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Distance formula for direct-co-direct product in the case of disconnected factors*

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Abstract

Direct-co-direct product $G \circledast H$ of graphs G and H is a graph on the vertex set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent if $gg' \in E(G)$ and $hh' \in E(H)$ or $gg' \notin E(G)$ and $hh' \notin E(H)$. We show that if at most one factor of $G \circledast H$ is connected, then the distance between two vertices of $G \circledast H$ is bounded by three unless some small number of exceptions. All the exceptions are completely described which yields the distance formula.

Keywords: Direct-co-direct product, distance, eccentricity, disconnected graphs.

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1 Introduction

Recently Wilfried Imrich complained (personal communication) that the modular product is the only associative and commutative graph product (up to their complementary products) for which he was not successful in finding a polynomial time algorithm for prime factorization with respect to the modular product. His best approach to that can be found in [6], while polynomial algorithms for factoring the Cartesian product is in [8], for the strong product in [5] and for the direct product in [7]. The reason for this could be hidden in the fact that modular products are often diameter two graphs, see [9]. This means that any polynomial algorithm for a prime factorization of modular product would yield a powerful tool how to connect (some) diameter two graphs with two or more smaller graphs

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usually of higher diameter. Namely, the connections between a product and its factors often give rise to several connections between graph invariants on the products. Let us mention here just the two most known: the Vizing's conjecture, see [2], which is still open and the Hedetniemi's conjecture that was recently disproved in [13], but had also inspired a broad palette of innovative results and new approaches.

The direct-co-direct product $G \circledast H$ of graphs G and H can be viewed as a subgraph of modular product with important difference, that as a product the direct-co-direct product is not associative. Much less is known about products that are not associative. Up to the best of our knowledge, the first use of non-associative products is due to Watkins [14] to construct certain automorphism groups of graphs. This was later generalized to directed graphs by Grech et al. in [3]. In between we are aware of only one publication on direct-co-direct product by Kozen [12] where it is shown that for two graphs G and H of order n, the problem of finding a clique of order n in $G \circledast H$ is equivalent to isomorphism problem between G and H. We expect that non-associative products hide several pleasant surprises from many different aspects: structural, algorithmic and to derive connections between products and their factors.

Recently, in [10], the distance formula was presented for the direct-co-direct product for the case of connected factors. In this work we continue to study the distance in direct-co-direct product where at least one factor is not connected. Also here one can observe some similarities between direct-co-direct and modular product. The distance (when neither factor is complete) is again limited, usually by three, often even by two, but there are some exceptions where the distance can be four or even five. For connected factors observe [10] and the case where at least one factor is not connected is treated in the rest of this paper.

2 Preliminaries

Let G be a finite, simple and undirected graph and $v \in V(G)$. The set $N_G(v)$ is the *open neighborhood* of v and contains all vertices adjacent to v in G. The *closed neighborhood* $N_G[v]$ of v is $N_G(v) \cup \{v\}$. We also use the notation $\overline{N}_G[g]$ for the complement $V(G) - N_G[g]$ of $N_G[g]$. The cardinality of $N_G(v)$ is the *degree* of v and is denoted by $\delta_G(v)$. An *isolated vertex* of G has $\delta_G(v) = 0$, a *leaf* of G has $\delta_G(v) = 1$ and a *universal vertex* of G has $\delta_G(v) = |V(G)| - 1$. The *complement* $\overline{G}$ of G is a graph with $V(\overline{G}) = V(G)$ and two vertices are adjacent in $\overline{G}$ whenever they are not adjacent in G. The subgraph of G induced by $S \subseteq V(G)$ is denoted by $G[S]$. By $G \cup H$ we mean the disjoint union of graphs G and H. As usual K_n is a complete graph on n vertices and $K_{p,q}$ is a complete bipartite graph with partitions of cardinality p and q.

By a product of two graphs G and H we mean a graph on a vertex set $V(G) \times V(H)$. Different products have then different definitions of their edge set. Two vertices (g, h) and (g', h') are adjacent in the *direct product* $G \times H$ if $gg \in E(G)$ and $hh \in E(H)$. All such edges are then called the *direct edges*. The edge set of *direct-co-direct product*, or DcD product for short, $G \circledast H$ can now be expressed as

$$E(G \circledast H) = E(G \times H) \cup E(\overline{G} \times \overline{H}). \tag{2.1}$$

In other words, $(g, h)(g', h') \in E(G \circledast H)$ if $gg \in E(G)$ and $hh \in E(H)$ or $gg' \in E(\overline{G})$ and $hh' \in E(\overline{H})$. Direct edges fulfill the first condition while the edges that correspond to the second condition are called the *co-direct* edges, because we can see them as the edges of direct product of complements of G and H, that is $\overline{G} \times \overline{H}$. Notice by (2.1) that $K_n \circledast H \cong K_n \times H$.

DcD product is clearly commutative because of its symmetric definition. On the other hand, it is not hard to see that DcD product is not associative, see Introduction section of [10]. From definition it follows that

$$G \circledast H \cong \overline{G} \circledast \overline{H}. \tag{2.2}$$

Clearly, we have $\overline{K}_n \circledast H \cong K_n \circledast \overline{H} \cong K_n \times \overline{H}$ by (2.2) and (2.1).

2.1 Connectivity

The result about connectivity of DcD product was given already in [10] and it is as follows. The proof strongly rely on connectivity of direct product due to Wiechsel in [15], see also [4].

Theorem 2.1 ([10, Theorem 2])**.** *Let G and H be two graphs on at least two vertices. Direct-co-direct product $G \circledast H$ is not connected if and only if*

- *one factor has a universal vertex and the other an isolated vertex, or*
- *one factor is K_2 and the other is bipartite, or*
- *one factor is $\overline{K}_2$ and the complement of the other is bipartite, or*
- *one factor is K_t and the other is not connected, or*
- *one factor is $\overline{K}_t$ and the complement of the other is not connected, or*
- *both factors are disjoint union of two complete graphs, or*
- *both factors are complete bipartite graphs.*

We are interested in disconnected factors. As already mention, totally disconnected graphs $\overline{K}_t$ can be treated in the same manner as complete graphs. Hence we are left only with two possible choises from above theorem that needed to be avoid: first and for-last item.

2.2 Distance

The *distance* $d_G(u,v)$ between vertices u and v in a graph G is the minimum number of edges on a path between u and v; if there is no such path, then we have $d_G(u,v) = \infty$. The maximum distance between v and any vertex of G is the *eccentricity* $\mathrm{ecc}_G(v)$ of v in G and the maximum eccentricity of a vertex in graph G is called the *diameter* $\mathrm{diam}(G)$ od G. By a distance formula in a graph product we usually describe a rule that completely describe the distance between two vertices (g,h) and (g',h') in a product. Such formulas are well known for Cartesian, strong and lexicographic product, see [4]. Here we rely more on the distance formula in direct product due to its connection with DcD product. This formula is a bit more complicated and contains odd distance $d^o_G(u,v)$ and even distance $d^e_G(u,v)$ between two vertices u and v in a graph G. More detailed, $d^o_G(u,v)$ is the minimum odd number of edges on a walk between u and v if it exists and ∞ otherwise. Similarly, $d^e_G(u,v)$ is the minimum even number of edges on a walk between u and v if it exists and ∞ otherwise. The following distance formula was proven for the direct product by Kim [11]

$$d_{G\times H}((g,h),(g',h')) = \min\{\max\{d_G^e(g,g'), d_H^e(h,h')\}, \max\{d_G^o(g,g'), d_H^o(h,h')\}\} \tag{2.3}$$

and an alternative approach can be found in [1]. By a direct use of this formula we can get the distance formula for $K_n \circledast H$ as well as for $\overline{K}_n \circledast H$ as observed already in [10]. We state only the formula for $\overline{K}_n \circledast H$ because we are interested in connected DcD products with at least one disconnected factor, see Theorem 2.1. Moreover, we divide this case to $n = 2$ and to $n \geq 3$. For $n = 2$ we need such a graph H, that $\overline{H}$ is connected and non-bipartite to assure connectedness of $\overline{K}_2 \circledast H$ by Theorem 2.1. We have $\overline{K}_2 \circledast H \cong K_2 \times \overline{H}$ and by (2.3) we immediately obtain

$$d_{\overline{K}_2\circledast H}((g,h),(g',h')) = \begin{cases} d^o_{\overline{H}}(h,h') & : \quad g \neq g' \\ d^e_{\overline{H}}(h,h') & : \quad g = g'. \end{cases}. \tag{2.4}$$

For $n \geq 3$ graph H must be such that $\overline{H}$ is connected to assure connectedness of $\overline{K}_n \circledast H$ by Theorem 2.1. We have $\overline{K}_n \circledast H \cong K_n \times \overline{H}$ and by (2.3) we obtain

$$d_{\overline{K}_n\circledast H}((g,h),(g',h')) = \begin{cases} \min\{d^o_{\overline{H}}(h,h'), \max\{2, d^e_{\overline{H}}(h,h')\}\} & : \quad g \neq g' \\ \min\{d^e_{\overline{H}}(h,h'), \max\{3, d^o_{\overline{H}}(h,h')\}\} & : \quad g = g'. \end{cases} \tag{2.5}$$

Another useful result from [10] is the following that describe all the pairs of vertices that are at distance two in $G \circledast H$. The idea of the proof is to check what happens if both edges on a shortest path are direct edges, both are co-direct edges or one is a direct and the other a co-direct edge.

Theorem 2.2 ([10, Theorem 9])**.** *Let G and H be non-complete graphs such that $G \circledast H$ is connected. The distance $d_{G\circledast H}((g,h),(g',h')) = 2$ if and only if at least one of the following possibilities holds for some $g'' \in V(G)$ and $h'' \in V(H)$*

(i) *(path $gg''g'$ is induced in G and $hh'h''$ is C_3 in H) or (path $hh''h'$ is induced in H and $gg'g''$ is C_3 in G);*

(ii) *(path $gg''g'$ is induced in $\overline{G}$ and $hh'h''$ is C_3 in $\overline{H}$) or (path $hh''h'$ is induced in $\overline{H}$ and $gg'g''$ is C_3 in $\overline{G}$);*

(iii) *($g = g'$ and $gg'' \in E(G)$ and $hh''h'$ is a path in H) or ($h = h'$ and $hh'' \in E(H)$ and $gg''g'$ is a path in G);*

(iv) *($g = g'$ and $gg'' \in E(\overline{G})$ and $hh''h'$ is a path in $\overline{H}$) or ($h = h'$ and $hh'' \in E(\overline{H})$ and $gg''g'$ is a path in $\overline{G}$);*

(v) *($g'gg''$ is induced in G and $hh'h''$ is induced in $\overline{H}$) or ($gg'g''$ is induced in $\overline{G}$ and $h'hh''$ is induced in H);*

(vi) *($g'gg''$ is induced in $\overline{G}$ and $hh'h''$ is induced in H) or ($gg'g''$ is induced in G and $h'hh''$ is induced in $\overline{H}$).*

3 Main result

We start with several conditions that describe all the situations when the distance exceeds three in $G \circledast H$. First three conditions are about two components in both factors G and H.

Condition 1. Let G and H be graphs with two components, $\{X,Y\} = \{G,H\}$, $X = X_1 \cup X_2$ and $Y \cong K_1 \cup K_t$, $t \geq 2$, and at most one of X_1 and X_2 is complete. Let x be the universal vertex of X_1, let x' be the universal vertex of X_2 and let $y = y'$ be the isolated vertex of Y.

Notice that if both X_1 and X_2 are complete in above condition, then $X \circledast Y$ is not connected by Theorem 2.1.

Condition 2. Let G and H be graphs with two components, $\{X,Y\} = \{G,H\}$, $X = X_1 \cup K_s$, where $X_1 \ncong K_r$ and $Y \cong K_1 \cup K_t$, $t \geq 2$. Let x and x' be the universal vertices of X_1 (they can be the same vertex), let y be the isolated vertex of Y and let $y' \in V(K_t)$.

Next condition is just a symmetric version of Condition 2 with respect to y and y'.

Condition 3. Let G and H be graphs with two components, $\{X,Y\} = \{G,H\}$, $X = X_1 \cup K_s$, where $X_1 \ncong K_r$ and $Y \cong K_t \cup K_1$, $t \geq 2$. Let x and x' be the universal verices of X_1 (they can be the same vertex), let y' be the isolated vertex of Y and let $y \in V(K_t)$.

We continue with a condition that we need when one factor has at least three components and the other is connected.

Condition 4. Let G and H be two graphs such that one has at least two components and the other is $K_{1,t}$, $t \geq 2$, $\{X,Y\} = \{G,H\}$, $X = X_1 \cup X_2 \cup \cdots \cup X_r$ and $Y \cong K_{1,t}$. Let $y = y'$ be a universal vertex of Y and let x and x' be vertices of X_1 such that $d_X(x,x') = 3$ or $d_X(x,x') = 1$ and such that for every $x_1 \in N_X(x)$ it holds that $N_X(x') \subseteq N_X(x_1)$.

All the following conditions are needed in the case where one factor is connected and the other has two components.

Condition 5. Let G and H be two graphs such that one has two components and the other is connected, $\{X,Y\} = \{G,H\}$, $X = K_2 \cup K_t$, $t \geq 1$, and $Y = K_{2,p}$, $p \geq 1$. Let $K_2 = xx'$ for X and $\{y,y'\}$ is one of bipartite sets of Y.

Condition 6. Let G and H be two graphs such that one has two components and the other is connected, $\{X,Y\} = \{G,H\}$, $X = K_2 \cup K_1$. Let $x = x' \in V(K_2)$ and let $y, y' \in V(Y)$ be such that $\mathrm{ecc}_Y(y) = 3$, $d_Y(y,y') = 3$ and $Y[N_Y[y]]$ and $Y[\overline{N}_Y[y]]$ are complete.

Condition 7. Let G and H be two graphs such that one has two components and the other is connected, $\{X,Y\} = \{G,H\}$, $X = K_t \cup K_1$, $t \geq 2$. Let $x \in V(K_t)$, x' is the component K_1 and let $y, y' \in V(Y)$ be such that $\mathrm{ecc}_Y(y) = 3$, y' is a universal vertex of $Y[N_Y[y]]$, y' is adjacent to all vertices that are at distance two to y and every vertex at distance three to y is a universal vertex of $Y[\overline{N}_Y[y]]$.

For the next condition we need a new notation. A vertex $g_1 \in N_G(g)$ is a *satellite* of g if $N_G[g_1] \subseteq N_G[g]$. Set B_1' contains further all the satellite vertices of g.

Condition 8. Let G and H be two graphs such that one has two components and the other is connected, $\{X,Y\} = \{G,H\}$, $X = K_1 \cup K_p$, $p \geq 2$. Let $x = x'$ be component K_1, let $y \in V(Y)$ be a vertex with $\mathrm{ecc}_Y(y) = 3$ and let $y' \in V(Y)$ be such that $d_Y(y,y') = 2$ and y' is adjacent to every vertex from $V(Y) - (B'_1 \cup \{y,y'\})$ where B'_1 contains all the satellite vertices of y.

Condition 9. Let G and H be two graphs such that one has two components and the other is connected, $\{X,Y\} = \{G,H\}$, $X = K_1 \cup K_p$, $p \geq 2$. Let x be a component K_1, let $x' \in V(K_p)$ and let $y \in V(Y)$ be such that $\mathrm{ecc}_Y(y) = 2$, $Y[\overline{N}_Y[y]]$ is complete and let y' be a satellite vertex of y such that $N_Y[y'] = B'_1 \cup \{y\}$ and there exists all edges between $\overline{N}_Y[y]$ and $N_Y(y) - B'_1$ where B'_1 contains all the satellite vertices of y.

Condition 10. Let G and H be two graphs such that one has two components and the other is connected, $\{X,Y\} = \{G,H\}$, $X = K_1 \cup K_p$, $p \geq 2$. Let $x = x'$ be a component K_1, let $y \in V(Y)$ be a vertex with $\mathrm{ecc}_Y(y) = 2$ and let $y' \in V(Y)$ be such that $d_Y(y,y') = 2$ and y' is adjacent to every vertex from $V(Y) - (B'_1 \cup \{y,y'\})$ where B'_1 contains all the satellite vertices of y.

Condition 11. Let G and H be two graphs such that one has two components and the other is connected, $\{X,Y\} = \{G,H\}$, $X = K_1 \cup K_p$, $p \geq 2$. Let $x = x'$ be a component K_1, let $y \in V(Y)$ be such that $\mathrm{ecc}_Y(y) = 2$ and let y' be adjacent to every vertex of $V(Y) - (B'_1 \cup \{y'\})$ and (there exists a vertex from $\overline{N}_Y[y']$ that is not adjacent to some vertex from B'_1 or $Y[\overline{N}_Y[y]]$ is not complete or there is an edge between $\overline{N}_Y[y']$ and $N_Y(y) - B'_1$) where B'_1 contains all the satellite vertices of y.

The only situation where G or H is not connected and we have distance five between two vertices of $G \circledast H$ is described by the following condition, see Theorem 4.5.

Condition 12. Let G and H be two graphs such that one has two components and the other is connected, $\{X,Y\} = \{G,H\}$, $X = K_1 \cup K_p$, $p \geq 2$. Let $x = x'$ be a component K_1, let $y \in V(Y)$ be such that $\mathrm{ecc}_Y(y) = 2$ and let y' be adjacent to every vertex of $V(Y) - (B'_1 \cup \{y'\})$ and every vertex from $\overline{N}_Y[y']$ is adjacent to every vertex from B'_1 and $Y[\overline{N}_Y[y]]$ is complete and there is no edge between $\overline{N}_Y[y']$ and $N_Y(y) - B'_1$ where B'_1 contains all the satellite vertices of y.

The main goal is to prove the distance formula for DcD product under the assumption that at least one factor of $G \circledast H$ is not connected and that the product is connected according to Theorem 2.1. The formula is as follows:

$$d_{G \circledast H}((g,h),(g',h')) = \begin{cases} 0 & : \quad g = g' \wedge h = h' \\ 1 & : \quad (g,h)(g',h') \in E(G \circledast H) \\ 2 & : \quad (g,h),(g',h') \text{ fulfills Theorem 2.2} \\ 3 & : \quad \text{otherwise} \\ 4 & : \quad \text{one of Conditions 1 – 11 holds} \\ 5 & : \quad \text{Condition 12 holds.} \end{cases} \tag{3.1}$$

The proof for this formula follows directly from the adjacency definition of $G \circledast H$, from Theorem 2.2 and from Theorems 4.1 – 4.5 that are stated in the next section. In the mentioned theorems we bound eccentricity of a vertex (g,h) by five in all different possible cases with respect to the number of components in factors G and H. In particular, we fully describe all the cases when the eccentricity is five by Condition 12 and when it is four by Conditions 1 – 11.

We end with two simple corollaries about diameter of DcD product. The first one is a direct consequence of Theorem 4.1 and the second one follows from situations described in Conditions 1 – 12.

Corollary 3.1. *If G and H are both graphs with at least three components, then* $\operatorname{diam}(G \circledast H) = 2$.

Corollary 3.2. *Let G and H be graphs different than $K_{1,t}$, $K_1 \cup K_t$ and $K_2 \cup K_t$, $t \geq 2$. If at least one of G and H is not connected, then* $\operatorname{diam}(G \circledast H) \leq 3$.

4 Eccentricity of a vertex in DcD product

In this section we bound the eccentricity of a vertex (g, h) in a connected DcD product $G \circledast H$ of graphs G and H, where at least one of G and H is disconnected. For this we use the following notation. Suppose that graph G has r components. We write $G = G_1 \cup G_2 \cup \cdots \cup G_r$, where G_i is a component of graph G for every $i \in \{1, 2, \ldots, r\}$.

We have to go through all possible numbers of components of factors. Namely, we study the eccentricity of vertices in DcD products where the numbers of components of both factors are all integer pairs (a, b), where $a \geq 1$ and $b \geq 1$, except the pair $(1, 1)$ which has already been studied in [10]. First we describe the situation, where both factors G and H have at least three components.

Theorem 4.1. *If G and H have at least three components, then* $\operatorname{ecc}_{G \circledast H}((g, h)) = 2$ *for any vertex* $(g, h) \in V(G \circledast H)$.

Proof. Without loss of generality suppose that $g \in V(G_1)$ and $h \in V(H_1)$. For any vertex (g', h'), where $g' \notin V(G_1)$ and $h' \notin V(H_1)$ we have an edge between (g, h) and (g', h') in DcD product $G \circledast H$. Next, suppose that $g' \in V(G_1)$ and h' is an arbitrary vertex from $V(H)$. Let $g'' \in V(G_2)$ be an arbitrary vertex. We can choose vertex h'' such that it is not in the same component as h and h', since we have at least three components in H. It follows that $(g, h)(g'', h'')(g', h')$ is a path of length two in $G \circledast H$ and $d_{G \circledast H}((g, h), (g', h')) \leq 2$. One can see that $d_{G \circledast H}((g, h), (g', h')) \leq 2$ also in the symmetric situation when $h' \in V(H_1)$ and g' is an arbitrary vertex from $V(G)$, since we have at least three components also in G. The equality follows because (g, h) is not adjacent to (g, h') for $h' \neq h$. □

Next, we study the situation, where one factor has two components and the other at least three components. We can suppose that the second factor has two components, since DcD product is commutative.

Theorem 4.2. *Let G and H be graphs. If one of them has at least three components and the other has two components, then* $\operatorname{ecc}_{G \circledast H}((g, h)) \leq 3$, *for any vertex* $(g, h) \in V(G \circledast H)$.

Proof. Without loss of generality suppose that G has at least three components and H has two components. Let $g \in V(G_1)$ and $h \in V(H_1)$. For any vertex (g', h'), where $g' \notin V(G_1)$ and $h' \in V(H_2)$ we have an edge between (g, h) and (g', h') in DcD product $G \circledast H$. Next, suppose that $h' \in V(H_1)$ and $g' \in V(G)$ are arbitrary vertices. We select $h'' \in V(H_2)$ arbitrarily. We can choose vertex g'' such that it is not in the same component as g and g', since we have at least three components in G. It follows that $(g,h)(g'',h'')(g',h')$ is a path of length two in $G \circledast H$ and $d_{G\circledast H}((g,h),(g',h')) \leq 2$. It remains to check the vertices (g',h'), where $g' \in V(G_1)$ and $h' \in V(H_2)$. Let $g_2 \in V(G_2)$ and $g_3 \in V(G_3)$. There exists a path $(g,h)(g_2,h')(g_3,h)(g',h')$ in $G \circledast H$ and $d_{G\circledast H}((g,h),(g',h')) \leq 3$. Therefore, $\mathrm{ecc}_{G\circledast H}((g,h)) \leq 3$. □

Now we study the eccentricity of vertices of DcD products where both factors G and H have two components. Notice that we are not studying cases where one factor is isomorphic to $\overline{K}_2$, since this was already covered by (2.4). The Conditions 1, 2 and 3 are essential to describe all the vertices of eccentricity four in this case.

Theorem 4.3. *If G and H are graphs with two components different than $\overline{K}_2$ such that $G \circledast H$ is connected, then $\mathrm{ecc}_{G\circledast H}((g,h)) \leq 4$. Moreover, we have $\mathrm{ecc}_{G\circledast H}((g,h)) = d_{G\circledast H}((g,h),(g',h')) = 4$ if and only if* Conditions 1 or *2 or 3 are fulfilled.*

Proof. Let $G = G_1 \cup G_2$ and $H = H_1 \cup H_2$. Without loss of generality suppose that $g \in V(G_1)$ and $h \in V(H_1)$. For any vertex (g',h'), where $g' \in V(G_2)$ and $h' \in V(H_2)$ we have an edge between (g,h) and (g',h') in DcD product $G \circledast H$. Next, suppose that $h' \in V(H_1)$ and $g' \in V(G_1)$ are arbitrary vertices, where $(g,h) \neq (g',h')$. There exists a path $(g,h)(g'',h'')(g',h')$ such that $g'' \in V(G_2)$ and $h'' \in V(H_2)$ and $d_{G\circledast H}((g,h),(g',h')) \leq 2$. Therefore, $d_{G\circledast H}((g,h),(g',h')) \leq 2$ for any $(g',h') \in (V(G_1) \times V(H_1)) \cup (V(G_2) \times V(H_2))$. Hence we deal in the remaining of the proof with vertices from $(V(G_1) \times V(H_2)) \cup (V(G_2) \times V(H_1))$. Suppose that vertex g is not a universal vertex of G_1 such that $gg_1 \notin E(G_1)$, $g_1 \in V(G_1)$. For any vertex (g',h'), where $g' \in V(G_2)$ and $h' \in V(H_1)$, there exists a path $(g,h)(g_1,h_2)(g',h')$ for $h_2 \in V(H_2)$. It follows that $d_{G\circledast H}((g,h),(g',h')) \leq 2$. For any vertex (g',h'), where $g' \in V(G_1)$ and $h' \in V(H_2)$, there exists a walk $(g,h)(g_1,h')(g_2,h)(g',h')$ for $g_2 \in V(G_2)$. It follows that $d_{G\circledast H}((g,h),(g',h')) \leq 3$. With similar construction we can bound the distance from (g,h) to the vertices of $(V(G_1) \times V(H_2)) \cup (V(G_2) \times V(H_1))$ in the case when vertex h is not a universal vertex of H_1. Therefore, we can assume that both g and h are universal vertices in their components.

In the rest of the proof we split cases according to the fact whether some components of factors are complete graphs or not. First we go through situations where at least two (out of four) components are not complete graphs and then we analyse cases where exactly one connected component is not a complete graph. Recall, from Theorem 2.1, that if all four components are complete graphs, then the DcD product is not connected.

Suppose first that G_1 and H_1 are not complete graphs, so there are $g_1g_1' \notin E(G_1)$ and $h_1h_1' \notin E(H_1)$. For each vertex $(g',h') \in V(G_1) \times V(H_2)$ there exists a path $(g,h)(g_1,h_1)(g_2,h_1')(g',h')$ for $g_2 \in V(G_2)$ and $d_{G\circledast H}((g,h),(g',h')) \leq 3$. Similarly for $(g',h') \in V(G_2) \times V(H_1)$ there exists a path $(g,h)(g_1,h_1)(g_1',h_2)(g',h')$ for $h_2 \in V(H_2)$ and $d_{G\circledast H}((g,h),(g',h')) \leq 3$.

Next suppose that G_1 and H_2 are not complete graphs, therefore, there exist non-edges $g_1g_1' \notin E(G_1)$ and $h_2h_2' \notin E(H_2)$. For each vertex $(g',h') \in V(G_2) \times V(H_1)$ there exists a path $(g,h)(g',h_2)(g,h_2')(g',h')$ and $d_{G\circledast H}((g,h),(g',h')) \leq 3$. Let (g',h') be an arbitrary vertex from $V(G_1) \times V(H_2)$. If $h' \in V(H_2)$ is not a universal vertex of H_2, then there is a non-edge $h'h'' \notin E(H_2)$. It follows that there exists a path $(g,h)(g_2,h'')(g',h')$ for $g_2 \in V(G_2)$ and $d_{G\circledast H}((g,h),(g',h')) = 2$. Otherwise, $h' \in V(H_2)$ is a universal vertex of H_2. If $g' \neq g$, then there exists a path $(g,h)(g_2,h_2)(g,h_2')(g',h')$ for $g_2 \in V(G_2)$ and

$d_{G \circledast H}((g,h),(g',h')) \leq 3$. If $g' = g$, then there exists a path $(g,h)(g_2,h_2)(g_1,h_2')(g',h')$ for $g_2 \in V(G_2)$ and again $d_{G \circledast H}((g,h),(g',h')) \leq 3$. Notice that the case where G_2 and H_1 are not complete graphs is symmetric to the case where G_1 and H_2 are not complete graphs, so we can derive the same conclusions.

Now suppose that G_2 and H_2 are not complete graphs, therefore, there exist non-edges $g_2g_2' \notin E(G_2)$ and $h_2h_2' \notin E(H_2)$. For each vertex $(g',h') \in V(G_2) \times V(H_1)$ there exists a path $(g,h)(g_2,h_2)(g,h_2')(g',h')$ and it holds that $d_{G \circledast H}((g,h),(g',h')) \leq 3$. Symmetrically we consider the case when $(g',h') \in V(G_1) \times V(H_2)$.

The last case where at least two components are not complete graphs is the case where G_1 and G_2 are not complete graphs. Notice that the case where H_1 and H_2 are not complete graphs is symmetrical. Therefore, there exist non-edges $g_1g_1' \notin E(G_1)$ and $g_2g_2' \notin E(G_2)$. For each vertex $(g',h') \in V(G_1) \times V(H_2)$ there exists a path $(g,h)(g_2,h')(g_2',h)(g',h')$ and $d_{G \circledast H}((g,h),(g',h')) \leq 3$. Let (g',h') be an arbitrary vertex from $V(G_2) \times V(H_1)$. If $g' \in V(G_2)$ is not a universal vertex of G_2, then there is a non-edge $g'g'' \notin E(G_2)$. It follows that there exists a path $(g,h)(g'',h_2)(g',h')$, for $h_2 \in V(H_2)$, and $d_{G \circledast H}((g,h),(g',h')) = 2$. Otherwise, $g' \in V(G_2)$ is a universal vertex of G_2. Suppose that there are at least two vertices h and h_1 in H_1. If $h' \neq h$, then there exists a path $(g,h)(g_2,h_2)(g_2',h)(g',h')$ for $h_2 \in V(H_2)$ and $d_{G \circledast H}((g,h),(g',h')) \leq 3$. If $h' = h$, then there exists a path $(g,h)(g_2,h_2)(g_2',h_1)(g',h')$ for $h_2 \in V(H_2)$ and again $d_{G \circledast H}((g,h),(g',h')) \leq 3$. Suppose now that there is only one vertex in H_1, namely $H_1 \cong K_1$. If H_2 is not isomorphic to a complete graph, then there exists a non-edge $h_2h_2' \notin E(H_2)$ and there is a path $(g,h)(g_2,h_2)(g,h_2')(g',h')$. It follows that $d_{G \circledast H}((g,h),(g',h')) \leq 3$. Otherwise, $H_2 \cong K_t$, $t \geq 2$ and Condition 1 is fulfilled for the pair of vertices (g,h) and (g',h'). With the path $(g,h)(g_2,h_2)(g_2',h)(g,h_2)(g',h')$ for $h_2 \in V(H_2)$ we get $d_{G \circledast H}((g,h),(g',h')) \leq 4$.

It remains to study the cases where exactly one component is not a complete graph. Suppose first that only G_2 is not a complete graph. Therefore, there exists a non-edge $g_2g_2' \notin E(G_2)$. We can make the same construction of the paths and we get the same conclusions as in the previous case where G_1 and G_2 were not complete graphs, since there were no usage of vertices g_1 and g_1' in the proof. The only difference is that there is no possibility that H_2 is not isomorphic to a complete graph. Notice that also Condition 1 is fulfilled since it requires that one connected component has to be non-isomorphic to a complete graph. The case where only H_2 is not a complete graph is symmetric.

Suppose that only G_1 is not a complete graph. There exists a non-edge $g_1g_1' \notin E(G_1)$. Suppose that $|V(H_1)| \geq 2$ and $|V(H_2)| \geq 2$. For each vertex $(g',h') \in V(G_2) \times V(H_1)$ there exists a path $(g,h)(g_1,h_1)(g_1',h_2)(g',h')$ for $h_1 \in N_{H_1}(h)$ and $h_2 \in V(H_2)$. It follows that $d_{G \circledast H}((g,h),(g',h')) \leq 3$. Let (g',h') be an arbitrary vertex from $V(G_1) \times V(H_2)$. If g' is not a universal vertex of G_1, then there is a non-edge $g'g'' \notin E(G_1)$. It follows that there exists a path $(g,h)(g'',h_1)(g',h')$ for $h_1 \in N_{H_1}(h)$ and $d_{G \circledast H}((g,h),(g',h')) \leq 2$. Otherwise, g' is a universal vertex of G_1. There exists a path $(g,h)(g_1,h_1)(g_1',h_2)(g',h')$ for $h_1 \in N_{H_1}(h)$ and $h_2 \in N_{H_2}(h')$. Again $d_{G \circledast H}((g,h),(g',h')) \leq 3$. Suppose now that $H_1 \cong K_1$. It follows that $H_2 \cong K_t$, $t \geq 2$. For (g,h) and $(g',h') \in V(G_2) \times V(H_1)$ the Condition 1 is fulfilled. With the path $(g,h)(g',h_2)(g_1,h)(g_1',h_2)(g',h')$ for $h_2 \in V(H_2)$ we get that $d_{G \circledast H}((g,h),(g',h')) \leq 4$. Let (g',h') be an arbitrary vertex from $V(G_1) \times V(H_2)$. If g' is not a universal vertex of G_1, then there is a non-edge $g'g'' \notin E(G_1)$. It follows that there exists a path $(g,h)(g_2,h')(g'',h)(g',h')$ for $g_2 \in V(G_2)$ and we have $d_{G \circledast H}((g,h),(g',h')) \leq 3$. Otherwise, g' is a universal vertex of G_1 and the Condition 2 is fulfilled for (g,h) and (g',h'). With the path $(g,h)(g_2,h')(g_1,h)(g'_1,h_2)(g',h')$ for $g_2 \in V(G_2)$ and $h_2 \in N_{H_2}(h')$ we get that $d_{G\circledast H}((g,h),(g',h')) \leq 4$. Finally suppose that $H_2 \cong K_1 = h_2$. It follows that $H_1 \cong K_t$, $t \geq 2$. For each vertex $(g',h') \in V(G_2) \times V(H_1)$ there exists a path $(g,h)(g_1,h_1)(g'_1,h_2)(g',h')$ for $h_1 \in N_{H_1}(h)$. It follows that $d_{G\circledast H}((g,h),(g',h')) \leq 3$. Let (g',h') be an arbitrary vertex from $V(G_1) \times V(H_2)$. If g' is not a universal vertex of G_1, then there is a non-edge $g'g'' \notin E(G_1)$. It follows that there exists a path $(g,h)(g'',h_1)(g',h')$ for $h_1 \in N_{H_1}(h)$ and $d_{G\circledast H}((g,h),(g',h')) \leq 2$. Otherwise, g' is a universal vertex of G_1 and the Condition 3 is fulfilled for (g,h) and (g',h'). With the path $(g,h)(g_1,h_1)(g'_1,h')(g_2,h)(g',h')$ for $h_1 \in N_{H_1}(h)$ and $g_2 \in V(G_2)$ we get that $d_{G\circledast H}((g,h),(g',h')) \leq 4$. The case where only H_1 is not a complete graph is symmetric.

For the equivalence notice from this proof until now, that if Conditions 1 or 2 or 3 are not fulfilled, then $\mathrm{ecc}_{G\circledast H}((g,h)) \leq 3 \neq 4$, which proves one implication of the equivalence. For the other implication, suppose that Conditions 1 or 2 or 3 are fulfilled for (g,h) and (g',h'). By the symmetry we may assume that $H \cong K_1 \cup K_t$, $t \geq 2$. Suppose first that Condition 1 holds. We will show that $d_{G\circledast H}((g,h),(g',h')) \geq 4$. The neighborhood of (g,h) in DcD product $G \circledast H$ is $V(G_2) \times V(H_2)$, since $H_1 \cong K_1$ and g is a universal vertex of G_1. Similarly, the neighborhood of (g',h') in DcD product $G \circledast H$ is $V(G_1) \times V(H_2)$, since $H_1 \cong K_1$ and g' is a universal vertex of G_2. There are no edges between vertex sets $V(G_1) \times V(H_2)$ and $V(G_2) \times V(H_2)$, since $H_2 \cong K_t$. Hence $d_{G\circledast H}((g,h),(g',h')) \geq 4$ and it follows that $\mathrm{ecc}_{G\circledast H}((g,h)) = d_{G\circledast H}((g,h),(g',h')) = 4$. Next, suppose that Condition 2 holds. Again, we will show that $d_{G\circledast H}((g,h),(g',h')) \geq 4$. The neighborhood of (g,h) in DcD product $G \circledast H$ is $V(G_2) \times V(H_2)$, since $H_1 \cong K_1$ and g is a universal vertex of G_1. The neighborhood of (g',h') in DcD product $G \circledast H$ is a subset of $(V(G_1) \times V(H_2)) \cup (V(G_2) \times V(H_1))$, since $H_2 \cong K_t$ and g' is a universal vertex of G_1. There are no edges between sets $V(G_2) \times V(H_2)$ and $(V(G_1) \times V(H_2)) \cup (V(G_2) \times V(H_1))$, since $G_2 \cong K_s$ and $H_2 \cong K_t$. Hence $d_{G\circledast H}((g,h),(g',h')) \geq 4$ and it follows that $\mathrm{ecc}_{G\circledast H}((g,h)) = d_{G\circledast H}((g,h),(g',h')) = 4$. Finally, suppose that Condition 3 holds. We will show that $d_{G\circledast H}((g,h),(g',h')) \geq 4$. The neighborhood of (g,h) in DcD product $G \circledast H$ is a subset of $(V(G_1) \times V(H_1)) \cup (V(G_2) \times V(H_2))$, since $H_1 \cong K_t$ and g is a universal vertex of G_1. The neighborhood of (g',h') in DcD product $G \circledast H$ is $V(G_2) \times V(H_1)$, since $H_2 \cong K_1$ and g' is a universal vertex of G_1. There are no edges between vertex sets $(V(G_1) \times V(H_1)) \cup (V(G_2) \times V(H_2))$ and $V(G_2) \times V(H_1)$, since $G_2 \cong K_s$ and $H_1 \cong K_t$. Hence $d_{G\circledast H}((g,h),(g',h')) \geq 4$ and $\mathrm{ecc}_{G\circledast H}((g,h)) = d_{G\circledast H}((g,h),(g',h')) = 4$ follows. □

Next we study the eccentricity of the vertices of DcD products where one factor is connected and the other has at least three components. We are not studying cases where one factor is isomorphic to $\overline{K}_n$, $n \geq 3$, since this is already covered by (2.5). In this case Condition 4 describes the only situation where the distance exceeds three.

Theorem 4.4. *Let G and H be graphs such that $G \circledast H$ is connected. If one of them has at least three components but is not isomorphic to $\overline{K}_n, n \geq 3$, and the other is connected, then* $\mathrm{ecc}_{G\circledast H}((g,h)) \leq 4$. *Moreover, we have* $\mathrm{ecc}_{G\circledast H}((g,h)) = d_{G\circledast H}((g,h),(g',h')) = 4$ *if and only if* Condition 4 *is fulfilled.*

Proof. Without loss of generality suppose that H is connected, $G = G_1 \cup G_2 \cup \cdots \cup G_k$, $k \geq 3$, and $g \in V(G_1)$. In the first part of the proof we split two main cases regarding the neighborhood of the vertex h.

Case 1. Vertex $h \in V(H)$ is not a universal vertex.
For any vertex (g', h'), where $g' \in V(G) - V(G_1)$ and $h' \in \overline{N}_H[h]$ we have an edge between (g, h) and (g', h') in DcD product $G \circledast H$. Next, suppose that $h' = h$ and g' is an arbitrary vertex from $V(G)$ different than g. We select $h'' \in \overline{N}_H[h]$ arbitrarily. We can choose vertex g'' such that it is not in the same connected component as g and g', since we have at least three components in G. It follows that $(g, h)(g'', h'')(g', h')$ is a path of length two in $G \circledast H$ and $d_{G\circledast H}((g, h),(g', h')) = 2$. For any vertex (g', h'), where $g' \in V(G_1)$ and $h' \in \overline{N}_H[h]$ there exists a path $(g, h)(g'', h')(g''', h)(g', h')$, for arbitrary vertices $g'' \in V(G_2)$ and $g''' \in V(G_3)$. Hence, $d_{G\circledast H}((g, h),(g', h')) \leq 3$.
It remains to check the vertices (g', h') where $h' \in N_H(h)$ and g' is an arbitrary vertex from $V(G)$. If h' is not adjacent with at least one vertex h'' from the set $\overline{N}_H[h]$, then there exists a path $(g, h)(g'', h'')(g', h')$, where g'' is a vertex from different component than g and g'. Otherwise, h' is adjacent with all the vertices from $\overline{N}_H[h]$. If g' is not an isolated vertex in graph G, then there exists $g'' \in V(G)$, such that $g'g'' \in E(G)$. If g' is additionally not from $V(G_1)$, then there exists a path $(g, h)(g'', h'')(g', h')$, for an arbitrary vertex $h'' \in \overline{N}_H[h]$. Otherwise, $g' \in V(G_1)$ and there exists a walk $(g, h)(g_2, h'')(g'', h)(g', h')$, where $g_2 \in V(G) - V(G_1)$ and $h'' \in \overline{N}_H[h]$. So let g' be an isolated vertex. In this case h' is not a universal vertex by the fact that $G \circledast H$ is a connected graph. Therefore, there exists $h'' \in N_H(h)$ such that $h'h'' \notin E(H)$. Recall also that $G \ncong \overline{K}_n$ and there exist g'' and g''' such that $g''g''' \in E(G)$. Suppose that $g' = g$. If $h''h''' \in E(H)$, for some $h''' \in \overline{N}_H[h]$, then there exists a path $(g, h)(g''', h''')(g_0, h'')(g', h')$ where g_0 is from different component than g'. Otherwise, h'' is not adjacent to h''' for every vertex $h''' \in \overline{N}_H[h]$ and there exists a path $(g, h)(g_x, h''')(g'', h'')(g', h')$, where $g_x \in V(G)$ is neither from connected component that contains g'' nor from G_1. Suppose that $g' \neq g$. If $h''h''' \notin E(H)$, for some $h''' \in \overline{N}_H[h]$, then there exists a path $(g, h)(g', h''')(g'', h'')(g', h')$. Otherwise, h'' is adjacent to every $h''' \in \overline{N}_H[h]$. If $g''' \notin N_G[g]$, then there exists a path $(g, h)(g''', h''')(g'', h'')(g', h')$. If $g''' \in N_G(g)$, then there exists a path $(g, h)(g''', h'')(g', h')$. If $g''' = g$, then there exists a path $(g, h)(g'', h'')(g', h')$. Therefore, for all vertices (g', h'), where $h' \in N_H(h)$ and g' is an arbitrary vertex from $V(G)$, it holds that $d_{G\circledast H}((g, h),(g', h')) \leq 3$.

Case 2. Vertex h is a universal vertex.
In this case none of the components of G is an isolated vertex by Theorem 2.1, since graph $G \circledast H$ is connected. Let $g' \in V(G) - V(G_1)$. If h' is not a universal vertex in graph H, then there exists a path $(g, h)(g_1, h_1)(g', h')$, where $g_1 \in N_G(g)$ and $h_1h' \notin E(H)$. If h' is a universal vertex in H (it can also be h), then there exists a path $(g, h)(g_1, h_1)(g'', h_2)(g', h')$, where $g_1 \in N_G(g)$, $h_1h_2 \notin E(H)$ and $g'' \in N_G(g')$. Notice that such h_1 and h_2 exist, since H is not a complete graph by Theorem 2.1 as $G \circledast H$ is connected. Next, let $g' \in V(G_1)$. If h' is not a universal vertex in H, then there exists a walk $(g, h)(g_1, h')(g_2, h_1)(g', h')$, where $g_1 \in N_G(g)$, $g_2 \in V(G_2)$ and $h_1h' \notin E(H)$. If h' is a universal vertex in H different than h, then there can appear five different options with respect to g'. First, if $g' = g$, then there exists a path $(g, h)(g_1, h_1)(g', h')$, where $g_1 \in N_G(g)$ and h_1 an arbitrary vertex different that h and h'. Second, if $g' \in N_G(g)$, then (g, h) and (g', h') are adjacent. Third, if $d_G(g, g') = 2$, then there exists a path $(g, h)(g_1, h_1)(g', h')$, where $g_1 \in N_G(g) \cap N_G(g')$ and h_1 an

arbitrary vertex different that h and h'. Fourth, if $d_G(g, g') = 3$, then there exists a path $(g, h)(g_1, h')(g_2, h)(g', h')$, where $g_1 \in N_G(g)$ and $g_2 \in N_G(g')$, $g_1g_2 \in E(G)$, are two mid-vertices from some shortest path between g and g'. Fifth, if $d_G(g, g') \geq 4$, then there exists a path $(g, h)(g_1, h_1)(g_2, h_2)(g', h')$, where $g_1 \in N_G(g)$, $h_1h_2 \notin E(H)$ and $g_2 \in N_G(g')$ (notice that $g_1g_2 \notin E(G)$). Therefore, for all vertices $(g', h') \in V(G \circledast H) - (V(G_1) \times \{h\})$ it holds that $d_{G\circledast H}((g, h), (g', h')) \leq 3$. It remains to check the vertices (g', h'), where $g' \in V(G_1)$ and $h' = h$. If there exists an edge $h_xh_y \in E(H)$, where $h_x \neq h$ and $h_y \neq h$ (with this $H \ncong K_{1,t}, t \geq 2$), then there can appear four different options with respect to g'. First, if $g' \in N_G(g)$, then there exists a path $(g, h)(g', h_x)(g, h_y)(g', h')$. Second, if $d_G(g, g') = 2$, then there exists a path $(g, h)(g_1, h_x)(g', h')$, where $g_1 \in N_G(g) \cap N_G(g')$. Third, if $d_G(g, g') = 3$, then there exists a path $(g, h)(g_1, h_x)(g_2, h_y)(g', h')$, where $g_1 \in N_G(g)$ and $g_2 \in N_G(g')$, $g_1g_2 \in E(G)$, are two mid-vertices from some shortest path between g and g'. Fourth, if $d_G(g, g') \geq 4$, then there exists a path $(g, h)(g_1, h_1)(g_2, h_2)(g', h')$, where $g_1 \in N_G(g)$, $h_1h_2 \notin E(H)$ (recall that $H \ncong K_p$ as $G \circledast H$ is connected) and $g_2 \in N_G(g')$. Again, for all vertices (g', h') it holds that $d_{G\circledast H}((g, h), (g', h')) \leq 3$. We are left with the case that $H \cong K_{1,t}, t \geq 2$. If $d_G(g, g') = 2$ or $d_G(g, g') \geq 4$, then we can construct the same paths than before, since they are not dependent on the edge h_xh_y. Let $g' \in V(G_1)$, such that $d_G(g, g') = 3$ or $d_G(g, g') = 1$. If there exists a vertex $g_1 \in N_G(g)$ such that $g_1g'' \notin E(G)$, for some $g'' \in N_G(g')$, then there exists a path $(g, h)(g_1, h_1)(g'', h_2)(g', h')$, where h_1 and h_2 are different leaves of H, and $d_{G\circledast H}((g, h), (g', h')) \leq 3$. Otherwise, for every $g_1 \in N_G(g)$ it holds that $N_G(g') \subseteq N_G(g_1)$ and the Condition 4 is fulfilled. There exists a path $(g, h)(g_1, h_1)(g_2, h_2)(g'', h_1)(g', h')$, where h_1 and h_2 are different leaves of the star graph H, $g_1 \in N_G(g)$, $g_2 \in V(G_2)$ and $g'' \in N_G(g')$. It follows that $d_{G\circledast H}((g, h), (g', h')) \leq 4$ and Case 2 is completed.

For the equivalence notice from this proof until now, that if Condition 4 is not fulfilled, then $\mathrm{ecc}_{G\circledast H}((g, h)) \leq 3 \neq 4$, which proves one implication of the equivalence. For the other implication, suppose that Condition 4 is fulfilled for (g, h) and (g', h'). By the symmetry we may assume that $H \cong K_{1,t}, t \geq 2$. We will show that $d_{G\circledast H}((g, h), (g', h')) \geq 4$. The neighborhood of (g, h) in DcD product $G \circledast H$ is $N_G(g) \times (V(H) - \{h\})$, since h is a universal vertex in H. Similarly, the neighborhood of (g', h') in DcD product $G \circledast H$ is $N_G(g') \times (V(H) - \{h\})$, since h' is a universal vertex in H. There are no edges between sets $N_G(g) \times (V(H) - \{h\})$ and $N_G(g') \times (V(H) - \{h\})$, since all vertices $g_1 \in N_G(g)$ are adjacent to all neighbors g'' of the vertex g' in the graph G and there are no edges between vertices from $(V(H) - \{h\})$ in H. Hence $d_{G\circledast H}((g, h), (g', h')) \geq 4$ and it follows that $\mathrm{ecc}_{G\circledast H}((g, h)) = d_{G\circledast H}((g, h), (g', h')) = 4$. □

By a careful reading of Case 2 from the above proof, one can observe that we did not use three or more components of G when constructing the desired paths in $G \circledast H$. A fact that will come handy in the proof of the next theorem.

Theorem 4.5. *Let G and H be graphs not isomorphic to K_2 nor to $\overline{K}_2$ and such that $G \circledast H$ is connected. If one of them has two components and the other is connected, then $\mathrm{ecc}_{G\circledast H}((g, h)) \leq 5$. Moreover, we have $\mathrm{ecc}_{G\circledast H}((g, h)) = d_{G\circledast H}((g, h), (g', h')) = 5$ if and only if* Condition 12 *holds. Furthermore, we have $\mathrm{ecc}_{G\circledast H}((g, h)) = d_{G\circledast H}((g, h), (g', h')) = 4$ if and only if* Condition 4 or 5 or 6 or 7 or 8 or 9 or 10 or 11 *holds.*

Proof. We may assume that G has two components G_1 and G_2 and that H is connected. Let $g \in V(G)$ and $h \in V(H)$ be arbitrary. By symmetry we may further assume that $g \in V(G_1)$. We split the proof into three parts with respect weather G_1 or G_2 are isomorphic to K_1 or not. Notice that the case $G_1 \cong K_1$ and $G_2 \cong K_1$ yields a contradiction with the assumption.

Case 1. $G_1 \ncong K_1$ and $G_2 \ncong K_1$.
This is the only case when H can have a universal vertex, say that h is such. As it is mentioned in a comment before this theorem, we can use the same proof as in Case 2 of Theorem 4.4 including Condition 4. Hence we may assume that h is not universal and with this $\overline{N}_H[h]$ is not an empty set. Vertices from $A_1 = N_G(g) \times N_H(h)$, from $A_2 = \overline{N}_{G_1}[g] \times \overline{N}_H[h]$ and from $A_3 = V(G_2) \times \overline{N}_H[h]$ are adjacent to (g,h). Notice, that A_2 can be empty when g is a universal vertex of G_1. So, assume first that g is not a universal vertex of G_1. Vertices from $A_4 = V(G_2) \times \{h\}$ are adjacent to all vertices of A_2 by co-direct edges and we have $d_{G \circledast H}((g,h),(a,h)) = 2$ for every $a \in V(G_2)$. Every vertex a from $V(G_2)$ has a neighbor a' in G_2 because $G_2 \ncong K_1$. But then we have a direct edge $(a,b)(a',h)$ for any $b \in N_H(h)$. Hence, we have $d_{G \circledast H}((g,h),(a,b)) \leq 3$ for any $(a,b) \in V(G_2) \times N_H(h)$ and we are done with vertices from $V(G_2) \times V(H)$. Further every vertex $(c,b') \in N_G[g] \times \overline{N}_H[h]$ is adjacent to every vertex of A_4, which gives $d_{G \circledast H}((g,h),(c,b')) \leq 3$. Vertices from $A_5 = (V(G_1) - \{g\}) \times \{h\}$ are adjacent to all vertices of A_3 by co-direct edges and we have $d_{G \circledast H}((g,h),(c',h)) = 2$ for any $(c',h) \in A_5$. Every vertex $c'' \in \overline{N}_G[g]$ has a neighbor $c' \in V(G_1) - \{g\}$, say that c' is a neighbor of c'' on a shortest path between g and c''. But then $(c'',b)(c',h)$ is a direct edge for any $b \in N_H(h)$ where $(c',h) \in A_5$. This means that $d_{G \circledast H}((g,h),(c'',b)) \leq 3$ for any $(c'',b) \in \overline{N}_G[g] \times N_H(h)$. Finally, a vertex (g,b), $b \in N_H(h)$, is adjacent to every vertex from $N_G(g) \times \{h\} \subseteq A_5$ by a direct edge. Again we have $d_{G \circledast H}((g,h),(g,b)) \leq 3$ for any $(g,b) \in \{g\} \times N_H(h)$ and we are done when g is not universal in G_1.

Let now g be a universal vertex of G_1 and with this $A_2 = \emptyset$. Every vertex from A_3 is adjacent to every vertex $(g_1,h) \in N_{G_1}(g) \times \{h\}$ and we have $d_{G \circledast H}((g,h),(g_1,h)) = 2$. Further, $(g_1,h) \in N_{G_1}(g) \times \{h\}$ is adjacent to every vertex from $(g,h_1) \in \{g\} \times N_H(h)$ and $d_{G \circledast H}((g,h),(g,h_1)) \leq 3$. In what follows we need the following notation. Let $N_H(h) = B_1' \cup B_1''$ where B_1' contains all satellite vertices of h and $B_1'' = N_H(h) - B_1'$. Notice that vertices from B_1' have no neighbors in $\overline{N}_H[h]$ and can be empty set while $B_1'' \neq \emptyset$ since h is not universal in H. Also let $B_2 = \{x \in V(H) : d_H(h,x) = 2\}$ and $B_3 = \{x \in V(H) : d_H(h,x) \geq 3\}$. Let $h_1 \in B_1''$ be a common neighbor of $h_2 \in B_2$ and h. For $g' \in V(G_2)$ we have $d_{G \circledast H}((g,h),(g',h_1)) = 2$ by the path $(g,h)(g_2,h_2)(g',h_1)$ where $g'g_2 \in E(G_2)$ and we are done with $V(G_2) \times B_1''$. Further, we have $d_{G \circledast H}((g,h),(g',h)) \leq 3$ by the path $(g,h)(g',h_2)(g_2,h_1)(g',h)$ where again $g'g_2 \in E(G_2)$ and we are done with $V(G_2) \times \{h\}$. For $(g_1,h_3) \in V(G_1) \times B_3$ a path $(g,h)(g_2,h_2)(g_3,h_1)(g_1,h_3)$ yields $d_{G \circledast H}((g,h),(g_1,h_3)) \leq 3$ where $g_2g_3 \in E(G_2)$. Let $h_2 \in B_2$ and let h_1 be a common neighbor of h and h_2. A path $P = (g,h)(g_1,h_1)(g,h_2)$, $g_1 \in V(G_1)$ shows that $d_{G \circledast H}((g,h),(g,h_2)) = 2$ and we are done with $\{g\} \times B_2$. We can extend P to a vertex $(g_2,h') \in V(G_2) \times B_1'$ by a co-direct edge since vertices from B_1' have no neighbors in B_2. Hence, $d_{G \circledast H}((g,h),(g_1,h')) \leq 3$ and this concludes $V(G_2) \times B_1'$. We are left with the vertices from $N_{G_1}(g) \times B_2$. If $g_1 \in N_{G_1}(g)$ has a neighbor $g_4 \in N_{G_1}(g)$, then we have a path $(g,h)(g_4,h_1)(g_1,h_2)$ where hh_1h_2 is a shortest path in H and $d_{G \circledast H}((g,h),(g_1,h_2)) = 2$ follows for $(g_1,h_2) \in N_{G_1}(g) \times B_2$. So, we may assume that g is the only neighbor of $g_1 \in N_{G_1}(g)$. If $|V(G_1)| > 2$, then we have a path $(g,h)(g_2,h_2)(g_4,h)(g_1,h_2)$ for $h_2 \in B_2$, $g_2 \in V(G_2)$ and $g_4 \in V(G_1) - \{g, g_1\}$. Therefore, $d_{G \circledast H}((g,h),(g_1,h_2)) \leq 3$ for $(g_1,h_2) \in N_{G_1}(g) \times B_2$. So, let finally $|V(G_1)| = 2$ and with this $G_1 \cong K_2 = gg_1$. If there exists two non-adjacent vertices $g_3, g_4 \in V(G_2)$, then $(g,h)(g_3,h_2)(g_4,h)(g_1,h_2)$ is a path and we have $d_{G \circledast H}((g,h),(g_1,h_2)) \leq 3$. So, we may further assume that $G_2 \cong K_t$ for some $t \geq 2$. If there exists $h_3 \in B_3$, then a path $(g,h)(g_2,h_3)(g,h_1)(g_1,h_2)$ where $g_2 \in V(G)$ and h_1 is a common neighbor of h and $h_2 \in B_2$ assures that $d_{G \circledast H}((g,h),(g_1,h_2)) \leq 3$. So, we may assume that $B_3 = \emptyset$ and with this $\mathrm{ecc}_{\mathrm{H}}(\mathrm{h}) = 2$. Further, if there exist $h_2' \in B_2 - \{h_2\}$, then $(g,h)(g_1,h_1)(g,h_2')(g_1,h_2)$ is a path when $h_2h_2' \in E(H)$ where h_1 is a common neighbor of h and h_2'. Thus, $d_{G \circledast H}((g,h),(g_1,h_2)) \leq 3$ again. If $h_2h_2' \notin E(H)$, then the path $(g,h)(g_2,h_2')(g_1,h_2)$, for $g_2 \in V(G_2)$, forces $d_{G \circledast H}((g,h),(g_1,h_2)) = 2$. So, let $B_2 = \{h_2\}$ and every vertex from B_1'' is adjacent to h_2. If some $h_1 \in B_1''$ is adjacent to $h_1' \in N_H(h) - \{h_1\}$, then $(g,h)(g_1,h_1')(g,h_1)(g_1,h_2)$ is a path and we have $d_{G \circledast H}((g,h),(g_1,h_2)) \leq 3$ again. So, we are left with a situation where there are no edges between vertices from B_1' and B_1'' and no edges in $H[B_1'']$. If there exist $h_1' \in B_1'$, then $h_1h_1' \notin E(H)$ for $h_1 \in B_1''$ and $(g,h)(g_1,h_1)(g_2,h_1')(g_1,h_2)$ is a path for $g_2 \in V(G_2)$. This means that $d_{G \circledast H}((g,h),(g_1,h_2)) \leq 3$. We end with assumption that $B_1' = \emptyset$. But then $H \cong K_{2,p}$ for some $p \geq 1$ and Condition 5 is fulfilled for $(x,y) = (g,h)$ and $(x',y') = (g_1,h_2)$. We have $d_{G \circledast H}((g,h),(g_1,h_2)) \leq 4$ by the path $(g,h)(g_2,h_2)(g_1,h)(g,h_1)(g_1,h_2)$ where $g_2 \in V(G_2)$ and $h_1 \in B_1''$.

Case 2. $G_1 \ncong K_1$ and $G_2 \cong K_1$.
Let $G_2 = a$. The proof of this case is in the beginning very similar as the first paragraph of the previous case with two exceptions. Firstly, h is not a universal vertex now by Theorem 2.1 and we can ignore the first part of first paragraph of Case 1. Secondly, when g is not a universal vertex of G_1 is the sentence: "Every vertex a from $V(G_2)$ has a neighbor a' in G_2 because $G_2 \ncong K_1$". Now such an a' does not exists and we need to take care of all vertices from $V(G_2) \times N_H(h)$. Since there are no universal vertices in H there exists b' for every $b \in N_H(h)$ such that b' is not adjacent to b. Now (a,b) is adjacent to (c,b') by a co-direct edge, where $c \in N_G(g)$ when $b' \in N_H(h)$ or $c \in \overline{N}_G[g]$ when $b' \in \overline{N}_H[h]$. In first option (c,b) belongs to A_1 and in the second option it belongs to A_2 (as defined in the previous case). Hence, $d_{G \circledast H}((g,h),(a,b)) = 2$ and we are done if g is not universal in G_1.

There are more differences when g is a universal vertex of G_1 and we write all the details. Sets $A_1, A_3, B_1', B_1'', B_2$ and B_3 have the same meaning as in the Case 1. Every vertex from A_3 is adjacent to every vertex $(g_1,h) \in N_{G_1}(g) \times \{h\}$ and we have $d_{G \circledast H}((g,h),(g_1,h)) = 2$. Further, $(g_1,h) \in N_{G_1}(g) \times \{h\}$ is adjacent to every vertex from $(g,h_1) \in \{g\} \times N_H(h)$ and $d_{G \circledast H}((g,h),(g,h_1)) \leq 3$. If $g_1, g_2 \in V(G_1)$ are non-adjacent, then $(g,h)(a,h_2)(g_2,h)(g_1,h_2)$ is a path for $h_2 \in \overline{N}_H[h]$ and $d_{G \circledast H}((g,h),(g_1,h_2)) \leq 3$ follows. In particular, if $h_2 \in B_3$ we have a path $(g,h)(g_2,h_1)(g_1,h_2)$ for $h_1 \in N_H(h)$ and we have $d_{G \circledast H}((g,h),(g_1,h_2)) = 2$. Last path can be extended to any vertex (a,h_1) for $h_1 \in B_1''$ and we have $d_{G \circledast H}((g,h),(a,h_1)) \leq 3$ for every $(a,h_1) \in \{a\} \times B_1''$ when G_1 is not complete and $B_3 \neq \emptyset$. If $B_3 = \emptyset$, then every vertex $h_1 \in B_1''$ is non-adjacent to some h_1' because H has no universal vertices. If $h_1' \in N_H(h)$, then the path $(g,h)(g_1,h_1')(a,h_1)$ for $g_1 \in N_G(g)$ gives $d_{G \circledast H}((g,h),(a,h_1)) = 2$. If $h_1' \in \overline{N}_H[h]$, then we have $d_{G \circledast H}((g,h),(a,h_1)) \leq 3$ by the path $(g,h)(g_1,h_1'')(g,h_1')(a,h_1)$ for $g_1 \in N_G(g)$ and h_1'' is a common neighbor of h and h_1'. Hence, vertices from $\{a\} \times B_1''$ remains open only when G_1 is complete and

$B_3 \neq \emptyset$. Let g' be a universal vertex of G_1 and let g_1 be a neighbor of g'. Notice that g' can be equal to g. For $h_2 \in B_2$ we have a path $P = (g,h)(g_1,h_1)(g',h_2)$ for a common neighbor h_1 of h and h_2 which gives $d_{G \circledast H}((g,h),(g',h_2)) = 2$ for every universal vertex g' of G_1 when $|V(G_1)| > 2$ and only for g when $G_1 \cong K_2$. Path P can be extended to $(g,h)(g_1,h_1)(g',h_2)(a,h')$ for $h' \in \{h\} \cup B_1'$ and we have $d_{G \circledast H}((g,h),(a,h')) \leq 3$. For $h_3 \in B_3$ with a neighbor $h_2 \in B_2$ we have a path $(g,h)(g',h_1)(g_1,h_2)(g',h_3)$ where h_1 is a common neighbor of h and h_2. Notice that if $G_1 \cong K_2$, then $g_1 = g$ and vertex (g,h_3) is still open. Hence, $d_{G \circledast H}((g,h),(g',h_3)) \leq 3$ for every vertex h_3 at distance three from h and every universal vertex g' of G_1, with the exception of g when $G_1 \cong K_2$. If $d_H(h,h_4) \geq 4$, then we have a path $(g,h)(a,h_2)(g',h_4)$ where $h_2 \in B_2$. This gives $d_{G \circledast H}((g,h),(g',h_4)) = 2$ for every universal vertex g' of G_1 and $h_4 \in B_3$ with $d_H(h,h_4) \geq 4$.

At this stage we still need to check the vertices from $\{a\} \times B_1''$ when $B_3 \neq \emptyset$ and $G_1 \cong K_t$ for some $t \geq 2$ and in addition when $G_1 \cong K_2 = gg'$ we need to take care also for (g,h_3) and (g',h_2) where $d_H(h,h_3) = 3$ and $h_2 \in B_2$. Suppose first that there exist $h_4 \in B_3$ with $d_H(h,h_4) \geq 4$. The path $(g,h)(a,h_4)(g',h_2)$ shows that $d_{G \circledast H}((g,h),(g',h_2)) = 2$ for every $h_2 \in B_2$. Last path can be extended to (g,h_3) and we have $d_{G \circledast H}((g,h),(g,h_3)) \leq 3$ for every $h_3 \in B_3$ with $d_H(h,h_3) = 3$ since h_3 has a neighbor in B_2. In addition we have a path $(g,h)(a,h_2)(g,h_4)(a,h_1)$ for every $h_1 \in B_1''$ where $h_2 \in B_2$ and $d_{G \circledast H}((g,h),(a,h_1)) \leq 3$ follows. Therefore, we may assume in the rest of this case that all vertices from B_3 are at distance three to h.

First we concentrate on vertices from $\{g'\} \times B_2$ when $G_1 \cong K_2 = gg'$. If there exist $h_3 \in B_3$, then we have a path $(g,h)(a,h_3)(g,h_1)(g',h_2)$ where $h_1 \in B_1''$ is a neighbor of $h_2 \in B_2$ and $d_{G \circledast H}((g,h),(g',h_2)) \leq 3$ follows. So, we may assume that $B_3 = \emptyset$. Let $h_2, h_2' \in B_2$ be different vertices. If they are non-adjacent, then the path $(g,h)(a,h_2')(g',h_2)$ yields $d_{G \circledast H}((g,h),(g',h_2)) = 2$. If $h_2h_2' \in E(H)$, then the path $(g,h)(g',h_1)(g,h_2')(g',h_2)$, where h_1 is a common neighbor of h and h_2', shows that $d_{G \circledast H}((g,h),(g',h_2)) \leq 3$. It remains that $B_2 = \{h_2\}$. Since $G \circledast H$ is connected, H contains no universal vertices by Theorem 2.1. If $h_1 \in B_1'$ is not adjacent to $h_1' \in N_H(h)$, then we have $d_{G \circledast H}((g,h),(g',h_2)) \leq 3$ by the path $(g,h)(g',h_1')(a,h_1)(g',h_2)$ where $h_1h_2 \notin E(H)$ as $h_1 \in B_1'$. Hence, next $h_1 \in B_1'$ is adjacent to all the vertices in $N_H(h)$ and with this also to a neighbor $h_2' \in B_1''$ of h_2. This means that $d_{G \circledast H}((g,h),(g',h_2)) \leq 3$ by the path $(g,h)(g',h_1)(g,h_2')(g',h_2)$. Finally, we may assume that $B_1' = \emptyset$. If there exists $h_1h_1' \in E(H[B_1''])$, then the path $(g,h)(g',h_1)(g,h_1')(g',h_2)$ shows that $d_{G \circledast H}((g,h),(g',h_2)) \leq 3$. If $H[B_1'']$ is without edges, then $H \cong K_{2,p}$, $p \geq 1$, and Condition 5 holds. We have $d_{G \circledast H}((g,h),(g',h_2)) \leq 4$ by the path $(g,h)(a,h_2)(g',h)(g,h_1)(g',h_2)$ for $h_1 \in B_1''$.

Next we deal with an arbitrary vertex (g,h_3) from $\{g\} \times B_3$ when $G_1 \cong K_2 = gg'$. If there exist two non-adjacent vertices h_1, h_1' from $N_H(h)$, then we have the path $(g,h)(g',h_1)(a,h_1')(g,h_3)$ and $d_{G \circledast H}((g,h),(g,h_3)) \leq 3$ holds. So, we may assume that $H[N_H[h]]$ is a complete subgraph. If h_3 is non-adjacent with some $h_3' \in \overline{N}_H[h] - \{h_3\}$, then $d_{G \circledast H}((g,h),(g,h_3)) = 2$ follows from the path $(g,h)(a,h_3')(g,h_3)$. Therefore, we suppose that h_3 is a universal vertex of $H[\overline{N}_H[h]]$. If h_3', h_3'' are non-adjacent vertices of $\overline{N}_H[h]$, then the path $(g,h)(a,h_3')(g',h_3'')(g,h_3)$ ensures that $d_{G \circledast H}((g,h),(g,h_3)) \leq 3$. Hence, we can also assume that $H[\overline{N}_H[h]]$ is a complete subgraph of H. With this Condition 6 holds and we have $d_{G \circledast H}((g,h),(g,h_3)) \leq 4$ by the path $(g,h)(a,h_3)(g,h_1)(g',h_2)(g,h_3)$ for $h_2 \in B_2$ and his neighbor $h_1 \in B_1''$.

We end with vertices from $\{a\} \times B_1''$ when $B_3 \neq \emptyset$ $G_1 \cong K_t$ for some $t \geq 2$. Let $(a, h_1) \in \{a\} \times B_1''$ and let g_1 be a neighbor of g in G_1. If h_1 is non-adjacent to some $h_1' \in N_H(h) - \{h_1\}$, then $d_{G \circledast H}((g, h), (a, h_1)) = 2$ by the path $(g, h)(g_1, h_1')(a, h_1)$. So, we may assume that h_1 is a universal vertex of $H[N_H[h]]$. If h_1 is non-adjacent to some $h_2 \in B_2$, then for a common neighbor h_1' of h and h_2 the path $(g, h)(g_1, h_1')(g, h_2)(a, h_1)$ gives $d_{G \circledast H}((g, h), (a, h_1)) \leq 3$. Hence, we may suppose that h_1 is adjacent to all vertices of B_2. (Notice that $B_3 \neq \emptyset$ because otherwise h_1 is a universal vertex of H which is not possible.) If there exists $h_3 \in B_3$ that is not a universal vertex of $H[\overline{N}_H[h]]$, that is h_3 is non-adjacent to $h_2 \in \overline{N}_H[h] - \{h_3\}$, then we have a path $(g, h)(a, h_2)(g, h_3)(a, h_1)$ and $d_{G \circledast H}((g, h), (a, h_1)) \leq 3$ follows. Finally, if every vertex from B_3 is a universal vertex in $\overline{N}_H[h]$, then Condition 7 is fulfilled and we have $d_{G \circledast H}((g, h), (a, h_1)) \leq 4$ by the path $(g, h)(g_1, h_1)(g, h_2)(g_1, h_3)(a, h_1)$ where $hh_1h_2h_3$ is a shortest path between h and $h_3 \in B_3$ in the graph H.

Case 3. $G_1 \cong K_1$ and $G_2 \ncong K_1$.

If there exists a universal vertex of H, then $G \circledast H$ is not connected by Theorem 2.1. Hence H is without universal vertices and in particular $\mathrm{ecc}_H(h) \geq 2$. We partition $V(H)$ into sets $B_1 = N_H(h)$, $B_2 = \{x \in V(H) : d_H(h, x) = 2\}$, $B_3 = \{x \in V(H) : d_H(h, x) = 3\}$, $B_4 = \{x \in V(H) : d_H(h, x) \geq 4\}$ and $\{h\}$. Further we partition B_1 into B_1' that contains all vertices that have no neighbor in B_2, that is B_1' contains all satellite vertices of h, and $B_1'' = B_1 - B_1'$. Clearly, B_1' can be empty, but B_1'' is non-empty because $\mathrm{ecc}_H(h) \geq 2$. For any vertex $h_4 \in B_4$ with $d_H(h_4, h) = 4$ (if it exists) let $hh_1h_2h_3h_4$ be a shortest h, h_4-path in H. Clearly that $h_i \in B_i$ for $i \in \{1, 2, 3, 4\}$ (if it exists for $i \in \{3, 4\}$). Notice that $N_{G \circledast H}((g, h)) = V(G_2) \times (B_2 \cup B_3 \cup B_4)$. Let $g_1g_2 \in E(G_2)$ which exists since $G_2 \ncong K_1$. Every vertex $(g_1, h_1) \in V(G_2) \times B_1''$ is at distance two to (g, h) since $(g_2, h_2)(g_1, h_1)$ is a direct edge. Hence $d_{G \circledast H}((g, h), (g_1, h_1)) = 2$ for any $(g_1, h_1) \in V(G_2) \times B_1''$. In addition there is a path $(g, h)(g_1, h_2)(g_2, h_1)(g_1, h)$ which means that $d_{G \circledast H}((g, h), (g_1, h)) \leq 3$ for any $(g_1, h) \in V(G_2) \times \{h\}$. Also every vertex (g, h_0), $h_0 \in B_1'$ is adjacent to (g_1, h_2) by a co-direct edge and $d_{G \circledast H}((g, h), (g, h_0)) = 2$ for any $(g, h_0) \in \{g\} \times B_1'$. A path $(g, h)(g_2, h_2)(g_1, h_1)(g, h^*)$ for any $h^* \in B_3 \cup B_4$ (if it exists) implies that $d_{G \circledast H}((g, h), (g, h^*)) \leq 3$ for any $(g, h^*) \in \{g\} \times (B_3 \cup B_4)$. So, in what follows we only need to consider vertices from $V(G_2) \times B_1'$ and $\{g\} \times (B_1'' \cup B_2)$.

To end this case we need further to distinguish the options when G_2 is complete or not. Suppose first that $G_2 \ncong K_p$ and let g_3 and g_4 be non-adjacent vertices of G_2. There exist a path $(g, h)(g_3, h_2)(g_4, h)(g, h_2)$ and for any vertex $(g, h_2) \in \{g\} \times B_2$ we have $d_{G \circledast H}((g, h), (g, h_2)) \leq 3$. Let next $h_0 \in B_1'$. A path $(g, h)(g_3, h_2)(g_4, h_0)$ assures that $d_{G \circledast H}((g, h), (g_4, h_0)) \leq 2$ for any $(g_4, h_0) \in V(G_2) \times B_1'$ where g_4 is not a universal vertex of G_2. If g_5 is a universal vertex of G_2, then we have $d_{G \circledast H}((g, h), (g_5, h_0)) \leq 3$ by the path $(g, h)(g_3, h_2)(g_4, h)(g_5, h_0)$ where the last edge is a direct one since g_5 is universal and h_0 is adjacent to h. To end with $G_2 \ncong K_p$ let $h_1 \in B_1''$. As mentioned before, H is without universal vertices and also h_1 is not universal. If $B_3 \neq \emptyset$, then $d_{G \circledast H}((g, h), (g, h_1)) = 2$ by the path $(g, h)(g_1, h^+)(g, h_1)$ for any $h^+ \in B_3$. If $B_3 = \emptyset$, then h_1 is not adjacent to some $h_2' \in B_2$ or to some $h_1' \in B_1$, since it is not universal vertex in graph H. For the first option we have a path $(g, h)(g_1, h_2')(g, h_1)$ and for the second option there exists a path $(g, h)(g_1, h_2')(g_2, h_1')(g, h_1)$ when $h_1' \in B_1''$ and $h_2' \in B_2$ is a neighbor of h_1' or a path $(g, h)(g_3, h_2)(g_4, h_1')(g, h_1)$ when $h_1' \in B_1'$. In all cases we have $d_{G \circledast H}((g, h), (g, h_1)) \leq 3$ and we are done when $G_2 \ncong K_p$.

We continue with $G_2 \cong K_p$, $p \geq 2$. If there exists $h_4 \in B_4$, then we have a path $(g,h)(g_1,h_4)(g,h^*)$ for any $g_1 \in V(G_2)$ and $h^* \in B_1'' \cup B_2$, which means that $d_{G \circledast H}((g,h),(g,h^*)) = 2$ and we are done with vertices from $\{g\} \times (B_1'' \cup B_2)$. Similar there exists a path $(g,h)(g_1,h_2)(g,h_4)(g_1,h_0)$ that consist only of co-direct edges and where $g_1 \in V(G_2)$ and $h_0 \in B_1'$ are arbitrary and $h_2 \in B_2$. Hence, $d_{G \circledast H}((g,h),(g_1,h_0)) \leq 3$ for any $(g_1,h_0) \in V(G_2) \times B_1'$.

Next we may assume that $B_4 = \emptyset$ and with this $2 \leq \mathrm{ecc}_H(h) \leq 3$ because h is not universal in H. Let first $\mathrm{ecc}_H(h) = 3$ and there exists $h_3 \in B_3$. For $(g,h_1) \in \{g\} \times B_1''$ we have a path $(g,h)(g_1,h_3)(g,h_1)$ for $g_1 \in V(G_2)$ and we have $d_{G \circledast H}((g,h),(g,h_1)) = 2$ for any $(g,h_1) \in \{g\} \times B_1''$. Let now $(g,h_2) \in \{g\} \times B_2$. If h_2 is not adjacent to some $h_0 \in B_2 \cup B_3$, $h_0 \neq h_2$, then a path $(g,h)(g_1,h_0)(g,h_2)$ where $g_1 \in V(G_2)$ implies $d_{G \circledast H}((g,h),(g,h_2)) = 2$. Furthermore, last path can be extended to $(g,h)(g_1,h_0)(g,h_2)(g',h')$ where $(g',h') \in V(G_2) \times B_1'$ and we have $d_{G \circledast H}((g,h),(g',h')) \leq 3$ for any $(g',h') \in V(G_2) \times B_1'$ when $H[B_2 \cup B_3]$ is not a complete graph. Therefore we may assume that h_2 is adjacent to all vertices from $(B_2 \cup B_3) - \{h_2\}$. Moreover, if h_2 is not adjacent to $h_1' \in B_1''$, then we have a path $(g,h)(g_1,h_2')(g_2,h_1')(g,h_2)$ where $g_1g_2 \in E(G_2)$, $h_2' \in B_2$ and $h_1'h_2' \in E(H)$ exists because $h_1' \in B_1''$. This means that $d_{G \circledast H}((g,h),(g,h_2)) \leq 3$. Finally, let h_2 be adjacent to every vertex from $(B_1'' \cup B_2 \cup B_3) - \{h_2\}$. Notice that for $g' = g$ and $h' = h_2$ Condition 8 is fulfilled and we have $d_{G \circledast H}((g,h),(g',h')) \leq 4$ by the path $(g,h)(g_1,h_2)(g_2,h_1)(g_1,h)(g',h')$ where $g_1g_2 \in E(G_2)$ and h_1 is a common neighbor of h and h_2 in H. We still need to take care of vertices in $V(G_2) \times B_1'$ when $H[B_2 \cup B_3]$ is a complete graph. Let (g',h') be arbitrary vertex from $V(G_2) \times B_1'$. If h' is not adjacent to $h'' \in B_1 - \{h'\}$, then we have a path $(g,h)(g_1,h_3)(g,h'')(g',h')$ where $g_1 \in V(G_2)$ and $h_3 \in B_3$ and clearly $d_{G \circledast H}((g,h),(g',h')) \leq 3$. So, we may assume also that h' is adjacent to every other vertex from B_1 or in other words $N_H[h'] = N_H[h]$. We have $d_{G \circledast H}((g,h),(g',h')) \leq 3$ by the path $(g,h)(g',h_2)(g_1,h_1)(g',h')$ where $g_1 \in V(G_2) - \{g'\}$ and hh_1h_2 is a shortest path in H (notice that h_1 is adjacent also to h' as $N_H[h'] = N_H[h]$).

We end this case with $\mathrm{ecc}_H(h) = 2$ and with this $B_3 = \emptyset$. If $h_2, h_2' \in B_2$ are not adjacent, then a path $(g,h)(g_1,h_2')(g,h_2)$ where $g_1 \in V(G_2)$ yields $d_{G \circledast H}((g,h),(g,h_2)) = 2$. Last path can be extended to $(g,h)(g_1,h_2')(g,h_2)(g',h')$ for any $(g',h') \in V(G_2) \times B_1'$ and we have $d_{G \circledast H}((g,h),(g',h')) \leq 3$. Hence we are done with $V(G_2) \times B_1'$ when $H[B_2]$ is not complete. So let $H[B_2]$ be complete. Suppose that h' is adjacent to some $h_1' \in B_1''$. We have $d_{G \circledast H}((g,h),(g',h')) \leq 3$ by the path $(g,h)(g',h_2')(g_1,h_1')(g',h')$ where $g'g_1 \in E(G_2)$ and $h_2' \in B_2$ is a neighbor of h_1'. Therefore, we can assume that h' is not adjacent to any vertex from B_1''. Further suppose that $h_1' \in B_1''$ is not adjacent to some $h_2 \in B_2$. A path $(g,h)(g_1,h_2)(g,h_1')(g',h')$ shows that $d_{G \circledast H}((g,h),(g',h')) \leq 3$ again. Assume now that h' is not adjacent to $h_1' \in B_1' - \{h'\}$. We have $d_{G \circledast H}((g,h),(g',h')) \leq 3$ by the path $(g,h)(g',h_2)(g,h_1')(g',h')$ where $h_2 \in B_2$. We are left with h' is a satellite of h that is adjacent to all the satellites of h, that is $N_H[h'] = B_1' \cup \{h\}$, $H[B_2]$ is a complete subgraph and there exist all edges between B_1'' and B_2. But then Condition 9 holds for (g,h) and (g',h') and we have a path $(g,h)(g_1,h_2)(g',h_1)(g_1,h)(g',h')$ where $g_1 \in V(G_2) - \{g'\}$ and hh_1h_2 is a shortest path in H. Therefore, $d_{G \circledast H}((g,h),(g',h')) \leq 4$ and we are done with vertices from $V(G_2) \times B_1'$. To end with vertices from $\{g\} \times B_2$ recall that we need to consider only those (g,h_2) where h_2 is adjacent to all vertices from $B_2 - \{h_2\}$. If such a vertex h_2 is not adjacent to $h_1' \in B_1''$, then we have a path $(g,h)(g_1,h_2')(g_2,h_1')(g,h_2)$ where $g_1g_2 \in E(G_2)$ and $h_2' \in B_2$ is a neighbor of h_1'. This path forces $d_{G\circledast H}((g,h),(g,h_2)) \leq 3$. Hence, we may assume further that $N_H[h'] = B_1'' \cup B_2$ for some $h' \in B_2$. But then Condition 10 is fulfilled for $g' = g$ and we have $d_{G\circledast H}((g,h),(g',h')) \leq 4$ by the path $(g,h)(g_1,h')(g_2,h'')(g_1,h)(g,h')$ where $h'' \in N_H(h) \cap N_H(h')$ and $g_1g_2 \in E(G_2)$.

We finish with the vertices from $\{g\} \times B_1''$. Let (g',h') be an arbitrary vertex from $\{g\} \times B_1''$. If h' is not adjacent to some $h_2 \in B_2$, then $d_{G\circledast H}((g,h),(g',h')) = 2$ by the path $(g,h)(g_1,h_2)(g',h')$ for $g_1 \in V(G_2)$. Similar, if h' is not adjacent to some $h_1 \in B_1'' - \{h'\}$, then $d_{G\circledast H}((g,h),(g',h')) \leq 3$ by the path $(g,h)(g_1,h_2)(g_2,h_1)(g',h')$ where $g_1g_2 \in E(G_2)$ and $h_2 \in B_2$ is a neighbor of h_1. Therefore we may assume that h' is adjacent to all vertices from $(B_2 \cup B_1'') - \{h'\}$. If this happens, notice that there exists $h_0 \in B_1'$ that is non-adjacent to h' because H has no universal vertices by Theorem 2.1. In particular, $h_0 \in \overline{N}_H[h'] \subseteq B_1'$. Hence the only edges with end-vertex (g',h') are now edges $(g_i,h_0)(g',h')$ for $g_i \in V(G_2)$. If h_0 is not adjacent to some $h_0' \in B_1'$, then Condition 11 holds and we have $d_{G\circledast H}((g,h),(g',h')) \leq 4$ by the path $(g,h)(g_1,h_2)(g,h_0')(g_1,h_0)(g',h')$ where $g_1 \in V(G_2)$ and $h_2 \in B_2$. Similar, if $H[\overline{N}_H[h]]$ is not complete, that is $h_2, h_2' \in B_2$ are not adjacent, then Condition 11 holds again and we have $d_{G\circledast H}((g,h),(g',h')) \leq 4$ by the path $(g,h)(g_1,h_2)(g,h_2')(g_1,h_0)(g',h')$ where $g_1 \in V(G_2)$. Suppose now that there exists an edge between between $\overline{N}_H[h']$ and $B_1'' = N_H(h) - B_1'$, say $h_3h_4 \in E(H)$ is such where $h_3 \in B_1''$ and $h_4 \in \overline{N}_H[h']$. Again Condition 11 if fulfilled and we have $d_{G\circledast H}((g,h),(g',h')) \leq 4$ by the path $(g,h)(g_1,h_2)(g_2,h_3)(g_1,h_4)(g',h')$ where $g_1g_2 \in E(G_2)$ and $h_2 \in B_2$ is a neighbor of h_3. Further, if Condition 12 holds, then we have $d_{G\circledast H}((g,h),(g',h')) \leq 5$ by the path $(g,h)(g_2,h_2)(g_1,h_1)(g_2,h)(g_1,h_0)(g',h')$ where $g_1g_2 \in E(G_2)$ and hh_1h_2 is a shortest path in H. With this Case 3 is concluded.

For the first equivalence, notice that if Condition 12 does not hold, then $d_{G\circledast H}((g,h),(g',h')) \leq 4$ by the proof so far, which gives one implication. For the second implication assume that Condition 12 holds for $G = X, H = Y, (g,h) = (x,y)$ and $(g',h') = (x',y')$. Clearly $N_{G\circledast H}((g,h)) = V(G_2) \times B_2 = D$ and $N_{G\circledast H}((g',h')) = V(G_2) \times \overline{N}_H[h'] = D' \subseteq V(G_2) \times B_1'$. Clearly there are no edges between vertices of D and vertices of D' because G_2 is complete and there are no edges between B_1' and B_2 in H. Vertices from D have their neighbors in $\{g\} \times B_1'$ and in $V(G_2) \times B_1''$. In particular, there is no edge between D and $\{g\} \times B_2$ because $H[B_2] = H[\overline{N}_H[h]]$ is a complete subgraph of H. Also any vertex from D' has no neighbor in $V(G_2) \times B_1''$ because there is no edge between $\overline{N}_H[h']$ and $B_1'' = N_H(h) - B_1'$ and because G_2 is complete. Finally, there are no edges between any vertex from $\{g\} \times B_1'$ and D' because every vertex from $\overline{N}_Y[y']$ is adjacent to every vertex from B_1'. All together we have $d_{G\circledast H}((g,h),(g',h')) \geq 5$ and equality follows which settles the first equivalence.

For the second equivalence, notice that the proof until now shows that when Condition 4 does not hold and Condition 5 does not hold and Condition 6 does not hold and Condition 7 does not hold and Condition 8 does not hold and Condition 9 does not hold and Condition 10 does not hold and Condition 11 does not hold, then $d_{G\circledast H}((g,h),(g',h')) \leq 3$ or $d_{G\circledast H}((g,h),(g',h')) = 5$ and we are done with one implication. For the second implication assume first that Condition 4 holds. We can use the same arguments as in the proof of Theorem 4.4.

Suppose now that Condition 5 holds for $G = X, H = Y, (g, h) = (x, y)$ and $(g', h') = (x', y')$. We know already that $d_{G\circledast H}((g, h), (g', h')) \leq 4$. Clearly $N_{G\circledast H}((g, h)) = (V(G_2) \times \{h'\}) \cup (\{g'\} \times N_H(h)) = D$ and $N_{G\circledast H}((g', h')) = (V(G_2) \times \{h\}) \cup (\{g\} \times B_1'') = D'$. There are no edges between vertices from $V(G_2) \times \{h'\}$ and D' because G_2 is complete or because h' is adjacent to all vertices from B_1''. Also, there are no edges between vertices from $\{g'\} \times N_H(h)$ and D' because all vertices of B_1'' are adjacent to h or because there are no edges in $H[B_1'']$. Hence, $d_{G\circledast H}((g, h), (g', h')) \geq 4$ and the equality follows.

If Condition 6 holds for $G = X, H = Y, (g, h) = (x, y)$ and $(g', h') = (x', y')$, then we have $d_{G\circledast H}((g, h), (g', h')) \leq 4$ by the proof until now. For $K_2 = gg_1$ and $K_1 = a$ we have $N_{G\circledast H}((g, h)) = (\{g_1\} \times N_H(h)) \cup (\{a\} \times \overline{N}_H[h]) = D$ and $N_{G\circledast H}((g', h')) = (\{a\} \times N_H[h]) \cup (\{g_1\} \times (\overline{N}_H[h] - \{h'\})) = D'$. Vertices from D' are not adjacent to $\{g_1\} \times N_H(h)$ because $N_H[h]$ is complete or because they have the same first coordinate g_1. Similar, vertices from D' are not adjacent to $\{a\} \times \overline{N}_H[h]$ because they have the same first coordinate a or because $\overline{N}_H[h]$ is complete. Therefore, $d_{G\circledast H}((g, h), (g', h')) \geq 4$ and the equality follows.

Let now Condition 7 be true for $G = X, H = Y, (g, h) = (x, y)$ and $(g', h') = (x', y')$. As usual $d_{G\circledast H}((g, h), (g', h')) \leq 4$ by the proof until now. We have $N_{G\circledast H}((g, h)) = (N_{G_1}(g) \times N_H(h)) \cup (\{g'\} \times \overline{N}_H[h]) = D$ and $N_{G\circledast H}((g', h')) = V(G_1) \times B_3 = D'$. Vertices from D' are not adjacent to $N_{G_1}(g) \times N_H(h)$ because G_1 is complete and vertices of B_3 are not adjacent with vertices from $N_H(h)$. Similar, vertices from D' are not adjacent to $\{g'\} \times \overline{N}_H[h]$ because every vertex from B_3 is a universal vertex of $H[\overline{N}_H[h]]$ and g' is not adjacent to vertices of G_1. Again $d_{G\circledast H}((g, h), (g', h')) \geq 4$ and equality follows.

Next we assume that Condition 8 holds for $G = X, H = Y, (g, h) = (x, y)$ and $(g', h') = (x', y')$. We know already that $d_{G\circledast H}((g, h), (g', h')) \leq 4$. Clearly $N_{G\circledast H}((g, h)) = V(G_2) \times (B_2 \cup B_3) = D$ and $N_{G\circledast H}((g', h')) = V(G_2) \times \overline{N}_H(h') = D' \subseteq V(G_2) \times (\{h\} \cup B_1')$. It is easy to see that there is no edge between a vertex from D and a vertex from D' because G_2 is complete and there is no edge between $B_2 \cup B_3$ and $\{h\} \cup B_1'$ in H. Hence, $d_{G\circledast H}((g, h), (g', h')) \geq 4$ and equality follows.

Next we assume that Condition 9 holds for $G = X, H = Y, (g, h) = (x, y)$ and $(g', h') = (x', y')$. We know already that $d_{G\circledast H}((g, h), (g', h')) \leq 4$. Clearly $N_{G\circledast H}((g, h)) = V(G_2) \times B_2 = D$ and $N_{G\circledast H}((g', h')) = ((V(G_2) - \{g'\}) \times N_H(h')) \cup (\{g\} \times (B_1'' \cup B_2))$. There is no edge between D and a vertex in $\{g\} \times B_2$ because $H[B_2] = H[\overline{N}_H[h]]$ is complete. Similar, there is no edge between D and a vertex in $\{g\} \times B_1''$ because there are all possible edges between $B_2 = \overline{N}_H(h)$ and $B_1'' = N_H[h] - B_1'$ in H. Finally, there are no edges between vertices from D and from $(V(G_2) - \{g'\}) \times N_H(h')$ because G_2 is complete and there are no edges between B_2 and $\{h\} \cup B_1'$ in H. Therefore, $d_{G\circledast H}((g, h), (g', h')) \geq 4$ and equality follows.

Next we assume that Condition 10 holds for $G = X, H = Y, (g, h) = (x, y)$ and $(g', h') = (x', y')$. We know already that $d_{G\circledast H}((g, h), (g', h')) \leq 4$. Clearly $N_{G\circledast H}((g, h)) = V(G_2) \times B_2 = D$ and $N_{G\circledast H}((g', h')) = V(G_2) \times (B_1' \cup \{h\}) = D'$. There is no edge between D and a vertex in D' because G_2 is complete and there are no edges between B_2 and $B_1' \cup \{h\}$ in H. Therefore, $d_{G\circledast H}((g, h), (g', h')) \geq 4$ and equality follows.

We end when Condition 11 holds for $G = X, H = Y, (g, h) = (x, y)$ and $(g', h') = (x', y')$. We know already that $d_{G\circledast H}((g, h), (g', h')) \leq 4$. Clearly $N_{G\circledast H}((g, h)) = V(G_2) \times B_2 = D$ and $N_{G\circledast H}((g', h')) = V(G_2) \times \overline{N}_H[h'] = D' \subseteq V(G_2) \times B_1'$.

There are no edges between vertices of D and vertices of D' because G_2 is complete and there are no edges between vertices of B_2 and vertices of B_1' in H. Therefore, $d_{G \circledast H}((g,h),(g',h')) \geq 4$ and equality follows and the proof is completed. □

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