b, c, x are fixed by every non-trivial automorphism of $\widetilde{T}$. We now colour the remaining edges in $\widetilde{T}$ starting from the edges incident to a, b, c in such a way that two uncoloured adjacent edges obtain two different colours 1 and 2. This 2-colouring breaks all non-trivial automorphisms of $\widetilde{T}$. Hence, $D'(G) \leq 3$ by Lemma 3.1.

Let T be a bisymmetric tree $T''_{h,3}$. Denote a central edge by xy and its neighbours by a, b adjacent to x, and c, d adjacent to y. We colour xy, xa and yc with 1, and xb and yd with 2. Since G is claw-free, there exists in G either at least one of the edges by, cx (or symmetrically dx or ay) or both ab and cd. We define a subgraph $\widetilde{T}$ obtained from the tree T by adding either one of the edges by, cx (or symmetrically, dx or ay) or both ab and cd. In the first case we colour by or cx (or symmetrically, dx or ay) with 1, in the second case we colour ab with 1 and cd with 2. Now all vertices a, b, c, d, x, y are fixed by every non-trivial automorphism of $\widetilde{T}$. We then colour the remaining edges of $\widetilde{T}$ as above, and we obtain the claim.

If a graph G is not 2-connected, then its graph of blocks and cut-vertices is a path, since G is claw-free. We colour every block according to the rules described above. Then to break all non-trivial automorphisms of G, it is enough to break a possible automorphism $\psi \in \mathrm{Aut}(G)$ that exchanges two terminal blocks. Let z be a cut-vertex that belongs to a terminal block B_0. It follows that z and its neighbours in B_0 induce a clique K of order $k \geq 2$. We have three colours in our disposal, so it is easily seen that we can permute the colours to obtain a nonisomorphic colouring of K, thus breaking ψ. □

The theorem is sharp for graphs of order at most 5. We conjecture that the distinguishing index of claw-free graphs of order big enough is 2.

3.3 Planar graphs

First, recall that by the famous Theorem of Tutte [19], every 4-connected planar graph G is Hamiltonian. Hence, its distinguishing index is at most 2, by Theorem 3.2, whenever $|G| \geq 7$. A similar result as for claw-free graphs we obtain for 3-connected planar graphs. In the proof, we use the following result of Barnette about spanning trees of such graphs.

Theorem 3.5 ([3])**.** *Every* 3*-connected planar graph has a* 3*-tree.*

Using a similar method as in the proof of Theorem 3.4, we obtain the following.

Theorem 3.6. *If G is* 3*-connected planar graph, then $D'(G) \leq 3$.*

Proof. Let T be a 3-tree of G. It follows from Theorem 2.6 that $D'(T) \leq 2$ and hence, $D'(G) \leq 3$ by Lemma 3.1, if T is neither a symmetric nor a bisymmetric tree.

Let then T be a symmetric tree $T_{h,3}$. Denote the central vertex by x, and by T_a, T_b and T_c the connected components of $T - x$ which are trees rooted at the neighbours a, b, c of a vertex x, respectively. Since G is 3-connected, there exist an edge e between T_a and T_b in G. Consider a spanning subgraph $\widetilde{T} = T + e$. Then we colour xa and xc with 1, and xb with 2, and extend this colouring as in the proof of Theorem 3.4 to a colouring of $\widetilde{T}$ breaking all non-trivial automorphisms of $\widetilde{T}$ (the colour of e is irrelevant). Consequently, $D'(G) \leq 3$ by Lemma 3.1.

If T is a bisymmetric tree $T''_{h,3}$ with the central edge xy, then we can add to T one edge in a subtree of $T - xy$ rooted at x, and such a graph can be easily distinguished by two colours. Again, our claim follows from Lemma 3.1. □

3.4 2-connected graphs

For a 2-connected planar graph G, the distinguishing index may attain $1 + \left\lceil \sqrt{\Delta(G)} \right\rceil$ as it is shown by the complete bipartite graph $K_{2,q}$ with $q = r^2$ for a positive integer r. In this case, $D'(K_{2,q}) = r + 1$ as it follows from the result obtained independently by Fisher and Isaak [11] and by Imrich, Jerebic and Klavžar [14]. They proved the following theorem. Actually, they formulated it for the distinguishing number $D(K_p \square K_q)$ of the Cartesian product of complete graphs, but $D'(K_{p,q}) = D(K_p \square K_q)$.

Theorem 3.7 ([11, 14])**.** *Let p, q, d be integers such that $d \geq 2$ and $(d-1)^p < q \leq d^p$. Then*

$$D'(K_{p,q}) = \begin{cases} d, & \text{if } q \leq d^p - \lceil \log_d p \rceil - 1, \\ d+1, & \text{if } q \geq d^p - \lceil \log_d p \rceil + 1. \end{cases}$$

If $q = d^p - \lceil \log_d p \rceil$ then the distinguishing index $D'(K_{p,q})$ is either d or $d+1$ and can be computed recursively in $O(\log^*(q))$ time.

In the next section, we make use of the following immediate corollary.

Corollary 3.8. *If $p \leq q$, then $D'(K_{p,q}) \leq \lceil \sqrt[p]{q} \rceil + 1$.*

In the proof of Proposition 3.10 we also make use of an earlier result of Imrich and Klavžar [15] which is a slightly weaker version of Theorem 3.7 for $d = 2$.

Theorem 3.9 ([15])**.** *If $2 \leq p \leq q \leq 2^p - p + 1$, then $D'(K_{p,q}) = 2$.*

Proposition 3.10. *If $p \leq q \leq 2^p - p + 1$ and $p + q \geq 7$, then there exists a distinguishing edge 2-colouring of $K_{p,q}$ such that the edges in one of colours induce a connected spanning or almost spanning, asymmetric subgraph of $K_{p,q}$.*

Proof. The assumptions imply that $p \geq 3$, and $D'(K_{p,q}) = 2$ by Theorem 3.9. Let P and Q be the two sets of bipartition of $K_{p,q}$ with $|P| = p$ and $|Q| = q$. If $p = q$, then $p \geq 4$, and there exists a spanning asymmetric tree of $K_{p,p}$ (see [17]). If $p < q \leq 2^p - p + 1$, then for the proof of Theorem 3.9, Imrich and Klavžar in [15] constructed a distinguishing vertex 2-colouring of $K_p \square K_q$ that corresponds to a distinguishing edge 2-colouring f of $K_{p,q}$, where a colouring of vertices in a K_q-layer can be represented by a sequence from $\{1,2\}^q$ and it corresponds to a colouring of edges incident to a vertex in P (the same is true for K_p-layers and vertices in Q). We wish to show that this colouring yields a connected asymmetric subgraph of $K_{p,q}$ which is spanning or almost spanning.

First assume that $q = 2^p - p + 1$. In the coloring f, every vertex in P has distinct positive number of edges coloured with 1, and there exists a vertex v_1 with all incident edges coloured with 1. Moreover, distinct vertices from Q have distinct sets of neighbours joined by edges coloured with 1, and there exists a vertex, say v_2, with all incident edges coloured with 2. Let S be a subgraph induced by edges coloured with 1. Then S is an almost spanning subgraph since v_2 is the only vertex outside S. The graph S is connected because v_1 is adjacent to every vertex in Q, and every vertex in P is joined to a vertex in Q by an edge coloured with 1. Moreover, S is also asymmetric since f breaks all non-trivial automorphisms of $K_{p,q}$ and any automorphism interchanging some parts of the sets P and Q does not preserve distances in S.

Following [15] for $p < q < 2^p - p + 1$, we exclude a relevant number of such pairs of sequences of colours that the sum of them is a sequence $(3, \ldots, 3)$. Additionally, if both q and p are odd, we exclude the sequence $(0, \ldots, 0)$. Again, we obtain a connected spanning (or almost spanning) asymmetric subgraph S of $K_{p,q}$ induced by the edges coloured with 1. □

Proposition 3.10 and Lemma 3.1 immediately imply the following.

Corollary 3.11. *If a graph G of order at least 7 is spanned by $K_{p,q}$ and $p \leq q \leq 2^p - p + 1$, then $D'(G) \leq 2$.*

In general, for 2-connected graphs we conjecture that the complete bipartite graph K_{2,r^2} is the worst case, i.e. attains the highest value of the distinguishing index.

Conjecture 3.12. *If G is a 2-connected graph, then*

$$D'(G) \leq 1 + \left\lceil \sqrt{\Delta(G)} \right\rceil .$$

4 Nordhaus-Gaddum inequalities for D'

In this section, we discuss Conjecture 1.6, formulated at the end of Introduction, stating that

$$2 \leq D'(G) + D'(\overline{G}) \leq \Delta + 2$$

for every admissible graph G of order $n \geq 7$, where $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$.

The left-hand inequality is obvious. Indeed, if a graph G is asymmetric, then so is $\overline{G}$. Thus we are only interested in the right-hand inequality $D'(G) + D'(\overline{G}) \leq \Delta + 2$. Note also that at least one of the graphs G and $\overline{G}$ is connected.

The bound $\Delta + 2$ cannot be improved. To see this, consider a star $K_{1,n-1}$ of any order $n \geq 7$. As $\overline{K_{1,n-1}}$ is a disjoint union of a complete graph K_{n-1} and an isolated vertex, it follows from Proposition 2.3 that $D'(\overline{K_{1,n-1}}) = 2$. Therefore, $D'(K_{1,n-1}) + D'(\overline{K_{1,n-1}}) = n - 1 + 2 = \Delta + 2$.

If T is a tree, then $\Delta(T)$ can be much smaller than $\Delta = \Delta(\overline{T}) = n - 1$. However, the following holds.

Proposition 4.1. *If T is a tree of order $n \geq 7$, then*

$$D'(T) + D'(\overline{T}) \leq \Delta(T) + 2.$$

Proof. As it was shown above, the conclusion holds for stars. If T is not a star, then $D'(\overline{T}) \leq 2$ by Lemma 3.1. Indeed, as it was proved by Hedetniemi et al. in [13], a complete graph K_n contains edge disjoint copies of any two trees of order n distinct from a star $K_{1,n-1}$. Thus, the complement $\overline{T}$ contains a spanning asymmetric tree. By Theorem 2.6, we have the inequality $D'(T) + D'(\overline{T}) \leq \Delta(T) + 2$. □

This fact emboldened us to formulate the following stronger conjecture.

Conjecture 4.2. *Every connected admissible graph G of order $n \geq 7$ satisfies the inequality*

$$D'(G) + D'(\overline{G}) \leq \Delta(G) + 2.$$

Now we show that Conjecture 1.6 holds not only for trees, but also for some other classes of graphs. To do this we use the following fact.

Theorem 4.3. *Let G be a connected admissible graph of order $n \geq 7$. If either G or every connected component of $\overline{G}$ has the distinguishing index at most 3, then*

$$D'(G) + D'(\overline{G}) \leq \Delta + 2,$$

where $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$.

Proof. Our claim is true for trees by Proposition 4.1. Observe also, that it is true if G is a path or a cycle of order at least 7 since its complement $\overline{G}$ is Hamiltonian, and $D'(G) + D'(\overline{G}) \leq 4$. So, now we can assume that $\Delta(G) \geq 3$ and neither G nor $\overline{G}$ is a tree. We consider two cases.

Case A. Every component H of $\overline{G}$ satisfies $D'(H) \leq 3$.
Then $D'(G) \leq \Delta(G) - 1$ by Theorem 2.8, and if $\overline{G}$ is connected, then our claim holds. Assume now that $\overline{G}$ is disconnected. Then G is spanned by $K_{p,q}$ with $p \leq q$ and $\Delta \geq q$, where $p + q = |V(G)|$. Suppose that the graph $\overline{G}$ has t isomorphic components. If we had a distinct set of three colours for every component, then $D'(\overline{G}) \leq \lceil \sqrt[3]{6t} \rceil$. We then consider two cases:

a) If $q \leq 2^p - p + 1$, then $D'(G) = 2$ by Corollary 3.11. Moreover, we then have at most $\frac{n}{3}$ components of $\overline{G}$, so $D'(\overline{G}) \leq \lceil \sqrt[3]{2n} \rceil$. And we can easily see that

$$\lceil \sqrt[3]{2n} \rceil + 2 \leq \frac{n}{2} + 2$$

for every $n \geq 4$.

b) If $q \geq 2^p - p + 1$, then there exists a big component (of order q) in $\overline{G}$ and we can assume that $t \leq \frac{p}{3}$ remaining components are isomorphic. In this case, by assumptions we have $p \leq \lceil \log_2(q + p - 1) \rceil$, therefore

$$D'(\overline{G}) \leq \lceil \sqrt[3]{6t} \rceil \leq \sqrt[3]{2\lceil \log_2(q + p - 1) \rceil}.$$

On the other hand, $D'(G) \leq \lceil \sqrt[p]{q} \rceil + 2$ by Corollary 3.8 and Theorem 3.1. Then it is not difficult to check that for $q \geq 2^p - p + 1$

$$\sqrt[3]{2\lceil \log_2(q + p - 1) \rceil} + \lceil \sqrt[p]{q} \rceil + 2 \leq q + 2$$

what finishes the proof in Case A.

Case B. $D'(G) \leq 3$.
If graph $\overline{G}$ is connected, then the claim follows immediately from Theorem 2.7 whenever $D'(G) = 2$ or $D'(\overline{G}) = 2$, and it follows from Theorem 2.8 if $D'(G) = 3$. Assume now that $\overline{G}$ has $t \geq 2$ components. Then $\Delta \geq \frac{n}{2}$ and, in the worst case, all components of $\overline{G}$ are isomorphic. Observe that maximal degree of every component is at most $\frac{n}{t} - 1$. If we assign one extra colour to every component, then we need at most $\frac{n}{t} - 1 + (t-1)$ colours to distinguish $\overline{G}$. Hence, if

$$\frac{n}{t} + t \leq \frac{n}{2} - 1,$$

then $D'(\overline{G}) \leq \Delta - 1$, and our claim is true. The above inequality holds unless $t = 2$.

If there exist two isomorphic components in $\overline{G}$, then $D'(G) \leq 2$ due to Corollary 3.11 since G is spanned by $K_{\frac{n}{2},\frac{n}{2}}$. Then $D'(\overline{G}) \leq \frac{n}{2}$, and finally $D'(G) + D'(\overline{G}) \leq \frac{n}{2} + 2$. □

Now we can formulate some consequences of Theorem 4.3 and suitable results proved in Section 3.

Corollary 4.4. *Let G be an admissible graph of order $n \geq 7$. If G satisfies at least one of the following conditions:*

i) G is a traceable graph, or

ii) G is a claw-free graph, or

iii) G is a triangle-free graph, or

iv) G is a 3-connected planar graph,

then

$$D'(G) + D'(\overline{G}) \leq \Delta + 2,$$

where $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$.

Proof. It suffices to apply Theorem 4.3 together with Theorem 3.2, Theorem 3.4 and Theorem 3.6, respectively. Observe also that if the girth of a graph G is at least 4, i.e., G is triangle-free, then its complement $\overline{G}$ is claw-free. □

Finally, it has to be noted that there exist graphs of order less than 7 such that the right-hand inequality in Conjecture 1.6 is not satisfied. For example, for the graph $K_{3,3}$ we have $D'(K_{3,3}) = 3$, $D'(\overline{K_{3,3}}) = D'(2K_3) = 4$ and $\Delta = 3$, hence $D'(K_{3,3}) + D'(\overline{K_{3,3}}) = \Delta + 4$. Also, $D'(C_5) + D'(\overline{C_5}) = 3 + 3 = \Delta + 4$, and $D'(K_{1,i}) + D'(\overline{K_{1,i}}) = \Delta + 3$ for $i = 3, 4, 5$.

References

[1] M. O. Albertson and K. L. Collins, Symmetry breaking in graphs, *Electron. J. Combin.* **3** (1996), #R18, `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v3i1r18`.

[2] M. Aouchiche and P. Hansen, A survey of Nordhaus-Gaddum type relations, *Discrete Appl. Math.* **161** (2013), 466–546, doi:10.1016/j.dam.2011.12.018.

[3] D. W. Barnette, Trees in polyhedral graphs, *Canad. J. Math.* **18** (1966), 731–736, doi:10.4153/cjm-1966-073-4.

[4] D. L. Boutin, The cost of 2-distinguishing Cartesian powers, *Electron. J. Combin.* **20** (2013), #P74, http://www.combinatorics.org/ojs/index.php/eljc/article/view/v20i1p74.

[5] J. O. Choi, S. G. Hartke and H. Kaul, Distinguishing chromatic number of Cartesian products of graphs, *SIAM J. Discrete Math.* **24** (2010), 82–100, doi:10.1137/060651392.

[6] K. L. Collins, M. Hovey and A. N. Trenk, Bounds on the distinguishing chromatic number, *Electron. J. Combin.* **16** (2009), #R88, http://www.combinatorics.org/ojs/index.php/eljc/article/view/v16i1r88.

[7] K. L. Collins and A. N. Trenk, The distinguishing chromatic number, *Electron. J. Combin.* **13** (2006), #R16, http://www.combinatorics.org/ojs/index.php/eljc/article/view/v13i1r16.

[8] K. L. Collins and A. N. Trenk, Nordhaus-Gaddum theorem for the distinguishing chromatic number, *Electron. J. Combin.* **20** (2013), #P46, http://www.combinatorics.org/ojs/index.php/eljc/article/view/v20i3p46.

[9] E. Estaji, W. Imrich, R. Kalinowski, M. Pilśniak and T. W. Tucker, Distinguishing Cartesian products of countable graphs, *Discuss. Math. Graph Theory* **37** (2017), 155–164, doi:10.7151/dmgt.1902.

[10] R. Faudree, E. Flandrin and Z. Ryjáček, Claw-free graphs — a survey, *Discrete Math.* **164** (1997), 87–147, doi:10.1016/s0012-365x(96)00045-3.

[11] M. J. Fisher and G. Isaak, Distinguishing colorings of Cartesian products of complete graphs, *Discrete Math.* **308** (2008), 2240–2246, doi:10.1016/j.disc.2007.04.070.

[12] R. Hammack, W. Imrich and S. Klavžar, *Handbook of Product Graphs*, Discrete Mathematics and Its Applications, CRC Press, Boca Raton, Florida, 2nd edition, 2011, doi:10.1201/b10959.

[13] S. M. Hedetniemi, S. T. Hedetniemi and P. J. Slater, A note on packing two trees into K_n, *Ars Combin.* **11** (1981), 149–153.

[14] W. Imrich, J. Jerebic and S. Klavžar, The distinguishing number of Cartesian products of complete graphs, *European J. Combin.* **29** (2008), 922–929, doi:10.1016/j.ejc.2007.11.018.

[15] W. Imrich and S. Klavžar, Distinguishing Cartesian powers of graphs, *J. Graph Theory* **53** (2006), 250–260, doi:10.1002/jgt.20190.

[16] W. Imrich, S. M. Smith, T. W. Tucker and M. E. Watkins, Infinite motion and 2-distinguishability of graphs and groups, *J. Algebraic Combin.* **41** (2015), 109–122, doi:10.1007/s10801-014-0529-2.

[17] R. Kalinowski and M. Pilśniak, Distinguishing graphs by edge-colourings, *European J. Combin.* **45** (2015), 124–131, doi:10.1016/j.ejc.2014.11.003.

[18] E. A. Nordhaus and J. W. Gaddum, On complementary graphs, *Amer. Math. Monthly* **63** (1956), 175–177, doi:10.2307/2306658.

[19] W. T. Tutte, A theorem on planar graphs, *Trans. Amer. Math. Soc.* **82** (1956), 99–116, doi:10.2307/1992980.

[20] V. G. Vizing, The chromatic class of a multigraph, *Kibernetika* **1** (1965), 29–39, doi:10.1007/bf01885700.

[21] S. Win, On a connection between the existence of k-trees and the toughness of a graph, *Graphs Combin.* **5** (1989), 201–205, doi:10.1007/bf01788671.

[22] A. A. Zykov, On some properties of linear complexes, *Mat. Sbornik (N. S.)* **24** (1949), 163–188, http://mi.mathnet.ru/eng/msb5974.