UNIVERSITY OF LJUBLJANA
FACULTY OF MATHEMATICS AND PHYSICS
DEPARTMENT OF MATHEMATICS

Andrej Vodopivec

Embeddings of snarks into closed surfaces

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ADVISER: Prof. Dr. Bojan Mohar

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Andrej Vodopivec

Vložitve snarkov v sklenjene ploskve

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Abstract

In the thesis we study embeddings of cubic graphs of class 2. Cubic graphs of class 2 with some additional connectivity requirements are called snarks. The motivation for the study of these graphs comes from attempts to prove the four color theorem. The four color theorem states that the vertices of every simple planar graph can be colored with four colors such that any two adjacent vertices are colored with different colors. The theorem is equivalent to the statement that the edges of every simple planar 3-connected cubic graph can be colored with three colors such that every two adjacent edges are colored with different colors. The edges of every simple cubic graph can be colored with either three or four colors. Graphs whose edges can not be colored with three colors are said to be of class 2. The four color theorem states that 3-connected cubic graphs of class 2 are not planar. One generalization of this statement is that if a cubic graph has a polyhedral embedding into an orientable surface, then it is edge 3-colorable. This generalization is known as the Grünbaum conjecture and was proposed by Grünbaum in 1967. Although 40 years have passed not much progress has been made toward resolving it.

We start with the study of some known families of snarks. We determine the orientable and non-orientable genus of flower snarks and Goldberg snarks. We prove some results about the genus of dot products of graphs and in particular dot products of the Petersen graph.

We then study polyhedral embeddings of known families of snarks. We prove that short cycles in graphs are facial cycles in polyhedral embeddings of cubic graphs. Using this we prove that some known families of snarks do not have polyhedral embeddings into orientable surfaces. We prove that flower snarks do not have polyhedral embeddings (into orientable or non-orientable surfaces) and that Goldberg snarks do not have polyhedral embeddings. We construct for every non-orientable surface N a snark which has a polyhedral embedding into N.

In the last section we study Kochol snarks and superposition. We prove that Kochol snarks do not have polyhedral embeddings into orientable surfaces. We define the defect of a graph as a measure for how far a cubic graph is from having a polyhedral embedding into an orientable surface. In case the Grünbaum conjecture is true we give a strong connection between the defect and the resistance of cubic graphs. (Resistance is a measure for how far a cubic graph is from having a 3-edge-coloring). We prove that the Grünbaum Conjecture implies that snarks which are far from having a 3-edge-coloring are far from having a polyhedral embedding into an orientable surface.

Math. Subj. Class. (2000): 05C10 Topological graph theory, imbedding,
05C15 Coloring of graphs and hypergraphs.

Keywords: chromatic index, cubic graph, snark, polyhedral embedding,
flower snark, Goldberg snark, superposition, Kochol snark.

Povzetek

V disertaciji obravnavamo vložitve kubičnih grafov razreda 2. Kubični grafi razreda 2 z nekaj dodatnimi pogoji na povezanost so znani kot snarki. Motivacija za študij vložitev snarkov prihaja iz poskusov dokaza izreka štirih barv. Izrek štirih barv trdi, da je mogoče točke vsakega enostavenega ravninskega grafa pobarvati s štirimi barvami tako, da so sosednje točke pobarvane z različnima barvama. Izrek je ekvivalenten trditvi, da je mogoče povezave vsakega enostavnega 3-povezanega kubičnega grafa povarvati s tremi barvami tako, da sta dve sosednji povezavi pobarvani z različnima barvama. Povezave enostavnega kubičnega grafa lahko pobarvamo s tremi ali pa s štirimi barvami. Kubični grafi, katerih povezave ne moremo pobarvati s tremi barvami, so grafi razreda 2. Izrek štirih barv pravi, da 3-povezani kubični grafi razreda 2 niso ravninski. Ena izmed posplošitev izreka štirih barv je trditev, da so kubični grafi, ki imajo poliedrsko vložitev v kako orientabilno ploskev, razreda 1. Posplošitev je znana kot Grünbaumova hipoteza in je bila podana leta 1969 in je po skoraj 40 letih še vedno odprta.

Študij začnemo s študijem znanih družin snarkov. Določimo orientabilni in neorientailni rod cvetnih snarkov in Goldbergovih snarkov. Potem študiramo rod 4-vsote grafov, posebej se posvetimo rodu 4-vsot Petersenovega grafa.

Nato študiramo poliedrske vložitve znanih družin snarkov. Pokažemo, da so kratki cikli v kubičnih grafih lica v poliedrskih vložitvah. Pokažemo, da cvetni snarki nimajo poliedrskih vložitev niti v orientabilne niti v neorientabilne ploskve in da Goldbergovi snarki nimajo poliedrskih vložitev v orientabilne ploskve. Za vsako neorientabilno ploskev N konstruiramo snark, ki ima poliedrsko vložitev v N.

V zadnjem poglavju študiramo poliedrske vložitve grafov dobljenih s superpozicijo. Za Kocholove snarke pokažemo, da nimajo poliedrskih vložitev v orientabilne ploskve. Definiramo degeneriranost grafa kot mero kako daleč je kubičen graf od tega, da ima poliedrsko vložitev. V primeru, da Grünbaumova hipoteza drži, pokažemo povezavo med degeneriranostjo in odpornostjo grafa. Odpornost meri, kako daleč je kubičen graf od tega, da ima 3-barvanje povezav. Pokažemo, da so v primeru, da Grünbaumova hipoteza drži, kubični grafi, ki so daleč od tega, da imajo 3-barvanje povezav, tudi daleč od tega, da imajo poliedrske vložitve.

Math. Subj. Class. (2000): 05C10 Topološka teorija grafov, vložitve,
05C15 Barvanja grafov in hipergrafov.

Ključne besede: kromatični indeks, kubičen graf, snark, poliedrska vložitev, cvetni snark, Goldbergov snark, superpozicija, Kocholov snark.

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Contents

Chapter 1

Introduction

In the thesis we study the embeddings of snarks into closed surfaces. The study is motivated by a conjecture of Grünbaum which states that no snark has a polyhedral embedding into an orientable surface. This is a generalization of the Four Color Theorem and is one of the most interesting and long standing conjectures in graph theory. In the Introduction we define basic graph theory and topological notions which are required in later chapters.

1.1 Graphs

A *graph* G is a structure defined by a pair of sets $(V(G), E(G))$. The set $V(G)$ is a non-empty set and its elements are called the *vertices* of G. The set $E(G)$ is a set of 2-element subsets of $V(G)$ and its elements are called the *edges* of G. A set $\{u, v\}$, representing an edge, will be denoted by uv. We will investigate only finite graphs, that is graphs for which the set $V(G)$ is finite. Also note that graphs are simple, that is there are no parallel edges and no loops. The number of vertices $n = |V(G)|$ is called the *order* of the graph. For an edge $e = uv$ in $E(G)$ we call vertices u and v the *ends* of the edge e. If for vertices $u, v \in V(G)$ there is an edge $e = uv \in E(G)$ we say that the vertices u and v are adjacent and that the edge e connects vertices u and v. If v is an end of an edge e we say that v is incident with e and if vertices u and v are connected by the edge e we say that v is a *neighbor* of u. The set of neighbors of a vertex v is denoted by $N(v)$. The *degree* $\deg_G(v)$ of a vertex $v \in V(G)$ is the number of edges incident with v. The minimum degree of a vertex in the graph G is denoted by $\delta(G)$ and the maximum degree of a vertex in the graph G is denoted by $\Delta(G)$. If all degrees of vertices in the graph G are equal to k, the graph is *k-regular*. A *cubic graph* is a 3-regular graph.

A generalization of a simple graph is *multigraph*. A multigraph M is defined as a triple $(V(M), E(M), \delta)$ where $V(M)$ is the set of vertices, $E(M)$ is the set of edges and δ is a mapping which assigns each edge $e \in E(M)$ a pair of its ends, where we allow the ends to be the same vertex. In the latter case the edge is called a *loop*. We allow that two edges have the same ends in which case we say that the edges are *parallel*. The degree of a vertex v in a multigraph is the number of edges such that v is its end where we count loops incident to v twice.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$ then H is a *spanning subgraph* of G. If H is a subgraph of G this will be denoted by $H \subseteq G$. If $V(H) \subseteq V(G)$ and if for each pair of vertices $u, v \in V(H)$ the edge $uv \in E(H)$ if and only if $uv \in E(G)$, then H is an *induced subgraph* of G.

A bijection $\psi : V(G) \to V(H)$ is an isomorphism if it maps adjacent vertices into adjacent vertices and non-adjacent vertices into non-adjacent vertices. If there exists an isomorphism between graphs G and H they are said to be *isomorphic*. We will not distinguish between isomorphic graphs and will write $G = H$ if G and H are isomorphic.

A path P_n of length $n-1$ is a graph with vertices $V(P_n) = \{v_1, \ldots, v_n\}$ and edges $E(P_n) = \{v_i v_{i+1} \mid i = 1, \ldots, n-1\}$. Vertices v_1 and v_n are the *ends* of the path P_n and we say that the path P_n connects its ends. A cycle C_n of length n is a graph with vertices $V(C_n) = \{v_1, \ldots, v_n\}$ and edges $E(C_n) = \{v_i v_{i+1} \mid i = 1, \ldots, i-1\} \cup \{v_1 v_n\}$. A subgraph $P \subseteq G$ isomorphic to a path P_n is called a path in G and we say that P connects its ends in G. If for each pair of vertices $u, v \in V(G)$ there exists a path P in G connecting u and v we call the graph G *connected*. A maximal connected subgraph in G is called a *connected component* of G.

A *walk* W in a graph G is a sequence of vertices $(v_1, v_2, \ldots, v_n)$ where vertices v_i and v_{i+1} are incident for $i = 1, \ldots, n-1$. Vertices $v_1, \ldots, v_n$ need not be all distinct. If v_1 and v_n are connected then W is called a *closed walk* in G. Instead of defining a walk by a sequence of vertices $(v_1, v_2, \ldots, v_n)$ we will sometimes define it with the sequence of edges $(e_1, \ldots, e_{n-1})$, where $e_i = v_i v_{i+1}$, $i = 1, \ldots, n-1$.

For a subset $S \subseteq E(G)$ we denote by $G - S$ the graph H with vertices $V(H) = V(G)$ and with edges $E(H) = E(G) \setminus S$. If the number of connected components of $G - S$ is larger than the number of connected components of G we call the set S a *cut*. A minimal set S which is a cut is called a *minimal cut*. A connected graph G is k-edge-connected if every cut contains at least k edges. A cut of size k will be called a *k-cut*.

For a subset $U \subseteq V(G)$ we denote by $G - U$ the graph H with vertices $V(H) = V(G) \setminus U$ and in H two vertices are connected if and only if they are connected in G. A graph G is k-connected if every set U, for which the graph $G - U$ is not connected, contains at least k vertices. A cubic graph is k-connected if and only if it is k-edge-connected.

A *k-edge-coloring* of a graph G is a mapping $c : E(G) \to \{1, 2, \dots, k\}$ such that each pair of adjacent edges is mapped into distinct elements of $\{1, 2, \dots, k\}$. The minimum number k, for which there exist a k-edge-coloring of G, is the *chromatic index*, $\chi'(G)$, of G. Vizing proved the following theorem

Theorem 1.1 (Vizing). *Every (simple) graph G satisfies*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$$

.

Vizing's theorem divides graphs into two groups. Graphs for which $\chi'(G) = \Delta(G)$ are called *class 1 graphs* and graphs for which $\chi'(G) = \Delta(G) + 1$ are called *class 2 graphs*. As a special case cubic graphs of class 1 are those for which $\chi'(G) = 3$ and cubic graph of class 2 are those for which $\chi'(G) = 4$.

1.2 Surfaces and graph embeddings

In this section we give basic definitions for closed surfaces and graph embeddings. We do not define basic topological objects. We follow the book [1]. A *closed surface* is a connected compact Hausdorff topological space S which is locally homeomorphic to an open disc in the plane $\mathbb{R}^2$. To simplify some arguments we will assume that graphs in this section do not have vertices of degree one or two. All results hold if we allow vertices of degree one or two also.

Examples of surfaces are obtained as follows. Suppose $\mathcal{F}$ is a collection of polygons with all sides of length 1 which altogether have an even number of sides $\sigma_1, \dots, \sigma_{2k}$. Arbitrarily orient each side σ_i by choosing one of its endpoints as the initial endpoint and choose a partition of sides into pairs. Form a topological space S by identifying two sides in each pair so that the orientations are respected (that is for a pair σ_i, σ_j we identify the initial endpoint of σ_i with the initial endpoint of σ_j). We get a compact Hausdorff topological space S and if S is connected then S is a surface.

The sides of polygons in $\mathcal{F}$ and their endpoints define a multigraph G'. We say that G' is *2-cell embedded* in the surface S. The collection of polygons $\mathcal{F}$ is called the collection of faces of G'.

Take a triangulated surface S and on a face T two disjoint triangles T_1 and T_2. If we orient the sides of T_1 and T_2 so that the orientations are clockwise, remove T_1 and T_2 from S and identify triangle T_1 with T_2 we obtain a surface

S'. We say S' is obtained from S by *adding a twisted-handle*. If we orient the sides of T_1 clockwise and the sides of T_2 anticlockwise, remove T_1 and T_2 from S and identify triangles T_1 and T_2 we obtain a surface S''. We say S'' is obtained from S by *adding a handle*. Let Q be a equilateral quadrangle in T. If we delete Q from T and identify opposite points on the boundary of Q we obtain a surface S'''. We say S''' is obtained by *adding a cross-cap* to S. When we add handles and cross-caps we will usually use discs instead of triangles in T.

Now start with a sphere S_0 which is a tetrahedron. If we add n handles to S_0 we obtain a surface S_n which is called the *orientable surface of genus n*. If we add $n > 0$ cross-caps to S_0 we obtain a surface N_n which is called the *non-orientable surface of genus n*. The surface S_1 is called the *torus* and the surface S_2 is called the *double torus*. The surface N_1 is called the *projective plane* and the surface N_2 is called the *Klein bottle*. Instead of embedding graphs into the sphere we will usually embed graphs into the plane, which is equivalent by the stereographic projection of the sphere into the plane. The torus will be represented as a quadrangle with corners a, b, c, d where we orient sides as ab, bc, dc, ad and identify sides ab and dc and sides bc and ad. A projective plane will be represented by a disc in which we identify antipodal vertices.

It turns out that by adding handles and cross-caps to a sphere we can construct all possible examples of surfaces. This is established by the following theorem.

Theorem 1.2 (Classification of surfaces). *Every surface S is homeomorphic to precisely one of the surfaces S_n, $n \geq 0$ or N_n, $n > 1$.*

For surfaces S_n we define the Euler characteristic $\kappa(S_n) = 2 - 2n$ and for surfaces N_n we define the Euler characteristic $\kappa(N_n) = 2 - n$. For arbitrary surface S we define $\kappa(S)$ as the Euler characteristic of the unique surface S_n or N_n which is homeomorphic to S. For S_n we define the *orientable genus* $g(S_n) = n$ and for N_n we define the *non-orientable genus* $\tilde{g}(N_n) = n$. A surface S is an *orientable surface* if it is homeomorphic to S_n for some $n \geq 0$ and it is a *non-orientable surface* if it is homeomorphic to some N_n, $n > 1$. The *genus* $g(S)$ of an orientable surface S is n if S is homeomorphic to S_n. The *non-orientable genus* $\tilde{g}(S)$ of a non-orientable surface S is n, if S is homeomorphic to N_n. The *Euler genus* of an orientable surface S is $\epsilon(S) = 2g(S)$ and the Euler genus of a non-orientable surface N is $\epsilon(N) = \tilde{g}(N)$.

A 2-cell embedding of a graph G into a surface S is graph G' which is 2-cell embedded in S and isomorphic to G. Faces of the embedding of G are faces of G'.

Let G be 2-cell embedded in S. Put a small disc D_v on each vertex v of G such that D_v intersects G only in v and edges incident with v and so that the intersection of D_v with each edge incident with v is a segment. Choose an orientation of the boundary of D_v. Intersections of edges $\{e_1, \ldots, d_k\}$ incident with v and the boundary of D_v define a *clockwise* ordering of edges incident with v around v. This ordering defines a permutation π_v of edges incident with v for which $\pi_v(e) = e'$ if e' follows e in the ordering. For an edge $e = uv$ we say that orderings π_v and π_u are consistent if for an orientation of e the discs D_v and D_u with orientations which define π_v and π_u cross e one from left to right and the other from right to left. If π_v and π_u are consistent than we set $\lambda(e) = 1$ and if they are not consistent we set $\lambda(e) = -1$. The mapping λ is called the *signature of edges* (see Figure 1.1). It turns out that if S is orientable then we can choose the orderings around vertices so that for each edge $e \in E(G)$ we have $\lambda(e) = 1$.

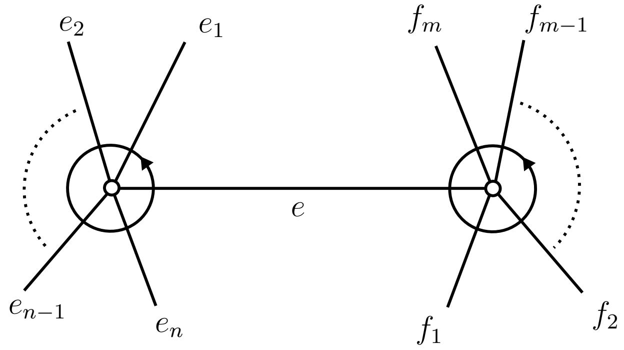

Figure 1.1: An edge $e = uv$ in an embedded graph with chosen clockwise orderings at its ends and rotations $\pi_v = (ee_1e_2 \cdots e_{n-1}e_n)$ and $\pi_u = (ef_1f_2 \cdots f_{m-1}f_m)$ and $\lambda(e) = 1$.

Denote by $\pi = \{\pi_v \mid v \in V(G)\}$ the collection of clockwise permutations around vertices of the embedded graph G. The pair $\Pi = (\pi, \lambda)$ is is called a *rotation system* of the embedded graph G. Two rotation systems Π and Π' are equivalent if Π' can be obtained from Π by a sequence of transformations where in each transformation we reverse the clockwise ordering around a vertex v and change the signs of all signatures of edges incident with v. It turns out that a 2-cell embedding of G is completely determined by its rotation system and that each rotation system defines a 2-cell embedding. A rotation system is called a combinatorial embedding. From now on whenever we say that Π is an embedding of a graph G we mean that Π is a rotation system which defines the embedding.

A sequence of vertices of and embedded graph G which appears along a face of G is called a *facial walk*. If all vertices along W are distinct then W is called a *facial cycle*.

Given a rotation system Π of G the collection of facial walks is obtained as follows. Choose a vertex v_0 and an edge $e = v_0v_1$ incident with v_0. Traverse the edge e. From v_1 continue on the edge $\pi_{v_1}(e)$ and repeat this until an edge $f = v_{i-1}v_i$ is traversed from v_{i-1} to v_i for which $\lambda(f) = -1$ (it could be that $f = e$). Now traverse the edge which follows f in the anticlockwise order around v_i, $\pi_{v_i}^{-1}(f)$, and repeat this until an edge with negative signature is traversed again. From there on traverse edges in clockwise order around vertices and so on. Repeat this until e is traversed again in the same order from v_0 to v_1. When this happens we have obtained a facial walk of the embedding of G. To get other facial walks repeat this procedure starting with another vertex u_0 and an edge u_0u_1 which has not been traversed from u_0 to u_1. When no such edges remain (that is all edges have been traversed in both directions) we get all facial walks of the embedding. Two equivalent rotation systems define the same collection of facial walks.

A rotation system is determined by the collection of facial walks. Suppose $\mathcal{F}$ is a collection of facial walks. Choose a vertex v and an edge e_1 incident with v. There is a facial F_1 walk which contains the edge e_1. This walk also contains another edge incident with v, say e_2, so that the edges e_1 and e_2 are consecutive along F_1. There is a facial walk F_2 which contains e_2 and a third edge e_3 such that e_2 and e_3 are consecutive along F_2. We continue this until we come back to the edge e_1. We define the clockwise order around v to be $e_1, e_2, \ldots$. Once we have clockwise orderings around each vertex we can define the signatures of edges. Of course not every collection of walks is a collection of facial walks of some embedding. For a cubic graph a sufficient condition that a collection of closed walks $\mathcal{F}$ is a collection of facial walks of some embedding is that each path of length 3 appears along exactly one walk in $\mathcal{F}$.

Suppose we have an embedding Π of a graph G into a surface S. Denote with $F(G)$ the collection of facial walks of the embedding. The number of facial walks can be determined by the following relation.

Proposition 1.3 (Euler formula). *The following equation holds*

$$|V(G)| - |F(G)| + |F(G)| = 2 - \epsilon(S).$$

If Π is an embedding of G into an orientable surface S we define the *orientable genus* of Π as $g(\Pi) = g(S)$. If Π is an embedding of G into a non-orientable surface S we define the *non-orientable genus* of Π as $\tilde{g}(\Pi) = \tilde{g}(S)$.

The *(orientable) genus* of a graph G is the minimum

$$g(G) = \{g(\Pi) \mid \Pi \text{ orientable embedding of } G\}$$

and the *non-orientable genus* of a graph G is the minimum

$$\tilde{g}(G) = \{\tilde{g}(\Pi) \mid \Pi \text{ non-orientable embedding of } G\}.$$

Let Π be an embedding of G into a surface S. We define the geometric dual G^* of G in S as follows. The vertices of G^* correspond to facial walks of the embedding of G. The edges of G^* are in bijective correspondence with the edges of G. An edge e^* joins vertices w and v in G^* if the edge e appears on facial walks corresponding to vertices w and v. For a facial walk $W = e_1 e_2 \cdots e_n$ define the rotation around the vertex w in G^* corresponding to W as $\pi_w = (e_1, e_2, \ldots, e_n)$. We define $\lambda(e^*) = 1$ if facial walks W and V corresponding to vertices w and v, $e = vw$, traverse e in opposite directions and $\lambda(e^*) = -1$ otherwise. It is easy to verify using the Euler formula that Π and Π^* are embeddings into the same surface. Note that for a graph G the dual G^* can be a multigraph (that is there could be parallel edges or loops in G^*).

A graph G embedded into a surface S such that all facial walks are of length 3 is called a *triangulation* of S. The geometric dual of a triangulation is a cubic graph (see Figure 1.7).

1.3 Snarks

In this thesis we will mostly be interested in cubic graphs of class 2. Before we start with the introduction to class 2 cubic graphs we state a very useful Lemma about 3-edge-colorings of cubic graphs.

Lemma 1.4 (Parity lemma). *Let c be a 3-edge-coloring of a cubic graph G and S a cut in G. Denote by S_i the set of edges in S colored with color i. Then*

$$|S_1| \equiv |S_2| \equiv |S_3| \equiv |S| \pmod 3.$$

Snarks are non-trivial cubic graphs of class 2. A cubic graph G of class 2 is trivial if there is a reduction of G to a smaller snark or if there is an obvious obstruction for G which prevents it to have a 3-edge-coloring. We now explain what are trivial class 2 cubic graphs which will be excluded in the definition of snarks.

Suppose $S = \{e\}$ is a cut of size 1 in a cubic graph G. The edge e is called a *bridge* of G. If c is a 3-edge-coloring of G then we can assume that $c(e) = 1$ which implies that $|S_1| = 1$ and $|S_2| = |S_3| = 0$ which is a contradiction to the Parity lemma 1.4. Therefore if a cubic graph contains a bridge it can not be 3-edge-colorable. We will therefore require that snarks must be bridgeless graphs.

Suppose $S = \{e, f\}$ is a 2-cut in a bridgeless cubic graph G. Let $G - S$ be composed of graphs G_1 and G_2. Suppose $e = v_1v_2$ and $f = u_1u_2$ and suppose that $v_1, u_1 \in V(G_1)$ and $v_2, u_2 \in V(G_2)$. Add edges u_1v_1 to G_1 obtain a cubic graph G'_1 and u_2v_2 to G_2 to obtain a cubic graph G'_2. If G'_1 and G'_2 are 3-edge-colorable then we have a coloring c' of graphs G'_1 and G'_2 and further we can assume that $c'(v_1u_1) = c'(v_2u_2) = 1$. Now we define a coloring c of G as follows. For an edge $g \notin \{e, f\}$ define $c(g) = c'(g)$ and $c(e) = c(f) = 1$. It is easy to check that c is a 3-edge-coloring of G. Therefore if there is a 2-cut in a class 2 cubic graph G, we can reduce G to smaller cubic graphs G_1 and G_2 such that at least one of them is of class 2. Therefore we will require that snarks are 3-connected.

A cut S in G such that $G - S$ has at least two components containing a cycle is called a *cyclic cut*. A graph is *cyclically k-edge-connected* if every cyclic cut contains at least k edges. Suppose that G is a 3-connected cubic graph containing a cyclic cut $S = \{e_1, e_2, e_3\}$. Then $G - S$ consists of two connected components G_1 and G_2 each containing a cycle. Graphs G_1 and G_2 each contain three vertices of degree 2 which are the ends of edges in S. If we add a vertex v_1 to G_1 and connect it to the degree 2 vertices in G_1 and add a vertex v_2 to G_2 and connect it to the degree 2 vertices in G_2 we get cubic graphs G'_1 and G'_2. Suppose we have a 3-edge-coloring c' of G_1 and G_2. We can assume that $c'(v_1u_i) = c'(v_2w_i) = i$ where u_i and w_i are the ends of e_i. We can define a coloring c of G by defining $c(e) = c'(e)$ if $e \notin \{e_1, e_2, e_3\}$ and $c(e_i) = i$. This is a 3-edge-coloring of G. So if there is a cyclic 3-edge-cut in a class 2 graph G, we can reduce G to smaller cubic graphs G'_1 and G'_2 at least one of which is of class 2. Therefore we will require that snarks are cyclically 4-edge-connected. Note that a 3-connected cubic graph is cyclically 4-edge-connected if every 3-cut separates the graph into two components, one of which is a vertex.

Suppose we have a cubic graph G which contains a 3-cycle C_3 on vertices 0, 1, 2 (see Figure 1.2). If we replace C_3 with a vertex v we obtain a cubic graph G'. Suppose c' is a 3-edge-coloring of G'. Define a mapping $c : E(G) \to \{0, 1, 2\}$ as follows. If an edge is not incident with any of 0, 1, 2 then $c(e) = c'(e)$. Further define $c(v_ii) = c(v_{i+1}v_{i+2}) = c'(v_iv)$, $i = 0, 1, 2$, where incides are modulo 3. Then c is a 3-edge-coloring of G. We see that if G is of class 2 then G' is also of class 2. Therefore if we have a 3-cycle in a class 2 cubic graph we can reduce it to a smaller cubic graph of class 2. We will therefore require that snarks have no cycles of length 3.

Suppose we have a cubic graph G which contains a 4-cycle C_4 on vertices 0, 1, 2, 3 (see Figure 1.3). If we replace C_4 with two edges $e_0 = v_0v_1$ and $e_1 = v_2v_3$ we obtain a cubic graph G'. We can assume that G' is bridegles, otherwise we add edges v_0v_3 and v_1v_2. Suppose c' is a 3-edge-coloring of G'. Define a

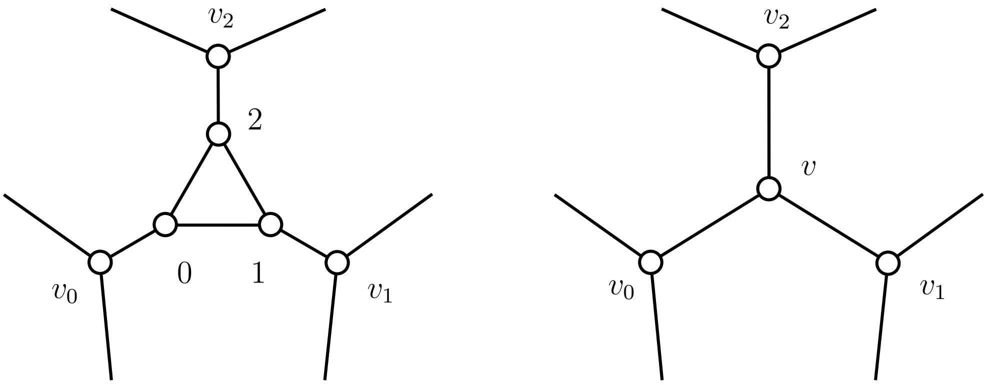

Figure 1.2: Removing a 3-cycle from a graph.

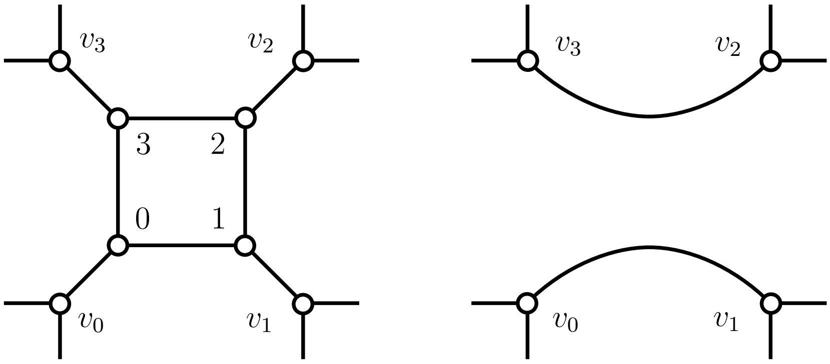

Figure 1.3: Removing a 4-cycle from a graph.

mapping $c : E(G) \to \{0, 1, 2\}$ as follows. If an edge is not incident with any of 0, 1, 2, 3 then $c(e) = c'(e)$. If $c'(e_0) = c'(e_0) = 1$ then color $c(v_i i) = 1$, $c(01) = c(23) = 2$ and $c(12) = c(30) = 3$. Otherwise $c(e_0) = 1$ and $c(e_1) = 2$ and we color $c(v_0 0) = c(v_1 1) = c(23) = 1$, $c(v_2 2) = c(v_3 3) = c(01) = 2$ and $c(03) = c(12) = 3$. In both cases c is a 3-edge-coloring of G. Therefore if we have a 4-cycle in a class 2 cubic graph we can reduce it to a smaller cubic graph of class 2. We will therefore require that snarks have no cycles of length 4.

The length of the shortest cycle in G is called the *girth* of G. Since we will not allow cycles of length 3 or 4 in snarks, snarks will be required to have girth at least 5. We now ready to give the formal definition of a snark. A *snark* is a 3-connected, cyclically 4-edge-connected cubic graph of class 2 with girth at least 5.

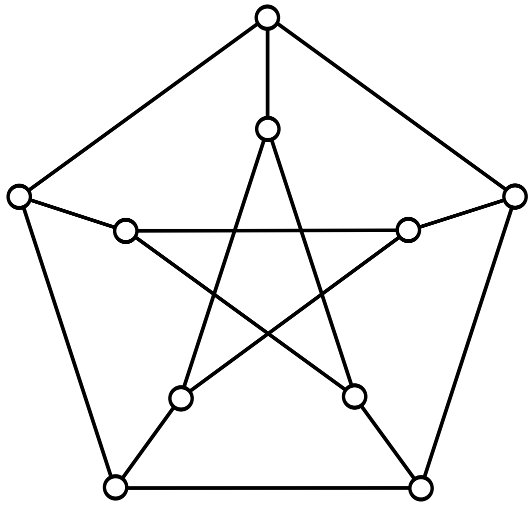

Figure 1.4: The Petersen graph.

The smallest snark is the Petersen graph found by Petersen at the end of 19th century [2]. The Petersen graph is one of the most important graphs in graph theory. It is shown in Figure 1.4.

Although the Petersen graph was found very early finding other snarks proved to be a difficult task. This is where snarks get their name. The name comes from the song *The Hunting of the Snark* by Lewis Carroll in which snarks are monsters which are very hard to find.

The Petersen graph is the only snark on 10 vertices. The are no other snarks on less than 18 vertices. In 1940's Croatian mathematician Blanuša discovered two snarks on 18 vertices, now known as Blanuša snarks [3]. They are shown in Figure 1.5 and are the only two snarks on 18 vertices.

The first infinite family of snarks was discovered in 1970's. Isaacs published a paper [7] in which he describes a dot product of graphs which constructs a snark G as a product of two smaller snarks G_1 and G_2. Although the dot product is attributed to Isaacs the construction was published earlier by a Russian mathematician Titus but this paper is unknown to many people working on snarks.

The dot product of graphs G_1 and G_2 is constructed as follows. Choose an edge $e = uv$ in G_1 and two non-adjacent edges $f_1 = v_1v_2$ and $f_2 = v_3v_3$ in G_2. Denote the neighbors of u distinct from v with u_1 and u_2 and the neighbors of u distinct from v with u_3 and u_4. The *dot product* $G = G_1 \cdot G_2$ of graphs G_1 and G_2 is constructed by removing the vertices u and v from G_1 and edges f_1 and f_2 from G_2 and adding edges v_iu_i for $i = 1, 2, 3, 4$. Note that if a

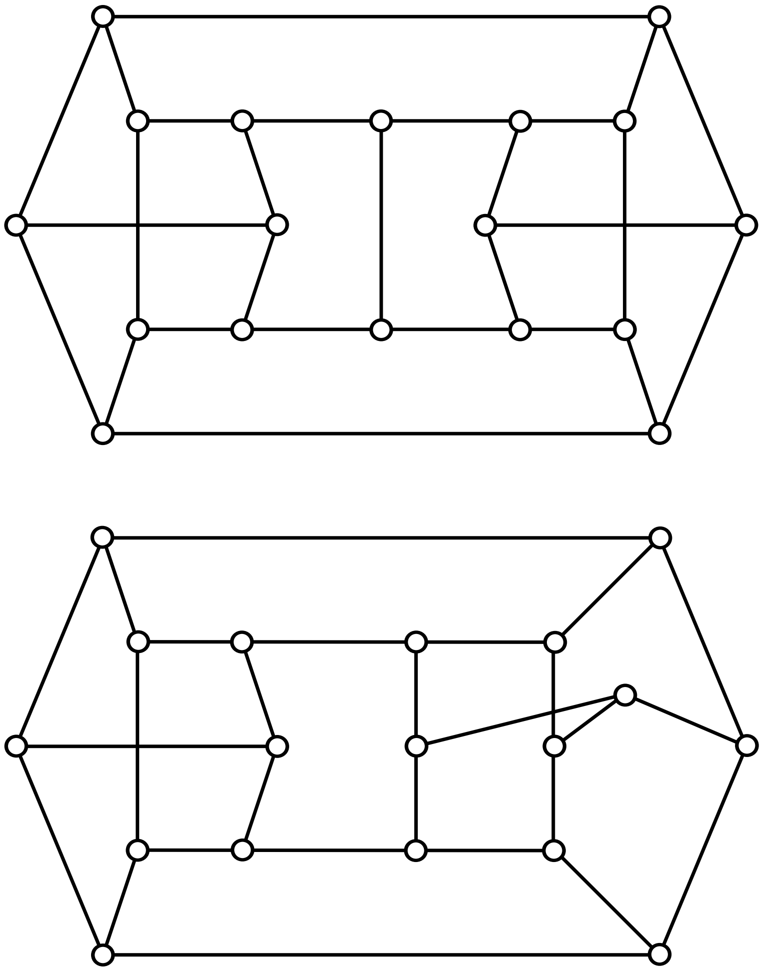

Figure 1.5: Blanuša graphs.

graph is a dot product of two smaller graphs, then it is (at most) cyclically 4-edge-connected. The cut consisting of edges added to G_1 and G_2 is called the *product cut*. It is easy to prove using the Parity lemma that if G_1 and G_2 are snarks then G is also a snark. A reverse of previous statement also holds. If G is a snark with a cyclic 4-cut S then there are two smaller graphs G_1 and G_2 so that G is obtained as a dot product of G_1 and G_2, at least one of G_1 and G_2 is a snark and that S is the product cut of the dot product.

It is clear from the definition of the dot product that the dot product of G_1 and G_2 is not uniquely defined by G_1 and G_2 but it depends on the choice of edges and vertices in G_1 and G_2. If we take two copies of the Petersen graph for G_1 and G_2 there are two possible non-isomorphic dot product we can construct. These two non-isomorphic dot products are exactly the Blanuša snarks.

By starting with the Petersen graph and constructing bigger snarks from smaller it is possible to construct the first infinity family of snarks. All snarks in this family are cyclically 4-edge-connected. Isaacs also described an infinite family of cyclically 6-edge-connected snarks which are known as flower snarks. A flower snark J_{2k+1}, $k > 1$, is a snark on vertices

$$V(J_{2k+1}) = \{a_i, b_i, c_i, d_i \mid i = 0, \ldots, 2k\}$$

and with edges

$$E(J_{2k+1}) = \{a_i a_{i+1}, a_i b_i, b_i c_i, b_i d_i, c_i d_{i+1}, d_i c_{i+1} \mid i = 0, \ldots, 2k\}$$

where indices are modulo $2k + 1$. The subgraphs Y_i induced on vertices $\{a_i, b_i, c_i, d_i\}$ are called *tiles of flower snarks*. The flower snark J_{2k+1} is obtained by putting tiles Y_i on a circle and then appropriately adding three edges between tiles Y_i and Y_{i+1} for $i = 0, \ldots, 2k$. The flower snark J_5 is shown in Figure 1.6.

We note that the graph J_3 is of class 2 but is not a snark since it contains a 3-cycle. If we remove the 3-cycle in J_3 and replace it with a vertex, we obtain the Petersen graph.

Another well known infinite family of snarks was given by Goldberg. Goldberg snark G_{2k+1}, $k > 1$, is the graph with vertices

$$V(G_{2k+1}) = \{a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i \mid i = 0, \ldots, 2k\}$$

and with edges

$$\begin{aligned} E(G_{2k+1}) &= \{a_i a_{i+1}, a_i b_i, b_i c_i, b_i d_i, c_i e_i, c_i g_i, \\ &\quad d_i f_i, d_i h_i, g_i h_i, e_i f_i, f_i e_{i+1}, g_i h_{i+1} \mid i = 0, \ldots, 2k\} \end{aligned}$$

where indices are modulo $2k+1$. The subgraphs T_i induced on vertices $\{a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i\}$ are called *tiles of the Goldberg snarks*. Similarly as flower snark the Goldberg snarks are obtained by putting tiles T_i on a circle and appropriately adding three edges between tiles T_i and T_{i+1} for $i = 0, \ldots, 2k$. The Goldberg snark G_5 is snown in Figure 1.6. If we do not require that there are an odd number of tiles, we can define graphs J_k and G_k for all $k \geq 3$. Graphs J_{2k} and G_{2k} are of class 1.

Snarks described so far all have girth at most 6 (flower snarks J_{2k+1}, $k > 1$, have girth 6 and Goldberg snarks have girth 5). If there exist snarks with arbitrary large girth has been an open question for some time. In 1980 Jaeger and Swart [10] conjectured that all snarks have girth at most 6. This conjecture was disproved by Kochol [17] in 1997 when he constructed an infinite family of snarks which contain snarks with arbitrary large girth. Kochol's construction called superposition is the most general construction of snarks known. A special class of snarks constructed by superposition for which Kochol proved that it contains snarks with arbitrarily large girth is called Kochol snarks.

There are some other constructions of snarks known. For example Goldberg snarks are a special case of the Loupekhine construction of snarks. Also all snarks with at most 28 vertices are known [24].

1.4 Superposition

We give a short description of the superposition of graphs. Superposition is the most general known construction of snarks. It generalizes many previously known constructions, for example the dot product. It was introduced by Kochol in [17] where he disproved the girth conjecture for snarks. The girth conjecture stated that snarks have bounded girth (in particular that for any snark G, the girth of G is at most 6). Kochol proved that a special class of snarks obtained as a superposition of the Petersen graph contains snarks with arbitarilly large girth which disproves this conjecture.

Superposition is a construction of snarks in which we replace the edges and vertices of snarks by cubic graphs (with pending edges) called *supervertices* and *superedges*. There are almost no requirements for supervertices, all that is required is that superedges satisfy certain properties. Because there are almost no requirements for supervertices we can construct a very rich family of snarks using superposition. We give a short description of the superposition, for more details see [17].

A multipole $M = (V, E, S)$ consists of a set of vertices V, edges E and semiedges S. A semiedge s is incident to one vertex v and denoted by $s = (v)$. We assume that the degrees of vertices in a multipole are all 3 (the degree of

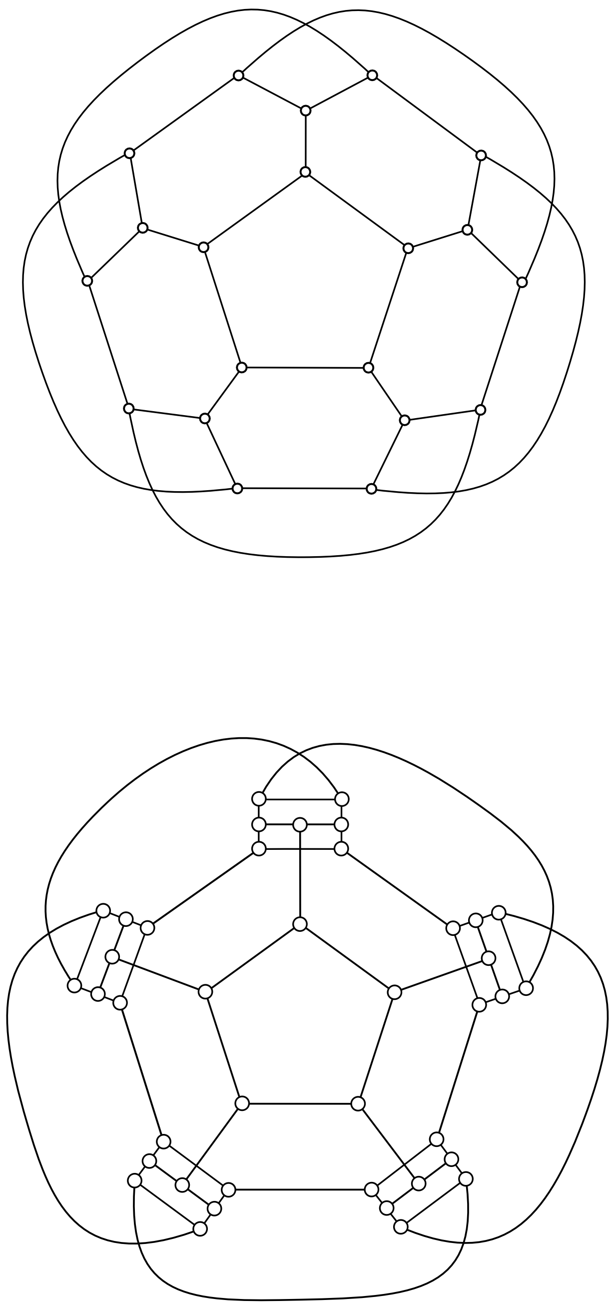

Figure 1.6: The flower snark J_5 (above) and the Goldberg snark G_5 (below).

a vertex v in a multipole is the number of edges and semiedges incident with v).

A $(k_1, \ldots, k_n)$-pole is a multipole (V, E, S) with a partition of semiedges into sets $S = S_1 \cup \cdots \cup S_n$ with $|S_i| = k_i$, $i = 1, \ldots, n$. The sets $S_1, \ldots, S_n$ are called the *connectors* of the multipole. A (k_1, k_2)-pole is called a *superedge* and a (k_1, k_2, k_3)-pole is called a *supervertex*. A $(1, 1, 1)$-pole consisting of a single vertex v and three semiedges incident with v is called a *trivial supervertex*.

Let G be a snark. We remove two non-adjacent vertices v and u from G and replace all edges vx_i incident with v with semiedges (x_i), $i = 1, 2, 3$, and all edges uy_i with semiedges (y_i), $i = 1, 2, 3$. We define $S_1 = \{(x_1), (x_2), (x_3)\}$ and $S_2 = \{(y_1), (y_2), (y_3)\}$ and we obtain a $(3, 3)$-multipole with connectors S_1 and S_2 called a *proper superedge*. We say we obtained this superedge by removing vertices v and u from G. An empty multipole will be considered as a special $(1, 1)$-multipole and a proper superedge. For a broader definition of a proper superedge see [17].

Let $G = (V, E)$ be a cubic graph. To each vertex $v \in V$ we assign a supervertex $\mathcal{S}(v)$ and additionally to each edge incident to v we assign one of the connectors of $\mathcal{S}(v)$. To each edge $xy \in E$ we assign a (proper) superedge $\mathcal{E}(xy)$ and additionally we assign one of the connectors to x and the other to y (unless $\mathcal{E}(xy)$ is an empty multipole).

Assume that for each edge $e = xy \in E$ the following holds. If $\mathcal{E}(xy)$ is an empty multipole, then the connectors assigned e in supervertices $\mathcal{S}(x)$ and $\mathcal{S}(y)$ have cardinality 1. Otherwise the connector assigned to edge e in supervertex $\mathcal{S}(x)$ ($\mathcal{S}(y)$) has the same cardinality as the connector assigned to x (y) in superedge $\mathcal{E}(xy)$.

We can then construct a new graph as follows. If the superedge assigned to $e = xy$ is an empty multipole, then we remove semiedge (v) in the connector of $\mathcal{S}(x)$ assigned to e and the semiedge (u) in the connector of $\mathcal{S}(y)$ assigned to e and add an edge uv. Otherwise we have semiedges $\{(u_1), (u_2), (u_3)\}$ in the connector of $\mathcal{S}(x)$ and semiedges $\{(x_1), (x_2), (x_3)\}$ in the connector of e assigned to x. We remove them and add edges $\{u_1x_1, u_2x_2, u_3x_3\}$ and do the same for vertex y. By repeating the procedure for all edges $e \in E$ we get a cubic graph G' called a superposition of G. If to all edges we have assigned proper superedges, the graph G' is called a proper superposition of G.

Kochol proved the following result [17]

Theorem 1.5. *For a snark G a proper superposition G' is a snark.*

Snarks are important in graph theory because they appear as possible minimal counter-examples for some of the most important open problems in graph theory. One of the most interesting open problems is the Cycle Double Cover conjecture. A collection $\mathcal{C}$ of cycles in a graph G is called a double cover if every edge of G is contained in exactly two cycles from $\mathcal{C}$. The Cycle Double Cover conjecture states that for every 2-edge-connected graph there exists a cycle double cover. It is not too hard to show that every minimal counter-example to this conjecture would be a cubic graph. Now suppose that c is a 3-edge-coloring of a cubic graph G. A subgraph $H_{i,j}$ induced on the edges colored with colors i and j, $1 \leq i < j \leq 3$ is a union of cycles. The collection of cycles in graphs $H_{1,2}$, $H_{1,3}$ and $H_{2,3}$ covers each edge twice since an edge colored for example with color 1 is contained in a cycle in the graph $H_{1,2}$ and a cycle in the graph $H_{1,3}$. Therefore we see that the minimal counter-example to the Cycle Double Cover Conjecture would be a snark. Another well known conjecture is the Tutte's 5-flow conjecture. It states that every bridgless graph admits a 5-flow. Again, every minimal counter-example to the Tutte's conjecture would be a snark.

1.5 Embeddings of cubic graphs

One of the most famous solved problem in graph theory is the Four Color Theorem. In its earliest form the Four Color Theorem states that regions of every map in the plane can be colored with four colors such that two regions which share a boundary are colored with two different colors. In the language of graph theory the Four Color Theorem states that vertices of every graph embedded into the sphere S_0 can be colored with four colors such that any two adjacent vertices are colored with different colors. The Four Color Theorem was first proposed 1852 and various attempts were made to prove it but the first proof was by Appel and Haken in 1977 using a computer ([8], [9]). Another proof was published by Robertson, Sanders, Seymour and Thomas in 1996 [18], also using a computer. It is still an interesting question if a proof without using a computer is possible.

Suppose we have a graph G embedded into a sphere and we want to color its vertices with 4 colors. We can add to G all edges possible so that the graph is still embedded into the sphere. We get a graph T for which all faces are of size 3 (since otherwise we could still add some edges). If we can color the vertices of T with 4 colors then the coloring also defines a coloring of vertices of G with 4 colors. So to prove the four color theorem we can assume that the graph is a triangulation of the sphere. Now take the dual T^* of T in the sphere (see Figure 1.7). Since T is a triangulation, T^* is a cubic graph. Taitte observed that if T^* has a 3-edge-coloring than the vertices of T can be colored with 4 colors. The Four Color Theorem therefore states that snarks can not be embedded into the plane.

Before the Four Color Theorem was proved by Apel and Haken, many at-

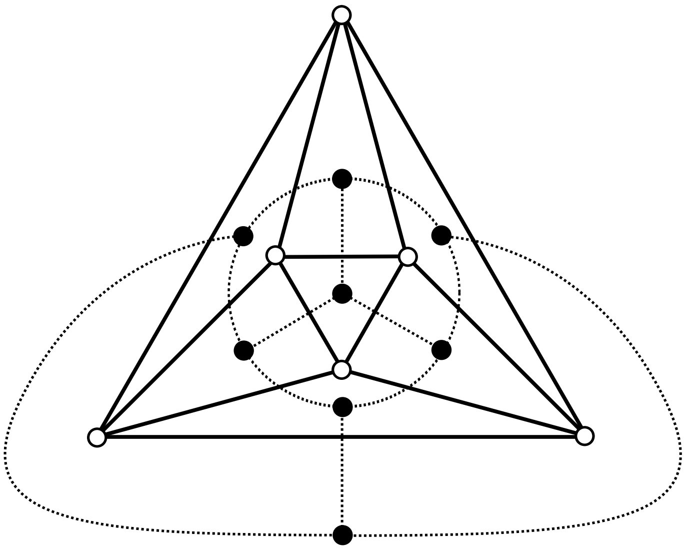

Figure 1.7: A triangulation of the plane with its dual.

tempts have been made and some proofs have been published but were later shown to be incomplete. Many of the attempts to prove the Four Color Theorem opened new direction of research in graph theory. One possible approach is to generalize the Four Color Theorem and maybe prove the generalization. One of the interesting generalization is to generalize the statement that snarks can not be embedded into the plane. The Petersen graph can be embedded into the torus (see Figure 4.1). However in the embeddind there are two facial walks that have more than one edge in common. This is true for all known embeddings of snarks.

An embedding of a graph in called *polyhedral embedding* if all facial walks are cycles and two facial walks are either disjoint, intersect in precisely one vertex or intersect in precisely one edge. An embedding of a cubic graph is polyhedral if all facial walks are cycles and two facial walks are either disjoint or share precisely one edge.

Suppose G is embedded in a surface S. A cycle on the surface (a closed simple curve on the surface) is *contractible* if it bounds a region isomorphic to an open disc in the plane and *non-contractible* otherwise. We say that the embedding of G has *face-width* k if every non-contractible cycle on S intersects G at least k times. Using face-width we can describe polyhedral embeddings of G using the following proposition.

Proposition 1.6. *An embedding of a graph G is polyhedral if and only if G is 3-connected and the embedding has face-width at least 3.*

If 1967 Grünbaum proposed a far-reaching generalization of the Four Color Theorem (which had not yet been proved at that time). The lack of orientable polyhedral embeddings of the Petersen graph and other non 3-edge-colorable cubic graphs known at that time led Grünbaum to the following

Conjecture 1.7 (Grünbaum [4]). *If a cubic graph admits a polyhedral embedding in an orientable surface, then it is 3-edge-colorable.*

Another way of stating the Grünbaum conjecture is that cubic graphs which are not 3-edge-colorable do not admit polyhedral embeddings. The conjecture is not true for non-orientable surfaces since the Petersen graph has a polyhedral embedding into the projective plane (see Figure 3.1). Later on we will construct for each non-orientable surface N a snark which has a polyhedral embedding in N.

Even though almost 40 years have passed since it was proposed, not much progress has been made toward resolving the Grünbaum conjecture. The conjecture has been verified for flower snarks by Szekeres in [5] where he proves that graphs J_{2k+1} do not have orientable polyhedral embeddings. The proof does not rely on the fact that graphs J_{2k+1} are snarks and later we show that indeed none of the graphs J_k, $k > 3$, have polyhedral embeddings into any (orientable or non-orientable) surfaces. We also show that the conjecture is true for Goldberg snarks and Kochol snarks.

Besides the Szekeres' paper [5], not much has been published about polyhedral embeddings of snarks. Tinsley and Watkins studied the genus of flower snarks [12]. They observe that the genus of snarks they study increases with the order of the graph. In the next chapter we extend their results. We find the genus of flower snarks and Goldberg snarks and prove some results about the genus of dot products of the Petersen graph. In the third chapter we study polyhedral embeddings of flower snarks and Goldberg snarks into orientable and non-orientable surfaces. We show some obstructions for existance of polyhedral embeddings and construct polyhedral embeddings of snarks into non-orientable surfaces. In the last chapter we prove that Kochol snarks do not have polyhedral embeddings into orientable surfaces. We define the defect of a graph which is a measure for how far a graph is from having a polyhedral embedding into an orientable surface and prove some results connecting the Grünbaum conjecture, defect and resistance of cubic graphs.

Chapter 2

Genus of snarks

In this part of the thesis we give some results about the genus of snarks. The genus of snarks has been studied in a paper of Tinsley and Watkins [12] in which they determine the orientable genus of flower snarks. They give an upper bound for the orientable genus of Goldberg snarks and make a conjecture about the genus of dot products of the Petersen graph. Based on these results they observe that the genus of the snarks they studied increases with the order of the snark.

The method Tinsley and Watkins used to prove their results on the genus of J_{2k+1} are topological. We first prove their result on the orientable genus of J_{2k+1} using a combinatorial method. This method extends to the non-orientable case aswell. Using the same idea we determine orientable and non-orientable genus of Goldberg snarks.

Next we study the orientable genus of dot products. We first disprove the conjecture of Tinley and Watkins about the orientable genus of P^n. We show that there are infinitely many graphs P^n which can be embedded in the torus. Further for each g, $1 \leq g \leq n$, we show that there is a product P^n such that the orientable genus of P^n is equal to g. Finally we give tight bounds for the orientable genus of a dot product of two cubic graphs.

2.1 Flower snarks and Goldberg snarks

Tinsley and Watkins determined the orientable genus of flower snarks. They use topological methods to prove the lower bound and used a different approach for the non-orientable genus. In this section we give a short combinatorial proof of their results which works for both orientable and non-orientable genus. The proof works by counting arguments and uses the Euler formula. A similar approach also works for Goldberg snarks.

Theorem 2.1 (Tinsley, Watkins). *The orientable genus of the flower snark is $g(J_{2k+1}) = k$ and the non-orientable genus is $\tilde{g}(J_{2k+1}) = 2k - 1$.*

Proof. An embedding of J_{2k+1} in an orientable surface of genus k is described by a list of facial cycles

- $a_0 a_1 \cdots a_{2k}$,
- $c_0 d_{2k} c_{2k-1} d_{2k-2} \cdots c_1 d_0 c_{2k} \cdots d_1 c_0$,
- $d_0 b_0 c_0 d_1 b_1 c_2 \cdots d_{2k} b_{2k} c_{2k} d_0$,
- $F_i = a_i b_i d_i c_{i+1} b_{i+1} a_{i+1} a_i$ for $i = 0, \ldots 2k$,

which gives $g(J_{2k+1}) \leq k$ (see also Figure 2.1 which show an embedding of J_5 into an orientable surface of genus 2).

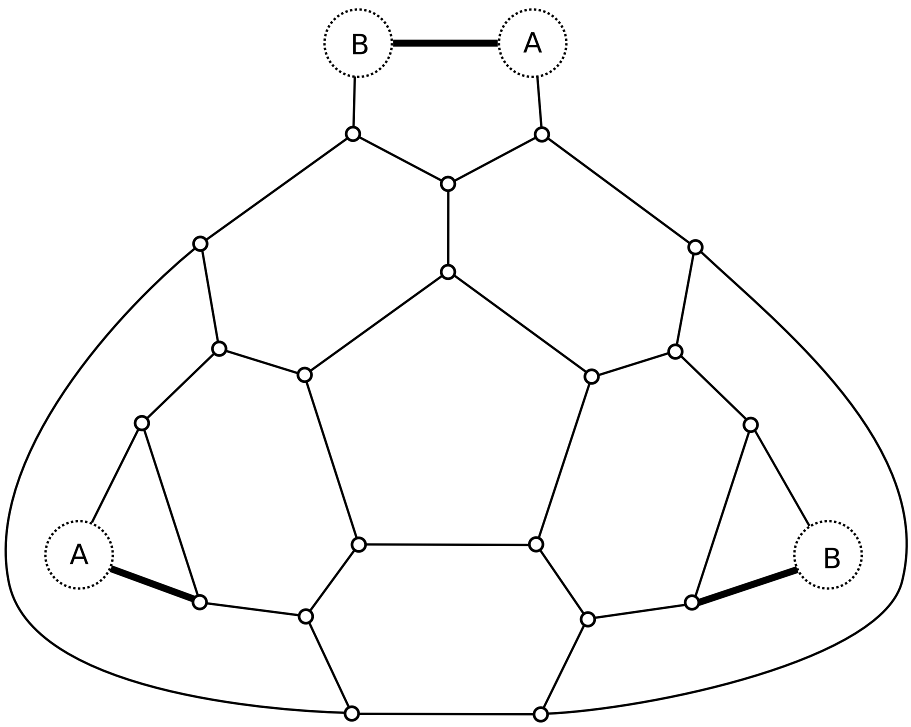

Figure 2.1: The flower snark J_5 embedded into an orientable surface of genus 2.

A non-orientable embedding of J_{2k+1} in a surface of genus $2k-1$ is described by a list of facial cycles

- $a_0a_1 \cdots a_{2k}$,
- $c_0d_1b_1c_1d_2b_2c_2 \ldots d_{2k-1}b_{2k-1}c_{2k-1}d_{2k}c_0$,
- $c_0d_1c_2d_2 \ldots d_{2k-1}c_{2k}b_{2k}d_{2k}c_0$,
- $d_0c_1d_2c_2 \ldots d_{2k}b_{2k}c_{2k}d_0$,
- $F_i = a_ib_id_ic_{i+1}b_{i+1}a_{i+1}a_i$ for $i = 0, \ldots 2k$,

which gives the upper bound $\tilde{g}(J_{2k+1}) \leq 2k - 1$. See also Figure 2.2 which shows the embedding of the flower snark J_5 into the non-orientable surface N_3. Is is easy to see from the figure how to embed snarks J_{2k+1} into surfaces N_{2k-1} for $k \geq 3$.

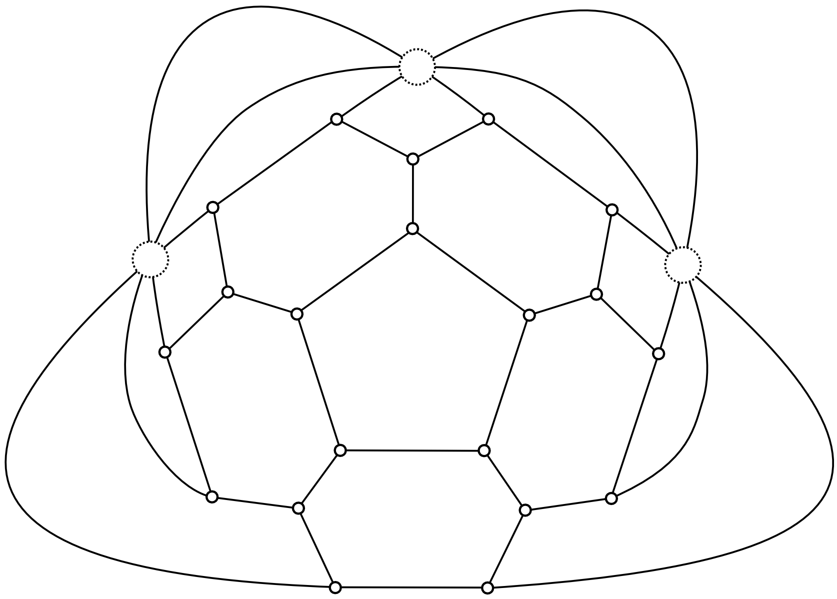

Figure 2.2: The flower snark J_5 embedded into the non-orientable surface of genus 3.

By contracting each tile Y_i of J_{2k+1} to a vertex i we get a cycle Q of length $2k+1$. Each facial walk W in an embedding Π of J_{2k+1} induces a walk W' in Q. We define the winding number $w(W)$ of W to be the winding number of W' in Q. A facial walk in Π is *local* if $w(W)$ is zero and *global* otherwise.

We show that in the embedding of Π we can have at most $2k+1$ local facial walks. For each local facial walk W there exists an index i, such that W contains a path $P = x_0x_1 \ldots x_{l-1}x_l$, where vertices x_0 and x_l are in the tile

Y_{i-1} and vertices $x_1, \ldots, x_{l-1}$ are in the tile Y_i. To the walk W we assign the vertex i of Q.

There are three paths of the form $x_0x_1b_i$ where x_0 is in the tile Y_{i-1}. Since each walk assigned to the vertex i contains two such paths and each path of length tree can appear at most once along facial walks of Π, we see that to each vertex of C_{2+1} we assigned at most one facial walk. So we can have at most $2k+1$ local facial walks in the embedding of J_{2k+1}.

In the embedding of J_{2k+1} we can either have 6 global facial walks or at most $2k+1$ local facial walks and 4 global walks. This implies that there are at most $2k+5$ facial walks in an embedding of J_{2k+1}.

Suppose Π is an embedding of J_{2k+1} into a non-orientable surface of minimum possible genus $\tilde{g}(J_{2k+1})$. By Euler formula

$$\begin{aligned} 2-\tilde{g}(J_{2k+1}) &= |V(J_{2k+1})| - |E(J_{2k+1})| + |F_\Pi(J_{2k+1})| \\ &\leq 4(2k+1) - 6(2k+1) + 2k + 5 = 3 - 2k \end{aligned}$$

the non-orientable genus is $\tilde{g}(J_{2k+1}) \geq 2k-1$.

Suppose Π is an embedding of J_{2k+1} into an orientable surface of minimum possible genus $g(J_{2k+1})$. Since $|V(J_{2k+1})| = 4(2k+1)$ and $|E(J_{2k+1})| = 6(2k+1)$, by Euler formula $|V(J_{2k+1})| - |E(J_{2k+1})| + |F_\Pi(J_{2k+1})| = 2 - 2g(J_{2j+1})$, there are an even number of facial walks in Π. Therefore there can be at most $2k+4$ facial walks in Π. Now the Euler formula

$$\begin{aligned} 2-2g(J_{2k+1}) &= |V(J_{2k+1}| - |E(J_{2k+1}| + |F_\Pi(J_{2k+1})| \\ &\leq 4(2k+1) - 6(2k+1) + 2k + 4 = 2 - 2k \end{aligned}$$

implies that $g(J_{2k+1}) \geq k$. □

The same argumentats also work for graphs J_{2k}. We can show that in every embedding of J_{2k} there can be at most $2k$. If there are $2k$ local facial walks, then there are four global facial walks. Since every embedding can have at most $2k+4$ facial walks we get a lower bound for genera of J_{2k}.

Theorem 2.2. *The orientable genus of the flower graph J_{2k} is $g(J_{2k}) = k-1$ and the non-orientable genus is $\tilde{g}(J_{2k}) = 2k-2$.*

Proof. The lower bound is obtained in the paragraph before the theorem. From Figure 2.4 it is easy to obtain embeddings of J_{2k} into non-orientable surfaces of genus $2k-2$. An embedding of J_{2k} into an orientable surface of genus $k-1$ is given by the following list of facial cycles:

- $a_0a_1 \cdots a_{2k-1}$,

- $d_0c_{2k-1}d_{2k-2}c_{2k-3}\dots d_0$,
- $c_0d_{2k-1}d_{2k-2}d_{2k-3}\dots c_0$,
- $d_0b_0c_0d_1b_1c_1\dots d_{2k-1}b_{2k-1}c_{2k-1}d_0$,
- $F_i = a_ib_id_ic_{i+1}b_{i+1}a_{i+1}a_i$ for $i = 0, \dots 2k-1$,

□

Tinsley and Watkins obtained an upper bound for the orientable genus of Goldberg snark G_{2k+1} by showing an embedding into the orientable surface of genus $2k$. Using ideas similar to those used in the proof of the previous theorem we show that this bound is the correct value for the orientable genus of G_{2k+1}. We also determine the non-orientable genus of G_{2k+1}.

Theorem 2.3. *The orientable genus of the Goldberg graph is $g(G_k) = k-1$ and the non-orientable genus is $\tilde{g}(G_k) = k$.*

Proof. We first look at orientable genus. An embedding of the Goldberg graph G_k in the orientable surface of genus k is described by facial cycles

- $a_0a_1\cdots a_{k-1}a_0$,
- $C_i = a_ib_id_if_ie_{i+1}c_{i+1}b_{i+1}a_{i+1}a_i$ for $i = 0, \dots, k-1$,
- $D_i = b_ic_ig_ih_id_ib_i$, for $i = 0, \dots, k-1$,
- $f_0e_0f_{k-1}e_{k-1}\cdots f_1e_1f_0$,
- $h_0g_0h_1g_1\cdots h_{k-1}g_{k-1}h_0$,
- $f_0d_0h_0g_{2k}c_{k-1}e_{k-1}f_{k-2}d_{k-2}h_{k-2}\cdots g_0c_0e_0f_{k-1}d_{k-1}h_{k-1}\cdots g_1c_1e_1f_0$.

See also Figure 2.3.

For the lower bound for the orientable genus we use the Euler formula. We have $|V(G_k)| = 8k$, $|E(G_k)| = 12k$ and in the embedding into the orientable surface of genus k there are $2k+2$ facial walks. We show that if Π is an orientable embedding of G_k, then there are at most $2k+2$ facial walks in Π, which gives the lower bound k for the genus of the surface.

We group facial walks in the embedding Π of G_k into three groups. A facial walk is *short* if it is contained in a tile T_i of G_k and *long* otherwise. By contracting tiles T_i of G_k into vertices i we obtain a cycle Q of length k. Each facial walk W in the embedding Π defines a walk W' in Q. The winding number of W' in Q defines the winding number $w(W)$ of W. A long facial

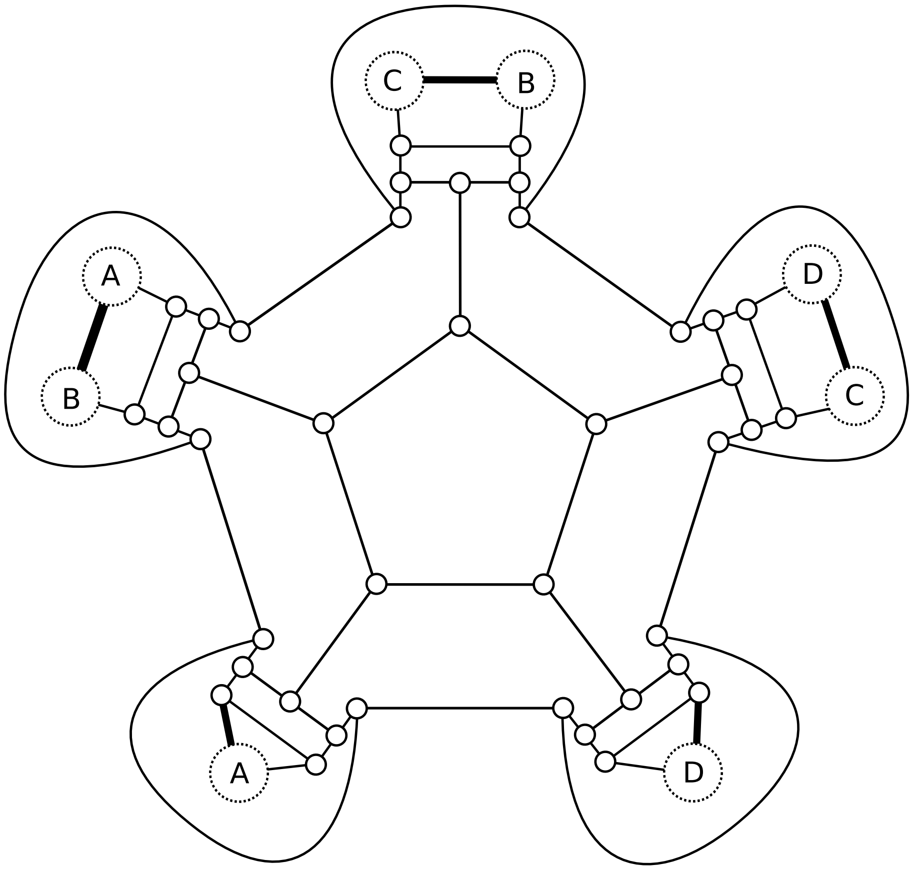

Figure 2.3: The Goldberg snark G_5 embedded in the orientable surface of genus 4.

walk is *local* if the winding number is zero and *global* otherwise. With this we have grouped facial walks of Π into three groups: short and long local walks and global walks.

We show that we can have at most $2k + 2$ local walks. To each local walk we assign a vertex in Q as follows. To a short walk in a tile T_i we assign the vertex i. If W is a long walk, there exists an index i and a sub-walk $P = x_0 x_1 \ldots x_{l-1} x_l$ on W such that x_0 and x_l are in the tile T_{i-1} and all vertices $x_1, \ldots x_{l-1}$ are in the tile T_i, since otherwise the winding number of W could not be zero. To W we assign the vertex i in Q (if there are more than one possibilities for i we arbitrarily choose one of them). We now prove that to each vertex i we can assign at most two facial walks which implies that we have at most $2k$ local walks.

Suppose we have assigned three long local walks W_1, W_2 and W_3 to i. Since there are only three edges from tile T_{i-1} to T_i, all are contained twice in walks W_1, W_2 and W_3, and in particular edge $a_{i-1}a_i$ is contained twice in them. But since we assigned all of W_1, W_2 and W_3 to i we see that if W_1 contains $a_{i-1}a_i$, it must contain $a_{i-1}a_ib_i$. But in the embedding Π it is not possible that a path of length 3 appears twice along facial walks in Π.

Suppose we have assigned three local walks W_1, W_2 and W_3 to i, where W_1 is short and W_2 and W_3 are long. There are two possibilities for W_1. Either it contains the cycle $h_ig_ic_ib_id_ih_i$ or $h_ig_ic_ie_if_id_ih_i$ (the case when it contains $d_ib_ic_ie_if_id_i$ is symmetric to the first case).

Suppose W_1 contains $h_ig_ic_ie_if_id_ih_i$. We have facial walks which contain paths $h_{i-1}g_ih_ig_{i+1}$, $e_{i-1}f_ie_if_{i+1}$ and $a_{i-1}a_ia_{i+1}$. This is a contradiction with the fact that $a_{i-1}a_i$, $e_{i-1}f_i$ and $h_{i-1}g_i$ appear twice on each of W_2 and W_3. Suppose that the consistent orientation of facial walks W_1 contains the path $h_ig_ic_ib_id_ih_i$. We have facial walks which contain paths (in some orientation) $h_{i-1}g_ih_ig_{i+1}$ and $a_{i-1}a_ia_{i+1}$. It follows that both W_2 and W_3 contain the edge $e_{i-1}f_i$. Now W_2 must contain $e_{i-1}f_id_ib_ia_ia_{i-1}$. The walk W_3 must contain edges f_ie_{i-1} and g_ih_{i-1} in these orientations. But this is a contradiction.

Finally assume that we have assigned two short local facial walks W_1 and W_2 to i. Since a short local walk at i contains one of three cycles $h_ig_ic_ib_id_ih_i$, $d_ib_ic_ie_if_id_i$ or $h_ig_ic_ie_if_id_ih_i$ it follows that at least one path of length 3 is contained twice along facial walks of the embedding, which is a contradiction. So in an embedding of G_k there can be at most $2k$ local facial walks. In particular we have shown that there can be at most k short local walks in an embedding of G_k.

Now suppose we have an embedding Π into an orientable surface of genus less than $k-1$. By Euler formula

$$\begin{aligned} 2-2g(G_k) &= |V(G_k)| - |E(G_k)| + |F(G_k)| \\ &= 8k - 12k + |F(G_k)| \\ &= |F(G_k| - 4k \end{aligned}$$

we get

$$|F(G_k)| = 4k + 2 - 2g(G_k) \geq 4k + 2 - 2(k-2) = 2k + 6.$$

Since at most $2k$ of them can be local walks, we have at least 6 global walks. Since each global walk contains at least one edge connecting the tile T_{i-1} with the tile T_i and there are three edges connecting tile T_{i-1} and T_i, we see that no local walk can contain an edge between two tiles. So all local walks are short. But we can have at most k short local walks, a contradiction. We have shown that the genus of G_k is at least $k-1$.

We now prove that the non-orientable genus of the Goldberg snark is $\tilde{g}(G_k) = k$. An embedding of G_k into a non-orientable surface of genus k is described by facial cycles

- $a_0 a_1 \cdots a_{k-1k} a_0$,
- $C_i = a_i b_i d_i f_i e_{i+1} c_{i+1} b_{i+1} a_{i+1} a_i$ for $i = 0, \ldots, k-1$,
- $D_i = b_i c_i g_i h_i d_i b_i$ for $i = 0, \ldots, k-1$,
- $E_i = f_i e_{i+1} f_{i+1} d_{i+1} h_{i+1} g_i c_i e_i f_i$ for $i = 0, \ldots, k-1$,
- $F_i = h_0 g_0 h_i g_i \ldots h_{k-1} g_{k-1} h_0$.

An embedding of G_5 into the non-orientable surface of genus 5 is shown in Figure 2.4. Is is easy to see how to get an embedding of arbitrary G_k into a non-orientable surface of genus k.

To prove the lower bound we show that in a non-orientable embedding of G_{2k+1} there can be at most $3k$ local facial walks which will give an upper bound for the number of facial walks to be $3k+2$. This implies that the genus of the surface is at least k. Again as before we can assign to each local facial walk a vertex of Q. We show that to each tile T_i we can assign at most three local facial walks. As in the case of the orientable embedding there can be at most one short facial cycle in each tile. If we assigned three long local walks to a tile T_i, then all three edges between tiles T_{i-1} and T_i must appear on them, each twice. But this implies that the path $a_{i-1} a_i b_i$ is contained in two facial walks, which is a contradiction. To each tile we can assing at most three local facial walks (one short and two long), which implies that there can be at most $3k$ local facial walks. If we assigned three local facial walks to any tile, then there can be at most two global facial walks. So the biggest possible number of facial walks is $3k+2$ and by Euler formula

$$\begin{aligned} 2 - \tilde{g}(G_k) &= 8k - 12k + |F(G_k)| \\ &\leq 8k - 12k + 3k + 2 = 2 - k \end{aligned}$$

we have $\tilde{g}(G_k) \geq k$. □

In particular case, the last theorem states that for Goldberg snarks we have $g(G_{2k+1}) = 2k$ and $\tilde{g}(G_{2k+1}) = 2k+1$.

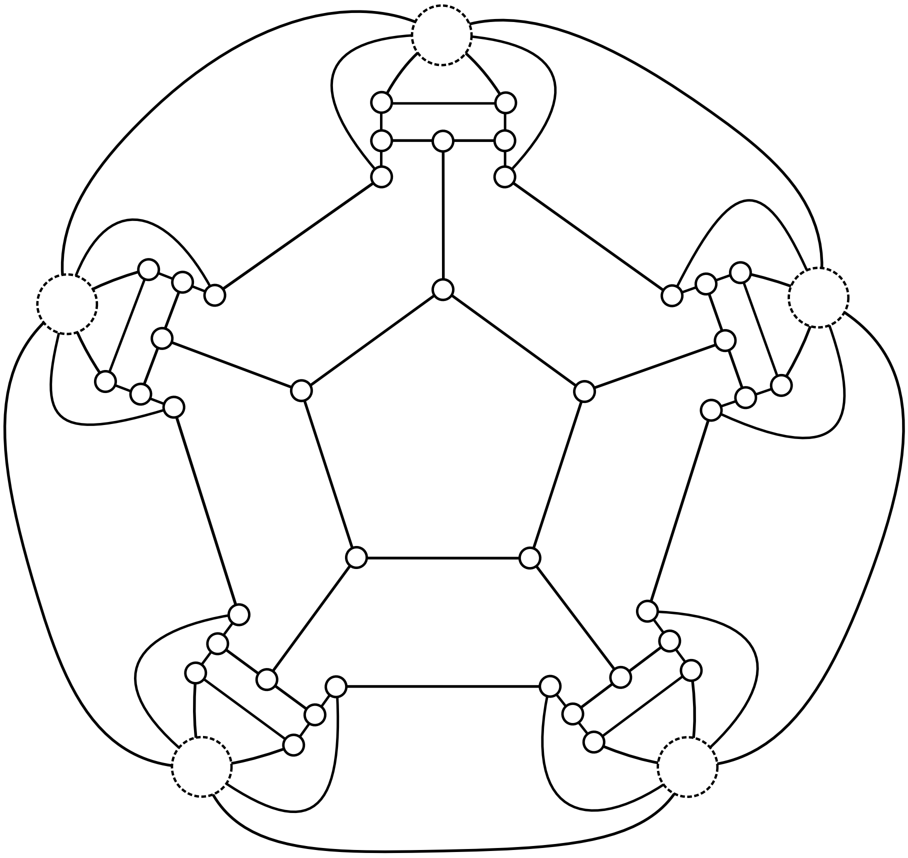

Figure 2.4: The Goldberg snark G_5 embedded in the non-orientable surface of genus 5.

2.2 Toroidal snarks

Let P^n denote a dot product of n copies of the Petersen graph. In [12] authors proposed a conjecture, that a graph P^n has orientable genus precisely $n - 1$. In the construction of P^2 there are two non-equivalent ways to choose edges e_1 and e_2 in the first copy of P, so there are two non-isomorphic dot products of two copies of the Petersen graph (which are the only two snarks on 18 vertices). The previous conjecture was disproved in [21], where it was shown that one of the two possible dot products P^2 has orientable genus 2, so that the genus can be bigger than conjectured.

In this section we show that for every positive integer n a dot product of n copies of the Petersen graphs exists, which can be embedded in the torus and has therefore genus 1, so there exists and infinite family of counter-examples for which the value of the genus can also be (much) smaller than the conjectured value. We also show that for each g there are infinitely many snarks with orientable genus precisely g.

Let G_1 be a cubic graph embedded into an orientable surface S_g and G_2 be a cubic graph embedded in the torus T. Let $e_1 = x_1x_2$ and $e_2 = x_3x_4$ be two edges of G_1 such that in the embedding of G_1 there are two facial walks $C_1 = x_1x_2P_1x_3x_4P_2x_1$ and $C_2 = x_2x_1P_4x_4x_3P_3x_2$. Then we say that edges e_1 and e_2 satisfy property $\mathcal{P}$. Let $f = uv$ be an edge in G_2 such that the neighbors of u, distinct from v, are y_1, y_2, the neighbors of v, distinct from u are y_3, y_4 and in the embedding of G_2 there are distinct facial walks $D_1 = y_1uvy_4R_4y_1$, $D_2 = y_3vuy_2R_3y_3$ and $D_3 = y_2uy_1R_2y_4vy_3R_1y_2$.

Lemma 2.4. *Let G_1 and G_2 be as above. Then a dot product $G = G_1 \cdot G_2$ exists which has an embedding into the surface S_1. Furthermore, the edges $e_i' = x_iy_i$ and $e_j' = x_jy_j$ in G have property $\mathcal{P}$.*

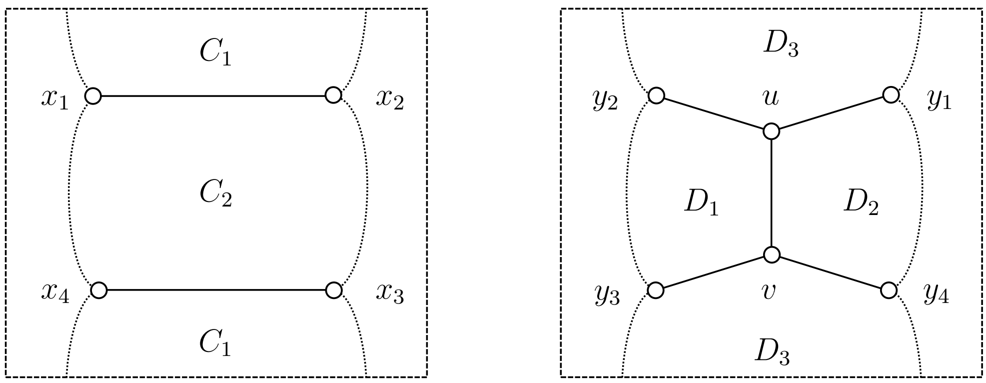

Figure 2.5: The configuration of faces in G_1 and G_2.

Proof. Let G_1 and G_2 be embedded as in the Lemma. Let G be the dot product as described in the paragraph above the Lemma. We define the embedding of G by specifying vertex rotations. Denote with X the set $\{x_i, y_i \mid i = 0, 1, 2, 3, 4\}$. The rotations at vertices in $V(G) \setminus X$ are the same as the rotations in the embeddings of G_1 and G_2. The rotations at vertices in X are the same as the rotations in the embeddings of G_1 and G_2 where we naturally replace the deleted edges with the added ones. This is clearly an embedding into an orientable surface. To prove that this surface is S, we count the facial walks of the embedding. The facial walks which do not contain any of the vertices from X are facial walks in the embedding of G_1 or G_2. The facial walks, which contain vertices from X are $x_2P_1x_3y_3R_1y_2x_2$, $x_1y_1R_2y_4x_4P_2x_1$, $x_2P_1x_3y_3R_1y_2x_2$

and $x_1P_4x_4y_4R_4y_1x_1$. So we have replaced five facial walks with four. We have $|V(G)| = |V(G_1)| + |V(G_2)| - 2$, $|E(G)| = |E(G_1)| + |E(G_2)| - 3$ and $|F(G)| = |F(G_1)| + |F(G_2)| - 1$. So

$$\begin{aligned} |V(G)| - |E(G)| + |F(G)| &= |V(G_1)| - |E(G_1)| + |F(G_1)| + \\ &\quad |V(G_2)| - |E(G_2)| + |F(G_2)| - 1 \\ &= 1 + 2 - 2g - 1 = 2 - 2g \end{aligned}$$

which shows that this is an embedding in S_g. It is also easy to see that edges x_1y_1 and y_4x_4 satisfy the property $\mathcal{P}$ □

Corollary 2.5. *For every positive integer n there exists a dot product of n copies of the Petersen graph, that can be embedded in the torus.*

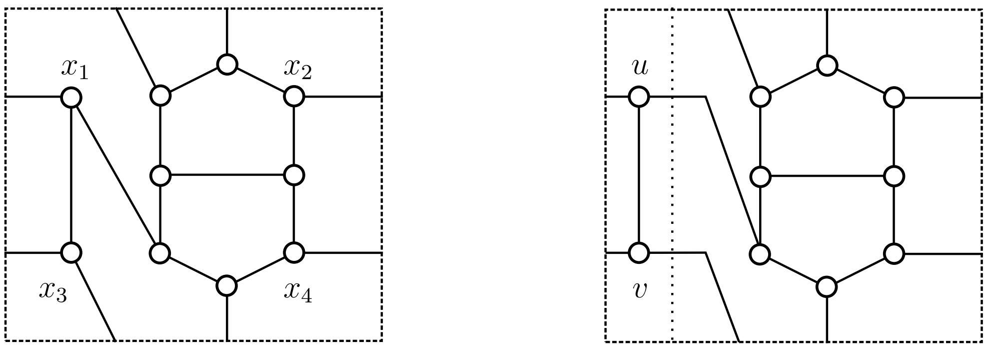

Figure 2.6: The Petersen graph in the torus.

Proof. An embedding of the Petersen graph in the torus is shown in Figure 2.6. It is easy to check that if we take the edges x_1x_2 and x_3x_4 in one copy and the edge uv in the other, the conditions of Lemma 2.4 are satisfied for both copies. The corollary follows. □

As an immediate corollary of this result we show that for each $g > 0$ there are infinitely many snarks with orientable genus precisely g. This result will also follow from Corollary 2.9.

Corollary 2.6. *For each $g > 0$ there exist infinitely many snarks with orientable genus g.*

Proof. We already constructed infinitely many snarks embedded in the torus.

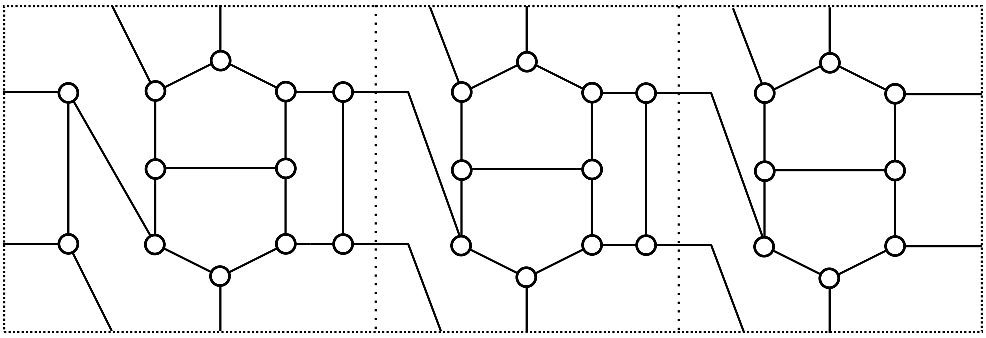

Figure 2.7: A graph P^3 in the torus.

For $g > 1$ we start with the snark J_{2g+1} which has orientable genus g. In the embedding described in the proof of theorem 2.1 edges c_0d_1 and c_2d_3 satisfy property $\mathcal{P}$. By Lemma 2.4 we have infinitely many snarks $G_0 = J_{2g+1}$, $G_1 = G_0 \cdot P$, $G_2 = G_1 \cdot P$, …, embedded in S_g. There are two disjoint paths P_1 connecting y_1 and y_2 and P_2 connecting y_3 and y_4 in $P - \{u, v\}$. Therefore there is a subgraph in G_{i+1} which is isomorphic to a subdivision of G_i. This implies that in G_i there is a subgraph which is a subdivision of J_{2g+1} and therefore G_i can not be embedded in a surface of genus less than g. $\square$

2.3 The genus of P^n

In Corollary 2.5 we have described products P^n which are embeddable in the torus. In this section we describe products P^n which have genus g, $1 \leq g \leq n$. We need the following lemma for the construction.

Lemma 2.7. 1. *If two adjacent vertices u and v are removed from the Petersen graph P then in a drawing of $P - \{u, v\}$ in the plane, the degree 2 vertices can not be drawn on the boundary of the same face.*

2. *If we remove two edges e, f as indicated in Figure 2.8 from the Petersen graph, then the graph $P - \{e, f\}$ is not planar.*

3. *For any vertex $x \in V(P)$ the graph $P - \{x\}$ is not planar.*

Proof. For the first part note that if we have an embedding of $P - \{u, v\}$ in the plane such that the degree two vertices are on the boundary of one face, then we can add a vertex in that face and connect it to the degree two vertices.

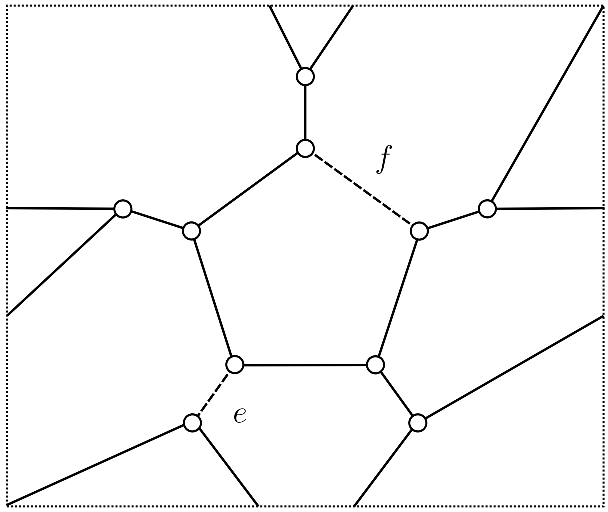

Figure 2.8: The Petersen graph in torus with dashed edges e and f.

We get an embedding of the P/uv in the plane, which is a contradiction since P and P/uv are not planar graphs.

For the second and third part note that in graphs $P - \{e, f\}$ and $P - \{x\}$ there are subdivisions of the graph $K_{3,3}$ which implies they are not planar. □

Theorem 2.8. *For each genus $n \geq 1$, there exists a dot product P^n of n copies of the Petersen graph, whose genus is equal to n.*

Proof. We construct products P^n together with their embeddings Π_n with the following properties.

- The genus of P^n is $g(P^n) = g(\Pi_n) = n$
- In the embedding Π_n there are two edges $e, f \in E(P^n)$ on the same facial walk $\mathcal{F}$ such that the genus of the graph $P - \{e, f\}$ is $g(P^n - \{e, f\}) = n$. Further there are two distinct facial walks $\mathcal{F}_1$ and $\mathcal{F}_2$, both distinct from $\mathcal{F}$, such that $\mathcal{F}_1$ contains e and $\mathcal{F}_2$ contains f (or equivalently there is exactly one facial cycles $\mathcal{F}$ which contains both e and f).

For $n = 1$ we have $g(P) = 1$ and edges e, f from Lemma 2.7 (See Figure 2.8) satisfy the stated conditions.

Let u, v be adjacent vertices in P and denote the neighbors of v distinct from u with v_1 and v_2 and the neighors of u distinct from v with u_1 and u_2.

Now suppose we have an embedding Π_n of P_n, edges $e = x_1x_2$ and $f = y_1y_2$ and a facial walk $\mathcal{F}$ with required properties. We can assume that vertices

x_1, x_2, y_1, y_2 appear in this order along the walk $\mathcal{F}$. Denote the walks which contain edges e and f by $\mathcal{F} = x_1x_2P_1y_1y_1P_2x_1$, $\mathcal{F}_1 = x_2x_1R_1x_2$ and $\mathcal{F}_2 = y_2y_1R_2y_2$. We construct P^{n+1} by removing edges e, f from P^n and vertices u, v from P and adding product edges $e_1 = x_1v_1, e_2 = x_2v_2, f_1 = y_1u_1, f_2y_2v_2$. We claim that $g(P^{n+1}) = g(P^{n+1} - \{e_1, e_2\}) = n + 1$.

Since $P^n - \{e, f\}$ has genus n it follows that $g(P^{n+1}) \geq n$. Suppose that $g(P^{n+1}) = n$. Since the embedding of $P^n - \{e, f\}$ induced by the embedding of P^{n+1} has genus n it follows that the embedding of P^{n+1} also induced an embedding of $P - \{u, v\}$ into the plane so that the degree two vertices are on the same face. But this is a contradiction to Lemma 2.7.

Now suppose that $g(P^{n+1} - \{e_1, e_2\}) = n$. Again, since the induced embedding of $P^n - \{e, f\}$ has genus n, the induced embedding of $P - \{u, v\}$ is in the plane such that two vertices u_1, u_2 are on the same face. But this would induce an embedding of $P - \{v\}$ in the plane, a contradition to Lemma 2.7.

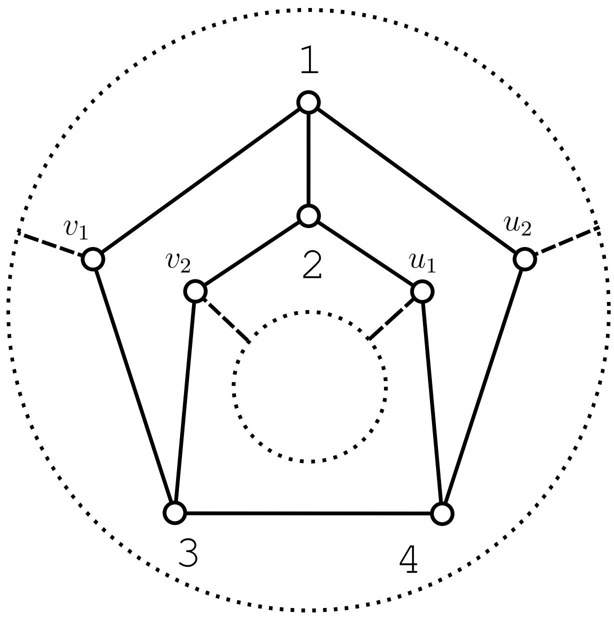

Figure 2.9: The graph $P - \{u, v\}$ embedded on the cylinder.

We have shown that $g(P^{n+1}) \geq n + 1$. Let $P - \{u, v\}$ be embedded into the cylinder Z as shown on the Figure 2.9. In the embedding Π_n remove a disc from the face $\mathcal{F}$ and join S_n with the cylinder Z using a sphere with three discs removed to obtain a surface S_{n+1}. We can add product edges on S_{n+1} to obtain an embedding Π_{n+1} into S_{n+1} (see Figure 2.10).

Facial cycles containing product edges are $x_2P_1y_1u_112v_2x_2$, $y_2P_2x_1v_13u_2y_2$ and $x_1R_1x_2v_24u_1y_1R_2y_2u_221v_1x_1$ so the embedding Π_{n+1} satisfies all require-

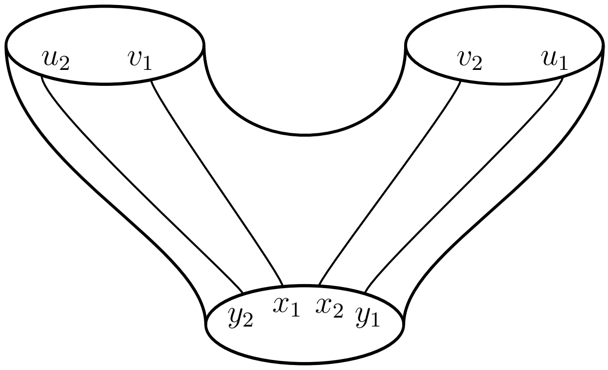

Figure 2.10: The sphere minus three discs with the product edges.

ments. □

Corollary 2.9. *For each g, $1 \leq g \leq n$ there exists a product P^n with orientable genus $g(P^n) = g$.*

Proof. Suppose $1 \leq g \leq n$. By Theorem 2.8 we can construct a product P^g with orientable genus $g(P^g) = g$. By construction there is an embedding of P^g into S_g such that all product edges are on the same face. This implies that there are two edges e and f which satisfy property $\mathcal{P}$ of Lemma 2.4. From P^g we can then construct a product P^n with orientable genus $g(P^n) = g$ by successively applying Lemma 2.4. □

2.4 Genus of the dot product

In this section we give general bounds for the genus of the dot product.

Theorem 2.10. *Let G_1 and G_2 be two cubic graphs with orientable genera $g(G_1) = g_1$ and $g(G_2) = g_2$. Then the genus of the dot product $G_1 \cdot G_2$ satisfies*

$$g_1 + g_2 - 2 \leq g(G_1 \cdot G_2) \leq g_1 + g_2 + 1.$$

The bounds are best possible, even if G_1 and G_2 are required to be snarks.

Proof. First we show the upper bound. Let G_1 be embedded into the surface S_1 of genus g_1 and G_2 into the surface S_2 of genus g_2. Suppose that in the construction of the dot product we remove edges $e = x_1x_2$ from G_1 and vertices

u and v with neighbors $\{u_1, u_2, v\}$ and $\{v_1, v_2, u\}$ respectively from G_2. Remove a small disc D around the edge uv in S_2 which intersects G_2 only in vertices u_1, u_2, v_1, v_2. Note that the vertices appear in this order around the disc D. Remove two discs D_1 and D_2 from S_1 around edges e and f which intersect G_1 only in end vertices of edges e and f. Now join the surfaces S_1 and S_1 by a sphere with three discs removed. It is possible to add the product edges on the surface to get an embedding of G_1 and G_2 (see Figure 2.11).

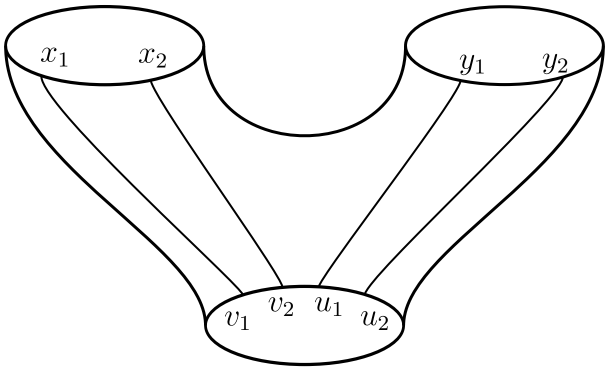

Figure 2.11: The sphere minus three discs with the produce edges.

For the lower bound let $G = G_1 \cdot G_2$ be embedded into a surface S_g of genus g. The product edges form a cut in G hence in the dual G^* of G in S_g the edges corresponding to product edges in G form a union of cycles (we consider a loop to be a cycle of length 1). If we cut the surface S_g along these cycles, the surface is split into two surfaces S_1' and S_2' without some discs so that $G_1 - \{e, f\}$ is embedded into S_1' and $G_2 - \{u, v\}$ is embedded into S_2'. We can assume that the vertices of degree 2 in G_1 and G_2 are on the boundaries of S_1' and S_2'. We do a case analysis on the number of cycles in G^* (discs on the boundary of S_1' and S_2') corresponding to the cut formed by the product edges. Denote the set of these cycles by $\mathcal{C}$.

Suppose first that the boundary of S_1' is a cycle C and that the boundary of S_2' is a cycle D. Further assume that the vertices x_1, x_2, y_1, y_2 appear in this order along C and vertices u_1, u_2, v_1, v_2 appear in this order along D. Then we can add discs to S_1' and S_2' to get surfaces S_1 and S_2 so that $g(S_1)+g(S_2) = g(S)$ and we can also add edges e, f to obtain embeddings of G_1 into S_1 and vertices u, v to S_2 to obtain and embedding of G_2 into S_2. Therefore $g(G_1) + g(G_2) \leq g(G)$ in this case.

Assume that the order around C is $x_1y_1x_2y_2$ and that the order around D is $v_1u_1v_2u_2$. Now we can discs with handles to S_1' and S_2' to obtain embeddings of G_1' and G_2' into S_1 and S_1. We can add edges e, g to G_1' and vertices u, v

to S_2 to get embeddings of G_1 and G_2 into S_1 and S_2. Therefore in this case $g(G_1) + g(G_2) - 2 \leq g(G)$. Because of symmetry these are all possible cases if we have one cycle in $\mathcal{C}$.

Suppose we have two cycles in $\mathcal{C}$. Surfaces S_1' and S_2' have boundaries consisting of cycles C_1, C_2 and D_1, D_2 respectively. There are three possibilities for positions of vertices x_1, x_2, x_3, x_4 (x_1, x_2, x_3, x_4) around C_1 and C_2 (D_1 and D_2).

Assume that vertices x_1 and x_2 are on the cycle C_1 and y_1 and y_2 are on the cycle C_2. Then we can add two discs to S_1' to get a surface S_1 and product edges to get an embedding of G_1 into S_1. We can add a handle to S_2' to get a surface S_2 and vertices u, v to S_2 to get an embedding of G_2 into S_2. In this case we have $g(S_1) + g(S_2) - 1 = g(S) - 1$ and hence $g(G_1) + g(G_2) \leq g(G)$.

Assume that vertices x_1 and y_1 are on the cycle C_1 and x_2 and y_2 are on the cycle C_2. In this case we can add a handle to S_1' and a handle to S_2' to get surfaces S_1 and S_2 and add product edges to S_1 and vertices u, v to S_2 to get embeddings of G_1 and G_2 into S_1 and S_2. in this case we have $g(S_1) + g(S_2) - 2 = g(S) - 1$ and hence $g(G_1) + g(G_2) - 1 \leq g(G)$.

The last possible case is that there is a vertex x_1 on C_1 and vertices x_1, y_1, y_2 on C_2. In this case we again get $g(G_1)+g(G_2)-1 \leq g(G)$. Because of symmetry these are all possible cases when there are two cycles in $\mathcal{C}$.

Suppose that there are three cycles in $\mathcal{C}$. There are cycles C_1, C_2, C_3 on the boundary of S_1' and cycles D_1, D_2, D_3 on the boundary of S_2'. Up to symmetry there are two possibilities for arrangement of vertices x_1, x_2, y_1, y_2 around C_1, C_2 and C_3. First case is when vertices x_1 and x_2 are on C_1 and y_1 and y_2 are on C_2 and C_3. The second case is when vertices x_1 and y_1 are on C_1 and vertices x_2 and y_2 are on C_2 and C_3. In both cases we can add a sphere minus three discs to surfaces S_1' and S_2' to get surfaces S_1 and S_2 in which we can embed graphs G_1 and G_2. Therefore in this cases we have $g(S_1) + g(S_2) + 4 = g(S) - 2$ and hence $g(G_1) + g(G_2) - 2 \leq g(G)$.

The last possible case is that $\mathcal{C}$ consists of four cycles. In this case the boundares of S_1' consist of four cycles each containing one of the vertices x_1, x_2, y_1, y_4. In this case we can add two handles to S_1' to get a surface S_1 and a sphere with four discs removed to get a surface S_2 so that graphs G_1 and G_2 embed into S_1 and S_2. In this case we have $g(S_1) + g(S_2) - 5 = g(S) - 3$ and hence $g(G_1) + g(G_2) - 2 \leq g(G)$.

We only give a sketch of the proof that the bounds are best possible. Let C be a cycle in a graph G. A *relative C-component* of G is either an edge in $E(G) \setminus E(C)$ with end points on C or a connected component of $G - C$ together with all edges between $G - C$ and C with their endpoints. An edge between a relative component of C and C is called a *foot*. A sequence of cycles $C_1, C_2, \ldots, C_k$ is *planarly nested* if for each C_i there exist relative components

H_i of C_i such that $H_1 \supset H_2 \supset \cdots \supset C_k$ and that graphs obtained from G by contracting each edge of H_i except its feet are planar. We use the following theorem from [15].

Theorem 2.11 (Mohar). *If Π is an orientable embedding of G into a surface S of minimum genus g and $C_1, C_2, \ldots, C_k$, $k > g$ is a sequence of planarly nested cycles then cycles $C_1, C_2, \ldots, C_{k-g}$ bound discs in S.*

By using superposition we can construct a snarks G_1 and G_2 with an embedding of minimum genus g such that they contain planarly nested cycles $C_1, \ldots, C_k$ (with relative components $H_1, \ldots, H_k$) and $D_1, \ldots, D_k$ (with relative components $H'_1, \ldots, H'_k$ which are contained in subgraphs corresponding to supervertices of G_1 and G_2. Further we can add edges e and f a face in the relative component H_1 such that relative components $H_1, \ldots, H_k$ are no longer planar and similarly adjacent vertices u and v connected to four vertices of a face in H'_1 such that components $H'_1, \ldots, H'_k$. Denote obtained graphs by G'_1 and G'_2. Since we only changed parts of G_1 and G_2 corresponding to supervertices, graphs G'_1 and G'_2 are snarks. From Theorem 2.11 it follows that $g(G'_1) = g(G'_2) = g + 1$. If we construct the dot product by using edges e and f and vertices u and v we get a snark $G'_1 \cdot G'_2$ with genus $g(G'_1 \cdot G'_2) = 2g$ and so $g(G'_1) + g(G'_2) - 2 = g(G'_1 \cdot G'_2)$.

Using a similar idea we show that the upper bound is tight. □

Chapter 3

Polyhedral embeddings of snarks

In this chapter we look at polyhedral embeddings of cubic graphs. We first prove that short cycles in polyhedral embeddings must be facial cycles. Using this fact we show that Goldberg snarks and Szekeres snark do not have polyhedral embeddings into orientable surfaces. Szekeres showed that flower snarks do not have polyhedral embeddings into orientable surfaces. We give a simpler proof of this result which works also for graphs J_k where k is even. We also show that flower snarks do not have polyhedral embeddings into non-orientable surfaces. On the other hand we construct polyhedral embeddings of the Goldbers snarks into non-orientable surfaces. We prove that for each non-orientable surface N there exist snarks which have polyhedral embedding into N.

3.1 Short cycles

In this section we look at short cycles in polyhedral embeddings. Let G be a cubic graph with a short cycle C has a polyhedral embedding, then C is very likely to be a facial cycle. This is established by the following lemmas.

Lemma 3.1. *Let G be a cubic graph and C a 3-cycle of G. Then C is a facial cycle in every polyhedral embedding of G.*

Proof. Let $C = v_0v_1v_2v_0$ be a 3-cycle of G. Denote the neighbor of v_i not in C with v'_i, $i = 0, 1, 2$. A facial cycle in a polyhedral embedding of G cannot contain any of the paths $v'_iv_iv_{i+1}v_{i+2}v'_{i+2}$, $i = 0, 1, 2$, indices modulo 3, since it must be induced. This implies that we have three facial cycles at C, which contain $v'_iv_iv_{i+1}v'_{i+1}$, $i = 0, 1, 2$, indices modulo 3. Then C is a facial cycle. □

Lemma 3.2. *Let G be a cubic graph other than K_4 and let C be a 4-cycle of G. Then C is a facial cycle in every polyhedral embedding of G.*

Proof. If G has a polyhedral embedding and G is not K_4, then every 4-cycle of G is induced, since G is 3-connected by Proposition 1.6.

Let $C = v_0v_1v_2v_3v_0$ be a 4-cycle of G and let v_i' be the neighbor of v_i not in C, $i = 0, 1, 2, 3$.

Suppose that all facial cycles, which intersect C, intersect C in one edge only. For each edge v_iv_{i+1} there is a facial cycle C_i which contains the path $v_i'v_iv_{i+1}v_{i+1}'$ where indices are modulo 4. Therefore all edges v_iv_i' are contained twice and edges v_iv_{i+1} are contained once in facial cycles c_i, $i = 0, 1, 2$. Therefore C must be a facial cycle since edges v_iv_{i+1} must be covered twice by facial cycles.

Suppose there is at least one facial cycle $C_1 \neq C$ which intersects C in more than one edge. Facial cycles in polyhedral embeddings are induced. Hence we may assume that C_1 contains the path $v_0'v_0v_1v_2v_2'$. The other facial cycle C_2, which contains the edge $v_0'v_0$, must contain the path $v_0'v_0v_3v_3'$ in order not to intersect C_1 at v_2. The third facial cycle through v_0 then contains edges v_0v_1, v_0v_3 and v_3v_2, which is a contradiction. □

Let a graph G be embedded in a surface S, let F be a facial cycle and let C be a cycle of G. We say that F is *k-forwarding* at C, if F and C intersect precisely in k consecutive edges on C.

Lemma 3.3. *Let G be a cubic graph and C an induced 5-cycle of G. If G has a polyhedral embedding in a surface S, then the following holds.*

(a) *If S is orientable, then C is a facial cycle.*

(b) *If S is non-orientable, then either C is a facial cycle or all facial cycles that intersect C are 2-forwarding at C.*

Proof. Let $C = v_0v_1v_2v_3v_4v_0$ be a 5-cycle of G. Suppose that no facial cycle (other than possibly C) intersects C in more than one consecutive edge on C. Then it is easy to see that C is a facial cycle.

Now let F be a facial cycle that intersects C in at least two consecutive edges on C. Facial cycles in polyhedral embeddings are induced. Therefore F is either 3-forwarding or 2-forwarding at C.

If F is 3-forwarding, we can assume that the path $v_0'v_0v_1v_2v_3v_3'$ is in F. Then the facial cycle, which contains the path $v_0v_4v_3$, intersects twice with F. This contradiction implies that no facial cycle is 3-forwarding at C.

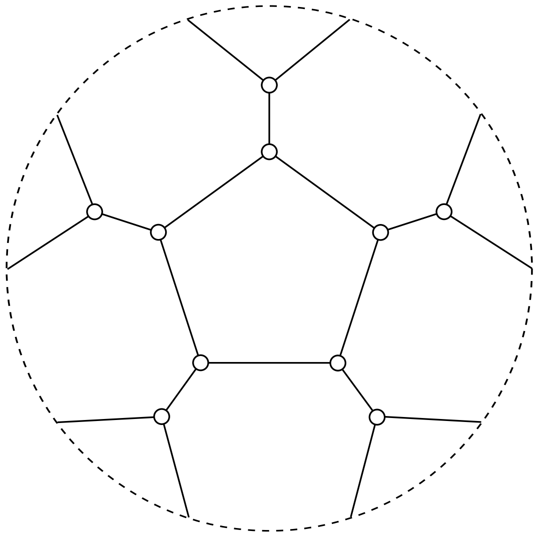

Figure 3.1: The Petersen graph embedded in the projective plane.

We may assume that F contains the path $v_0'v_0v_1v_2v_2'$. The facial cycle, which contains the path $v_1'v_1v_2$, must contain the path $v_1'v_1v_2v_3$ so it is 2-forwarding. If we continue along the cycle C, we see that all facial cycles at C are 2-forwarding at C.

To complete the proof, we will show that S is not orientable, if all facial cycles at C are 2-forwarding. Suppose that S is orientable and let C_i be the facial cycle, which contains the path $v_iv_{i+1}v_{i+2}$, $i = 0, 1, 2, 3, 4$, indices modulo 5. We can assume that in the orientation of C_0, induced by the orientation of S, vertices $v_0v_1v_2$ are in clockwise order. Then the vertices $v_3v_2v_1$ are in this clockwise order on C_1. If we continue along C, we see that in C_4 vertices $v_4v_0v_1$ are in clockwise order. But then C_0 and C_4 induce the same orientation of the edge v_0v_1, which is a contradiction with the assumption that S is orientable. □

Corollary 3.4. *If a cubic graph G contains two induced 5-cycles, whose intersection is nonempty and is not just a common edge, then G has no orientable polyhedral embeddings.*

Proof. Suppose we have an orientable polyhedral embedding of G. By Lemma 3.3 both 5-cycles are facial. This is a contradiction with the fact that their intersection contains more than just one edge. □

In the Petersen graph P every edge is contained in four induced 5-cycles. Lemma 3.3 therefore implies that P has no orientable polyhedral embeddings.

However, P has a polyhedral embedding in the projective plane (see Figure 3.1).

Lemma 3.3 and its Corollary 3.4 can be applied on many other snarks, for example the Szekeres snark that is shown in Figure 3.2.

Theorem 3.5 (Szekeres). *The Szekeres snark has no polyhedral embeddings.*

Proof. Each of the five "parts" of the Szekeres snark (see Figure 3.2) contains a path $v_1v_2\dots v_9$ on 9 vertices and a vertex v_0 that is adjacent with v_2, v_5, v_8 and further there are edges v_1v_6 and v_4v_9. There are four induced 5-cycles $C_1 = v_0v_2v_1v_6v_5v_0$, $C_2 = v_0v_2v_3v_4v_5v_0$, $C_3 = v_0v_8v_9v_4v_5v_0$ and $C_4 = v_0v_8v_7v_6v_5v_0$. Cycles C_1 and C_2 intersect at two edges adjacent to v_0. Therefore they are not both facial cycles. If none of C_1, C_2 is facial, then the 2-forwarding facial cycles at C_1 and C_2, which contain their intersection $C_1 \cap C_2$, are distinct and intersect in two edges. So one of them is facial and the other is not. Similarly, one of the cycles C_3, C_4 is facial and the other one is not.

Suppose the cycle C_2 is facial. Then it is 1-forwarding at C_4, so C_4 is facial and C_1 and C_3 are not facial. This implies that there is a facial cycle that contains the path $v_1v_6v_5v_4v_9$ and another facial cycle that contains the path $v_1v_2v_0v_8v_9$, which is a contradiction.

Suppose now that C_2 is not facial. Then C_1 is facial and is 1-forwarding at C_4. So C_4 is a facial cycle and C_3 is not. This implies that there is a facial cycle that contains the path $v_3v_2v_0v_8v_7$ and another facial cycle that contains the path $v_3v_4v_5v_6v_7$, which is a contradiction. □

Nonexistence of orientable polyhedral embeddings of the Szekeres snark has been proved earlier by Szekeres [5].

3.2 Small edge-cuts

Let G_1 and G_2 be cubic graphs and $v_1 \in V(G_1)$, $v_2 \in V(G_2)$. Denote the three neighbours of v_1 in G_1 by z_0, z_1, z_2 and the three neighbours of v_2 in G_2 by u_0, u_1, u_2. Let $G = G_1 * G_2$ be the cubic graph obtained from graphs G_1 and G_2 by deleting vertices v_1 and v_2 and connecting vertices u_i with z_i for $i = 0, 1, 2$. We call G the *star product* of G_1 and G_2. It is easy to see that the graph G is 3-edge-colorable if and only if both G_1 and G_2 are 3-edge-colorable.

Theorem 3.6. *The star product $G = G_1 * G_2$ has a polyhedral embedding in an (orientable) surface if and only if both G_1 and G_2 have polyhedral embeddings in some (orientable) surfaces.*

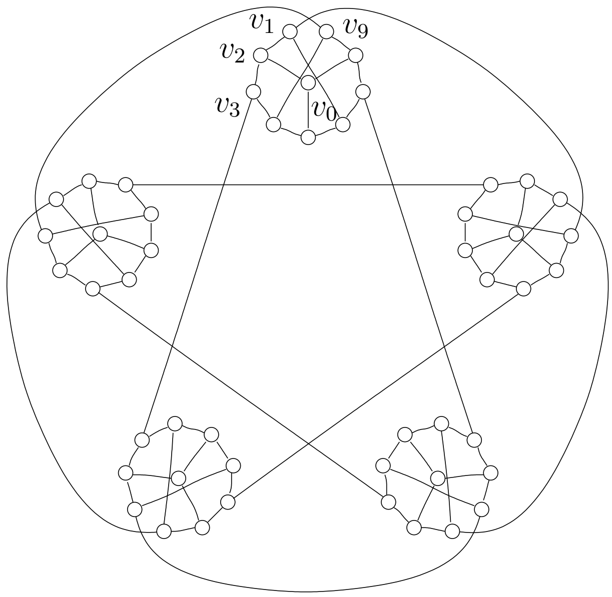

Figure 3.2: The Szekeres snark.

Proof. Suppose we have polyhedral embeddings of G_1 and G_2. At vertex v_1 we have three facial cycles $C_i = z_i v_1 z_{i+1} P_i z_i$ for $i = 0, 1, 2$, indices modulo 3. At vertex v_2 we have three facial cycles $D_i = u_i R_i u_{i+1} v_2 u_i$ for $i = 0, 1, 2$. Since the embeddings are polyhedral, paths P_0, P_1, P_2 and paths R_0, R_1, R_2 are pairwise disjoint. In the embedding of the star product $G = G_1 * G_2$ we keep all facial cycles from embeddings of G_1 and G_2, which do not contain vertices v_1 and v_2, and add three new facial cycles $F_i = z_i u_i R_i u_{i+1} z_{i+1} P_i z_i$, $i = 0, 1, 2$, indices modulo 3. Facial cycles in G, which are facial cycles in G_1 or G_2, intersect pairwise at most once. A facial cycle F, which is also a facial cycle in G_1 or G_2, intersects the facial cycle F_i, $i = 0, 1, 2$, only on the path P_i or only on the path R_i. So it intersects F_i at most once. Facial cycles F_i and F_{i+1} intersect only in the edge $u_{i+1} z_{i+1}$, $i = 0, 1, 2$, indices modulo 3, since the paths P_0, P_1, P_2 and R_0, R_1, R_2 are pairwise disjoint. So the embedding of G is polyhedral. It is easy to see that the embedding of G is orientable if and only if the embeddings of G_1 and G_2 are orientable.

Suppose now that G has a polyhedral embedding. The three edges $z_i u_i$, $i = 0, 1, 2$, form a 3-cut in G. Since the embedding is polyhedral, we have three facial cycles $F_i = u_i R_i u_{i+1} z_{i+1} P_i z_i u_i$, such that F_i and F_{i+1} intersect in the edge $z_{i+1} u_{i+1}$, $i = 0, 1, 2$, indices modulo 3. We may assume that there are no negative signatures on edges $z_i u_i$, $i = 0, 1, 2$. In the embedding of G_1 (and G_2) we keep all facial cycles, which do not intersect G_2 (respectively

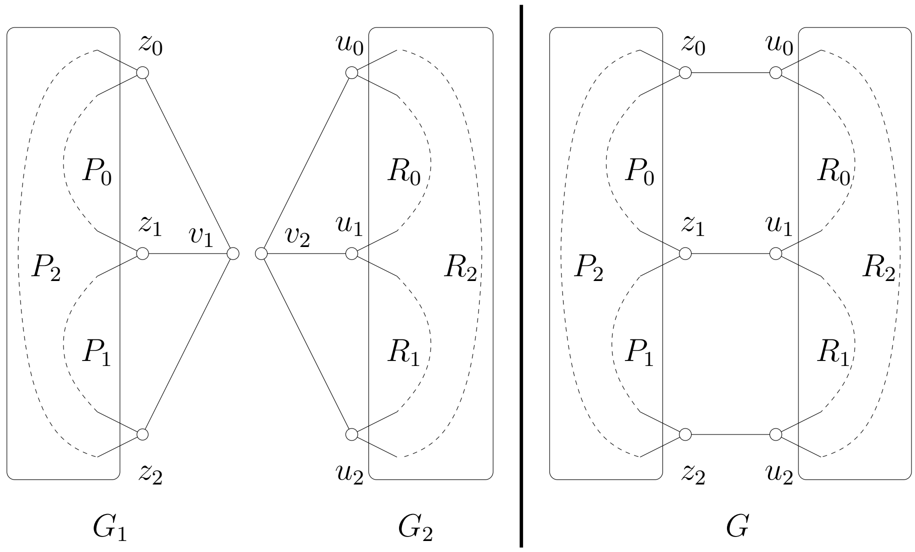

Figure 3.3: The star product G of graphs G_1 and G_2.

G_1), and add vertices v_1, v_2 with such local rotations that we obtain new facial cycles $C_i = z_i v_1 z_{i+1} P_i z_i$ in G_1 and $D_i = u_i R_i u_{i+1} v_2 u_i$ in G_2, $i = 0, 1, 2$, induces modulo 3. Since we have no new intersections between facial cycles (intersections on $z_i u_i$ become intersections on $z_i v_1$ and $u_i v_2$), the embeddings of G_1 and G_2 are polyhedral. It is also clear that both embeddings are in orientable surfaces if and only if the embedding of G is orientable, since we did not change local rotation at any vertex or change the signature of any edge. □

If the embedding of $G = G_1 * G_2$ in a surface S is constructed as in the proof of Theorem 3.6 from embeddings of G_1 and G_2 in surfaces S_1 and S_2 of Euler genus $\epsilon(S_1) = k_1$ and $\epsilon(S_2) = k_2$, respectively, then the Euler genus of S is $\epsilon(S) = k_1 + k_2$. This is easily proved by using Euler's formula for G, G_1 and G_2. Let G_1 and G_2 be cubic graphs. Choose an edge $e = xy$ in G_1 and two nonadjacent edges $f_1 = u_0 u_1$ and $f_2 = u_2 u_3$ in G_2. Denote the neighbors of x in G_1 by v_0, v_1, and the neighbors of y by v_2, v_3. Let G be the dot product of G_1 and G_2 obtained by deleting vertices x, y in G_1 and edges f_1, f_2 in G_2 and joining pairs $v_i u_i$, $i = 0, 1, 2, 3$.

Theorem 3.7. *Let G_1 and G_2 be cubic graphs. If G_1 and G_2 have polyhedral embeddings in (orientable) surfaces S_1 and S_2, such that the geometric dual of G_2 is not a complete graph, then a dot product $G = G_1 \cdot G_2$ exists, which has a polyhedral embedding in an (orientable) surface S. If the Euler genera of surfaces S_1 and S_2 are $\epsilon(S_1) = k_1$ and $\epsilon(S_2) = k_2$, then the Euler genus of*

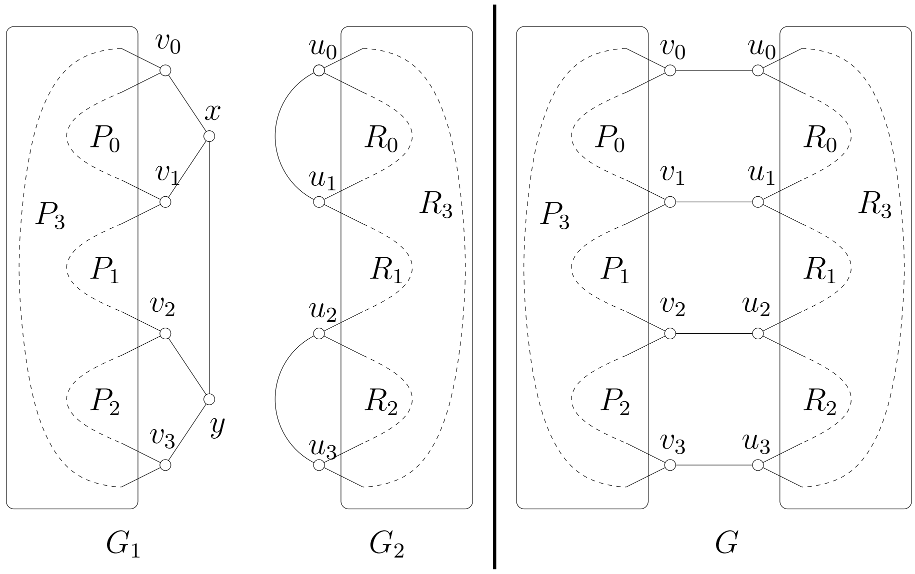

Figure 3.4: The dot product G of graphs G_1 and G_2.

S is $\epsilon(S) = k_1 + k_2$.

Proof. Suppose that we have polyhedral embeddings as described. We claim that G_2 contains facial cycles D_0, D_1, D_2, such that D_1 intersects D_0 and D_2 but D_0 and D_2 do not intersect. To see this, consider the dual graph R. Since it is not a complete graph, it has two vertices c_0 and c_2 that are at distance two in R. If c_1 is their common neighbor, then we can take D_0, D_1, D_2 to be the facial cycles corresponding to c_0, c_1 and c_2, respectively.

Let $f_1 = u_0u_1$ and $f_2 = u_2u_3$ be the intersections between D_0, D_1 and D_1, D_2, respectively, and choose an arbitrary edge $e = xy$ in G_1. Denote the neighbors of x and y in G_1 so that the facial cycles, which contain x or y, are $C_0 = v_0xv_1P_0v_0$, $C_1 = v_1xyv_2P_1v_1$, $C_2 = v_2yv_3P_2v_2$, and $C_3 = v_3yxv_0P_3v_3$. Since the embedding of G_1 is polyhedral, paths P_0, P_1, P_2, P_3 are pairwise disjoint, except that P_0 and P_2 may intersect. In G_2 we will use the following notation for facial cycles: $D_0 = u_0R_0u_1u_0$, $D_1 = u_0u_1R_1u_2u_3R_3u_0$ and $D_2 = u_2R_2u_3u_2$. The paths R_0, R_1, R_2, R_3 are pairwise disjoint. In the embedding of G we keep all local rotations at vertices of G_1 and G_2, which are not deleted (with added edges naturally replacing deleted edges), and all edge signatures. Instead of facial cycles C_i, D_i we get a facial cycle $F_i = v_iu_iR_iu_{i+1}v_{i+1}P_iv_i$, $i = 0, 1, 2, 3$, indices modulo 4. Since the paths P_i, R_i are pairwise disjoint, except for the possible intersection between P_0 and P_2, all intersections between facial cycles F_i, $i = 0, 1, 2, 3$, are the intersections of F_i and F_{i+1} in edges

$v_{i+1}u_{i+1}$, $i = 0, 1, 2, 3$, indices modulo 4, and possibly one more intersection between F_0 and F_2. It is clear that any facial cycle F that does not contain any of the vertices v_i, u_i intersects at most once with any F_i and that two such facial cycles intersect at most once. So the embedding of G is polyhedral. It is also clear that if the embeddings of G_1 and G_2 are in orientable surfaces, the embedding of G is also in an orientable surface.

The Euler genus of S is obtained from Euler's formula and equalities

$$\begin{aligned} |V(G)| &= |V(G_1)| + |V(G_2)| - 2 \\ |E(G)| &= |E(G_1)| + |E(G_2)| - 3 \\ |F(G)| &= |F(G_1)| + |F(G_2)| - 3 \end{aligned}$$

from which we conclude that $\epsilon(S) = k_1 + k_2$. □

Theorem 3.8. *Let G be a cubic graph and S a minimal cyclic 4-cut in G. If G admits a polyhedral embedding (in an orientable surface), then there exist graphs G_1 and G_2, such that $G = G_1 \cdot G_2$ and G_1 admits a polyhedral embedding (in an orientable surface).*

Proof. Suppose that the edges u_iv_i, $i = 0, 1, 2, 3$, form a 4-cut S in G. If a facial cycle contains more than two edges of S, the embedding of G can not be polyhedral. So we have four distinct facial cycles F_0, F_1, F_2, F_3 that contain edges of S. Since S is a cut, every cycle F_i, $i = 0, 1, 2, 3$, contains two edges of S.

Since the embedding is polyhedral, each of the F_i intersects two other F_j, F_k. In the dual a subgraph induced by the vertices corresponding to F_i, $i = 0, 1, 2, 3$, is a simple graph on four vertices in which all vertices are of degree 2. It must be a 4-cycle. Therefore we can assume that faces F_i and F_{i+1} intersect in the edge $v_{i+1}u_{i+1}$, $i = 0, 1, 2, 3$, indices modulo 4. Each facial cycle F_i is then of the form $F_i = v_iu_iR_iu_{i+1}v_{i+1}P_iv_i$. Since F_0 and F_2 intersect at most once, we can assume they do not intersect at the paths P_0 and P_2. Let G_1 be the component of $G - S$, which contains paths P_i. If we set rotations of all vertices in G_2 as they are in G (and replace deleted edges naturally with added edges), we can set rotations around vertices x and y so that the facial cycles in G_1, which do not contain x or y, remain unchanged and we have four new facial cycles $C_0 = v_0xv_1P_0v_0$, $C_1 = v_1xyv_2P_1v_1$, $C_2 = v_2yv_3P_2v_2$, and $C_3 = v_3yxv_0P_3v_3$. Since we added no new intersections between facial cycles, which were already in G, and facial cycles C_i, $i = 0, 1, 2, 3$ intersect pairwise only once, the embedding of G_1 is polyhedral. If the embedding of G is in an orientable surface, it is clear that the embedding of G_1 is in an orientable surface. □

Suppose we have polyhedral embeddings of cubic graphs G_1 and G_2, at least one of which is in a non-orientable surface. Let us construct the embedding of the dot product $G = G_1 \cdot G_2$ as in the proof of Theorem 3.7. If the embedding of G is in orientable surface, then we may assume that all signatures of edges are positive. Now we can construct embeddings of G_1 and G_2 similarly as the embedding of G_1 in the proof of Theorem 3.8, which are both in orientable surfaces and have the same set of facial cycles as the embeddings of G_1 and G_2 with which we started. Since at least one of these two is an embedding in a non-orientable surface, we have a contradiction. This shows

Corollary 3.9. *If we have polyhedral embeddings of G_1 and G_2, at least one of which is non-orientable, and construct a polyhedral embedding of $G = G_1 \cdot G_2$ as in the proof of Theorem 3.7, then the embedding of G is non-orientable.*

Let G_1 and G_2 be cubic graphs. Choose a vertex v in G_1, an edge v_3v_4 in G_1 and a vertex z_0 in G_2. Let the three neighbors of v be v_0, v_1, v_2 and let z_1, z_2, u_4 be the neighbors of z_0. Let the neighbors of z_1, z_2 other than u be u_0, u_1 and u_2, u_3, respectively. If all these vertices are distinct, remove the vertex v from G_1, vertices z_0, z_1, z_2 from G_2 and the edge v_3v_4 from G_1. If we join pairs v_iu_i, $i = 0, 1, 2, 3, 4$, we get a cubic graph $G = G_1 \Diamond G_2$, which is called a *square product* of graphs G_1 and G_2 (see also Figure 3.5). The cut $Q = \{v_iu_i \mid i = 0, \ldots, 4\}$ in G is said to be the *product cut*. It is claimed in [19] that if G_1 and G_2 are snarks, then G is also a snark, however this is not true in general. For results concerning 5-cuts in snarks, see [13].

Theorem 3.10. *Let G be a cubic graph with a matching Q, which is a 5-cut of G. If G admits a polyhedral embedding (in an orientable surface), then there exist graphs G_1 and G_2 such that $G = G_1 \Diamond G_2$ and Q is the corresponding product cut and such that G_2 admits a polyhedral embedding (in an orientable surface).*

Proof. Suppose that G has a polyhedral embedding. Since Q is a cut, every facial cycle contains an even number of edges in Q. It is easy to see that none of them contains four edges of Q (since the embedding is polyhedral). This implies that there are precisely 5 facial cycles $F_0, \ldots, F_4$ that intersect Q and that the edges v_iu_i of Q, $i = 0, \ldots, 4$, can be enumerated so that F_i contains edges v_iu_i and $v_{i+1}u_{i+1}$, indices modulo 5, and $v_0, \ldots, v_4$ are in the same component of $G - Q$. The facial cycles F_i are of the form $F_i = v_iu_iR_iu_{i+1}v_{i+1}P_iv_i$, $i = 0, \ldots, 4$, indices modulo 5. Since the embedding is polyhedral, every one of the pairs of paths P_i, P_{i+1} and R_i, R_{i+1} is disjoint.

Suppose that the facial cycles F_i and F_{i+2} are disjoint for some i. Then both pairs P_i, P_{i+2} and R_i, R_{i+2} are disjoint. One of the pairs P_{i+2}, P_{i+4} and

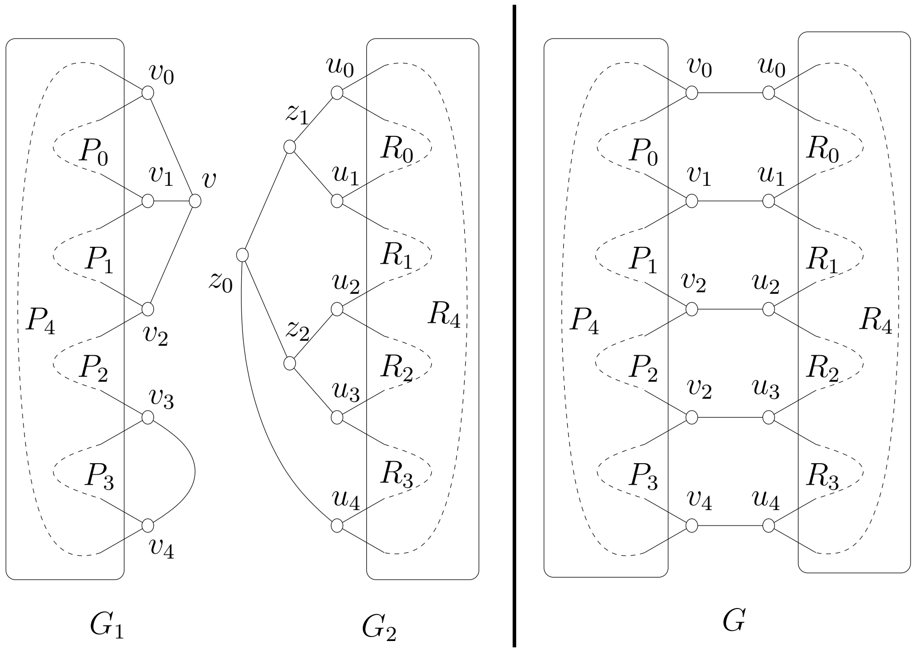

Figure 3.5: The square product of G_1 and G_2.

R_{i+2}, R_{i+4}, $i = 0, \ldots, 4$, is disjoint. Because of the symmetry, we can assume that the pair R_{i+2}, R_{i+4} is disjoint.

Suppose now that all pairs of cycles F_i, F_{i+2}, $i = 0, \ldots, 4$, intersect. In at least three out of five pairs, F_i and F_{i+2} intersect on the same "side" (P_i and P_{i+2} or R_i and R_{i+2}). By symmetry, we may assume that intersections are between P_i and P_{i+2}. Since facial cycles F_i and F_{i+2} intersect at most once, it follows that there exists an index j such that R_j, R_{j+2}, R_{j+4} are pairwise disjoint.

By above, we can assume that R_4, R_1, R_3 are pairwise disjoint. Now we can add to $G - Q$ new vertices v, z_0, z_1, z_2 and edges v_0v, v_1v, v_2v, v_3v_4 and u_0z_1, u_1z_1, u_2z_2 ,u_3z_2, z_1z_0, z_2z_0, u_4z_0 so that the graph G is a square product of G_1 and G_2. In the embedding of G_2 we keep all rotations and signatures of vertices and edges that were already in G and we naturally replace deleted edges with the added ones. Around vertices z_0, z_1, z_2 we can set rotations so that facial cycles in G_2, which were not already in G, are $D_0 = u_0R_0u_1z_1$, $D_1 = z_0z_1u_1R_1u_2z_2z_0$, $D_2 = z_2u_2R_2u_3z_2$, $D_3 = z_0z_2u_3R_3u_4z_0$ and $D_4 = z_0u_4R_4u_0z_1z_0$. The only new intersections of facial cycles of G_2 are between D_4 and D_1 and between D_1 and D_3. Hence the embedding of G_2 is polyhedral and if the embedding of G is in an orientable surface, so is the

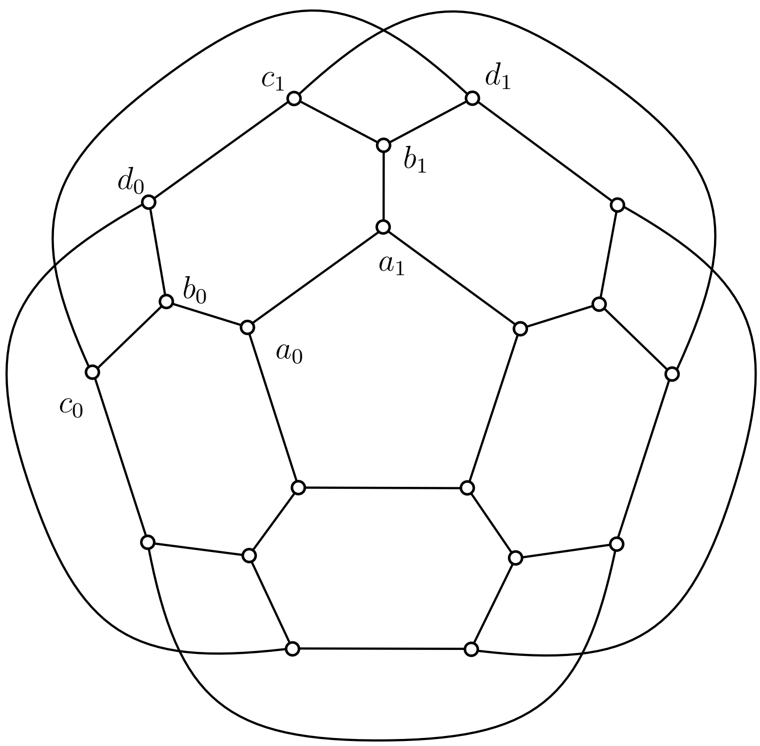

Figure 3.6: The Flower snark J_5.

embedding of G_2. □

3.3 Flower snarks

In this section we prove that Flower snarks J_{2k+1} do not have polyhedral embeddings. This was first proved by Szekeres using polyhedral decompositions. His proof only worked for graphs J_{2k+1} but not for graphs J_{2k} and only for orientable embeddings. We give a simpler proof which also works for all graphs J_k and also for non-orientable embedding.

The goal for this section is to prove the following theorem.

Theorem 3.11. *For $k \geq 4$ the flower graph J_k has no polyhedral embeddings.*

We first prove the theorem for larger k and then prove the theorem for smaller values of k. Note that the graph J_3 is obtained from the Petersen graph P by replacing one vertex in P by a triangle. Since Petersen graph has a polyhedral embedding into the projective plane so does J_3. Since there are no polyhedral embeddings of P into orientable surfaces it follows from the Lemma 3.1 that J_3 has no polyhedral embeddings.

Suppose that we have a polyhedral embedding of J_k. Let us look at how facial cycles can traverse Y_j. If we walk along a facial cycle C, come to Y_j from Y_{j-1} and then leave Y_j going back to the tile Y_{j-1}, we say that C is a *backward face at* Y_j. Similarly we define a *forward face at* j, which is a facial cycle that enters Y_j from Y_{j+1} and leaves it towards Y_{j+1}.

If a cubic graph G has a polyhedral embedding, then at every vertex $v \in V(G)$ with neighbours v_1, v_2, v_3, each path $P = v_i v v_j$, $j \neq i$, defines a unique facial cycle, which we will denote by $F(P)$.

Lemma 3.12. *If C is a facial cycle that contains at least two vertices of Y_j, then the intersection of C with Y_j is one of the three possible paths: $a_j b_j c_j$, $a_j b_j d_j$ or $c_j b_j d_j$.*

Proof. A cycle C can enter and exit Y_j only through vertices a_j, c_j or d_j. Suppose now that $a_j, c_j \in V(C)$. The facial cycle $C' = F(a_j b_j c_j)$ intersects C in two nonadjacent vertices a_j and c_j, so $C = C'$ and C' contains the path $a_j b_j c_j$. Similar conclusion holds if a_j and d_j are on C or if c_j and d_j are on C. Since all facial cycles are induced, the intersection $C \cap Y_j$ can consists only of one of the three paths. □

A facial cycle, which is neither forward nor backward at Y_j, is called a *cross face*. It follows from Lemma 3.12 that each facial cycle, which intersects Y_j, is either a backward, forward or a cross face.

Lemma 3.13. *At Y_j there can be at most one backward (forward) face. If there is one backward face, then there is also one forward face and four distinct cross faces. The backward face at Y_j is forward at Y_{j-1} and the forward face at Y_j is backward at Y_{j+1}.*

Proof. Suppose we have two backward (forward) faces. By Lemma 3.12 they intersect at an edge adjacent to b_j. If they intersect at $b_j a_j$, they also intersect at $a_{j-1} a_j$, which is a contradiction. Similarly we get a contradiction, if they intersect at $b_j c_j$ or $b_j d_j$. This shows that there is at most one backward (forward) face.

Suppose now that C is a backward face. The edges between Y_j and Y_{j+1} are traversed twice by C and four times by cross faces. The cross faces therefore traverse the edges between Y_j and Y_{j+1} at most four times, hence there must be a forward face at Y_j.

If C contains the path $a_j b_j c_j$, then $\{a_{j-1}, d_{j-1}\} \subseteq C \cap Y_{j-1}$. By Lemma 3.12, $C \cap Y_{j-1} = a_{j-1} b_{j-1} d_{j-1}$, so C is a forward face at Y_{j-1}. A similar conclusion holds if $C \cap Y_j$ is either $a_j b_j d_j$ or $c_j b_j d_j$. Similarly we also show that a forward face at Y_j is backward at Y_{j+1}.

Out of facial cycles $F(a_j b_j c_j)$, $F(a_j b_j d_j)$ and $F(c_i b_j d_j)$ one is a backward face, one is a forward face and one is a cross face. Since the one that is a cross face is the only cross face, which contains more than one vertex of Y_j, all cross faces are distinct. □

A backward face at j is called a *bottom face* if it contains the edge $a_{j-1}a_j$ and is called a *top face* if it does not contain $a_{j-1}a_j$. A top face at Y_j is of the form $c_{j-1}b_{j-1}d_{j-1}c_j b_j d_j c_{j-1}$. So it is clear that we cannot have backward top faces at Y_j and Y_{j+1} at the same time.

The tile Y_j is of *type 0*, if all facial cycles, which intersect it, are cross faces. It is of *type 1*, if there is one forward and one backward face at Y_j.

Lemma 3.12 implies that if the graph J_k has a polyhedral embedding, then all tiles are of type 0 or all tiles are of type 1.

Lemma 3.14. *If J_k has a polyhedral embedding, then $k \leq 6$ and all tiles are of type 1.*

Proof. By Lemma 3.12 every polyhedral embedding of J_k has at least four cross faces. For each $j = 0, \ldots, k-1$ we have at least one intersection between four selected cross faces on edges from Y_j to Y_{j+1}. Since we can have at most 6 such intersections, we have $k \leq 6$.

If all tiles are of type 0, then J_k has precisely 6 facial cycles. The geometric dual of G on S has 6 vertices and $\frac{4k \cdot 3}{2} = 6k$ edges. Since the dual is a simple graph, it has at most 15 edges, so $6k \leq 15$. This implies that $k \leq 2$. □

Lemma 3.15. *The graph J_4 has no polyhedral embeddings.*

Proof. Suppose we have a polyhedral embedding of J_4. All tiles are of type 1, so there are precisely 4 cross faces. We have three 4-cycles $C_1 = a_0a_1a_2a_3a_0$, $C_2 = d_0c_1d_2c_3d_0$, $C_3 = c_0d_1c_2c_3c_0$ in J_4, which are facial cycles by Lemma 3.2. These cycles are all cross faces. As in the proof of Lemma 3.14, we see that there are at least four intersections of cross faces. But since C_1, C_2, C_3 are pairwise disjoint, this is not possible. □

Lemma 3.16. *The flower snark J_5 has no polyhedral embeddings.*

Proof. Suppose we have a polyhedral embedding of J_5. Each tile must be of type 1. If all backward faces are bottom faces, then the inner cross face

$a_0a_1a_2a_3a_4a_0$ does not intersect any other cross faces. So we have 5 intersections between three cross faces, which is not possible.

Since we cannot have two consecutive top faces, we must have two consecutive bottom faces at tiles j and $j+1$ and a top face at tile $j+2$. We can assume $j = 1$. The facial cycle $F(a_0a_1a_2)$ contains the path $a_0a_1a_2a_3a_4$. If not, it would intersect twice with one of the bottom faces at tiles 1 or 2. So it must be $a_0 \dots a_4a_0$. The facial cycle, which contains b_2a_2 and is different from the backward face at tile 2, must contain the path $b_2a_2a_3b_3$. This facial cycle intersects twice with the facial cycle $d_2b_2c_2d_3b_3c_3d_2$, which is a contradiction. □

Lemma 3.17. *The graph J_6 has no polyhedral embeddings.*

Proof. All tiles in J_6 are of type 1. We have three 6-cycles $C_1 = a_0a_1 \dots a_5a_0$, $C_2 = c_0d_1c_2 \dots d_5c_0$ and $C_3 = c_0d_1c_2 \dots d_5c_0$. From previous proofs it follows that at each tile Y_j one of the four cross faces goes from one of C_1, C_2, C_3 to another. We say that this cross face has made a transition at Y_j. It is obvious that if a cross face makes at least one transition, it makes more than one transition. So one cross face makes no transitions, since we can have at most 6 transitions. Let the four cross faces be F_1, F_2, F_3, F_4 and let F_1 be the one, which does not make any transition. Because of the symmetry, we can assume that $F_1 = C_1$.

There are four cross faces and six intersections between them. This implies that they must all pairwise intersect and in particular, all cycles F_2, F_3, F_4 intersect F_1. All transitions of cross faces are transitions of F_i to C_1 and from C_1, $i = 2, 3, 4$. In particular, the cycle F_2 makes a transition to the cycle C_1 at some tile Y_j and a transitions from C_1 at the tile Y_{i+1}. But then F_2 is not induced, which is a contradiction. □

This completes the proof of Theorem 3.11.

3.4 Goldberg snarks

We now look at polyhedral embeddings of Goldberg snarks. We show that Goldberg snarks do not have polyhedral embeddings into orientable surfaces but they do have polyhedral embeddings into non-orientable surfaces.

Theorem 3.18. *No Goldberg graph has a polyhedral embedding in an orientable surface. On the other hand, every Goldberg graph G_k, $k \geq 3$, has a polyhedral embedding in the non-orientable surface of Euler genus k.*

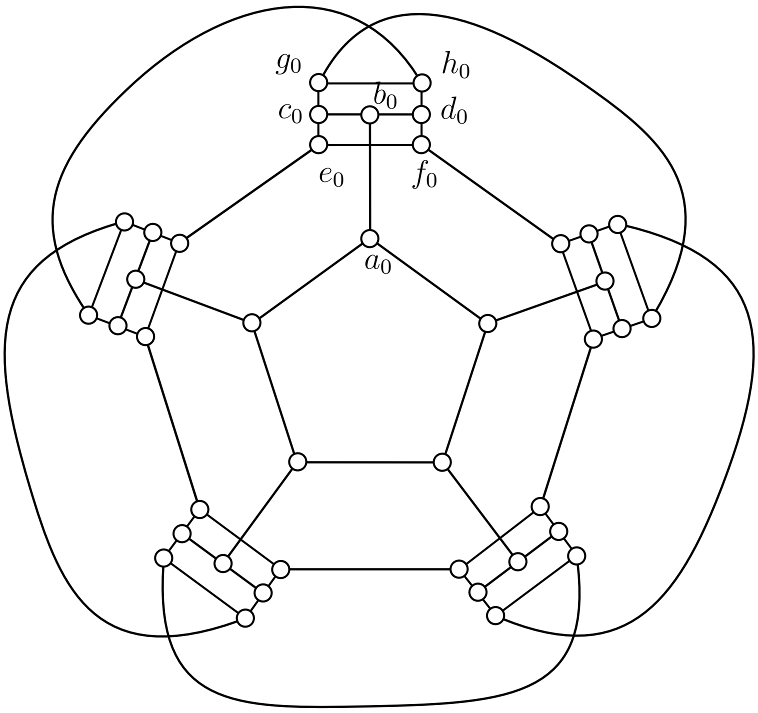

Figure 3.7: The Goldberg snark G_5.

Proof. Suppose that the graph G_k has a polyhedral embedding in an orientable surface. For every $i = 0, \dots, k-1$ we have have two 5-cycles $B_i = b_i d_i h_i g_i c_i b_i$ and $C_i = b_i d_i f_i e_i c_i b_i$. By Lemma 3.3 both are facial cycles. This is a contradiction, since B_i and C_i intersect in two edges $c_i b_i$ and $b_i d_i$.

An embedding in a non-orientable surface has the following facial cycles:

(a) $A = a_0 a_1 \dots a_{k-1} a_0$ and $B = f_0 e_0 f_1 e_1 \dots f_{k-1} e_{k-1} f_0$,

(b) $C_i = b_i d_i f_i e_i c_i b_i$, $i = 0, \dots, k-1$,

(c) $D_i = g_i h_i g_{i+1} h_{i+1} d_{i+1} f_{i+1} e_i c_i g_i$, $i = 0, \dots, k-1$,

(d) $E_i = a_i a_{i+1} b_{i+1} c_{i+1} g_{i+1} h_i d_i b_i a_i$, $i = 0, \dots, k-1$.

It is easy to see that this determines a non-orientable polyhedral embedding. The Euler genus of the underlying surface of the embedding is calculated from Euler's formula $2 - \epsilon(G_k) = |V(G_k)| - |E(G_k)| + |F(G_k)| = 8k - \frac{3}{2}8k + 3k + 2 = 2 - k$. □

Goldberg graphs have more than one polyhedral embedding, not all of the same genus. They can be described as follows.

Consider the subgraph T_i induced on vertices a_i, b_i, c_i, d_i, e_i, f_i, g_i and h_i. Let us look at how facial cycles can traverse it. There are (at least) two possibilities.

There is a facial 5-cycle $C^i = b_i d_i h_i g_i c_i b_i$ and there are facial cycles that contain paths $P_1^i = a_{i-1} a_i a_{i+1}$, $P_2^i = g_{i-1} h_i g_i h_{i+1}$, $P_3^i = g_{i-1} h_i d_i f_i e_i f_{i-1}$, $P_4^i = h_{i+1} g_i c_i d_i e_i f_i e_{i+1}$, $P_5^i = e_{i+1} f_i d_i b_i a_i a_{i+1}$ and $P_6^i = f_{i-1} e_i c_i b_i a_i a_{i-1}$, where P_1^i and P_2^i can possibly be part of the same facial cycle. In such case, we say that T_i is of *type 1*.

The second possibility is the following. There is a facial 5-cycle $D^i = b_i c_i e_i f_i d_i b_i$ and there are facial cycles that contain paths $R_1^i = a_{i-1} a_i a_{i+1}$, $R_2^i = f_{i-1} e_i f_i e_{i+1}$, $R_3^i = a_{i-1} a_i b_i d_i h_i g_{i-1}$, $R_4^i = a_{i+1} a_i b_i c_i g_i h_{i+1}$, $R_5^i = f_{i-1} e_i c_i g_i h_i g_{i-1}$ and $R_6^i = e_{i+1} f_i d_i b_i h_i g_i h_{i+1}$, where R_1^i and R_2^i can possibly be part of the same facial cycle. We say that T_i is of *type 2*.

We now choose arbitrary the types of all subgraphs T_i and join facial segments described above into facial cycles as follows. There is an automorphism of the graph G_k, which sends all cycles C^i into cycles D^i, so we can assume that the subgraph T_i is of type 1. If not, we join facial segments symmetrically according to this automorphism.

If subgraphs T_i and T_{i+1} are both of type 1, we join facial segments P_1^i and P_1^{i+1}, P_2^i and P_2^{i+1}, P_4^i and P_3^{i+1} and facial segments P_5^i and P_6^{i+1}.

If the subgraph T_i is of type 1 and T_{i+1} of type 2, we join facial segments P_1^i,R_3^{i+1} and P_2^i, facial segments R_1^{i+1}, P_5^i and R_2^{i+1} and facial segments P_4^i and R_5^{i+1}.

If all subgraphs T_i are of type 1 (or all are of type 2), then the embedding is the one described in the proof of Theorem 3.18. If there are two consecutive subgraphs T_i and T_{i+1} of different types, we say that there is a *transition at* i. It is easy to see that the embedding is polyhedral if we have at least 6 transitions. It is also easy to see that the number of facial cycles of the embedding is $3k$. In this manner we have obtained a large number of (combinatorially) different polyhedral embeddings of the graph G_k in a surface of Euler genus $k+2$.

This shows that Goldberg snarks admit polyhedral embeddings in distinct non-orientable surfaces (of Euler genera k and $k+2$) and that they admit combinatorially different polyhedral embeddings in the same non-orientable surface (of Euler genus $k+2$).

Corollary 3.19. *For every positive integer k there exists a snark which has a polyhedral embedding into N_k.*

Proof. The Petersen graph P has a polyhedral embedding in N_1. By Theorem 3.18 the Goldberg snark G_{2k+1} has a polyhedral embedding in N_{2k+1} for every $k \geq 1$. The graph G_3 is not a snark since it contains a 3-cycle $C = a_0a_1a_2a_0$. If we contract C to a vertex, we obtain a snark G_3', which polyhedrally embeds in N_3 (cf. Theorem 3.6). For $k > 1$ we have a snark $H_{2k+2} = G_{2k+1} \cdot P$, which polyhedrally embeds in N_{2k+2}, and $H_4 = G_3' \cdot P$, which polyhedrally embeds in N_4 (cf. Theorem 3.7). The dot product $H_2 = P \cdot J_3$ polyhedrally embeds in N_2. The graph H_2 is not 3-edge-colorable, but is not a snark, since the girth of H_2 is 4.

There are two non-isomorphic dot products of two copies of the Petersen graph P, but since the dual of P in the projective plane is K_6, we cannot use Theorem 3.7 to obtain a snark with polyhedral embedding into the Klein bottle. Indeed, it can be shown that they do not have such embeddings.

We construct a superposition G_{28} of the Petersen graph in the projective plane to get a snark embedded in the Klein bottle. Take an edge $e = uv$ in the Petersen graph. Replace vertices u and v with (1,1,3)-supervertices in the Figure 4.6 and the edge e with the superedge obtained from the Petersen graph by removing vertices x and y (see Figure 3.8). We claim that we get a snark with polyhedral embedding into the Klein bottle (see Figure 3.9).

G_{28} is clearly a snark since it was constructed as a superposition of the Petersen graph. In the embedding in Figure 3.9, facial cycles which cross cross-caps do not contain bad edges since these cycles come from embeddings of the Petersen graph into the projective plane. It is also clear from the figure that other cycles do not contain bad edges. □

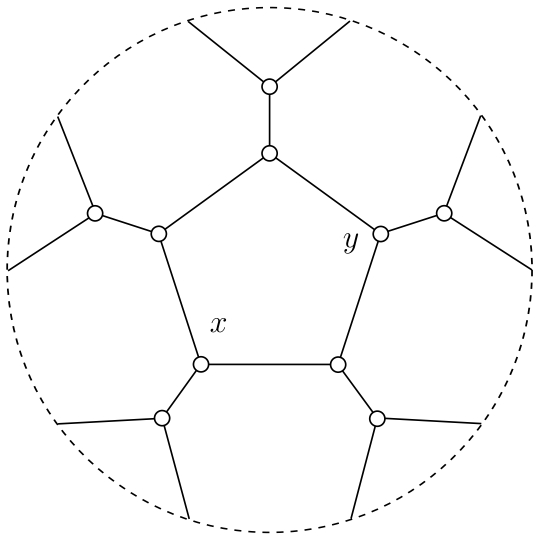

Figure 3.8: The Petersen graph in the projective plane.

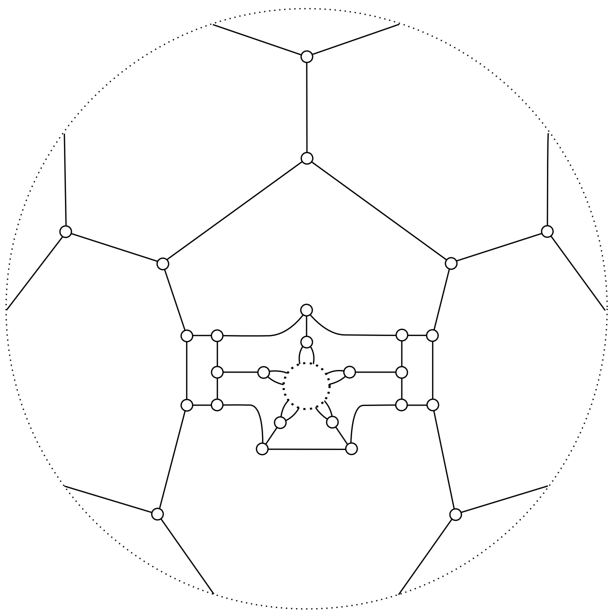

Figure 3.9: Polyhedral embedding of a snark into the Klein bottle.

Chapter 4

The defect of a graph

In this part of the thesis we define the defect of a graph which is a measure for how far a graph is from having a polyhedral embedding. The defect is defined so that for a given graph it is easy to compute. Using a computer and a database of snarks with up to 28 vertices we show that the Grünbaum conjecture is true for all snarks with up to 28 vertices.

Using the defect we show that the Grünbaum conjecture is true for Kochol snarks. The family of Kochol snarks is a rich family of snarks which includes for instance snarks with arbitrarily large girth.

We then prove some theoretical results about the defect. In particular we show that if Grünbaum conjecture is true than the defect for any snark is at least two, and for any $k \geq 2$ we construct an infinite family of snarks with defect precisely k.

We show that the Grünbaum conjecture implies a strong inequality between the defect and resistance of snarks. Resistance is a measure for how far a snark is from having 3-edge-coloring. We prove that if the Grünbaum conjectrure is true, graphs with high resistance have high defect.

4.1 Definition of defect and computer search

We define the *defect* of a graph as a measure for how far a (cubic) graph is from having a polyhedral embedding. Let Π be an embedding of a cubic graph G and let $\mathcal{F} = \{W_1, \ldots, W_k\}$ be the collection of facial walks of Π. For a walk $W_i \in \mathcal{F}$ we define the *defect* $d(W_i)$ of W_i to be the number of edges which appear twice along W_i. For two facial walks $W_i, W_j \in \mathcal{F}$, $i \neq j$, we define the *defect* $d(W_i, W_j)$ as

$$d(W_i, W_j) = \begin{cases} 0 & ; \ |E(W_i) \cap E(W_j)| = 0 \\ |E(W_i) \cap E(W_j)| - 1 & ; \ |E(W_i) \cap E(W_j)| > 0. \end{cases}$$

The defect of the embedding Π is defined as

$$d(\Pi) = \sum_{i=1}^{k} d(W_i) + \sum_{1 \leq i < j \leq k} d(W_i, W_j).$$

and the defect of the graph G is defined as

$$d(G) = \min\{d(\Pi) \mid \Pi \text{ an orientable embedding of } G\}.$$

In an embedding Π of G a pair of facial walks is a *bad pair* if they have more than one edge in common. An edge e is a *bad edge* if it appears twice along a facial walk of Π or if there is another edge f such that e and f both appear along two facial walks W_i and W_j.

It is clear from the definition of the defect that a graph G admits a polyhedral embedding into an orientable surface if and only if $d(G) = 0$. The Grünbaum conjecture is therefore equivalent to the statement that for any snark G the defect $d(G)$ is at least 1. We give a stronger implication in the last section of this chapter.

Using a computer program which examines all possible orientable embeddings of a graph we have determined the defects for snarks with up to 28 vertices. We found that the smallest defect among these snarks is two. The smallest snark with defect two has 26 vertices. It has two embeddings into the torus with defect two and it is the only snark on 26 vertices with defect 2. There are two snarks on 28 vertices with defect two. One of them has two embeddings of defect two and the other has one embedding of defect two. All these embeddings are into the double torus. There is one snark on 18 vertices with three distinct embeddings of defect three into the torus. There are two snarks on 24 vertices with defect three, one has a unique embedding and the other has two embeddings of defect three, all embeddings are into the double torus. There is one snark on 26 vertices with three embeddings of defect three into the double torus. There are 8 snarks on 28 vertices with defect three, 5 of them have unique embeddings of defect three and all embeddings are into the double torus.

We describe a snark G_{26} on 26 vertices with defect 2. The vertex set of G_{26} are integers between 1 and 26 and the adjacency lists are

```
1:   2  3  4
2:   1  5  6
3:   1  7  8
4:   1  9 10
5:   2  7  9
6:   2 11 12
```

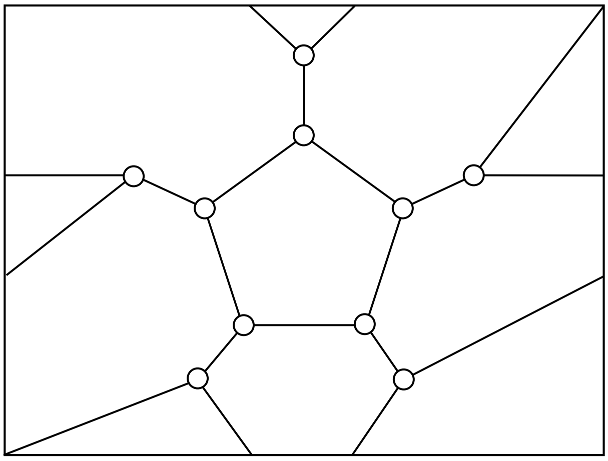

Figure 4.1: Embedding of the Petersen graph in the torus.

```
 7:    3  5 10
 8:    3 13 14
 9:    4  5 15
10:    4  7 16
11:    6 13 17
12:    6 18 19
13:    8 11 20
14:    8 18 21
15:    9 22 23
16:   10 24 25
17:   11 19 24
18:   12 14 26
19:   12 17 25
20:   13 21 22
21:   14 20 23
22:   15 20 24
23:   15 21 26
24:   16 17 22
25:   16 19 26
26:   18 23 25
```

The orientable embedding into the torus is described by the collection of facial walks

```
Face  1:     1  2  5  9  4
Face  2:     2  1  3  8 14 18 12  6
```

```
Face  3:   3  1  4 10  7
Face  4:   5  2  6 11 13  8  3  7
Face  5:   5  7 10 16 25 26 23 15  9
Face  6:   4  9 15 22 24 16 10
Face  7:  11  6 12 19 17
Face  8:   8 13 20 21 14
Face  9:  13 11 17 24 22 20
Face 10:  12 18 26 25 19
Face 11:  18 14 21 23 26
Face 12:  17 19 25 16 24
Face 13:  21 20 22 15 23
```

Another embedding of the same graph into the torus with defect two is described by the collection of facial walks

```
Face  1:   1  2  5  9  4
Face  2:   2  1  3  8 13 11  6
Face  3:   3  1  4 10  7
Face  4:   5  2  6 12 18 14  8  3  7
Face  5:   5  7 10 16 24 22 15  9
Face  6:   4  9 15 23 26 25 16 10
Face  7:   6 11 17 19 12
Face  8:  13  8 14 21 20
Face  9:  11 13 20 22 24 17
Face 10:  18 12 19 25 26
Face 11:  14 18 26 23 21
Face 12:  19 17 24 16 25
Face 13:  20 21 23 15 22
```

We note that G_{26} is cyclically 4-edge-connected. It can be constructed as a dot product of three copies of the Petersen graph.

We now describe embeddings of other snarks with low defect. We list facial cycles of all embeddings of snarks on less than 28 vertices of defect at most three.

Two embeddings of the first graph on 28 vertices with defect two.

```
Face  1:   1  2  5  9  4
Face  2:   2  1  3  8 13 17 12  6
Face  3:   3  1  4 10  7
Face  4:   5  2  6 11  8  3  7
Face  5:   5  7 10 15 22 25 18 14  9
Face  6:   4  9 14 20 23 16 21 15 10
```

```
Face  7:   11  6 12 18 25 26 28 21 16
Face  8:    8 11 16 23 19 13
Face  9:   12 17 24 27 20 14 18
Face 10:   17 13 19 26 25 22 24
Face 11:   15 21 28 27 24 22
Face 12:   19 23 20 27 28 26
```

```
Face  1:    1  2  5  9  4
Face  2:    2  1  3  8 11  6
Face  3:    3  1  4 10  7
Face  4:    5  2  6 12 17 13  8  3  7
Face  5:    5  7 10 15 21 16 23 20 14  9
Face  6:    4  9 14 18 25 22 15 10
Face  7:    6 11 16 21 28 26 25 18 12
Face  8:   11  8 13 19 23 16
Face  9:   17 12 18 14 20 27 24
Face 10:   13 17 24 22 25 26 19
Face 11:   21 15 22 24 27 28
Face 12:   23 19 26 28 27 20
```

The embedding of the second snark on 28 vertices with defect two.

```
Face  1:    1  2  5  9  4
Face  2:    2  1  3  8 13 20 27 25 18 12  6
Face  3:    3  1  4 10  7
Face  4:    5  2  6 11  8  3  7
Face  5:    5  7 10 15 19 24 28 21 14  9
Face  6:    4  9 14 17 12 18 22 15 10
Face  7:   11  6 12 17 23 16
Face  8:    8 11 16 24 19 13
Face  9:   17 14 21 25 27 23
Face 10:   13 19 15 22 26 20
Face 11:   22 18 25 21 28 26
Face 12:   16 23 27 20 26 28 24
```

The three embeddings of the Blanuša graph with defect three.

```
Face  1:    1  2  5  9  4
Face  2:    2  1  3  8 12 17 16 11  6
Face  3:    3  1  4 10  7
Face  4:    5  2  6  8  3  7
Face  5:    5  7 10 14 18 13  9
```

```
Face  6:    8  6 11 15 12
Face  7:    4  9 13 16 17 14 10
Face  8:   15 11 16 13 18
Face  9:   12 15 18 14 17
```

```
Face  1:    1  2  5  9  4
Face  2:    2  1  3  8 12 15 11  6
Face  3:    3  1  4 10  7
Face  4:    5  2  6  8  3  7
Face  5:    5  7 10 14 17 16 13  9
Face  6:    8  6 11 16 17 12
Face  7:    4  9 13 18 14 10
Face  8:   11 15 18 13 16
Face  9:   15 12 17 14 18
```

```
Face  1:    1  2  5  9  4
Face  2:    2  1  3  8  6
Face  3:    3  1  4 10  7
Face  4:    5  2  6 11 15 12  8  3  7
Face  5:    5  7 10 14 18 13  9
Face  6:    6  8 12 17 16 11
Face  7:    4  9 13 16 17 14 10
Face  8:   15 11 16 13 18
Face  9:   12 15 18 14 17
```

Two embeddings of the first snark on 24 vertices with defect 3.

```
Face  1:    1  2  6 12 18 23 21 15  9  4
Face  2:    2  1  3  7  5
Face  3:    3  1  4 10 16 11 13  8
Face  4:    2  5  9 15 14 20 24 19 13 11  6
Face  5:    7  3  8 14 15 21 22 17 16 10
Face  6:    5  7 10  4  9
Face  7:    6 11 16 17 12
Face  8:    8 13 19 23 18 20 14
Face  9:   12 17 22 24 20 18
Face 10:   22 21 23 19 24
```

```
Face  1:    1  2  6 12 18 20 14 15  9  4
Face  2:    2  1  3  7  5
Face  3:    3  1  4 10 16 11 13  8
Face  4:    2  5  9 15 21 22 24 19 13 11  6
```

```
Face  5:    7  3  8 14 20 24 22 17 16 10
Face  6:    5  7 10  4  9
Face  7:    6 11 16 17 12
Face  8:    8 13 19 23 21 15 14
Face  9:   12 17 22 21 23 18
Face 10:   20 18 23 19 24
```

The embedding of the second snark on 24 vertices with defect three.

```
Face  1:    1  2  6 11 17 23 24 19 15  9  4
Face  2:    2  1  3  7  5
Face  3:    3  1  4 10 16 21 20 14  8
Face  4:    2  5  9 15 14 20 22 18 12  6
Face  5:    7  3  8 13 11  6 12 16 10
Face  6:    5  7 10  4  9
Face  7:   13  8 14 15 19
Face  8:   11 13 19 24 18 22 17
Face  9:   16 12 18 24 23 21
Face 10:   20 21 23 17 22
```

Three embeddings of a snark on 26 vertices with defect three.

```
Face  1:    1  2  5  9  4
Face  2:    2  1  3  8 13 18 12  6
Face  3:    3  1  4 10  7
Face  4:    5  2  6 11 17 21 14  8  3  7
Face  5:    5  7 10 16 19 24 26 23 18 13 15  9
Face  6:    4  9 15 20 25 22 17 11 16 10
Face  7:   11  6 12 19 16
Face  8:   13  8 14 20 15
Face  9:   12 18 23 22 25 24 19
Face 10:   20 14 21 26 24 25
Face 11:   21 17 22 23 26
```

```
Face  1:    1  2  5  9  4
Face  2:    2  1  3  8 14 21 17 11  6
Face  3:    3  1  4 10  7
Face  4:    5  2  6 12 18 13  8  3  7
Face  5:    5  7 10 16 11 17 22 25 20 15  9
Face  6:    4  9 15 13 18 23 26 24 19 16 10
Face  7:    6 11 16 19 12
Face  8:    8 13 15 20 14
```

```
Face  9:   18 12 19 24 25 22 23
Face 10:   14 20 25 24 26 21
Face 11:   17 21 26 23 22

Face  1:    1  2  6 12 18 13 15  9  4
Face  2:    2  1  3  7  5
Face  3:    3  1  4 10 16 11 17 21 14  8
Face  4:    2  5  9 15 20 25 22 17 11  6
Face  5:    7  3  8 13 18 23 22 25 24 19 16 10
Face  6:    5  7 10  4  9
Face  7:    6 11 16 19 12
Face  8:   13  8 14 20 15
Face  9:   18 12 19 24 26 23
Face 10:   20 14 21 26 24 25
Face 11:   21 17 22 23 26
```

The defects of some particular snarks are summarized in the following Lemma.

Lemma 4.1. • *$d(P) = 5$.*

- *$d(B_1) = 3$ where B_1 is the Blanuša snark of genus 1.*
- *$d(G_{26}) = 2$.*

Figure 4.1 shows an embedding of the Petersen graph in the torus with defect 5 and Figure 4.5 show the graph B_1 embedded in the torus with defect 3.

4.2 Kochol snarks

We now prove the Grünbaum conjecture for Kochol snarks. Kochol snarks are a special class of snarks obtained as a superposition of the Petersen graph. To describe this superposition we will use the Petersen graph with the notation given in Figure 4.2.

Let G be a superposition of the Petersen graph P. If we assigned the trivial supervertex $\mathcal{S}(v)$ to a vertex $v \in V(P)$, we denote the only vertex in $\mathcal{S}(v)$ with v and call it *original vertex*. We call edges incident with original vertices *original edges*. A connected subgraph of G which is induced by nontrivial supervertices and superedges between them is called a *block*.

We will be describing cycles in G. If a cycle C contains a path $x_1 \dots x_k$ this will be denoted by $C = *x_1 \dots x_k*$. If a cycle enters a block in a supervertex $\mathcal{S}(x_2)$ from an original vertex x_1 and leaves this block from a supervertex $\mathcal{S}(y_1)$

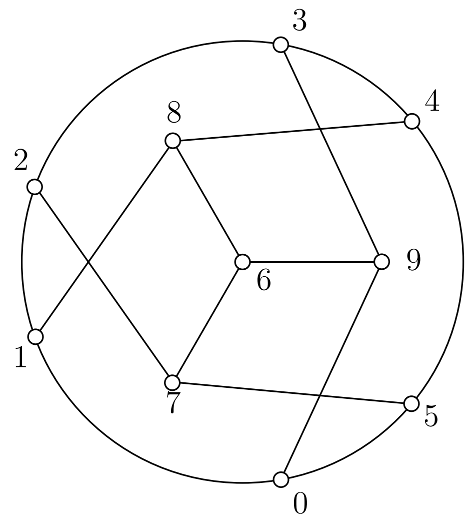

Figure 4.2: The Petersen graph.

to an original vertex y_2, this will be denoted by $C = *x_1x_2.y_1y_2*$. It is possible that $x_2 = y_1$ in which case we will sometimes write $C = *x_1x_2y_2*$. There are no original vertices on C between x_1 and y_2.

A *Kochol snark of type 1* is a proper superposition of the Petersen graph where we assign trivial supervertices to vertices 0, 3, 6, 7, 8, 9 of P (see also Figure 4.3).

Theorem 4.2. *Kochol snarks of type 1 have no orientable polyhedral embeddings.*

Proof. Let G be a Kochol snark of type 1 which is polyhedrally embedded into an orientable surface. Assume the notation from Figure 4.3.

Look at the facial cycles on edges 01 and 81. There are at least 3 distinct facial cycles on these two edges, otherwise the embedding would not be polyhedral.

We now show that there are exactly 3. Suppose we have four facial cycles $A = *01.23*$, $B = *01.27*$, $C = *81.27*$ and $D = *81.23*$. Since the embedding is polyhedral, the cycle C must be $C = 81.2768$ and the cycle A must be $A = 01.2390$. Since B already intersects cycles A and C it can not use the edge 43 or 48, therefore it must be $B = 01.2750$ and similarly $D = 81.2348$. There is another facial cycle which contains the vertex 3. It must be $F = 439675.4$ since the embedding is polyhedral. Since the embedding is orientable, we can consistently orient the facial cycles. Suppose that F is oriented so that the edges 43 and 67 are in the direction of orientation. Then the cycle D is oriented

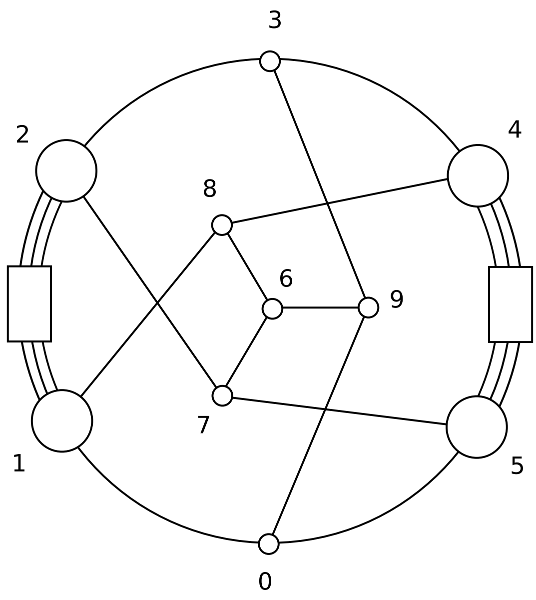

Figure 4.3: A Kochol graph of type 1.

so that the edges 34 and 81 are in the direction of the orientation. Finally the cycle C is directed so that edges 18 and 67 are in the direction of the orientation. This is a contradiction since facial cycles C and F are oriented in the same direction on the edge 67.

By symmetry we have exactly 3 facial cycles at edges from other supervertices. The facial cycles which contain original edges therefore induce an embedding of the underlying Petersen graph. Since the embedding of G is orientable we have a consistent orientation of cycles. We use this orientation in the induced embedding of P. Since facial walks are oriented consistently on original edges of G, this orientation is consistent on all edges of P and so the embedding is orientable.

Suppose that in the induced embedding of the Petersen graph we have two facial cycles A and B which have $k + 1$ edges in common. This implies that at least k of these edges correspond to superedges in G. It follows that the induced embedding of the Petersen graph has defect at most 2, since in G we have two superedges. This is a contradiction with Lemma 4.1. □

A *Kochol snark of type 2* is a proper superposition of the Petersen graph where we assign trivial supervertices to vertices 6, 7, 8, 9 and additionally trivial superedges to edges $(5, 0)$ and $(1, 2)$ (see also Figure 4.4). Note that Kochol snarks of type 1 have cyclic 4-cuts, but Kochol snarks of type 2 are cyclically 5-edge-connected.

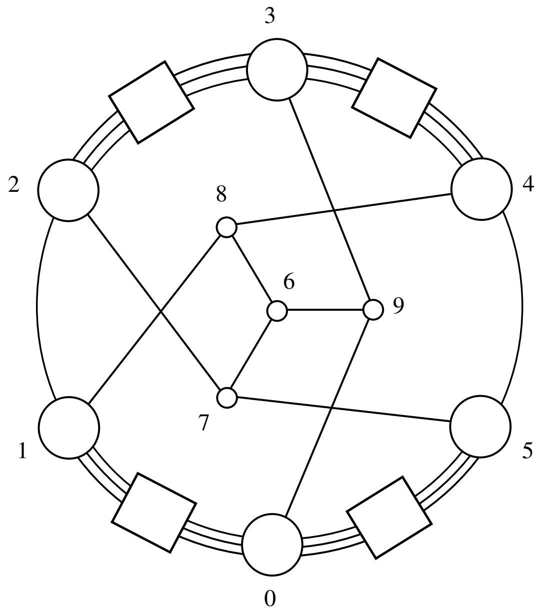

Figure 4.4: A Kochol snark of type 2.

If a cycle C enters a block on a supervertex x_2 from an original vertex x_1, then uses some vertices from a supervertex x_3 and then leaves the block from a supervertex x_3 to an original supervertex x_4, this will be denoted by $C = *x_1.x_2.x_3*$.

Theorem 4.3. *Kochol snarks of type 2 have no orientable polyhedral embeddings.*

Proof. Assume that a Kochol snark of type 2 has a polyhedral embedding into an orientable surface. Similarly as in the proof of the previous theorem we first show that this embedding induced an embedding of the underlying Petersen graph. Call supervertices $0, 1, 2$ with superedges between them the *lower block* and supervertices $3, 4, 5$ with superedges between them the *upper block*.

Assume that on edges 75 and 45 we have four distinct facial cycles, $A = *75.0*$, $B = *75.0*$, $C = *45.0*$ and $D = *45.0*$. Since the embedding is polyhedral, there must be two distinct facial cycles which enter the lower block on the edge 90. This implies that not all four of A, B, C, D can leave the lower block on edges 12 and 18.

CASE 1: Assume that only a facial cycle, which contains the edge 75, say A, leaves the lower block on the edge 09. Since the embedding is polyhedral, the face *967* can not be distinct from A, so we have $A = 75.0967$ and

$B = *275.0.1*$. We can assume $C = *45.0.12*$ and $D = *45.0.1.8*$. The cycle B can not leave the lower block on the edge 18 since then there would be a facial cycle at vertex 6 which would intersect it twice. So we have $B = 275.0.12$. The cycle C can not leave the upper block on edge 48 since it already intersects cycle D and also not on edge 39 since it would have to continue on the path 3968. Similarly it can't leave on the edge 27, so it must be $C = 45.0.12.3.4$. We have another cycle F which enters the lower block on the edge 81, $F = *81.093*$. This cycle will intersect with the cycle which contains the path 869 twice, a contradiction with the assumption that the embedding is polyhedral.

CASE 2: Assume that only a facial cycle, which contains the edge 45, say C leaves the lower block on the edge 09. So $C = *45.09*$, $D = *45.0.1*$, $A = *75.0.18*$ and $B = *75.0.12*$. Since the embedding is polyhedral we have $A = 75.0.1867$ and $B = 75.0.127$. If we have $D = *45.0.12*$ then we must have another facial cycle $F = *90.184*$ which will intersect the facial cycle which contains the path 869 twice, a contradiction. So we have $D = 45.0.184$ and $C = 45.093$. There is a facial cycle $F = *21.096*$. If we have $F = *21.0967*$, then F and B intersect twice, and if we have $F = *21.0968*$ then cycles A, B and F can not be consistently oriented.

CASE 3: Assume there that two cycles, say A and C, leave the lower block on the edge 09. Again we have $A = 75.0967$, $B = *275.0.1*$, $C = *45.0.93*$ and $D = *45.0.1*$. If B leaves the lower block on the edge 18, then it is $B = *275.0.184*$ and it intersects the facial cycle, which contains the path 867, twice. So we have $B = *275.0.12*$ and $D = 45.0.184$. Now we have a facial cycle $F = *218693*$ and we get a contradiction since cycles C, D and F can not be consistently oriented.

So we have that there are exactly 3 facial cycles on edges 45 and 75. By symmetry the same holds for edges at supervertices 1, 2 and 4. Since the embedding of G is polyhedral and orientable we get that facial cycles which contain the original edges of G induce an orientable embedding of P, which has defect at most 4. This is again a contradiction to Lemma 4.1. □

4.3 Defect and Grünbaum conjecture

Let $M = (V, E, S)$ be a multipole. A *combinatorial embedding* of M is an assignment of rotations to vertices V. As with combinatorial embeddings of graphs, we can define the collection of facial walks $\mathcal{F}$, which consists of closed walks and walks which start and end at a connector. Again we can describe the embedding of M by specifying $\mathcal{F}$. If in the definition of the defect we replace graphs with multipoles, we get the definition of a defect of a multipole.

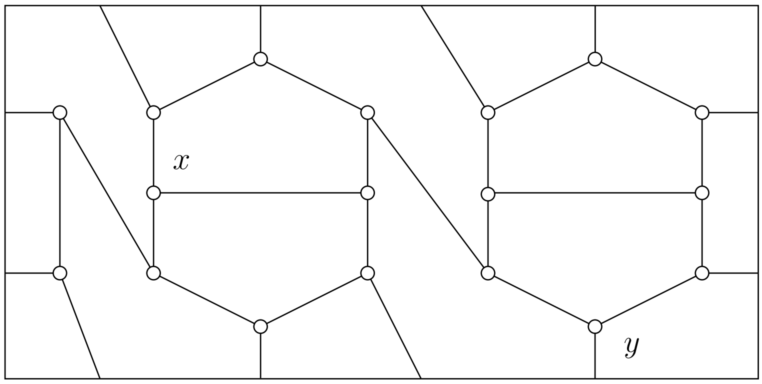

Figure 4.5: The Blanuša graph embedded in the torus with defect 3.

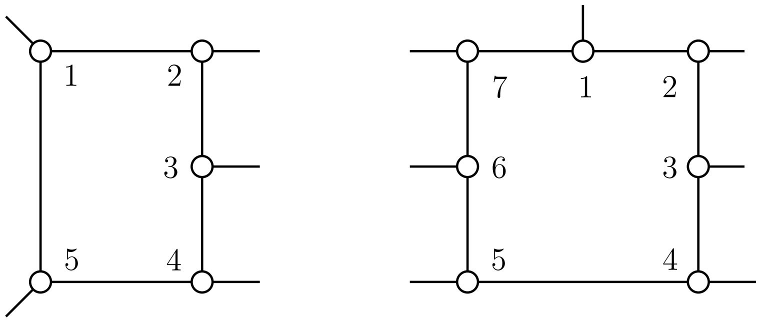

Figure 4.6: Supervertices used for replacing edges.

Suppose we have an orientable embedding of a superedge $M = (V, E, S_1 \cup S_2)$. Let the connectors be $S_1 = \{(u_1), (u_2), (u_3)\}$ and $S_2 = \{(v_1), (v_2), (v_3)\}$. Suppose that in the consistent orientation of facial walks we have walks $W_1 = u_1P_1v_1$, $U_1 = u_2R_1u_1$, $U_2 = u_3R_2u_2$, $W_2 = v_3P_2u_3$, $V_1 = v_1Q_1v_2$ and $V_2 = v_2Q_2v_3$. Suppose further that walks P_1 and P_2 are disjoint. An embedding as described is called a *nice embedding* of a superedge.

Take the Blanuša snark B_1 embedded in the torus and remove vertices x and y (see Figure 4.5) to obtain a proper superedge B_1'. Note that the embedding of B_1 in the torus induces a nice embedding of B_1' with defect 1. Using a computer we find that

Lemma 4.4. *Blanuša superedge B_1' obtained by removing vertices x and v from B_1 has defect 1.*

We now describe what we mean by replacing an edge in an embedded graph with a nicely embedded superedge. Suppose Π is an embedding of G

and $e = (x, y) \in E(G)$ is an edge. Denote the neighbors of x with $\{y, x_1, x_2\}$ and the neighbors of y with $\{x, y_1, y_2\}$ so that in the embedding Π there are facial walks $C_1 = *x_1xyy_1*$, $C_2 = *y_1yy_2*$, $C_3 = *y_2yxx_2*$ and $C_4 = *x_2xx_1*$.

We will use the (1,1,3)-supervertex $\mathcal{V}$ from the left of Figure 4.6 where the connectors are $\{(1)\}, \{(5)\}$ and $\{(2), (3), (4)\}$. To vertices x and y we assign $\mathcal{V}(x)$ and $\mathcal{V}(y)$, both copies of $\mathcal{V}$, to e we assign the nicely embedded superedge (with the notation defined at the beginning of this section) and to all other vertices and edges we assign trivial supervertices and superedges. We denote the vertices in $\mathcal{V}(y)$ with $1'$, $2'$, ... to distinguish them from the vertices in $\mathcal{V}(x)$. In $\mathcal{V}(x)$ we assign connectors $\{(1)\}, \{(5)\}, \{(2), (3), (4)\}$ to xx_1, xx_2, e and in $\mathcal{V}(x)$ we assign connectors $\{(1')\}, \{(5')\}, \{(2'), (3'), (4')\}$ to yy_1, yy_2, e. In the superposition we add edges $(2, u_1), (3, u_2), (4, u_3)$ and $(v_1, 2'), (v_2, 3'), (v_3, 4')$.

This superposition has an induced embedding defined by facial walks $\mathcal{F}$ defined as follows. Take all facial walks of Π which do not contain vertices x and y and modify facial walks C_i, $i = 1, 2, 3, 4$, to get walks C'_i, $i = 1, 2, 3, 4$, as follows: $C'_1 = *x_121u_1P_1v_12'1'y_1*$, $C'_2 = *y_11'5'y_2*$, $C'_3 = *y_25'4'v_3P_2u_345x_2*$ and $C'_4 = *x_251x_1*$. Add walks 543215 and $1'2'3'4'5'1'$. Add all closed walks in the embedding of the superedge M. Add walks $23u_2R_1u_12$, $34u_3R_2u_23$, $3'2'v_1Q_1v_23'$ and $4'3'v_2Q_2v_34'$. We have described an orientable embedding of G'. If in the embedding Π the cycles C_1 and C_2 are distinct then the bad edges in the induced embedding of G' are bad edges of Π minus possibly e and bad edges in the embedding of the superedge M.

Using the $(3, 1, 3)$-supervertex from Figure 4.6 we can similarly replace all edges on a facial cycle C in G. Again the bad edges in the induced embedding of the superposition are bad edges in the original graph minus possibly the edges of C and the bad edges in superedges.

Lemma 4.5. *The following statements are equivalent:*

1. *Grünbaum conjecture is true,*
2. *all snarks have defect at least 2,*
3. *all nicely embedded proper superedges have defect at least 1.*

Proof. First we prove that 1 is equivalent to 3.

If the Grünbaum conjecture is false, then there exists an embedding of a snark with defect 0. If we remove two vertices from one facial cycle in the embedding we get a nicely embedded superedge with an induced embedding of defect 0.

Suppose we have a nicely embedded superedge with defect 0. Take the embedding of P in the torus and replace each edge along the unique 9-cycle with the nicely embedded proper superedge to get a snark with defect 0.

It is clear that 2 implies 1. The Grünbaum conjecture implies that snarks have defect at least 1. We show that 3 implies that there is no snark with defect precisely 1, which completes the proof.

Suppose Π is an embedding of a snark G with defect 1. First we show that all facial walks are cycles and that there are two facial cycles C and D which have two edges $e = xy$ and $f = uv$ in common and that e and f are on distance at least 2 along C and D.

If there is a vertex v in G which appears twice along a facial walk W, then there is an edge incident with v which appears twice along W and contributes 1 to the defect of Π. There is another facial walk which contains v and it intersects W in at least two edges incident with v. So the defect of Π is at least 2, which shows that all facial walks are cycles.

There are two facial cycles C and D which intersect at two edges e and f. Suppose that e and f are at distance at most 2 on C. Edges e and f can not be adjacent since in this case C and D could not be facial cycles in an embedding of G. If they are at distance 1 on C, assume y and u are adjacent and there are vertices $x_1 \neq x, u$ and $v_4 \neq y, v$ such that x_1 is adjacent to y and v_1 is adjacent to u. Cycle C contains the path $xyuv$ and cycle D contains paths x_1yx and vuv_1. There is another facial cycle which contains the path v_1uyx_1 and we get that the defect of the embedding is more than 1.

Now we can choose two vertices u and v on C which are not incident with e or f and u and v separate e and f on C. Since the defect is 1, vertices u and v are not on the cycle D. Remove vertices u and v from G to obtain a superedge. This is a nicely embedded superedge with defect 0. □

If the Grünbaum conjecture is true then we get lower bounds for the defect of snarks or superedges. We now prove that these bounds are best possible since we can construct infinitely many snarks (superedes) with defect k for any $k \geq 2$ ($k \geq 1$).

Theorem 4.6. *For each $k \geq 2$ there exist infinitely many snarks with defect precisely k. For each $k \geq 1$ there exist infinitely many nicely embedded superedges with defect precisely k.*

Proof. Suppose we have an embedding Π of a snark G with defect k in which all facial walks are cycles and there are k bad edges which form an independent set. Let B_1' be the nicely embedded superedge obtained from the Blanuša snark by removing vertices x and y. Replace each bad edge in G by B_1' to obtain an embedded snark G'. By construction we see that the defect of G' is at most k. By lemma 4.4 each superedge contributed at least 1 to the defect of G', so we get that the defect of G' is precisely k.

Suppose that in G we can choose $k+1$ edges such that k of them are bad and one of them is good and they form an independent set of edges. If we replace each edge with B_1' we get a snark with the defect precisely $k+1$.

Note that if we take the snark G_{26} embedded into the torus we can perform both operations. Also it is easy to see that after we have performed one operation, the embedding of the superposition is such that allows us to perform both operations again. Thus for any $k \geq 2$ we can generate infinitely many snarks with defect precisely k.

Let M be a nicely embedded superedge such that all semiedges are good. Then we can perform above operations on M to obtain a nicely embedded superedge M' such that all semiedges of M' are good. Thus starting with the nice embedding of B_1' we can for each $k \geq 1$ construct infinitely many nicely embedded proper superedges with defect precisely k. □

Since the defect is a measure for how far a cubic graph is from having a polyhedral embedding, the last theorem shows that there are arbitrarily large snarks with nice embeddings (that is with embeddings with low defect). Similar measures have been introduced in the literature (for instance [20]) to measure how far a snark is from having a 3-edge-coloring. In the following we introduce resistance which is a measure for how far a graph is from having a 3-edge-coloring and prove an implication of the Grünbaum conjecture to the relation of defect and resistance. We show that if resistance is high then the defect is high. This implies that graphs which are far from having a 3-edge-coloring are do not have nice embeddings.

Suppose G is a cubic graph and let c be a 4-edge-coloring of G where we allow two edges of color 4 to be adjacent. The coloring c is minimum coloring if the number of edges colored with the color 4 is minimum possible among all such 4-edge-colorings of G. The number of edges colored with the color 4 in a minimum coloring is called the *resistance*, $r(G)$, of G, [20]. Note that in the minimum coloring the edges of color 4 can not be adjacent (since in this case the coloring is not minimum) and so the minimum coloring is a proper 4-edge-coloring of G. A cubic graph is not 3-edge-colorable if and only if its resistance is at least 1.

Suppose Π is an embedding of a cubic graph G. A vertex is called a *bad vertex* if in the embedding Π it appears three times along a facial walk. Denote the number of bad vertices in the embedding Π with $d_v(\Pi)$. We define the *modified defect* $d'(\Pi)$ of the embedding Π with

$$d'(\Pi) = d(\Pi) + 2d_v(\Pi).$$

and the modified defect of the graph G with

$$d'(G) = \min\{d'(\Pi) \mid \Pi \text{ an orientable embedding of } G\}.$$

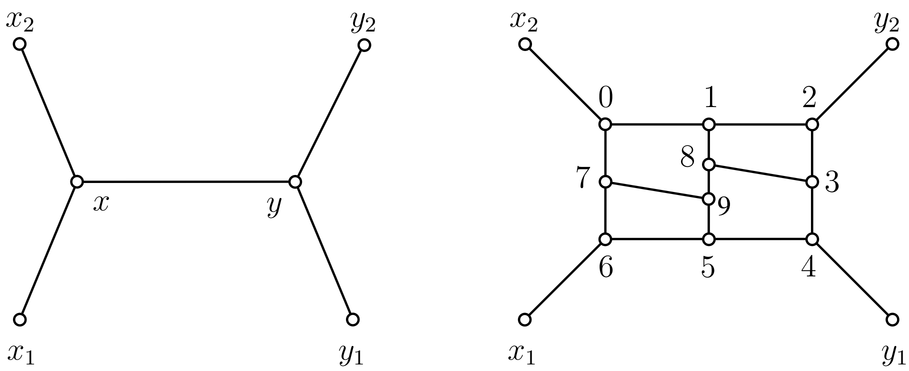

Figure 4.7: Thickening an edge.

Obviously for each graph G we have $d'(G) \geq d(G)$ and the Grünbaum conjecture is equivalent to the statement that $d'(G) > 0$ for every snark G. Stated with resistance, the Grünbaum conjecture is equivalent to the statement that for every graph G, $d'(G) > 0$ if $r(G) > 0$. The following theorem gives a stronger implication.

Theorem 4.7. *The following statements are equivalent:*

1. *the Grünbaum conjecture is true,*
2. *for all snarks G we have $d'(G) \geq \frac{r(G)}{2}$.*

Proof. It is clear that 2 implies 1. We show that 1 implies 2.

Suppose 2 is false. We have a we have a snark G which has a polyhedral embedding into an orientable surface with defect $2d'(G) < r(G)$.

We will construct a sequence of graphs $G_0 = G, G_1, G_2, \ldots, G_k$ such that $d'(G_i) > 0$ for $i < k$, $d'(G_k) = 0$, $d'(G_i) \leq d'(G_{i-1}) - 1$ for $i = 1, \ldots, k$ and $r(G_i) \geq r(G_{i-1}) - 2$ for $i = 1, \ldots, k$. The inequality $d'(G_i) \leq e'(G_{i-1}-1$ implies that $d'(G) \geq k$. By $2d'(G) < r(G)$ we have $r(G) > 2k$. Now the inequality $r(G_i) \geq r(G_{i-1}) - 2$ implies $r(G_k) > 0$. So G_k is a snark which has a polyhedral embedding and is therefore a counter-example for the Gruünbaum conjecture.

Suppose we have an embedding of G_i. We replace a bad edge $e = (xy)$ in the embedding of G_i with a graph on 10 vertices to get a graph G_{i+1} with an induced embedding of smaller modified defect (see Figure 4.7). In the embedding of G_i we can assume we have facial walks W_1, W_2, W_3, W_4 which contain paths x_1xyy_1, y_1yy_2, y_2yxx_2 and x_2xx_1 respectively, where some of W_1, W_2, W_3, W_4 may be equal. To define an embedding of G_{i+1} we take facial walks of the embedding of G_i, replace paths x_1xyy_1, y_1yy_2, y_2yxx_2 and x_2xx_1 on walks W_1, W_2, W_3, W_4 with paths x_1654y_1, y_1432y_2, y_2210x_2 and x_2076x_1

and add facial cycles 01870, 123981, 34593 and 567895. By appropriately choosing the bad edge e we can guarantee that the modified defect decreases by at least one.

We distinguish 4 choices for the bad edge e. At each step we can make choice 3 only if we can not make choices 1 or 2 and can make choice 4 if we can not make choices 1, 2, or 3. As long as the defect of the embedding is more than 0 we can make one of the choices.

Choice 1: bad edge $e = (x, y)$ where x and y are bad vertices. In this case $W_1 = W_2 = W_3 = W_4$.

To calculate the modified defect of the embedding of G_{i+1} observe that bad edges in the embedding of G_{i+1} are bad edges of the embedding of G_i minus e plus bad pairs $\{(70), (01)\}$, $\{(12), (23)\}$, $\{(34), (45)\}$ and $\{(56), (67)\}$. So $d(\Pi(G_{i+1})) = d(\Pi(G_i)) - 1 + 4 = d(G_i) + 3$. Since we removed two bad vertices x and y and created no new bad vertices we have $d_v(\Pi(G_{i+1})) = d_v(\Pi(G_i)) - 2$ and therefore the modified defect is $d'(\Pi(G_{i+1})) \leq d'(\Pi(G_i)) - 1$. We conclude that $d'(G_{i+1}) \leq d'(G_i) - 1$.

Choice 2: bad edge with $e = (x, y)$ where x is a bad vertex and y is not. In this case $W_1 = W_3 = W_4$ and $W_2 \neq W_1$.

The defect of the induced embedding of G_{i+1} is $d(\Pi(G_{i+1}) = d(\Pi(G_i)) - 1 + 2 = d(G_i) + 1$ and $d_v(\Pi(G_{i+1}) = d_v(\Pi(G_i)) - 1$. Therefore the modified defect is $d'(\Pi(G_{i+1})) = d'(\Pi(G_i)) - 1$. We conclude that $d'(G_{i+1}) \leq d'(G_i) - 1$.

Choice 3: bad edge $e = (x, y)$ which appears twice along one facial walk. Since we can not make choices 1 or 2 we can assume that $W_1 = W_3$ and $W_2 \neq W_1$ and $W_4 \neq W_1$ (but it is possible that $W_2 = W_4$).

In the embeddings of G_i and G_{i+1} there are no bad vertices. The defect of the embedding of G_{i+1} is $d(\Pi(G_{i+1}) = d(\Pi(G_i)) - 1$ and therefore $d'(G_{i+1}) \leq d'(G_i) - 1$.

Choice 4: $e = (x, y)$ which does not appear twice along one facial walk. Since we can not make choices 1, 2, or 3 it is only possible that maybe $W_2 = W_4$.

In the embeddings of G_i and G_{i+1} there are no bad vertices. The defect of the embedding of G_{i+1} is $d(\Pi(G_{i+1}) = d(\Pi(G_i)) - 1$ and therefore $d'(G_{i+1}) \leq d'(G_i) - 1$.

It remains to show that $r(G_{i+1}) \geq r(G_i) - 2$. Suppose we have a minimum coloring c of the graph G_{i+1}. We define a coloring c' of G_i as follows: $c'(e) = c(e)$ if e is not incident with x or y, and we let $c'(x_1x) = c(x_16)$, $c'(yy_2) = c(2y_2)$. We can color the edge e with one of the colors 1, 2, 3 and color edges x_20 and y_14 with color 4. So $r(G_i) \leq r(G_{i+1}) + 2$. □

The last theorem implies that if Grünbaum conjecture is true, we can bound $d'(G)$ from below with $r(G)$, which would be a very strong connection between the defect, which is a topological property, and resistance, which is a coloring property. We conclude with the following problems, which could be considered as a weakening of the Grünbaum conjecture:

Problem 4.8. *Is there a nondecreasing function* f *with* $\lim_{x\to\infty} f(x) = \infty$, *such that* $d'(G) \geq f(r(G))$ *for all cubic graphs.*

Problem 4.9. *Find a constant* $c > 0$ *such that* $d'(G) \geq c\,r(G)$ *for all cubic graphs.*

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Razširjeni povzetek

Definicije

Graf G je podan s parom množic $V(G)$ in $E(G)$. Množica $V(G)$ je končna množica vozlišč ali točk grafa G, $E(G)$ pa množica povezav. *Povezava* grafa G je množica $\{u, v\}$, krajše uv, kjer sta $u, v \in V(G)$ vozlišči grafa G. Vozlišči u in v sta *povezani*, če je $e = uv \in E(G)$. Vozlišču v rečemo *soseda* točke u. Vozlišči u in v sta *krajišči* povezave e. Številu sosed vozlišča v rečemo *stopnja vozlišča*. Največjo stopnjo vozlišča grafa G označimo z $\Delta(G)$. Za povezavi, ki vsebujeta kako skupno vozlišče, rečemo da sta *sosednji vozlišči*. Vsi grafi so *enostavni*, torej ne vsebujejo večkratnih povezav niti zank. Če v grafu dovolimo večkratne povezave ali zanke, govorimo o *multigrafu*.

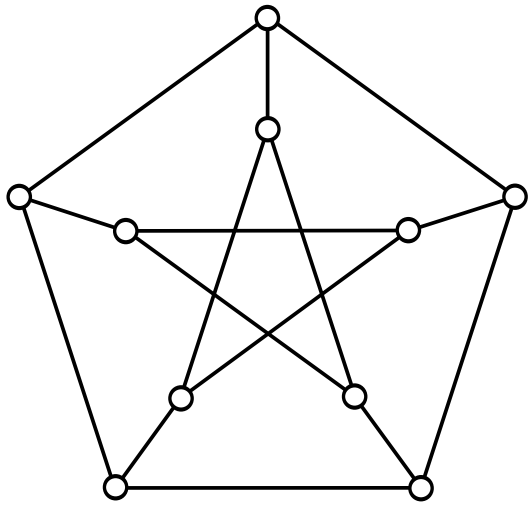
k*-barvanje povezav* grafa G je preslikava $c : E(G) \to \{1, 2, \ldots, k\}$, ki sosednjima povezavama priredi različni števili. Številom $\{1, 2, \ldots, k\}$ rečemo *barve*. Najmanjšemu številu k, za katerega obstaja k-barvanje povezav grafa G, rečemo *kromatični indeks* grafa G in ga označimo s $\chi'(G)$. Za enostavne grafe velja:

Izrek 1 (Vizing). *Za enostaven graf G je $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$.*

Grafom, za katere velja $\chi'(G) = \Delta$, rečemo *grafi razreda 1*, grafom, za katere velja $\chi'(G) = \Delta(G) + 1$, pa rečemo *grafi razreda 2*.

Če je stopnja vsakega vozlišča grafa G enaka k, je graf G k*-regularen*. 3-regularnim grafom rečemo *kubični grafi*.

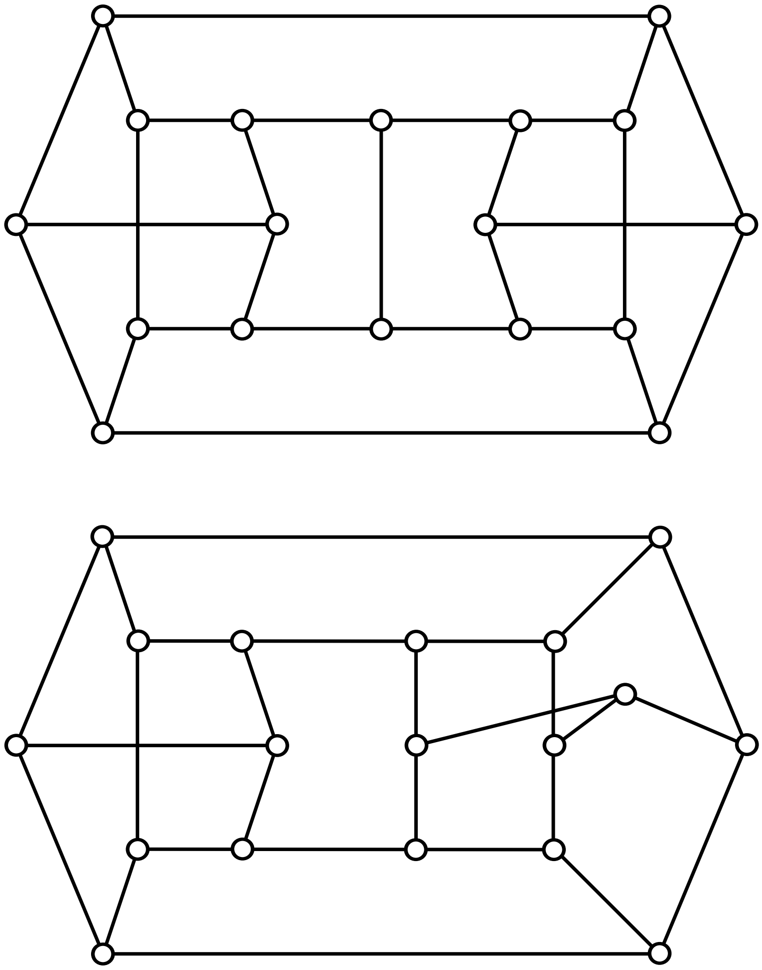
Če za vsaki vozlišči $u, v \in V(G)$ obstaja pot $P = v_0v_1 \cdots v_n$, kjer sta točki v_i in v_{i+1}, $i = 0, \ldots, n-1$ povezani, in je $v_0 = u$ ter $v_n = v$, je graf G *povezan*. Maksimalni povezani podmnožici grafa G rečemo *komponenta grafa* G. Za podmnožico $S \subset E(G)$ označimo z $G - S$ graf z množico vozlišč $V(G)$ in množico povezav $E(G) \setminus S$. Podmnožica $S \subset E(G)$ je *prerez*, če ima graf $G - S$ več komponent kot graf G. Če je velikost vsakega prereza povezanega grafa G vsaj k, je G *povezavno k-povezan*. Podmnožica $S \subset E(G)$ je *ciklični prerez*, če ima graf $G - S$ vsaj dve komponenti, ki vsebujeta cikel. Povezan graf G je *ciklično k-povezan*, če ima vsak ciklični prerez grafa G velikost vsaj k.



Slika 4.8: Petersenov graf.

Kubični graf razreda 2, ki je 3-povezan, ciklično 4-povezan z dolžino najkrajšega cikla vsaj 5, se imenuje *snark*. Ime so snarki dobili po pesmi *The Hunting of the Snark* avtorja Lewisa Carrolla, v kateri so snarki pošasti, ki jih je zelo težko najti. Najmanjši snark je Petersenov graf (glej sliko 4.8), ki ima 10 vozlišč. Odkrili so ga konec 18. stoletja [2]. Naslednja odkrita snarka sta Blanuševa snarka, ki ju je leta 1946 odkril hrvaški matematik Blanuša [3] (glej sliko 4.9). To so edini trije snarki z manj kot 20 točkami.

Prva znana neskončna družina snarkov so bili snarki, ki jih dobimo kot 4-vsote manjših snarkov, odkrita pa je bila v sedemdesetih letih prejšnjega stoletja [7]. Denimo da sta G_1 in G_2 kubična grafa. Naj bosta e, f nesosednji povezavi grafa G_1 in u, v sosednji vozlišči grafa G_2. Označimo z v_1, v_2 krajišči povezave e in z v_3, v_4 krajišči povezave f. Sosedi točke u, različni od v, označimo z u_1, u_2 in sosedi točke v, različni od u, označimo z u_3 in u_4. Grafu G_1 odstranimo povezavi e, f, grafu G_2 ostranimo vozlišči u, v in dodamo povezave v_iu_i, $i = 1, 2, 3, 4$. Dobimo kubičen graf $G = G_1 \cdot G_2$, ki ga imenujemo *4-vsota grafov* G_1 in G_2. Prerezu $\{v_iu_i \mid i = 1, 2, 3, 4\}$ rečemo *prerez 4-vsote*. Če sta G_1 in G_2 snarka, potem je njuna 4-vsota tudi snark. Velja tudi obrat: če ima snark G ciklični prerez S velikosti 4, potem obstajata taka grafa G_1 in G_2, da je $G = G_1 \cdot G_2$, vsaj eden od G_1 in G_2 je snark in S prerez 4-vsote G. Očitno 4-vsota grafov ni enolično določena. Če za G_1 in G_2 vzamemo dve kopiji Petersenovega grafa, lahko konstruiramo dve neizomorfni 4-vsoti. Izkaže



Slika 4.9: Blanuševa grafa.

se, da sta to ravno Blanuševa grafa. 4-vsoto pripisujejo Isaacsu, jo je pa pred njim opisal že ruski matematik Titus, a njegov članek na zahodu ni poznan.

Isaacs je opisal še družino ciklično 6-povezanih snarkov, ki jih imenujemo *cvetni snarki* (glej sliko 4.10). Cvetni snark J_{2k+1} je graf z množico vozlišč

$$V(J_{2k+1}) = \{a_i, b_i, c_i, d_i \mid i = 0, \ldots, 2k\}$$

in množico povezav

$$E(J_{2k+1}) = \{a_i a_{i+1}, a_i b_i, b_i c_i, b_i d_i, c_i d_{i+1}, d_i c_{i+1} \mid i = 0, \ldots, 2k\},$$

kjer so indeksi vzeti po modulu $2k + 1$.

Naslednjo neskončno družino je odkril Goldberg [11]. *Goldergov graf* G_{2k+1} (glej sliko 4.10) je graf z množico vozlišč

$$V(G_{2k+1}) = \{a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i \mid i = 0, \ldots, 2k\}$$

in množico povezav

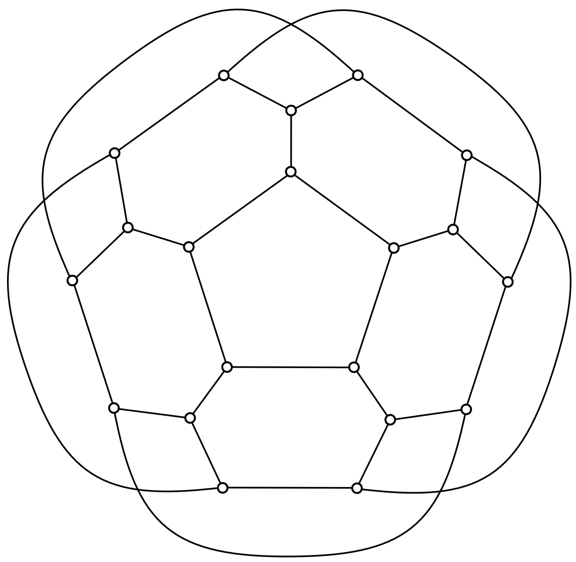
$$\begin{aligned} E(G_{2k+1}) &= \{a_i a_{i+1}, a_i b_i, b_i c_i, b_i d_i, c_i e_i, c_i g_i, \\ &\quad d_i f_i, d_i h_i, g_i h_i, e_i f_i, f_i e_{i+1}, g_i h_{i+1} \mid i = 0, \ldots, 2k\}, \end{aligned}$$

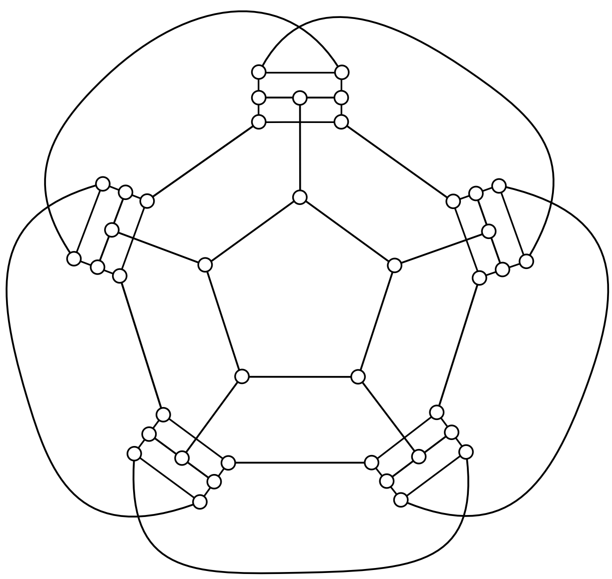
kjer so indeksi vzeti po modulu $2k + 1$.

Cvetni in Goldbergovi snarki so konstruirani tako, da liho število podgrafov Y_i oziroma T_i, induciranih na vozliščih $\{a_i, b_i, c_i, d_i\}$ oziroma $\{a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i\}$ ciklično povežemo med seboj. Če pri definiciji cvetnih oziroma Goldbergovih snarkov ne zahtevamo, da imamo liho število teh podgrafov, dobimo splošnejše grafe J_k in G_k. Grafi J_{2k} in G_{2k} so razreda 1.

Vzemimo družino poligonov s stranicami dolžine 1, ki imajo skupaj sodo število stranic $\sigma_1, \ldots, \sigma_{2n}$. Vsaki stranici izberemo orientacijo tako, da si izberemo začetno oglišče stranice. Izberemo si particijo stranic na pare. Konstruirajmo ploskev tako, da identificiramo stranice skladno z izbrano orientacijo (začetne točke identificiramo z začetnimi točkami). Dobimo ploskev S. Grafu, ki ga definirajo oglišča ploskve S in stranice kot povezave, rečemo *vložen graf*. *Celična vložitev* grafa G je je vložen graf G', izomorfen grafu G. Začetne poligone imenujemo *lica vložitve*. Lica identificiramo s sprehodi, definirani z obhodi lic.

Po klasifikaciji sklenjenih ploskev je vsaka ploskev izomorfna natanko eni od ploskev S_g (orientabilni ploskvi roda g) oziroma N_g (neorientabilni ploskvi roda g). *Orientabilni rod* $g(G)$ grafa G je najmanjši g, za katerega obstaja vložitev grafa G v ploskev izomorfno S_g. *Neorientabilni rod* $\tilde{g}(G)$ grafa G je najmanjši g, za katerega obstaja vložitev grafa G v N_g. *Eulerjeva karakteristika*





Slika 4.10: Cvetni snark J_5 (zgoraj) in Goldbergov snark G_5 (spodaj).

orientabilne ploskve S_g je $\epsilon(S_g) = 2g$, Eulerjeva karakteristika neorientabilne ploskve N_g pa je $\epsilon(N_g) = g$.

Vložitev grafa G je *poliedrska*, če je vsako lice cikel in če sta vsaki dve različni lici bodisi disjunktni, se sekata v natanko enem vozlišču ali pa se sekata v natanko eni povezavi. Vložitev kubičnega grafa je poliedrska, če je vsako lice cikel in če sta vsaki dve različni lici disjunktni ali pa se sekata v natanko eni povezavi.

Motivacija za študij vložitev snarkov prihaja iz poskusov dokaza izreka štirih barv. Izrek štirih barv pravi, da lahko vozlišča vsakega ravninskega grafa brez zank pobarvamo s štirimi točkami tako, da sta vsaki sosednji vozlišči pobarvani z različnima barvama. Tutte je pokazal, da je izrek štirih barv ekvivalenten trditvi, da ima vsak 3-povezan kubičen graf G v ravnini kromatični indeks $\chi'(G) = 3$.

Izrek štirih barv trdi, da snarki niso ravninski grafi. Snarke lahko vložimo v ploskve višjega roda, vendar imajo vse znane vložitve lice, ki vsebuje kako povezavo dvakrat, ali pa dve lici, ki se sekata v več kot eni povezavi. Torej vložitve niso poliedrske. Grünbaum je leta 1969 podal hipotezo

Hipoteza 2 (Grünbaum). *Če ima kubičen graf poliedrsko vložitev v orientabilno ploskev, potem je razreda 1.*

Grünbaumova hipoteza je posplošitev izreka štirih barv.

Rod snarkov

Rod snarkov sta študirala Tinsley in Watkins [12]. Pokazala sta, da je orientabilni rod cvetnih snarkov enak $g(J_{2k+1}) = k$. V prvem poglavju podamo krajši dokaz njunega rezultata in hkrati izračunamo neorientabilni rod cvetnih snarkov.

Izrek 3. *Orientabilni rod cvetnega snarka J_{2k+1} je $g(J_{2k+1}) = k$. Neorientabilni rod cvetnega snarka J_{2k+1} je $\tilde{g}(J_{2k+1}) = 2k-1$. Orientabilni rod grafa J_{2k} je $g(J_{2k}) = k-1$, neorientabilni rod pa $\tilde{g}(J_{2k}) = 2k-2$.*

Tinsley in Watkins sta podala zgornjo mejo za orientabilni rod Goldbergovih snarkov. Pokažemo, da je njuna meja v resnici orientabilni rod Goldbergovih snarkov. Določimo še neorientabilni rod Goldbergovih grafov.

Izrek 4. *Orientabilni rod Goldbergovega grafa G_k je $g(G_k) = k-1$. Neorientabilni rod Goldbergovega grafa G_k je $\tilde{g}(G_k) = k$.*

Težji del dokaza zadnjih dveh izrekov je dokaz spodnje meje za rod. V obeh primerih pri dokazu omejimo število lic, ki jih lahko imamo v vložitvah, in tako dobimo mejo za rod s pomočjo Eulerjeve formule. Lica razdelimo na *lokalna* in *globalna* lica in pokažemo, da v vložitvah ne moremo imeti veliko lokalnih lic.

V istem članku sta Tinsley in Watkins postavila hipotezo o orientabilnem rodu 4-vsot Petersenovih snarkov. S P^n označimo 4-vsoto n kopij Petersenovega grafa. Tinsley in Watkins sta domnevala, da je $g(P^n) = n - 1$. Hipoteza je bila ovržena v [21], kjer so avtorji pokazali, da ima eden od Blanuševih snarkov rod 1, drugi pa 2. Rod je torej lahko višji od domnevanega. Pokažemo, da je lahko tudi veliko manjši od domnevanega.

Izrek 5. *Za vsak $n > 0$ obstaja 4-vsota n kopij Petersenovega grafa, ki ima rod 1.*

Pri konstrukciji $P^n = P \cdot P^{n-1}$ je lahko rod grafa P^n enak rodu grafa P^{n-1}, ali pa se rod poveča za 1. Raziščemo pogoje, pri katerih se rod 4-vsote poveča in pogoje, pri katerih se rod ne spremeni. Tako lahko konstruiramo 4-vsoto n kopij Petersenovega grafa, za katero lahko natančno povemo njen orientabilni rod.

Izrek 6. *Za vsako celo število k, $1 \leq k \leq n$ obstaja 4-vsota n kopij Petersenovega grafa P^n, ki ima rod $g(P^n) = k$.*

Na koncu pokažemo še meje za orientabilni rod 4-vsote poljubnih kubičnih grafov.

Izrek 7. *Za kubična grafa G_1 in G_2 je rod 4-vsote $G_1 \cdot G_2$ omejen z*

$$g(G_1) + g(G_2) - 2 \leq g(G_1 \cdot G_2) \leq g(G_1) + g(G_2) + 1.$$

Meje so najboljše možne, tudi če zahtevamo, da sta G_1 in G_2 snarka.

Poliedrske vložitve

Najprej pokažemo, da so kratki cikli v poliedrskih vložitvah lica.

Lema 8.

- *Če je C 3-cikel v kubičnem grafu G, potem je C obhod lica v vsaki poliedrski vložitvi grafa G.*
- *Če je C 4-cikel v kubičnem grafu G, potem je C obhod lica v vsaki poliedrski vložitvi grafa G.*

Če je C cikel v grafu in F obhod lica, potem rečemo da je F pri C *k-napredujoč obhod*, če se C in F sekata na k zaporednih povezavah na obhodu F.

Lema 9. *Če je C 5-cikel v kubičnem grafu G, potem je*

- *v vsaki poliedrski vložitvi grafa G v orientabilno ploskev cikel C obhod lica,*
- *v vsaki poliedrski vložitvi G v neorientabilno ploskev cikel C ali obhod lica ali pa je vsako lice 2-napredujoče pri C.*

Naj bosta G_1 in G_2 kubična grafa in $v \in V(G_1)$ ter $u \in V(G_2)$. Označimo sosede vozlišča v v G_1 z v_1, v_2, v_3 in sosede vozlišča u v G_2 z u_1, u_2, u_3. Grafu G_1 odstranimo vozlišče v skupaj z njenimi povezavami, grafu G_2 odstranimo vozlišče u skupaj z njenimi povezavami ter dodamo povezave u_iv_i, $i = 1, 2, 3$. Dobimo kubičen graf $G = G_1 * G_2$, ki ga imenujemo *3-vsota* grafov G_1 in G_2.

Izrek 10. *Naj bo G 3-vsota grafov G_1 ter G_2. Graf G ima poliedrsko vložitev (v orientabilno ploskev) natanko tedaj ko imata grafa G_1 ter G_2 poliedrski vložitvi (v orientabilni ploskvi).*

Posledica zadnjega izreka je, da je Grünbaumovo hipotezo dovolj pokazati za ciklično 4-povezane grafe. Po Lemi 8 je Grünbaumovo hipotezo dovolj pokazati za grafe z najkrajšim ciklom dolžine vsaj 4.

S pomočjo Leme 9 lahko za Goldbergove snarke pokažemo, da nimajo poliedrskih vložitev v orientabilne ploskve. To sledi iz dejstva, da imamo v Goldbergovih grafih 5 cikla na točkah $b_id_if_ie_ic_ib_i$ in $b_ic_ig_ih_id_ib_i$. V poliedrski vložitvi v orientabilno ploskev sta oba 5-cikla obhoda lic, to pa ni mogoče, saj je v tem primeru pot $c_ub_id_i$ dolžine 3 vsebovana v dveh različnih obhodih lic.

Da cvetni snarki nimajo poliedrskih vložitev v orientabilne ploskve je pokazal že Szekeres. Podamo enostavnejši dokaz te trditve. Hkrati pokažemo, da cvetni snarki J_{2k+1}, $k > 1$, nimajo poliedrskih vložitev v neorientabilne ploskve. Graf J_3 ima poliedrsko vložitev v projektivno ravnino, vendar ni snark, saj vsebuje cikel dolžine 3. Sledi izrek:

Izrek 11.
- *Cvetni snarki nimajo poliedrskih vložitev niti v orientabilne niti v neorientabilne ploskve.*
- *Goldbergovi snarki nimajo poliedrskih vložitev v orientabilne ploskve.*

Pri dokazu Izreka 11 ne uporabimo dejstva, da so grafi razreda 2. Isti dokaz pove, da tudi grafi J_{2k} nimajo poliedrskih vložitev.

Za Goldbergove snarke konstruiramo poliedrske vložitve v neorientabilne ploskve. Poliedrska vložitev grafa G_k v neorientabilno ploskev je podana z naslednjimi obhodi lic (indeksi so po modulu k)

- $A = a_0 a_1 \dots a_{k-1} a_0$ in $B = f_0 e_0 f_1 e_1 \dots f_{k-1} e_{k-1} f_0$,
- $C_i = b_i d_i f_i e_i c_i b_i$, $i = 0, \dots, k-1$,
- $D_i = g_i h_i g_{i+1} h_{i+1} d_{i+1} f_{i+1} e_i c_i g_i$, $i = 0, \dots, k-1$,
- $E_i = a_i a_{i+1} b_{i+1} c_{i+1} g_{i+1} h_i d_i b_i a_i$, $i = 0, \dots, k-1$.

Zgoraj opisana vložitev ima rod k. Za Goldbergove snarke konstruiramo tudi poliedrske vložitve v neorientabilne ploskve roda $k+2$.

Iz znanih poliedrskih vložitev lahko konstruiramo nove poliedrske vložitve snarkov s pomočjo 4-vsote.

Izrek 12. *Naj bosta G_1 in G_2 kubična grafa. Če imata G_1 in G_2 taki poliedrski vložitvi v (orientabilni) ploskvi S_1 in S_2, da dual grafa G_2 v S_2 ni poln graf, potem obstaja 4-vsota $G_1 \cdot G_2$, ki ima poliedrsko vložitev v (orientabilno) ploskev S. Če je Eulerjev rod ploskev $\epsilon(S_1) = k_1$ in $\epsilon(S_2) = k_2$, potem je Eulerjev rod ploskve S enak $\epsilon(S) = k_1 + k_2$.*

Velja tudi obrat:

Izrek 13. *Naj bo G kubičen graf s cikličnim 4-prerezom S ki ima poliedrsko vložitev. Potem obstajata taka kubična grafa G_1 in G_2, da je G 4-vsota grafov G_1 in G_2 ter da je S prerez 4-vsote. Vsaj eden od G_1 in G_2 ima poliedrsko vložitev.*

Goldbergovi snarki imajo poliedrske vložitve v neorientabilne ploskve roda $2k+1$, Petersenov graf pa ima poliedrsko vložitev v projektivno ravnino. S pomočjo Izreka 12 dobimo posledico:

Posledica 14. *Za vsako nenegativno celo število k obstaja snark s poliedrsko vložitvijo v neorientabilno ploskev N_k roda k.*

Pri dokazu posledice posebej obravnavamo Kleinovo steklenico, saj Izreka 12 ne moremo uporabiti za dve kopiji Petersenovega grafa, vloženi v projektivno ravnino. Snark s poliedrsko vložitvijo v Kleinovo steklenico dobimo kot superpozicijo Petersenovega grafa.

Degeneriranost

Naj bo G kubičen graf in Π vložitev grafa G v orientabilno ploskev. Za obhod lica F v vložitvi Π definiramo degeneriranost $d(F)$ kot število povezav grafa G, ki nastopajo dvakrat na obhodu F. Za dva različna disjunktna obhoda lic

F_i in F_j definiramo degeneriranost $d(F_i, F_j) = 0$. Če imata obhoda lic F_i in F_j kako skupno povezavo, definiramo degeneriranost $d(F_i, F_j)$ kot število povezav, ki nastopajo hkrati na obhodu F_i in F_j, minus 1. Naj ima vložitev Π množico obhodov lic $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$. Potem definiramo *degeneriranost vložitve* Π kot

$$d(\Pi) = \sum_{i=1}^{k} d(F_k) + \sum_{1 \le i < j \le k} d(F_i, F_j)$$

in *degeneriranost grafa* G kot

$$d(G) = \min\{d(\Pi) \mid \Pi \text{ vložitev grafa } G\}.$$

Povezavam, ki nastopajo več kot enkrat na kakem obhodu lica, rečemo *slabe povezave*. Paru povezav e, f, ki nastopata hkrati na dveh različnih obhodih lic, rečemo *slab par*. Točki, ki nastopa trikrat na obhodu kakega lica, rečemo *slaba točka*. Za vložitev Π označimo z $d_v(\Pi)$ število slabih točk v vložitvi Π. *Popravljena degeneriranost* vložitve Π je definirana kot

$$d'(\Pi) = d(\Pi) + 2d_v(\Pi).$$

Popravljena degeneriranost grafa je definirana kot

$$d'(G) = \min\{d'(\Pi) \mid \Pi \text{ vložitev grafa } G\}.$$

Degeneriranost meri, kako daleč je kubičen graf od tega, da ima poliedrsko vložitev. Očitno je Grünbaumova hipoteza ekvivalentna trditvi, da imajo kubični grafi razreda 2 degeneriranost vsaj 1. S pomočjo računalnika izračunamo degeneriranost snarkov z manj kot 30 točkami.

Izrek 15. *Snarki z manj kot 28 točkami nimajo poliedrskih vložitev.*

Najmanjša degeneriranost, ki jo imajo snarki na manj kot 30 točkah, je 2. Najmanjši snark z orientabilno vložitvijo degeneriranosti 2 ima 26 vozlišč. Dobimo ga kot 4-vsoto Blanuševega grafa in Petersenovega grafa in ima dve različni vložitvi z degeneriranostjo 2. Na 28 vozliščih obstajata dva snarka degeneriranosti 2. Prvi ima dve različni vložitvi, drugi pa eno vložitev degeneriranosti 2. Vse vložitve so vložitve v dvojni torus. Blanušev graf (roda 1) ima tri različne vložitve z degeneriranostjo 3 v torus. Na 24 vozliščih obtajata dva snarka degeneriranosti 3, eden z dvema, drugi pa z eno vložitvijo degeneriranosti 3 v dvojni torus. Na 26 vozliščih obstaja en snark degeneriranosti 3, ima tri vložive degeneriranosti 3 v dvojni torus. Na 28 vozliščih obstaja 8 snarkov degeneriranosti 3.

Izrek 16.
- *Če Grünbaumova hipoteza drži, potem imajo snarki razreda 2 degeneriranost vsaj 2.*
- *Za vsak $k \geq 2$ obstaja neskončna družina snarkov, ki imajo degeneriranost natanko k.*

Najsplošnejša znana konstrukcija snarkov je Kocholova superpozicija. Jaeger in Swart sta leta 1980 postavila hipotezo, da ima vsak snark cikel dolžine kvečjemu 6 [10]. Prvi je snarke brez kratkih ciklov s pomočjo superpozicije konstruiral Kochol leta 1996 [17]. Družini snarkov, ki jo je konstruiral Kochol in ki vsebuje snarke brez kratkih ciklov, rečemo *Kocholovi snarki*.

Kocholove snarke dobimo kot superpozicijo Petersenovega grafa. Obstajata dva tipa Kocholovih snarkov, prvi imajo ciklične 4-prereze, drugi so pa ciklično 5-povezani. Petersenov graf ima degeneriranost 5 in posledica tega je, da Kocholovi snarki nimajo poliedrskih vložitev.

Izrek 17. *Kocholovi snarki nimajo poliedrskih vložitev v orientabilne ploskve.*

Naj bo c 4-barvanje povezav kubičnega grafa G, kjer dovolimo, da so povezave pobarvane z barvo 4, sosednje. Barvanju c rečemo minimalno barvanje, če je število povezav, pobarvanih z barvo 4, minimalno možno med vsemi 4-barvanji grafa G. Vsako minimalno barvanje je pravo barvanje povezav (torej tudi povezave barve 4 niso sosednje). *Odpornost* grafa G, $r(G)$, je število povezav barve 4 v minimalnem barvanju. Očitno je kubični graf G graf razreda 1 natanko tedaj, ko je $r(G) = 0$. Pokažemo, da v primeru, da Grünbaumova hipoteza drži, obstaja povezava med odpornostjo in popravljeno degeneriranostjo.

Izrek 18. *Če Grünbaumova hipoteza drži, potem za vsak kubičen graf G velja*

$$d'(G) \geq \frac{r(G)}{2}.$$

Zadnji izrek pravi, da so grafi, ki so daleč od tega, da imajo 3-barvanje povezav, tudi daleč od tega, da imajo poliedrsko vložitev.

Izjava

Izjavljam, da je doktorska disertacija rezultat mojega raziskovalnega dela.

Ljubljana, junij 2007 Andrej Vodopivec