Bayesian ^{14}C-rationality, Heisenberg uncertainty, and Fourier transform: the beauty of radiocarbon calibration

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ABSTRACT – *Following some 30 years of radiocarbon research during which the mathematical principles of ^{14}C-calibration have been on loan to Bayesian statistics, here they are returned to quantum physics. The return is based on recognition that ^{14}C-calibration can be described as a Fourier transform. Following its introduction as such, there is need to reconceptualize the probabilistic ^{14}C-analysis. The main change will be to replace the traditional (one-dimensional) concept of ^{14}C-dating probability by a two-dimensional probability. This is entirely analogous to the definition of probability in quantum physics, where the squared amplitude of a wave function defined in Hilbert space provides a measurable probability of finding the corresponding particle at a certain point in time/space, the so-called Born rule. When adapted to the characteristics of ^{14}C-calibration, as it turns out, the Fourier transform immediately accounts for practically all known so-called quantization properties of archaeological ^{14}C-ages, such as clustering, age-shifting, and amplitude-distortion. This also applies to the frequently observed chronological lock-in properties of larger data sets, when analysed by Gaussian wiggle matching (on the ^{14}C-scale) just as by Bayesian sequencing (on the calendar time-scale). Such domain-switching effects are typical for a Fourier transform. They can now be understood, and taken into account, by the application of concepts and interpretations that are central to quantum physics (*e.g. *wave diffraction, wave-particle duality, Heisenberg uncertainty, and the correspondence principle). What may sound complicated, at first glance, simplifies the construction of ^{14}C-based chronologies. The new Fourier-based ^{14}C-analysis supports chronological studies on previously unachievable geographic (continental) and temporal (Glacial-Holocene) scales; for example, by temporal sequencing of hundreds of archaeological sites, simultaneously, with minimal need for development of archaeological prior hypotheses, other than those based on the geo-archaeological law of stratigraphic superposition. As demonstrated in a variety of archaeological case studies, just one number, defined as a gauge-probability on a scale 0–100%, can be used to replace a stacked set of subjective Bayesian priors.*

KEY WORDS – *radiocarbon calibration; Fourier transform; Born probability; Santorini*

Bayesova racionalnost ^{14}C, Heisenbergovo načelo nedoločenosti in Fourierjeva transformacija: lepota radiokarbonske kalibracije

IZVLEČEK – *Po približno 30 letih radiokarbonskih raziskav, v katerih si je Bayesova statistika izposojala matematične principe ^{14}C-kalibracije, le-te sedaj vračamo v kvantno fiziko. Ta vrnitev je osnovana na predpostavki, da lahko ^{14}C-kalibriranje opišemo kot Fourierjevo transformacijo. In to ima za posledico, da je potrebno ponovno konceptualizirati verjetnostno analizo ^{14}C. Poglavitna sprememba bo zamenjava tradicionalnega (enodimenzionalnega) koncepta verjetnosti ^{14}C-datiranja z dvodimenzionalno verjetnostjo. To je povsem analogono definiciji verjetnosti v kvantni fiziki, kjer kvadratna amplituda valovne funkcije, ki je definirana v Hilbertovem prostoru, zagotavlja merljivo*

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verjetnost iskanja ustreznega delca v določeni točki v času/prostoru, to je t.i. Bornovo pravilo. Fourierjeva transformacija, ki jo prilagodimo značilnostim ^{14}C-kalibracije, se nemudoma prilagodi tako rekoč vsem t.i. lastnostim kvantizacije arheoloških ^{14}C datumov, kot so hierarhične metode združevanja, spreminjanje starosti in popačenje amplitude. To velja tudi za pogosto opazovane kronološke lastnosti zaklepanja pri večjih podatkovnih bazah, če jih analiziramo z Gaussovim usklajevanjem krivulje (na lestvici ^{14}C) tako kot z Bayesovim zaporedjem (v koledarskem časovnem merilu). Takšni učinki prekopa domene so značilni za Fourierjevo transformacijo. Zdaj jih lahko razumemo in upoštevamo z vpeljavo konceptov in interpretacij, ki so osrednjega pomena v kvantni fiziki (npr. difrakcija valovnih dolžin, dvojnost valov in delcev, Heisengergovo načelo nedoločenosti in princip korespondence). Kar se na prvi pogled zdi zapleteno, v resnici poenostavi postavitev ^{14}C-kronologij. Nova ^{14}C analiza, ki temelji na Fourierjevi transformaciji, podpira kronološke študije na prej nedosegljivih geografskih (kontinentalnih) in časovnih (v glacialih v holocenu) lestvicah; npr. s časovno sekvenco na stotine arheoloških najdišč hkrati, z minimalno potrebo po razvoju predhodnih arheoloških hipotez, razen tistih, ki temeljijo na geo-arheoloških zakonih stratigrafske superpozicije. Kot je razvidno iz različnih arheoloških študijskih primerov lahko tudi le z eno številko, ki je definirana kot merilna verjetnost na lestvici od 0 do 100%, nadomestimo zložen nabor subjektivnih Bayesovih apriornih verjetnosti.

KLJUČNE BESEDE – *radiokarbonska kalibracija; Fourierjeva transformacija; Bornova verjetnost; Santorini*

Introduction

The analysis of radiocarbon data, aimed at construction of cultural chronologies at very high precision (even with the loss of accuracy), has been a continuous field of research for many decades. Much effort has been invested in the development of statistical models that allow incorporation of both qualitative and quantitative archaeological information in the construction of ^{14}C-chronologies. The large majority of these models utilize Bayesian theory (*e.g., Buck* et al. *1991; 1992; Nicholls, Jones 2000; Weninger F.* et al. *2000; Bayliss 2009; Blaauw, Christen 2011; Bronk Ramsey 2009; Steier, Rom 2000; Weninger F. 2011*) and applications of Bayesian age-modelling still have growing networks in archaeology, dendrochronology, terrestrial geomorphology, in ice-core studies and many other fields. In part this is due to the convenient availability of advanced software, such as OxCal (*Bronk Ramsey 2020*) BCal (*Buck* et al. *2020*), CALIB (*Stuiver* et al. *2020*), and Bacon (*Blaauw, Christen 2011*). But perhaps the main reason why Bayesian ^{14}C-analysis today represents the *dominant paradigm* (*Buck, Meson 2015. 567; Buck, Juarez 2017.5*) is that some 20 years ago the necessary statistical procedures were translated into computer language, so researchers today have at their disposal a well-established and now rigorously formalized statistical framework for chronological modelling (*e.g., Bronk Ramsey 2009; 2020; Buck, Meson 2015*). When applied to the construction of ^{14}C-based archaeological chronologies, aimed at the highest achievable dating precision, as mentioned above, the particular advantage of Bayesian modelling is that a wide spectrum of archaeological prior information can be included in the analysis. There is no question that Bayesian age-modelling can be recommended for virtually all fields of archaeological research. On the other hand, there have been persistent rumblings of discontent in parallel to the general acceptance of this statistics driven archaeological paradigm. One frequent expectation by archaeologists is that Bayesian modelling is faultless. It is seen to be capable of providing what appears to be universally coherent results, both in archaeological and mathematical terms, for all ^{14}C-based studies. Unfortunately, this expectation is in clear contradiction with the cautiousness, vigilance, and considerable experience needed for *good* Bayesian modelling, as described by Caitlin E. Buck and Bo Meson (*2015*). Similar concerns with regard to the widespread and naive belief in Bayesian-based chronologies are expressed by many of the authors of the *World Archaeology Special Issue*, Volume 47 (*2015*), that is dedicated to Bayesian radiocarbon chronology. The editorial conclusion of Paul Pettitt and João Zilhão (*2015.526*) is summarized as follows: "*many existing models are faulty*".

In the present paper, some five years later, we have reason to continue this discussion. Again, our aim is to optimize the application of radiocarbon dating in archaeological research. This time, however, it is not any particular archaeological case study, nor any assumption of maths-weakness within the archaeological user community, that drives the discourse, instead it is the very mathematical foundation of ^{14}C-calibration. As an alternative to its suggested *universal implementation* as a Bayesian paradigm (*Heaton* et al. *2020.4*), in our view the process of ^{14}C-calibration is best described as being a Fourier transform which thus has its mathematical foundation in Hilbert space theory. The main purpose of the present paper is to review the concepts of the Fourier transform and associated aspects of quantum theory, and describe it in an understandable manner, using an archaeological context to explain why these concepts can together provide a unique mathematical background to ^{14}C-age calibration.

Naturally, this program requires mathematical validation. Simultaneously we wish to avoid the large-scale ceremonial presentation of corresponding proofs, definitions, equations, formulas and theorems, although we derive a local minimum for all this in the Appendix. The first reason for this decision is that details of the Fourier transform are easily found in textbooks of quantum physics and optical or electronic signal processing. For these topics there are many online presentations, wherein the required mathematics is didactically well-developed. A further reason is that the translation of ^{14}C-calibration into the language of a Fourier transform is relatively straightforward, once a few points in the terminology have been clarified. Once this has been done, the very existence of so many telling analogies between ^{14}C-calibration and quantum physics is itself sufficient to prove the point. Given that under the Fourier transform the calibration algorithms of CalPal-software remain unchanged – one of us (BW) has always been using their mathematical background in quantum theory in his work – any attempt to rewrite the existing mathematics of the Fourier transform would be superfluous. On the other hand, given that our persistent observations of analogies between ^{14}C-ages and quantum particles have never received much attention in the radiocarbon community, perhaps due to deficits in mathematical formalization, or perhaps because it may sound like an outrageous idea, we now provide the reader with an appropriate selection of equations and so-called *engineering* rules (*e.g., the Fourier transform of a Gaussian is again a Gaussian*). Perhaps thankfully, the archaeological reader does not have to become deeply acquainted with the underlying mathematics, nor of Heisenberg's uncertainty principle. Instead, such a reader only needs to grasp the generalized result here, which is that practically all observed quantum properties of archaeological ^{14}C-ages can now be understood as what they are, the mathematical consequence of a Fourier transform.

A simple notion of mathematical foundation

To be as clear as possible, ^{14}C-calibration is so much a Fourier transform that we have taken the freedom, in one or the other figure captions below, to replace the very expression *^{14}C-calibration* with *Fourier transform*. It goes without saying that this equivalence in mathematical background applies equally to what is presently known as Bayesian ^{14}C-calibration, sequencing, age-depth modelling and so on, just as for non-Bayesian wiggle matching, barcode seriation, or construction of summed calibrated probability distributions. Despite the existence of these many different methods, variants, and names, and whether the calculations are technically designed to run on one (or the other) of the two scales, or switch between the two domains, or whether additional data, or hypotheses, are included in the analysis, all these methods are *fundamentally* identical in the sense that, ultimately, they are all based on a Fourier transform. The solution we adapt in describing this result is to a large part historical analysis in the main text, in combination with a technical description in the Appendix. Some carefully selected rules of translation, as provided in the Appendix, are themselves designed to mimic a Fourier transform: they translate backwards and forwards between radiocarbon dating in traditional terminology, and ^{14}C-quantum language in the new perspective. Our hope is that these translation rules may be helpful beyond the present study, in case the reader wishes to validate and extend these new concepts.

The report structure

The overall structure of our report is as follows. To begin, we take freedom to deconstruct the rather naive notion, upheld by many archaeologists, that (likelihood-ratio based) Bayesian ^{14}C-theory can be understood as the unconditional *ultima ratio* implementation of radiocarbon dating in archaeology. We do this by reference to the history of gravitational theory, which has shown wonderful advances over many centuries, but without need, necessity, or even knowledge of Bayes' theorem. This suggests that, since Bayes' probability theory has not been particularly fundamental for this kind of important scientific research in the past, this might also be the case for ^{14}C-analysis in the future. Following this prelude, we proceed towards the somewhat less trivial deconstruction of the – *supposed* – mathematical foundation of ^{14}C-calibration in Bayes' theorem, put forward some 27 years ago by Herold Dehling and Johannes van der Plicht (*1993*). The two studies, in combination, are aimed at providing a slow and protracted introduction of the new mathematical concepts, hence demonstrating at least *the possibility* that there might exist a world beyond Bayes. The point hereby is that, following some 30 years of dedicated, persistent, persuasive and sometimes even convincing Bayesian didactics, parts of the archaeological community have developed academic traits that future prehistorians could possibly describe as Bayesian entanglement. Having motivated both the necessity and possibility for a change in the mathematical foundation of the ^{14}C-calibration procedure, there comes the point where we must actually depart from the world of Bayesian ^{14}C-analysis.

Unfortunately, there does not appear to exist a slow and continuous transition from Bayes-based probability theory to the quantum theory needed in ^{14}C-analysis. Notwithstanding all that has been achieved in the last 30 years, Bayesian ^{14}C-analysis now appears to us to be an increasingly reckless journey, with routine passage yet moving at rather too high speed along a not well-constructed road. In comparison, and judging from the experience as formulated in practically all text-books, the transition from classical physics to quantum theory is inevitably abrupt. Quite simply, there is no slow and easy transition. One accustoms oneself to the new concepts, which takes a bit of time, and soon forgets how much the world has changed. To motivate the reader towards the realization that this transition is beneficial, and also in the context of ^{14}C-analysis, we provide a review of the paper by John Skilling and Kevin H. Knuth, entitled 'The Symmetrical Foundation of Measure, Probability, and Quantum Theory' (*Skilling, Knuth 2019*).

New concepts of probabilistic ^{14}C-analysis

Then, having introduced (in the Appendix) the ^{14}C-calibration as a Fourier transform, which is immediately operative in an abstract vector (so-called: Hilbert) space, we are prepared for the corresponding conceptual switch in probabilistic ^{14}C-analysis. The main change will be to replace the traditional (one-dimensional) concept of ^{14}C-dating probability by a two-dimensional probability. This is entirely analogous to the corresponding definition of probability as used in quantum physics, where the (measurable) probability of finding a wave/particle at a certain point in time/space is based on the squared magnitude of its amplitude: the so-called Born rule. When adapted to the needs of ^{14}C-analysis, application of the modulus-squared Born rule leads us to recognize the ^{14}C-dating probability is not well described as an area under the curve of the graph that shows the summed calibrated Gaussians. Instead, and in mathematically more satisfactory terms, it is possible (for the special purposes of archaeological ^{14}C-analysis) to define a Fourier-based dating probability as product of the amplitudes of two distributions given on the two scales (*i.e.* ^{14}C and calendric). Formulas for the ^{14}C-based Born probability are given in the Appendix [10]. In principle, what has up to now been termed a calibrated *probability* distribution (CPD) we now call a *wave-function*. This terminology is chosen due to the incomplete scaling properties of the calibrated distribution. The actually measurable (what we call *gauged*) dating probability is instead represented as a rectangular (two-dimensional) area that can be projected onto the calibration curve. But we can project such probability rectangles, if requested, onto other components of the calibration system. The flexibility of the new gauge-probability concept is illustrated first in Figure 1, by construction of the entire sequence of major calibration curve plateaus for INTCAL20, then again in Figure 2, by gauge-seriation of the recently published stratified Gravettian ^{14}C-data from Abri Pataud (*Douka* et al. *2019*), but without the need for stratification modelling, and finally in Figure 3, by showing the wave-function (*alias* CPD) for 2543 ^{14}C-ages from Greece and Crete (data: *Katsianis* et al. *2020*) in graphic comparison with the underlying (N=303) archaeological sites, with automated site-seriation.

As for the first example (Fig. 1), up to now the plateaus of the calibration curve have been known mainly for their precision-limiting properties, which were qualitatively defined. When quantified, as it turns out, the plateaus have all the necessary properties to be interpreted as measurable (modulus squared) dating probabilities in Born's sense. As for the second example (Fig. 2), we note that in a typical archaeological ^{14}C-data set there are usually many *open* (*i.e.* not automatically given) probability measures, but which can be chosen (*i.e. gauged*) by the observer.

As illustrated here for the Gravettian sequence at Abri Pataud, already by the simple measure of choosing one gauge for the total data we achieve a fourfold separation of the underlying archaeological

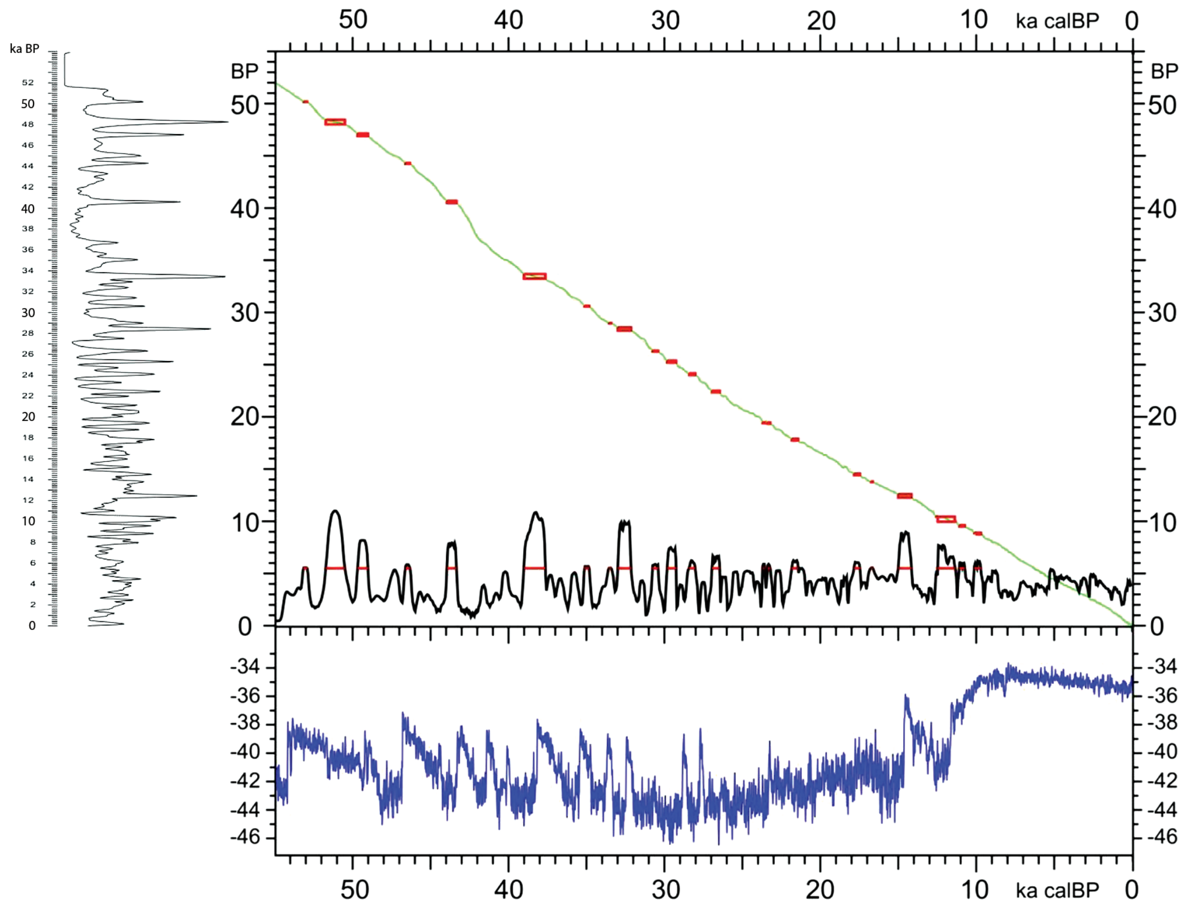

Fig. 1. Automated plateau-box construction for calibration curve INTCAL20 (green line) based on gauge integration in the age range 0–55 ka cal BP. Input: Dirac comb with N=55000 equidistant INTCAL20-derived samples measured at σ=±100 BP. Output: ^{14}C-histogram and back-calibrated summed probability distribution (SPD: black curve) with gauge-probabilities (red-lines) set at p=50% SPD-amplitude. Lower insert: NGRIP stable oxygen δ^{18}O isotope data on GICC05 timescale (*Anderson et al. 2006*). Note the interesting correlation of certain plateau rectangles with onset of major Greenland Interstadials (*cf. Weninger 2020*).

units. The methodological point of interest here is that the choice of a gauge meets the expectation of a probability definition, but without need for its interpretation as an archaeological hypothesis. The gauge is no more, nor less, than an explorative tool. What we have done, in this case, is to slowly (on-screen) move the *gauge-level* up and down an arbitrary gauge scale (0 ≤ g ≤ 100%), in search of a gauge-value for which the corresponding rectangles just start touching each other. The results are in immediate agreement with the thoroughly detailed chrono-stratigraphic analysis by Katerina Douka et al. (*2020*). Importantly, *qua* construction method, for whatever gauge we choose, the rectangles cannot overlap, nor can they occupy the same area. In measure theory, what we call 'gauging' is a standard procedure used to assign Lebesgue measures (in place of probabilities) to subsets of the study data. As a third illustration of the new Fourier-based dating concepts, Figure 3 shows the ^{14}C-demographic chronology of Greece (including Crete and the Aegean islands) for the last 12 000 years, based on the recently published database of Antonio Katsianis *et al.* (*2020*). Of particular interest, we note (1), the broadly synchronous end of the eastern Mediterranean wet period of Sapropel S1 and abrupt onset of Rapid Climate Change (RCC) conditions (~6.2 ka cal BP), with the major settlement gap (6.2–5.0 ka cal BP), that is very clearly observable not only in the summed ^{14}C-data, but also on site level (*cf. Weninger* et al. *2009*), as well as (2), the need for further studies on methods of automated multi-phase site discrimination.

To conclude this overview, all we are doing by introducing these new physics-based concepts into the world of ^{14}C-analysis is to fully account for the mathematical description of ^{14}C-calibration as a Fourier transform. Within this mathematical setting there may be a need for technical changes in the algorithms used in calibration software, other than in CalPal, which is a matter for each reader to judge for themselves. As for changing the underpinning theory, we can readily imagine a combined Fourier-Bayesian statistical approach, whereby the calibration is performed on the basis of the Fourier transform, and then archaeological age-modelling is run within a Bayesian framework. In an engineering context, such a combined approach is viable, if it is based on tried and proven research where the priors are well understood and therefore less subjective. In practice, the choice of specific likelihood functions needed to make Bayesian revisions to a given engineering risk assessment remains difficult to justify. This is especially so when the supporting information for a given prior does not originate from a relatively straightforward process of statistical sampling (*Winkler 1996*). For example, although NASA sometimes uses specific likelihood functions as part of their high-profile assessments to minimize potentially catastrophic space-flight related risks, the use of Bayes-based risk assessment usually occurs when the relevant engineering constraints involved are already clearly established, for instance after the thorough testing of key mechanical components. Somewhat surprisingly, however, our reading of the available literature suggests that probabilistic risk assessments made by NASA still do not rely heavily on Bayes-based statistical methods, despite huge increases in computational power now available since the Apollo program (*Lutomski 2013*).

Bayesian ^{14}C-rationality

As mentioned in the introduction, some five years have passed since Pettitt and Zilhão (*2015*) concluded that many existing Bayesian age models are faulty. Our present topic, however, is something very different. Do not be concerned, we will not be scanning the increasingly vast Bayesian ^{14}C-literature in search of some slightly imperfect age-model, let alone in pursuit of some unfortunate sub-optimal archaeological prior. Instead, we will be addressing the one important question that has not yet been widely studied: why must the application of radiocarbon dating always (under all conditions) be based on Bayesian research methodology?

Isaac Newton's *hypotheses hon fingo*

As stated above, the Fourier transform is a promising alternative to Bayes, not only in technical but also conceptual terms. An example, recently demonstrated in Bernhard Weninger (*2020*), is the possibility to perform automated seriation of hundreds of ^{14}C-dated archaeological sites (assuming the underlying data is truly representative of said sites), without need for development of further archaeological hypotheses post-excavation. In terms of in many cases expendable archaeological modelling, otherwise the very hallmark of Bayesian ^{14}C-analysis, this is reminiscent of the statement *hypotheses non fingo* by Isaac Newton, who wrote in an essay '*General Scholium*' that was added to the second edition (*1713*) of his *Mathematical Principles of Natural Philosophy* (first edition: 1686 – commonly abbreviated *Principia*), which included the following:

"*Hitherto I have not been able to discover the cause of those properties of gravity from phenomena, and I frame no hypotheses, hypotheses non*

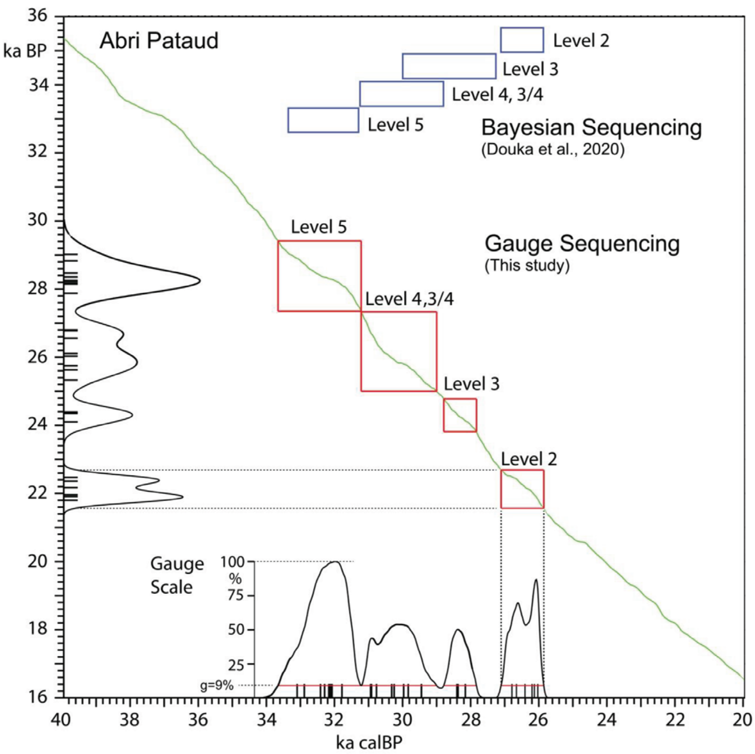

Fig. 2. Application of the probability-gauge sequencing method to a Gravettian ^{14}C-data set from Abri Pataud (**Douka et al. 2020*****). Data input: N=27 ^{14}C-ages from Doukas*** et al. ***(*2020.Fig. 4*). Red rectangles: 2D-Born-probabilities for applied g=9% gauge. Blue rectangles: Results of Bayesian sequencing based on sample-grouping by different levels and stratigraphical level-modelling.***

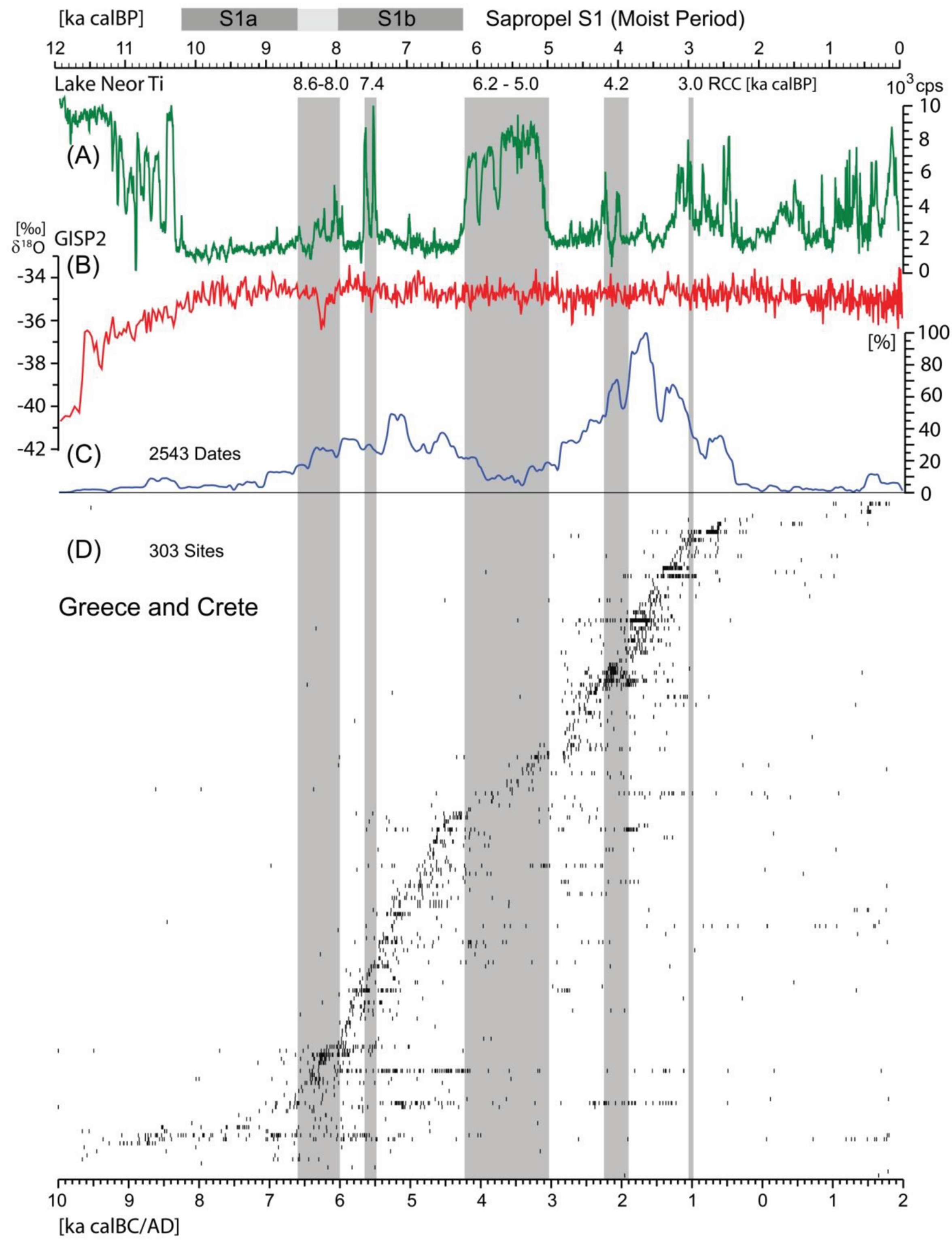

***Fig. 3.** ^{14}C-Demographic chronology of Greece and Crete based on data of Katsianis* et al. *(*2020*), in context with climate records,* A *Lake Neor (NW-Iran) dust flux as proxy for the Siberian High (*Sharafi *et al.* 2015; cf. Mayewski *et al.* 1997*) and for Rapid Climate Change (RCC) (most recently:* Rohling *et al.* 2019*);* B *GISP2 ice-core δ^{18}O as proxy for Greenland surface air temperature Grootes* et al. *(*1993*);* C *N=2543 summed calibrated ^{14}C-ages from Greece & Crete (database:* Katsianis *et al.* 2020*);* D *Barcode Seriation of 303 sites from same database (unfiltered; all sites with ≥3 dates shown) with gauge=40% and leading edge timing (*cf. Weninger 2020*). Vertical shading: periods with extreme climate variability. Upper horizontal shading: E-Mediterranean Sapropel S1 with subdivisions S1a & S1b as proxy for moist conditions, with dry/cold RCC-interruption 8.6–8.0 ka cal BP (*cf. Schmiedl *et al.* 2010*).*

fingo; and hypotheses, whether metaphysical or physical, whether of occult qualities or mechanical, have no place in experimental philosophy. In this philosophy particular propositions are inferred from the phenomena and afterwards rendered general by induction."

Interestingly, this quite unusual description of what makes science *rational* (and clearly based on Newton's long experience with the topic), was formulated by Newton some 100 years prior to Thomas Bayes' publication of '*An Essay Towards Solving a Problem in the Doctrine of Chances*' (*1763*). According to the historical analysis by Abigail E. Bell (*1942*), Newton's main intention with this statement is "*to keep science clear of metaphysical entanglements*". A more recent analysis of the *hypotheses non fingo* statement is by Scott Milner (*2018*), who makes the important point that certain disciplines may have good reason to not be hypothesis driven. Correspondingly, his paper is entitled 'Newton Didn't Frame Hypotheses – why should we?' As an example, the rationality of which presumably only few would find reason to doubt, Scott Milner reminds us of Albert Einstein's studies on gravitation. These were indeed (to begin with) so purely theoretical that – we may add – Einstein himself was apparently quite astonished that his theory could actually be used to forecast a measurable astronomical effect, the advance of Mercury's perihelion that was observed in 1915, where Einstein had not planned for his gravitation theory to measure such effects. Let us go a step further. The two approaches of Newton and Einstein are almost diametrically opposed. Applying modern terminology, whereas Newton's gravitational studies are largely experimental, Einstein's studies are primarily theoretical (unless we decide that *Gedankenexperiments* are experiments). Hence, we have in both comparisons the unique chance to study the true potential of Bayes' theorem, in terms of what would happen (Aristoteles': τελος), should we swap the two sets of paired input variables, namely θ=Theory and D=Data in Bayes' theorem (cf Eq. 1). The result (see below for further discussion of the structure of Bayes' theorem) is – nothing. Namely, for neither of the two aforementioned scientists does the application of Bayes' theorem appear to have been important. For Newton this is trivially true, by virtue of his having lived before Bayes. For Einstein we conclude the same, since for his research the application of Bayes' theorem was either unnecessary, or using it never occurred to Einstein, which amounts to the same. The following quote is what Einstein actually writes in a paper submitted to the Prussian Academy of Sciences in 1915 (*i.e.* prior to the famous experimental confirmation, in 1919, of Einstein's predicted bending of light in strong gravitational fields):

"*In der vorliegenden Arbeit finde ich eine wichtige Bestätigung dieser radikalsten Relativitätstheorie; es zeigt sich nämlich, daß sie die von LEVERRIER entdeckte säkulare Drehung der Merkurbahn im Sinne der Bahnbewegung, welche etwa 45" im Jahrhundert beträgt qualitativ und quantitativ erklärt, ohne daß irgendwelche besondere Hypothese zugrunde gelegt werden müßte.*" (*Einstein 1915*; taken from von *Meyenn 1990.234*).

"*In this paper I find an important confirmation of this most radical relativity theory; it is shown, namely, that this theory explains both qualitatively and quantitatively the secular advance of Mercury's orbital movement discovered by LEVERRIER, and which amounts to around 45" per century, without need for the formulation of any particular hypothesis.*" (Our translation).

Although Einstein's version of the *hypotheses non fingo* statement is clearly different from Newton's, it is similarly non-Bayesian. In physics, just as in biology and other disciplines, it is not *always* helpful to first devise hypotheses and subsequently test them (unless we decide that *Gedankenexperiments* are hypotheses). This applies similarly to quantum string theory (as is well known), as well as to prehistoric archaeology (as is less apparent), although one could argue that Bayes-based approaches to radiocarbon dating in archaeology are often more about parameter estimation than hypothesis testing, an interesting point that could be made much clearer in much more published research.

Bayes' theorem

According to Bayes' theorem (Eq. 1), in a formal representation taken from Skilling and Knuth (*2019*), the probability (*i.e.* scaled truth values 0–100%) of achieved (output) posterior results is not only dependent on the truth value of the empirical (input) evidence, but also on the validity of the (input) prior belief. Both can be seen in normal scientific procedures, because yes, we learn by experience. What is more remarkable is the formal structure of Bayes' theorem, and in particular its symmetry:

$$P(\theta) \cdot P(D|\theta) = P(D) \cdot P(\theta|D) \qquad \text{[Eq. 1] Bayes' theorem}$$

Prior·Likelihood = Evidence·Posterior

As it appears, the Bayes formula is entirely symmetric in terms of probabilities p(θ) and p(D) that are formulated for variables θ=Theory and D=Data, as well as for the conditional probabilities p(θ|D) and p(D|θ). As a consequence, it would be easily possible – with no methodological restriction – to reformulate the Bayes formula, with a commutative switch in the positions of variables θ and D. We would then not only have θ=Data and D=Theory, but the earlier priors would turn into evidence, the previous posteriors would become past likelihoods, and we could even swap their pairwise multiplication, with no change at all in what is termed 'Bayesian inference'. This wonderfully open structure of Bayes' theorem becomes yet more apparent when its four different components are introduced as symbolic variables. Let us call them A, B, C, D. It then follows that A·B = C·D. By solving Equation 1 for each of its variables, we recognize the existence of a total of four distinct possibilities of division by zero (A=(C·D)/B, or B=(C·D)/A, or C=D/(A·B), or D= A·B)/C), that we should avoid under all circumstances.

Beyond this seemingly minor caveat, the symmetric (commutative) Bayesian entanglement of 'Theory' and 'Experiment' is most remarkable and, to our knowledge, would find its closest analogue *not* in Popperian falsifiability (a widely assumed fundament of scientific rationality), but rather more in the more flexible (historiographic) scientific philosophies of Thomas Kuhn and Paul Feierabend. There are other philosophies of rationality that have accompanied the development – in our case – of quantum theory, ranging from the *Copenhagen interpretation* through *Many worlds* to Bohm's *Hidden-variable theory*, and others.

When applying Bayes' theorem to ^{14}C-applications in a single calibration the ^{14}C-measurement (on the ^{14}C-scale) is the likelihood function, and an updated *probability* is seen on the calendrical scale. In more complex Bayes-based ^{14}C models the calendar date-ranges become the likelihood functions in the parameter estimation process. Although this provides an extremely flexible methodological approach for many researchers, we urge that much more thought about this complex process is now required. The fundamental scientific rationality that underlines the expression of Bayes' theorem in radiocarbon calibration is best illustrated by the following story. Once, when asked whether he truly believed that the horseshoe hung above his door would bring him luck, Nils Bohr apparently replied: "*No, but I am told that they bring luck even to those who do not believe in them.*" Lurking in the shadow of Bayes-based ^{14}C-calibration, is the danger of wrongly applying multiple normalization to distributions that only look as if they were defined on two independent time-scales. In reality, there is only one distribution (or horseshoe) which is defined on two domains.

Symmetrical foundation of measure, probability, and quantum theories

Having updated Bayes' theorem in perhaps a somewhat idiosyncratic manner, but nevertheless in admiration of Bayesian rationality, we can make some further positive reference (despite remaining caveats) to the study of John Skilling and Kevin H. Knuth (*2019*), entitled *Symmetrical foundation of measure, probability, and quantum theories*. One of their claims – and this contrasts nicely with sometimes more sophisticated ones – is that "*there is no mystery or weirdness about quantum theory*". The supposedly simple reason for this finding is that quantum theory contains nothing but *ordinary probabilistic inference*. Well, although we would really like to accept such a mysterious statement, unfortunately Skilling and Knuth (*2019*), leave aside the discussion of the one single and altogether most important property that all quantum theories have, and that is the noncommutativity of certain paired variables, such as impulse/momentum and energy/time. We will return to the corresponding question of the ^{14}C-related noncommutativity below. As for Bayes' theorem, having stated its fundamental importance as being the *foundation of rational inference*, Skilling and Knuth (*2019*) conclude that it contains the "*same simple laws of proportion that apply widely elsewhere*". Indeed, that is exactly the structure A·B=C·D of the Bayes formula that we have recognized, above. Henceforth, and now accepting that both quantum theory and Bayes' theory have certain limits and restrictions (although in our view, Bayes may well have more of both), why not combine both methods?

Such was the very proposal we put forward some time ago (*Weninger* et al. *2011*). In the meantime, we must concede allowance for a change in opinion. Our recently derived notion is that a satisfactory mathematical foundation of ^{14}C-calibration can indeed be found in quantum theory, which no longer requires the expedient and computationally more expensive use of Bayes' theorem.

What is *non-classical* probability theory?

The earliest (detailed) suggestion that probabilistic ^{14}C-calibration can be based upon Bayes' theorem can be found in a seminal paper by Dehling and van der Plicht (*1993*), where it is based on the following arguments (*ebda.244*):

"*Mathematical pitfalls can cause calibration procedures to contradict classical formulas ... We show that these ambiguities can be understood in terms of classical and Bayesian approaches to statistical theory. The classical formulas correspond to a uniform prior distribution along the BP axis, the [Bayesian] calibration procedure to a uniform prior distribution along the calendar axis. We argue that the latter is the correct choice,* i.e. *the [Bayesian] computer programs used for radiocarbon calibration are correct.*"

We have here the proposal, that – thanks to Bayes – we may now relax and remain forever (as it were) reassured that the mathematical procedure of ^{14}C-calibration is Bayesian, and is furthermore best applied from the perspective of the calendar axis. From a technical perspective, we agree. This is an optimal and expedient approach. It avoids the problems of division by zero that inevitably occur when the calibration algorithm is run from a ^{14}C-scale perspective. To this point, however, we note that never once during some 35 years' experience with the *inverse* calibration from the perspective of the calendric time-scale have we experienced a problem with division by zero. This is despite the fact that, in ^{14}C-calibration, there are all kinds of potentially threatening problems of this type (*Bohr's Horseshoe*). The choice of an expedient algorithm, alone, does not make the underlying process Bayesian. Although it is described and even recommended as Bayesian by Dehling and van der Plicht (*1993*), in actual fact the algorithm is simply identical to the procedure described seven years earlier by Weninger (*1986*). In that paper, however, no mention is made of Bayes' theorem. Instead, it mentions the existence of an unresolved normalization problem, one which still exists today, but which is now so deeply concealed inside Bayes' theorem that is hiding in plain sight, producing one publication after another of curiously spiky summed calibrated radiocarbon probability distributions. Even more remarkable is how often authors and reviewers do not recognize these spikes as critically serious anomalies (Fig. 4)

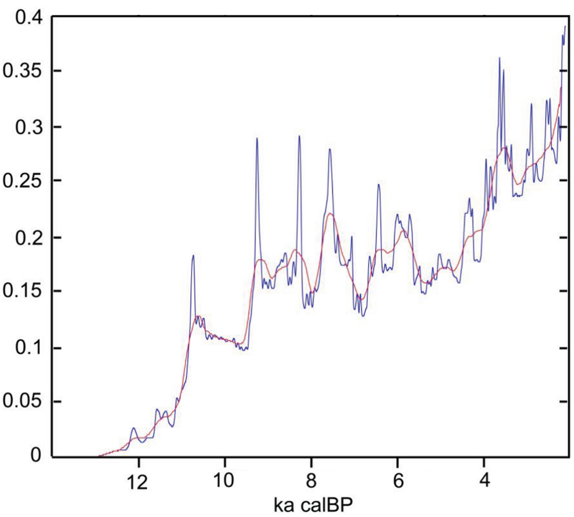

Fig. 4. Blue curve: Summed Calibrated Probability Distribution (SCPD) for 5464 ^{14}C-ages from 1147 archaeological sites in South America. Red curve: same data with 400-year moving average. Both graphs are redrawn from Goldberg* et al. *(*2016; ebda.Fig. 3b*). The authors suggest that the recurring mid-Holocene peaks and troughs (9 ka to 5.5 ka) cannot be explained by calibration artefacts. We suspect that the spikes are not real. They are anomalies caused by secondary normalization.

The cause of calibration spikes

The cause of the spikes is, however, easily understood using Fourier transform theory. Following the initial construction of the Gaussian on the ^{14}C-scale (which has an inbuilt area=1 normalization), during (or following) ^{14}C-calibration there is no need for further normalization. Under a Fourier transform there is only one function. Hence, once it is normalized, there is no reason to normalize the same function, that is already normalized, a second time. The unnecessary application of secondary SPD-normalization is confirmed by Enrico R. Crema and Andrew Bevan (*2020*).

Shape correction of ^{14}C-histograms

Many papers have been published with approaches that need either shape correction of archaeological and environmental ^{14}C-histograms (*e.g., Stolk* et al. *1994*), summed calibrated distributions (*e.g., Williams* et al. *2012*), or else – more recently – for their Bayesian counterparts in the form of kernel density plots (*Bronk Ramsey 2017; Feeser* et al. *2019; Loftus* et al. *2019; Capuzzo* et al. *2020; Mazzucco* et al. *2020*). An idea common to all these approaches is that since we can relate the existence of certain peaks, troughs, or spikes in the diagrams to the calibration curve shape, we expect it should be possible to apply an appropriate correction to the histogram shape (on one or the other scale). Some researchers prefer just to get rid of the spike-anomalies by smoothing. However, when smoothing the spikes to get rid of them they are still there, albeit obscured. In contrast, and quite simply, because with a Fourier transform the spikes are not produced, there is nothing that would require *smoothing* let alone *correction*. Even for the peaks and troughs many researchers are falling victim here to the misleading language developed long ago by radiocarbon specialists, who with good intention emphasize (sometimes even today) that measured ^{14}C-*ages* are *older* than expected by archaeologists, but which require *age-corrections* to allow for the secular variations in atmospheric ^{14}C-contents, and that these *corrections* can be achieved by a procedure called "*^{14}C-age calibration*". Unfortunately for the validity of this hypothesis, although fortunately for nature, nobody has yet demonstrated the feasibility of applying *corrections* to any of the many global physical processes that God (apparently with great wisdom) has undertaken great efforts so that humankind cannot understand them, at least not immediately. The conceptual difficulty in our view, is not the actual histogram correction. The problem is escaping from the illusion that this *normalization*/ correction really is useful, even when achieved.

Based on extensive modelling experiments, our previous (and still valid) conclusion (*Weninger* et al. *2015*) is that an (*apparent*) histogram shape *correction* is indeed possible, although it only *seemingly* works correctly, and only then under very limited *ideal* modelling conditions, and that is for extremely dense and exactly uniform sample distributions, and which only contain ^{14}C-ages with exactly equal ^{14}C-standard deviations. Yet, even under these ideal modelling conditions the actually achieved *perfect* correction of the ^{14}C-histogram shape, under closer scrutiny it is nothing more than a *chimera* or mathematical anomaly, as it were; we see no reason that this should only apply to the non-Bayesian histogram method. Instead – and this is what we now conclude – as a result of the Fourier transform there appears to exist some kind of fundamental mathematical rule that not only restricts, but actually forbids the histogram shape correction. This restriction, in the language of quantum theory, surely has to do with the uncertainty relation (see below) and is maybe even indicative for what physicists call the *second quantization*. This is a concept first introduced in 1927 by Paul Dirac, in order to generalize the application of quantum theory from single-particle to multi-particle systems. From the experiments described in Weninger *et al.* (*2015*), it looks as if – under the Fourier transform – we have yet to learn more about the statistical laws that large sets of ^{14}C-ages apparently follow. For data sets that contain indistinguishable *Bosons* (*i.e.* large sets of ^{14}C-ages with identical standard deviations; in the language of Fourier transform: waves with identical *frequency*, such as lasers), new phenomena emerge that took physicists considerable time to figure out. As is today well known in quantum theory, the properties of multi-particle systems that contain indistinguishable particles (bosons: examples are magnets, lasers, supra-conductivity) can be very different from systems that contain distinguishable particles (fermions: examples are electrons forced to occupy different atomic states under the Pauli exclusion principle). Under appropriate statistical conditions, as in ^{14}C-analysis under the Fourier transform, we may therefore confidently expect many of the larger (*multi-body*) assemblages of ^{14}C-bosons to have *exactly* the statistical properties as those described above, and which we unwittingly stumbled over, in our efforts to understand *why* it is so clearly impossible to find a *correct* histogram shape-correction. With this explanation we can replace the previous description, which Aristophanes may have termed "*perfect cloud-cuckoo-land*" (*Hall, Geldart 1906. 820*; βουλει Νεφελοκοκκυγιαν) by a somewhat more technical description: we are looking at a second (deeper) level of quantization.

Single Gaussian quantization

What we presently know, at least, is that the impossibility of *correct* histogram shape-correction not only applies to larger sets of ^{14}C-ages, but already to single ^{14}C-ages. This forecasting (again under non-Bayesian conditions) may sound curious if not outright wrong. Surely it is *obvious* that a single short-lived sample cannot possibly store the atmospheric ^{14}C-content for any of the years before (or after) it was actually growing, and even more impossible, if we haven't yet ^{14}C-dated the sample? Indeed. Nevertheless, we are in a quantum system. Therein, and whether we like it or not: ^{14}C-ages follow the rules of the Fourier transform. The point being, we do not know the sample age. The direct (and famous) analogy to quantum physics for this forecasting would be that, within a double-slit experiment, the wave/ particle can show interference with itself. An optical diffraction-pattern is observed, curiously, even when the intensity of the incoming particles is so strongly reduced that at any one moment there is only one single particle in the system. This particle, we suppose, can only pass through one, or the other, of the two slits, but not both at the same time. There are many online illustrations of optical diffraction-patterns that are produced when single particles are allowed to build up an interference pattern on a screen, even though they arrive one by one.

The Santorini dilemma

As an alternative to reproducing one of the easily accessible Thomas Young's interference graphs, in Figure 5 we have assembled from the literature some empirical data for the Santorini eruption. This may serve as an archaeologically more compelling illustration for the occurrence of wave diffraction patterns under a Fourier transform, and which also illustrates Young's interference.

What first emerges is a picture that shows how complicated single-event ^{14}C-dating can be, even under quasi-ideal research conditions. The unresolved radiometric problems for the Santorini dating include the possibility of interlaboratory offsets in the order of ±10 BP, in parallel to further issues concerning the existence of geographic and seasonal reservoir offsets, in the same order of magnitude (*Manning* et al. *2020*). In terms of actually dating the Santorini eruption, as illustrated in Figure 5 the scales are even today still well balanced between the alternative (high-middle-low) chronological hypotheses. This is not the place to advocate one or the other chronology. But it is interesting to see how the scales are now re-balancing in support of an intermediate date in the middle of the 16th century cal BC (*Fantuzzi 2018; Pearson* et al. *2020*), or even younger. Looking at Figure 5 we can see that the now 10-year old statement by Malcolm H. Wiener (*2009.203*) is as true as ever: "*Most radiocarbon measurements fall within the oscillating portion of the radiocarbon curve, which makes it impossible to distinguish dates between 1615 and 1525 BC*".

A pathway to the solution is proposed below. From the methodological perspective, and foremost apparent, is that the error-simulated calibrated distributions (corresponding to high-low shifted input ^{14}C-Gaussians) shown in Figure 5 have properties that are similar to the above-mentioned optical single-particle diffraction-pattern. *In principle*, although we are comparing here a physical system (optics) with a mathematical structure (^{14}C-calibration), both have the same underlying cause. Namely, when viewed from the perspective of a Fourier transform, both single-particle Young's interference as well as single-date ^{14}C-calibration can be mathematically described as transformations that decompose sharp/compact input signals (Gaussians) into output signals (dispersed waves) that have widely oscillating amplitudes at high frequencies

Under such conditions, a suitable research concept would be to temporarily refrain from further efforts to obtain a direct ^{14}C-based Santorini eruption date. It appears unlikely that the necessary natural scientific variables will be clarified in the near future (decadal scale).

Pottery dating by correspondence analysis

A promising solution to the *Santorini Dilemma*, from an archaeological perspective, would be to expand on the already now available dating precision of *c.* ±20 yrs (95% confidence). This is not any futuristic dating precision, but the starting precision for statistical seriation of Mycenaean pottery found in Helladic and Minoan deposits. As demonstrated in Figure 6 for the pottery data published by Arne Furumark (*1972a,b*), this dating precision is an order of magnitude (factor 10) better than achieved by single-particle ^{14}C-Fourier analysis at Santorini.

Naturally, Furumark's classification needs much updating and geographic extension in Helladic and Minoan pottery studies. His study was completed in 1940 (*Furumark 1972a.15*). It is 80 years old. Hence, when updated, the precision achieved with the CA is likely to increase (we hypothesize). Let's put it another way. Santorini is one site, but one which has attracted and focussed considerable attention for quite some long time. Of course, it is an important site. Yet, by CA-application, it is possible – based on an updated pottery database and classification that can be readily constructed from the literature – to provide a large number of archaeological sites with high-precision pottery dates, simultaneously, for many regions of the eastern Mediterranean. Furthermore, if this work is initiated, we may forecast the discovery of many more Santorini-type dating discrepancies than are presently known. Put differently, we forecast that the now well-studied Santorini Dilemma will be widely observable, and similarly CA-resolvable, at other sites in the eastern Mediterranean. As an important component of this dating program, there will be need for the critical combination of large numbers of ^{14}C-measurements and historical dates. There will be little need, however, for further distraction of the archaeological research by localized dating controversies.

The distinction between classical and Bayesian theory

Another rather knotty problem of the assumed ^{14}C-foundation in Bayes' theorem pertains to the distinction, also introduced by Dehling and van der Plicht (*1993*), that we should differentiate between classical and Bayesian approaches to statistical theory. This is easier to understand than the difference between a probability and likelihood ratio (function). In this case it is quite simply the terminology used by Dehling and van der Plicht (*1993*) that contrasts with much of the language used in physics. In physics the classical version of a theory is always the one that uses commutative variables (and which reminds us of Isaac Newton), whereby quantum theory supports the analysis of noncommutative variables (esteemed names like Werner Heisenberg, Paul Dirac, and John Neumann spring to mind). We will return to Heisenberg's uncertainty principle below.

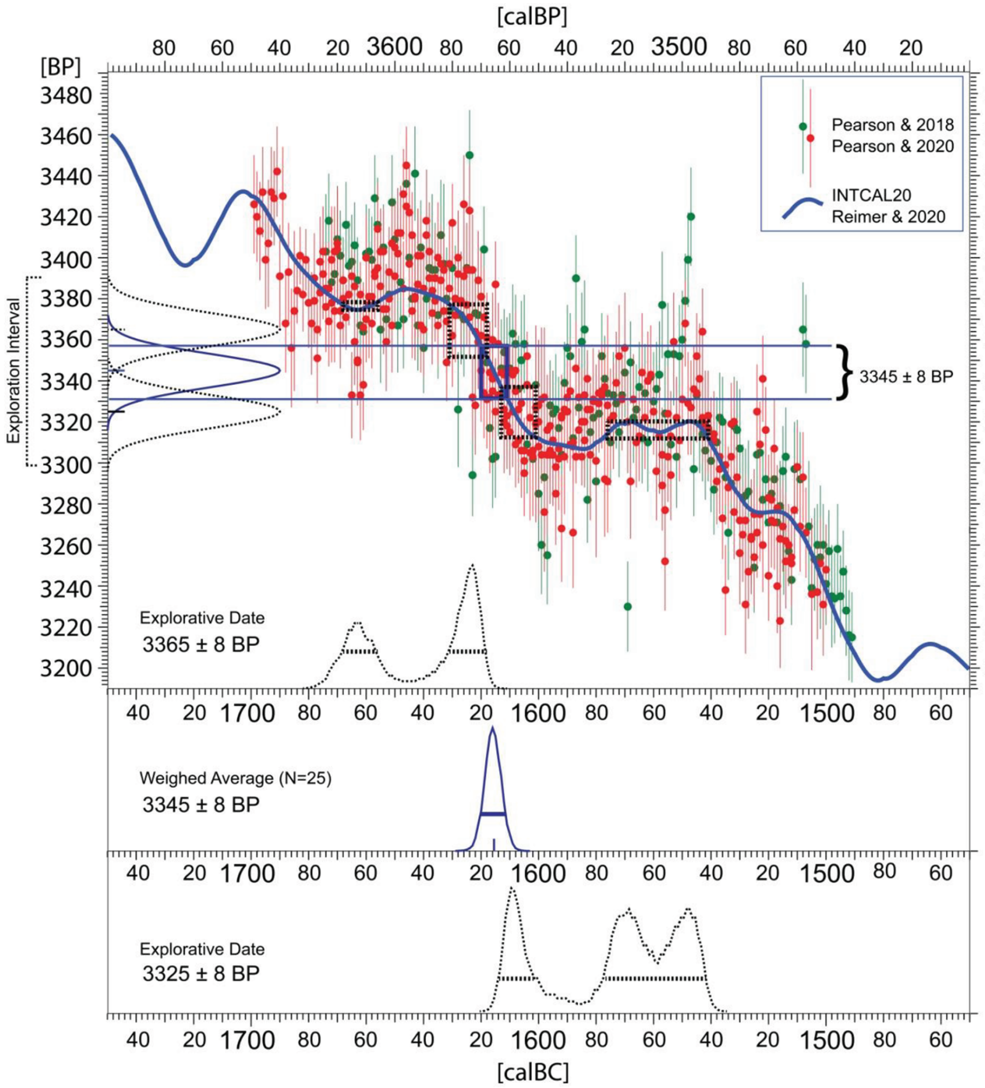

Fig. 5. Calibration of the ^{14}C-scale weighted average 3345±8 BP for the Minoan eruption (Akrotiri Volcanic Destruction Level) from Manning* et al. *(*2014*) (blue Gaussians) and two shifted (high-low) dates in the range of 20 BP (grey Gaussians) against INTCAL20 (*Reimer *et al.* 2020*). Graphic-overlay of annual tree-ring data from Pearson* et al. *(*2018; 2020*). Note: (1) The Arizona-lab data (green bars) by Pearson* et al. *(*2018*) is included in INTCAL20 construction, but not (red bars) the data by Pearson* et al. *(*2020*); (2) INTCAL20 is not only based on Arizona lab data (*cf. Reimer *et al.* 2020; Heaton *et al.* 2020*). Due to unresolved interlab variability, unknown growth-season, and possible geographic carbon-cycle differences, each in the range ±10 BP, this comparison indicates that even ±8 BP dating precision is not sufficient at the present stage of research to gain a final solution on the Minoan eruption dating. Ultimately, this is due to unknown error propagation in a Fourier transform that is very sensitive to high-frequency signal fluctuation. A good analogue for the Santorini dating dilemma, next to optical Young interference, is a faulty electrical connection (in German* Wackelkontakt*).

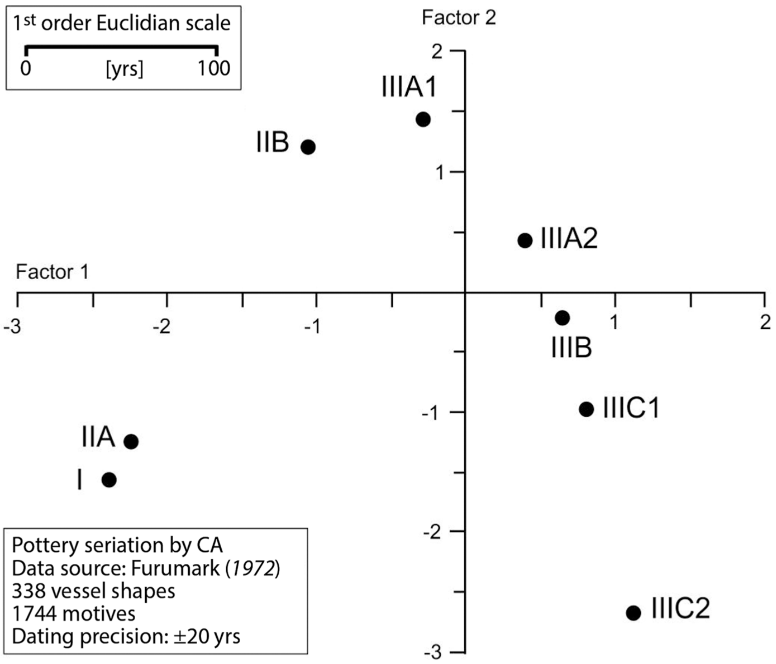

Fig. 6. Seriation of Furumark's pottery/decoration types by correspondence analysis. The* horseshoe *shape of the distribution allows to establish the relative order and relative chronology with a precision (1st order estimate) of ±20 yrs (95%-confidence), often better. Data: Furumark (*1972*). Note: the estimated dating precision is validated for excavation units (not shown) from Kalapodi, Lefkandi, and Mycenae (Granary).

but with later repetition B for the same sample (a=b) and with A=B [BP].

The comparison of Figure 7 (A,B,C, D) illustrates that even when we are truly desperate in a chronological study to achieve highest possible (calibrated) dating precision, it seldom helps to repeat the ^{14}C-measurement on the same sample. This is because the repeat measurement (performed on the ^{14}C-domain) does not provide any significant enhancement of the calibrated sample age (viewed on the calendric domain), even under ideal conditions. From the perspective of the Fourier transform, all that is achieved by repeating the measurement on the ^{14}C-scale is to replace an already existing particle with its identical copy.

In quantum physics, it is the uncertainty principle in particular that is associated with practically all known quantum properties of atomic and nuclear states, and of elementary particles, such that it is understood as responsible (generic) for all observed quantization effects. From a mathematical perspective, in ^{14}C-analysis just as in quantum physics, the uncertainty principle follows directly from the mathematical properties of the Fourier transform.

The noncommutativity of ^{14}C-ages

In analogy to the double-slit experiment, Figure 7 shows the ^{14}C-Fourier transform of a Gaussian-shaped ^{14}C-measurement. This figure recalls that when ^{14}C-ages are calibrated, different results are achieved, depending on whether the mathematical operation of averaging is performed on the ^{14}C-scale (before calibration) or calendric scale (after calibration). In mathematical terminology, this property of the ^{14}C-calibration operator is known as noncommutative. From the description of ^{14}C-calibration as a Fourier transform, it follows naturally that one and the same mathematical function can have different appearances, depending on the domain from which it is visualized. This is illustrated in Figure 7 (A-D) for initially only one Gaussian ^{14}C-measurement, called A,

A humorous illustration that demonstrates how deeply the Fourier transform is embedded within quantum theory is derived from the so-called *One-Electron Universe* hypothesis of John Wheeler and Richard Feynman.

According to the story told by Jagdis Mehra, in his splendid description of Richard Feynman's scientific and other achievements, it so happened that: "*... at about the same time, in the fall of 1940, Feynman received a telephone call from John Wheeler at the Graduate College in Princeton, in which he said that he knew why all electrons have the same charge and the same mass. 'Why?' asked Feynman, and Wheeler replied, 'Because they are all one and the same electron.*" (*Mehra 1996.113*)

This so-called *One-Electron Universe* is illustrated in Figure 8.A-B. It shows three wobbly lines drawn to represent the individual world-lines of three different electrons, called Particles 1, 2 and 3. But now, as shown in Figure 8.C, after turning the graph by 90° the same graph (with minor changes) has every appearance of the ^{14}C-age calibration curve, such that, even with only one ^{14}C-Gaussian to be calibrated, this one Gaussian may have any number of different calibrated ages. Note, however, that the analogy between the One-Electron-Joke (OEJ), the Calibration Curve (CC) and the Fourier Transform

(FT) is only valid for the CC-FT comparison, but strictly speaking not for OEJ-FT, for scaling reasons given in the Appendix Nr: [7].

First quantization properties (single and grouped ^{14}C-dates)

Such properties of ^{14}C-dates we call *first quantization*, and these are omnipresent in archaeological ^{14}C-analysis. They have an amplitude far beyond the statistical noise of the ^{14}C-measurements. The first quantization properties of ^{14}C-dates can be classified according to their occurrence for single dates, data groups, and data series.

Single ^{14}C-dates

- Lock-in of numeric age-values for confidence intervals (*e.g.,* 95% or 68%) that are used to abbreviate calendric-scale age distributions, also for multiple disjunct intervals
- Separation of calendric-scale confidence intervals into multiple disjunct regions
- Dispersal of the calibrated Gaussian and its separation into different components on the calendric timescale
- Lateral shift of the calibrated median along the calendric time-scale
- Dispersal and lateral shift of the area normalized Gaussian on the ^{14}C-scale
- *Probability* values assigned to multiple disjunct intervals seldom sum to 100%.

Data groups

For larger sets of radiocarbon ages (Data Groups) the properties assigned to the individual ^{14}C-ages are all similarly observable, but combine to produce the following new quantization effects:

- clustering of ^{14}C-ages on the ^{14}C-scale;
- clustering of readings on the calendric time-scale;
- attraction of ^{14}C-ages towards predefined intervals on the ^{14}C-scale;
- attraction of calendric readings toward as predefined intervals on the calendric scale.

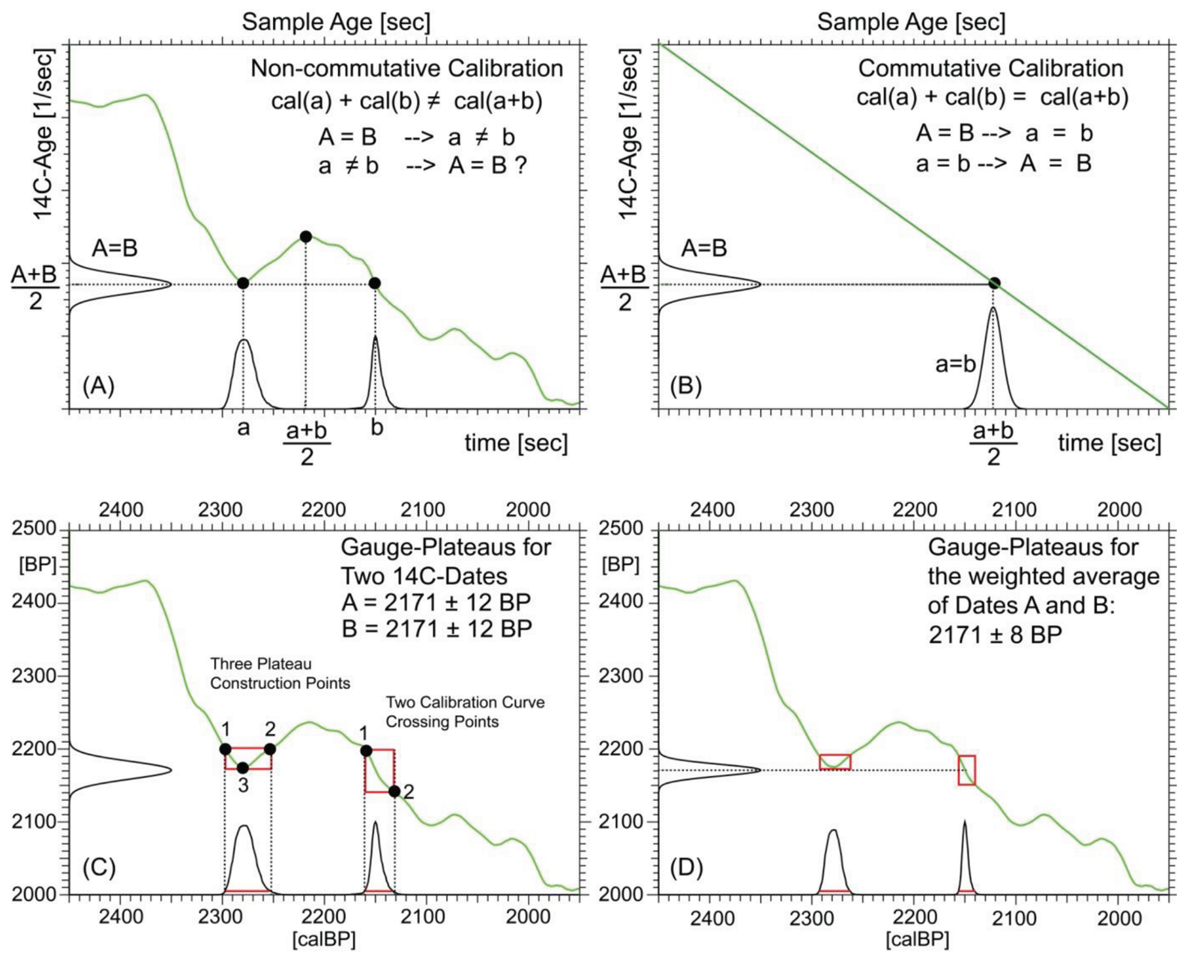

Fig. 7. Illustration of the noncommutative properties of ^{14}C-calibration and construction of gauge-plateaus for (input) single ^{14}C-domain Gaussian and (output) double-reading calendric domain quasi-Gaussians.

Second quantization properties (grouped ^{14}C-data)

For archaeologically sequenced or otherwise temporally seriated data sets (Structured Data Groups) all quantization effects noted above for Single ^{14}C-dates and data groups are known to occur, but under certain conditions some new effects – we call *second quantization* – can be observed. Above, we have already attributed the non-correctability of the histogram shape to so-called *second quantization effects*. As a reminder, these become observable for large assemblages of indistinguishable particles, in our case for large ^{14}C-data sets of closely packed and equidistant samples, best observable when all ^{14}C-measurements have equal standard deviations. We then have sets of bosons.

Second quantization properties (seriated ^{14}C-data)

Quite generally, the second (boson-analog) quantization is more difficult to recognize than the First, but not because it is weaker. Simply, it occurs mainly for larger data sets, and under certain conditions, but which are relatively rare. By theoretical consideration, we would expect such effects not only to occur for weakly coupled (*i.e.* grouped) ^{14}C-data, but to be even more visible for more strongly coupled (*i.e.* seriated) data, such as tree-ring sequences. In such cases, indeed, the pre-established sample order can be so strong as to completely prevent (inhibit) any changes in the structure of the data set. Nonetheless, since the quantization effects cannot be turned off, they are still operative. What happens?

Radiocarbon quantization: frozen data structures

Take by way of example a tree-ring sequence of ^{14}C-ages with precisely measured (say error-free) calendric-scale distances between the samples. In such a data set the sample order is so tightly restricted that the internal structure of the data set is – so to say – *frozen*. In this frozen state, when fitted to the calibration curve the remaining (only possible) quantization reaction is to increase the number of best-fit calendric ages *en bloc* for the entire data set. In consequence, the only remaining reaction is that the wiggle-matching will show multiple (logically alternative) best-fit solutions. We have observed this effect, and what we now call *Frozen Data Structures*, in many wiggle-matching studies. Although presently without formal proof, the occurrence of such *frozen* subset components is what we presently hypothesize has caused block-wise age distortion in Bayesian sequencing at Assiros (North Greece, Late Bronze Age), in this case for a mixture of strongly coupled

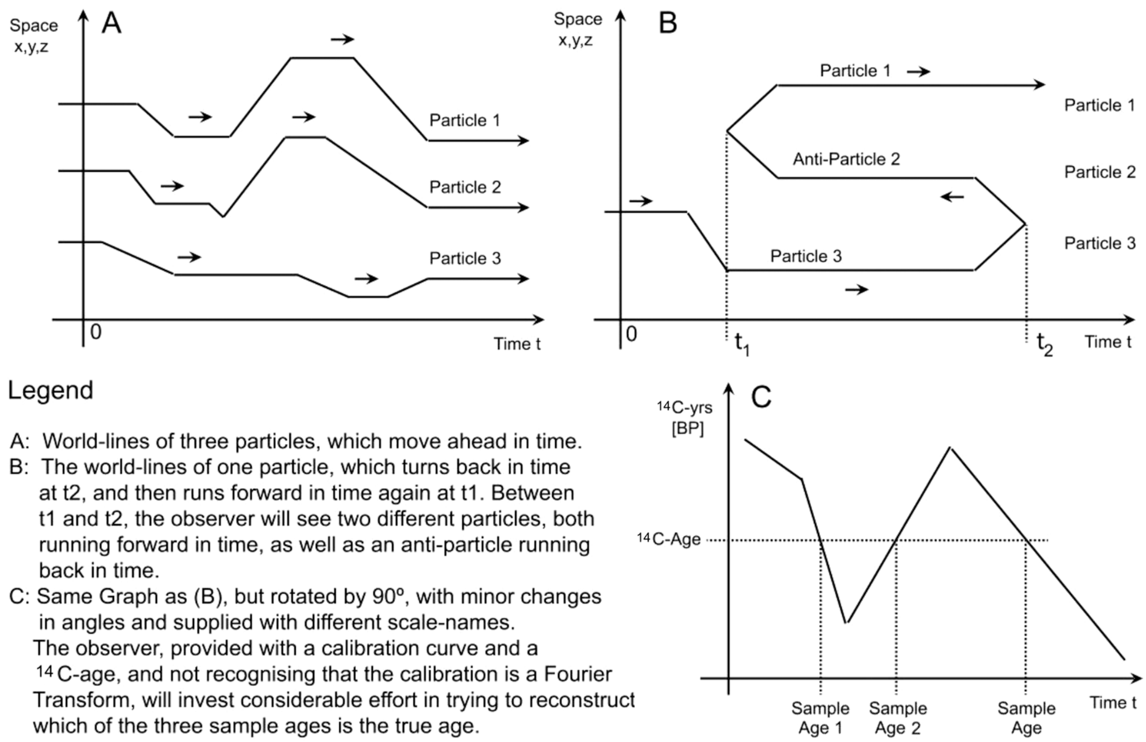

Fig. 8.* A-B *the* One-Electron Universe *hypothesis of Wheeler and Feynman. Redrawn from Mehra (*1996. Fig.5.1, 5.2*);* C *comparison with ^{14}C-Age Calibration System.

tree-ring sequences and weakly coupled bone data (*Gimatzidis, Weninger 2020.ebda.Fig. 2*). Nonetheless, we acknowledge, such properties of sequenced ^{14}C-data sets are presently at the limit of visibility.

Heisenberg's uncertainty principle

Given that even the most precisely fitting ^{14}C-data sets (*e.g.,* tree-ring sequences) show such strong calibration lock-in effects, we may expect that – ultimately – there must exist a generally applicable mathematical theorem that neatly forecasts all observable quantization properties of ^{14}C-data, whether for single dates, data groups, or data series. For the moment, we do not know how to formulate this theorem, but there are good chances that it will look similar to the Fourier transform uncertainty relation, shown in Equation 2. A shorter version of the same equation is found in many textbooks of quantum physics, where it is adapted to the properties of wave/particles and known as Heisenberg's uncertainty principle (Eq. 3).

$$\left(\int_{-\infty}^{+\infty} x^2 |f(x)|^2 dx\right) \cdot \left(\int_{-\infty}^{+\infty} \xi^2 |f(\xi)|^2 d\xi\right) \geq \frac{1}{16\pi^2} \quad \text{[Eq. 2]}$$

$$\Delta x \Delta p \geq \frac{\hbar}{2} \quad \text{[Eq. 3]}$$

The statement in common to both equations is that efforts to sharpen the study function in one domain will inevitably lead to its spread in the second domain. Therein we at last have a satisfactory mathematical explanation for the curious, and for dating experts the counter-intuitive observation that under Bayesian sequencing one may indeed achieve higher dating precision, but only at the loss of dating accuracy (*Steier, Rom 2000*). Please note that reasonably understandable surveys of the mathematical connections between the Fourier transform and the uncertainty principle are already provided by Gerald B. Folland and Alladi Sitaram (*1997*) and Dhiman Sen (*2014*). To these connections we may now add the process of ^{14}C-calibration.

Discussion

In the title of the paper our assertion is that the Fourier transform represents a *beautiful* foundation for ^{14}C-calibration. Yes, there may be concerns about the validity of our *beautiful calibration hypothesis*, which may in consequence produce some immediately critical debate. Whether for mathematical, physical, epistemological, or aesthetic reason, this is of no consequence, if our learned readers allow us to resolve one last question. That is: why do you describe such an apparently trivial, and certainly technical proposal as *beautiful*? The answer is four-fold. First, we attribute the *beauty* of ^{14}C-calibration to the sublime mathematical symmetry of the underlying Fourier transform; second, to the many remarkable analogies between ^{14}C-calibration and quantum theory, thirdly, to the unprecedented explanatory usefulness of Fourier-based ^{14}C-calibration, as well as – finally – to the curiously nonconformist yet immediately understandable appearance of the Fourier transform in an unexpected context. Perhaps our Fourier transform hypothesis is, in itself, not strikingly beautiful, but it is undeniably elegant.

Conclusion

In this paper we propose a rethinking of the mathematical foundation of archaeological ^{14}C-age calibration. We also suggest that archaeologists have at least as much to learn from physicists as they do from mathematicians and statisticians. Following many years of dedicated education, persistent technical support, and admirable instruction by radiocarbon dating experts, parts of the archaeological community are close to the erroneous conclusion that procedures underlying ^{14}C-calibration follow directly from Bayesian probability theory. The choice of a Bayesian framework in ^{14}C-analysis offers, indeed, highly luxurious analytical conditions for archaeological age-modelling. Next to established luxury and acclaimed beauty, the process of ^{14}C-calibration is better described as the Fourier transform.

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Appendix: $_{14}$C-age calibration under the Fourier transform

[1] Translation of variables

The most important translation rule is that, whenever in the Fourier transform an angular frequency (symbol ω) or a wave frequency (symbol ν) is involved, under ^{14}C-analysis we translate the corresponding variable as ^{14}C-age, measured on the [BP]-scale. Schematically this means:

radiocarbon	translates to	Fourier transform
^{14}C-Age $\mu \pm \sigma$ [BP]	<=>	frequency $\nu \pm \sigma_\nu$ [sec^{-1}]
^{14}C-Age $\mu \pm \sigma$ [BP]	<=>	angular frequency $\omega \pm \sigma_\omega$ [sec^{-1}]

In consequence, under the Fourier transform we must change the dimension of ^{14}C-ages from [BP] to [1/sec]. This is only required as a *Gedankenexperiment*. At first sight it may seem curious if not outrightly *wrong* to see this translation of the well-known ['^{14}C-yrs'] dimension into an inverse time-scale: [sec^{-1}]. However, from the perspective of *archaeological* ^{14}C-dating this change in scaling is easily possible. It follows mathematically from the needs of the Fourier transform.

[2] Change of ^{14}C-scale under the Fourier transform

We emphasize that the change of ^{14}C-scale from [BP] to [sec^{-1}] necessary for the Fourier transform is a *Gedankenexperiment*. Traditionally, the Libby equation is used to define ^{14}C-ages. However, Libby ages are measured, and provided to the user, according to the *technical* needs of ^{14}C-measurement, which are different from the *mathematical* needs of archaeological ^{14}C-analysis. The required change in ^{14}C-scale can be motivated by the following. The physical dimension of all ^{14}C-dates is related to the amount of ^{14}C remaining in the sample after its separation from the atmospheric carbon reservoir. From the perspective of Fourier transform, however, since the amount of ^{14}C actually measured *today* in the sample (by whatever technique *e.g.,* beta-decay, ^{14}C-AMS, any other), is the prime reference value, and this value has the physical dimensions of [counts/sec], abbreviated [sec^{-1}], it follows that ^{14}C-measurements could – in principle – be given on the scale [sec^{-1}]. From an archaeological perspective the actual [BP]-scale used is historically motivated but is otherwise secondary to the needs and requirements of ^{14}C-dating (in contrast to ^{14}C-measurement).

[3] Notes on the mathematics of the Fourier transform

Applications of the Fourier transform are typically based on complex-valued functions, for technical reasons. Computations with complex functions are easier to handle than sines/cosines. Even then, Fourier transform equations are often so complicated that they do not support a symbolic solution, but require numerical approximation. However, there is one major exception, famously known to students, which is fundamental to our introduction of ^{14}C-calibration as a Fourier transform. This engineering rule is: the Fourier transform of a Gaussian is again a Gaussian. Furthermore, another rule is important in ^{14}C-analysis, namely: the Fourier transform is a linear operation. The Fourier transform of the sum of two functions is the sum of the transforms.

[4] Why is the Fourier transform fundamental to ^{14}C-age calibration?

Together, these two rules guarantee the applicability of the Fourier transform to the ^{14}C-histogram method, as introduced by Mebus A. Geyh (*1969*), and the calibration of ^{14}C-histograms, according to the algorithm of Bernhardt Weninger (*1986*). Given that ^{14}C-scale histogram construction is additive for measured Gaussians, and that the histogram is most easily and efficiently (inversely) calibrated by applying that algorithm (and which supports further Euclidian error analysis), then these two methods in combination constitute all that is needed for calibration of ^{14}C-ages. It follows that, although useful as an approximation, there is no mathematical necessity for the foundation of ^{14}C-calibration in Bayes' theorem.

[5] Axiomatic foundation of ^{14}C-calibration?

There have been ruminations in earlier radiocarbon literature that – in contrast to Bayesian ^{14}C-calibration – the method of histogram-calibration has no foundation in probability theory. Such statements suggest that the respective authors do not acknowledge the existence of non-classical probability and measure theory. As is well-known in both theoretical and experimental physics, all modern quantum theory – of which the Fourier transform (and then in consequence ^{14}C-calibration) constitutes an important component – has its accepted and to some large extent axiomatic foundation in Hilbert Space theory. This quite remarkable result was achieved within the very short time span of two years (1927–1929) by John von Neumann (*cf. van Hove 1958*). The earliest version of his studies was published in 1927 in German (*von Neumann, 1927*).

[6] What makes ^{14}C-calibration a Fourier transform (FT)?

A brief but complete answer is that, as in every Fourier transform, in ^{14}C-analysis we have two scales (^{14}C and calendric), between which the function under study can switch backwards and forwards. Accordingly, two formulas are needed to describe the ^{14}C-calibration. Most typically, in archaeology the task is to calibrate a Gaussian shaped ^{14}C-age. The basic equation for calibration of a Gaussian ^{14}C-age $\mu \pm \sigma_\mu$ [BP} with median μ and standard deviation σ_μ on a calibration curve $r(t,\tau)$ for calendar ages with values t [cal BP] and ^{14}C-scale with values τ [BP] is given in Equation 1. As shown in Equation 2, the measured ^{14}C-scale Gaussian $\hat{g}(\tau)$ is area-normalized to unit 1. This supports interpretation of the underlying ^{14}C-measurements as (observable) probabilities. This interpretation is important in the laboratory for purposes of quality control and refinement of equipment and methods. The question of how to define (observable) probabilities for calibrated ^{14}C-ages, for similar purposes in archaeological chronology, is addressed below.

$$\hat{g}(\tau) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{1}{2}(\frac{\mu - r(t,\tau)}{\sigma})^2}$$

[Eq. 1] Gaussian distribution ^{14}C-measurement

t = radiocarbom scale [BP]
^{14}C-Age: $\mu \pm \sigma_\mu$ [BP]
$r(t,\tau) \pm \sigma(t,\tau)$ = calibration curve [a,BP]
$\sigma = \sqrt{\sigma_\mu^2 + \sigma^2(\tau)}$ [BP]

$$\int_{-\infty}^{+\infty} \hat{g}(\tau)d\tau = 1$$

[Eq. 2] single Gaussian area normalisation (radiocarbon domain normalisation)

The Fourier transform is widely applied in quantum physics (*e.g., Feynman 2010; Fließbach, Walliser 2012; Hermann 1970; Messiah 1976; Schenk* et al. *2014; Wichmann 1971*). It is typically based on two paired equations. In **Equation 3** and **Equation 4** we have chosen to show the most often used Fourier equations, with variables adapted to archaeological ^{14}C-analysis. In archaeology, we would traditionally be naming the (output) function g(t) as *calibrated ^{14}C-age* [cal BP]. We might also say that the *calibrated age* g(t) ***belongs*** to the *conventional ^{14}C-age* $\hat{g}(\tau)$ [BP]. Such expression of ownership (***belongs***) conforms with the fact that, under the Fourier transform, there only exists one function, although it looks different in the two domains. Also important is that, in agreement with the International System of Base Units (SI), in Equations 1–4 we have defined the function $\hat{g}(\tau)$ on the ***frequency axis τ*** with physical units [sec^{-1}], in replace of ^{14}C-yrs [BP]. Its twin function g(t) is defined on the ***time axis t*** with physical units [sec], in replace of calendric years [a]. The standard radiocarbon layout for the paired Fourier transform equations is then as follows:

$$\hat{g}(\tau) = \int_{-\infty}^{+\infty} g(t)e^{i\omega t}dt$$

[Eq. 3] inverse direction

Fourier transform paired equations

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{g}(\tau)e^{-i\omega\tau}d\tau$$

[Eq. 4] forward direction

τ = radiocarbon domain [BP] = [sec^{-1}]
t = calendar domain [sec]
i = imaginary unit
$\tau = 2\pi\nu$ = angular frequency [sec^{-1}]

[7] Scales with complementary dimensions

For the Fourier transform to work properly, the functions $\hat{g}(\tau)$ and g(t) – as defined on the two domains (^{14}C and cal-scale) – *must* have complementary dimensions. This guarantees that their product $\hat{g}(\tau) \cdot g(t)$ has ***unit = [1] dimension***. For example (as chosen here), if time t is measured in seconds [sec], then the same function but shown in the second domain must be measured using the inverted unit [sec^{-1}]. That is, for radioactivity measurements, the scale of *counts per sec*. When measuring frequencies, the corresponding dimension would be *cycles per second*. For the Fourier transform the only important thing is that the product of the two dimensions is equal to unit=1. When the Fourier transform is applied to ^{14}C-analysis, the calendric time scale automatically has the correct time-dimension of [sec], with trivial rescaling of calendar years to seconds. Hence – after some thought – it becomes clear that under the Fourier transform it is only the ^{14}C-scale that requires *non-trivial* rescaling (as mentioned above), from [BP] to [sec^{-1}]. The details of this rescaling need not bother us. The rescaling of [BP]-> [sec^{-1}] would imply rewriting the Libby equation, but in which neither we (as archaeologists) nor the ^{14}C-labs have any practical interest. To this dimensional point, one of the present authors (BW) must admit some long-standing mistaken thinking. The notion that the ^{14}C-scale is itself *dimensionless* with [BP]=unit [1] for the ^{14}C-scale is erroneous. This only applies to the dimensional *product* of the ^{14}C-scale and the calendar time scale.

[8] What is the difference between scale change and domain switch?

To preclude further misunderstanding we note that, in our terminology, the change of values between [BP] and Δ^{14}C is a *scale change, but this is a different concept than a domain switch*. We use the term domain switch according to Hilbert space theory, where it denotes a change between orthogonal scales. Again, language is important. As noted above, the use of the word *domain* to denote a change in scale for functions that are not orthogonal (*e.g.* [BP], Δ^{14}C, and F^{14}C) in recent radiocarbon literature (*Heaton* et al. *2020*) is definitely a correct use of such mathematical terms. It is nevertheless notable for leaving aside the possibility that ^{14}C-calibration can be described as a Fourier transform.

[9] Fourier transform ^{14}C-age calibration theorem

It is important that the structure of the paired equations (**Eq. 3** and **Eq. 4**) provides proof for the following theorem: following the initial Gaussian area normalization (**Eq. 2**), under the Fourier transform there is no need for further normalization of any of the ^{14}C-distributions. The validity of this theorem follows immediately from the internal structure of Equation 3 and Equation 4. Although we have here two equations, both contain the same function. This function is either called g(t) (in both equations), or else called $\hat{g}(t)$ (in both equations). This *proof* is surprisingly simple, even trivial. It is all the more important, however, in view of all the many studies (all the way back to the 1970's, but still continuing today) that erroneously apply secondary normalization (or weighting) to calibrated ^{14}C-ages. Also, in view of so many studies that have unsuccessfully attempted to *correct* the data for the slope-variability of calibration curve, it is perhaps useful that we formulate this theorem as a software programming rule: to avoid chronological distortion, additional normalization (beyond the initial norm=1 setting of the input Gaussians), is forbidden under the Fourier transform. It must be avoided (in both domains) under all circumstances.

[10] Definition of ^{14}C-dating probability under the Fourier transform

Under the Fourier transform, it is finally possible to introduce a satisfactory (two-dimensional) concept of ^{14}C-dating probability. The main underlying condition for this concept is that both domains contribute equally to the probability measure. In mathematical terminology, the *existence* of this *probability* is guaranteed because there exists a unique 2D-area in the calibration system that has a scaling dimension [sec] on the calendric scale, and scaling dimension of [sec^{-1}] on the ^{14}C-scale. When shown graphically, and in particular when projected as a *plateau-rectangle* onto the calibration curve, the 2D-area that can be derived – by different methods – as product function $\hat{g}(\tau)\cdot g(t)$ has all set-theoretical properties (in particular: additivity), as well as dimension [sec^{-1}]·[sec]=[1], that are needed to define a probability. This can be achieved by simultaneous re-interpretation of the ^{14}C-scale distribution and the corresponding (unnormalized) calibration distribution as *wave functions*. It is then possible to define what we call a *gauge probability* for calibrated ^{14}C-ages (*cf.* main text). This is in accordance with the Born rule, which is the standard procedure in quantum theory, where it is used to define the (measurable) probability P(x,t) of finding a wave-particle in location x and at time t based on the squared amplitude of wave function $\psi(x,t)$:

$$p(x,t) = N\int_{-\infty}^{+\infty}\left|\Psi(x,t)\right|^2 dt = 1 \qquad \text{[Eq. 5]}$$

Born rule: probability definition.
P = Normalised probability defined for squared amplitude of wave-function $\psi(x,t)$, with optional scale factor N to cover a finite number of wave-particles.

N is hereby an initially unknown constant that is independent of x, but which can be determined by a simple requirement: let us assume that the probability of finding the wave-particle *somewhere* is unity *i.e.* N=1. As indicated in Equation 6, under such conditions (when applied to Gaussian wave-functions) the Born rule cancels out the troublesome complex numbers $e^{i\omega t}$ and $e^{-i\omega t}$ contained in Equation 3 and Equation 4. This is because $e^{i\omega t} \cdot e^{-i\omega t} = 1$. In analogy for a set of N ^{14}C-dates, it is in consequence possible to assign a 'Dating Probability' based on the product of the two wave functions $\hat{g}(\tau)$ and g(t):

$$p(x,t) = \iint \Psi(x,t)\Psi^*(x,t)dxdt = 1 \qquad \text{[Eq. 6]}$$

Born rule: probability definition.
P = Normalised probability defined for product of wave-function $\psi(x,t)$ and its complex conjugate $\psi^*(x,t)$ under Fourier transform.

As a reminder, $\hat{g}(\tau)$ and $g(t)$ represent the same *date*, but viewed from the two complementary domains t (calendric scale) and τ (^{14}C-scale). In the end, it is the pleasing properties of the Fourier transform on which the Born rule is based.

[11] Definition of gauge probabilities for ^{14}C-dates under the Fourier transform

Functions that satisfy the conditions of Equation 6 are known as square-integrable. In quantum theory, this is one of the most important properties of the wave-function used to describe any wave-particle. As mentioned above, this condition guarantees the existence of the particle, in the sense that it can be found at some time, and measured at some place, even though the particle has wave-properties. Similarly, if we now define $\hat{g}(\tau)$ and $g(t)$ as *wave-functions*, the condition of square-integrability allows us to introduce a genuine (*i.e.* properly normalized) probability, if only for the product of the two wave functions $\hat{g}(\tau)\cdot g(t)$:

$$p(t,\tau) = \iint g(t)\hat{g}(\tau)dtd\tau = 1 \qquad \text{[Eq. 7]}$$

Born rule applied to radiocarbon dates:
P = Normalised probability defined for the product of the Gaussian function $\hat{g}(\tau)$ in the ^{14}C-domain and its twin associated cal-domain function $g(t)$ under Fourier transform. Note the symmetry of Equation 7 and Equation 6.

In contrast, for the functions $\hat{g}(\tau)$ and $g(t)$ when viewed separately (on their respective domains), it is not possible to define a measurable probability. For the very clear formulation of this important property, Max Born was awarded the Nobel Prize in Physics (although rather late, in 1954). Yet, that does not mean that all related questions are today satisfactorily resolved. As for ^{14}C-analysis, even under the Born rule, the different concepts of probability (*i.e.* quantum theoretical just as Bayesian: both gauged probabilities and likelihoods) still suffer under the same restriction. The problem is that the quadratic integrability in particular of the archaeological study function g(t), as requested in **Equation 5**, is often not ensured. When ensured, the task of this property is to guarantee that already (by itself) the calibrated wave function g(t) would represent a genuine probability. Unfortunately, we cannot resolve the many multiple readings of the ^{14}C-wave function on the wiggles of the calibration curve. The main problem to be resolved is that – using the perhaps better-known terminology of Bayesian ^{14}C-calibration – we have the existence of two very differently scaled concepts of *probability*, but which are typically used in parallel. These two concepts are, (1), the ^{14}C-measurement itself. For researchers in the lab, ^{14}C-ages have a well-defined unit=1 **probability**, namely as **measurement** represented on the ^{14}C-**scale**. However, (2) the very same Gaussian – from archaeological perspective – is seen as a **wave-function** that can be alternatively (or simultaneously) represented both on the ^{14}C-**domain** as well as on the calendric **domain**. When understood as representing an archaeological **date** (and not alone a ^{14}C-**measurement**) both wave-functions have ill-defined square-integrability. On both domains, this is of course mainly (*but not only*) due to the non-monotonous character of the calibration curve. It is particularly important to remember – one can easily overlook or even disbelieve this fact – that ^{14}C-calibration is a Fourier transform even for a linear calibration curve. In conclusion, and with great admiration and due respect for all researchers involved in radiocarbon dating, the ^{14}C-calibration does not have its foundation in Bayesian statistics but in mathematical physics.