Also available on http://amc.imfm.si
ARS MATHEMATICA CONTEMPORANEA 1 (2008) 44–50

A New Class of Movable (n_4) Configurations

Leah Wrenn Berman
Ursinus College, Department of Mathematics and Computer Science, P.O. Box 1000, Collegeville, Pennsylvania, USA

Received 30 July 2007, accepted 3 July 2008, published online 8 July 2008

Abstract

A geometric (n_4) configuration is a collection of n points and n lines, usually in the Euclidean plane, so that every point lies on four lines and every line passes through four points. This paper introduces a new class of movable $((5m)_4)$ configurations—that is, configurations which admit a continuous family of realizations fixing four points in general position but moving at least one other point—including the smallest known movable (n_4) configuration.

Keywords: Configurations.

Math. Subj. Class.: 52C30, 52C99

A geometric (n_4) configuration is a collection of n points and n lines, usually in the Euclidean plane, so that every point lies on four lines and every line passes through four points. In the last few years, there has been a fair amount of activity concerning (n_4) configurations, both in answering existence questions [4, 7] and classification questions [1, 2, 3, 6].

In [1], a class of symmetric (n_4) configurations (that is, configurations with non-trivial geometric symmetry) was introduced which were *movable*: they admit a continuous family of realizations fixing four points in general position but moving at least one other point. This paper introduces a second class of movable (n_4) configurations which are also quite symmetric and are smaller and simpler than the previous class; for example, the smallest example of the new class is a (30_4) configuration, which is the smallest known movable (n_4) configuration, while the smallest example of the old class is a (44_4) configuration. In addition, this new class provides examples of movable configurations which have m-fold rotational symmetry for odd m (the previous class required m to be even). Three members of a continuous family of (35_4) configurations are shown in Figure 1.

The class of configurations described here consists of $((5m)_4)$ configurations with cyclic symmetry; they have five symmetry classes of points and lines if m is odd, and five symmetry

E-mail address: lberman@ursinus.edu (Leah Wrenn Berman)

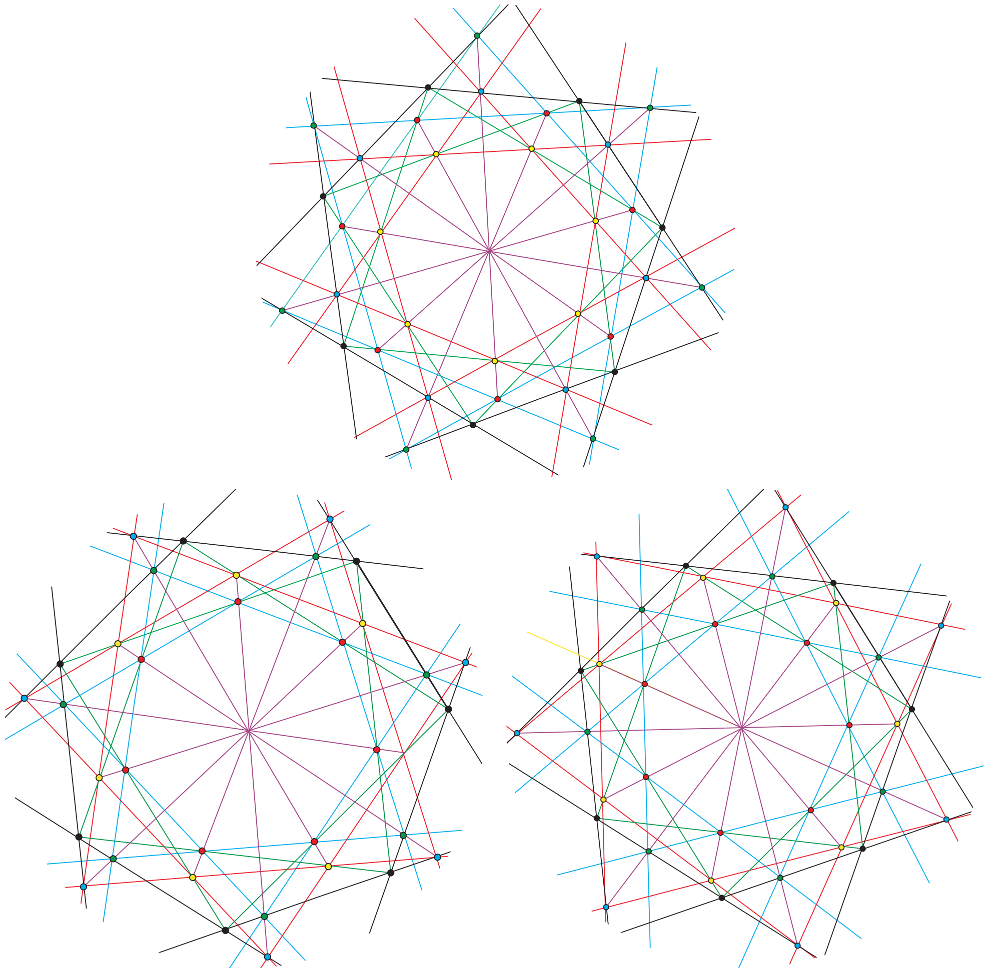

Figure 1: Three versions of a movable (35_4) configuration, with different choices of continuous parameter; the discrete parameters are $m = 7$, $a = 1$, and $b = 2$.

classes of points and six symmetry classes of lines if m is even. In the language introduced by Grünbaum in [5], if m is odd the configurations are (5)-astral and if m is even they are $(5,6)$-astral.

Given a regular m-gon $\mathcal{M}$ with vertices labelled cyclically as $v_0, \dots, v_{m-1}$, a *diagonal of span s* is a line that connects vertices v_i and v_{i+s}. Given a set of all diagonals of span s, we can label points formed by the intersection of a diagonal with other diagonals of the same span; again following Grünbaum [5], we say that the t-th intersection, counted from the center of a span s diagonal in some direction, is labelled $[[s,t]]$.

The construction for the new class of movable configurations depends on the following theorem from [1], listed there as Theorem 3 and given here in a restated form (See Figure 2):

Theorem 1 (Crossing Spans)**.** *Begin with a set of diagonals of span a and span b of an m-gon $\mathcal{M}$. Place an arbitrary point A on a diagonal of span a. Construct another m-gon $\mathcal{N}$ whose vertices are the rotated images of A through angles of $\frac{2\pi i}{m}$, and construct diagonals of span b using $\mathcal{N}$. Two of these span b diagonals intersect each other and a span b diagonal of $\mathcal{M}$ in the same point, and the intersection points are precisely the points labelled $[[b,a]]$ in $\mathcal{N}$.*

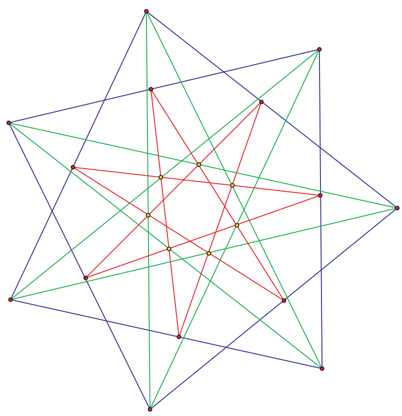

Figure 2: Crossing spans: Construct a regular m-gon (here, $m = 7$) and lines of span a and b (here, $a = 2$ and $b = 3$; the span a lines are blue and the span b lines are green). If a point is placed arbitrarily on a line of span a and rotated to make a regular m-gon, and lines of span b are constructed using this m-gon (the red lines), then two of these lines of span b and one of the original lines of span b intersect in a single point (here, the innermost polygon, which is formed as the intersections of two red lines and one green line).

The construction method is as follows.

Begin with the vertices of a regular m-gon $\mathcal{M}$ centered at the origin $\mathcal{O}$ (the black circles in Figure 3), and choose distinct integers a and b, with $1 \le a, b < \frac{m}{2}$, where if m is even, a and b are of opposite parity. Using the points in $\mathcal{M}$, construct all diagonals of span a (shown in black in Figure 3). Place a point A_0 arbitrarily on one of the diagonals, and construct the images under rotation by $\frac{2\pi i}{m}$ about the origin to form a new m-gon $\mathcal{A}$; we will label these points cyclically as $A_0, A_1, \dots, A_{m-1}$ (the blue squares in Figure 3). Construct the ray $\overrightarrow{\mathcal{O}A_0}$ and let B_0 be an intersection of this ray with one of the diagonals of span a in a point other than A_0. (If m is large, there may be many choices for which line to place B_0 on.) Rotate

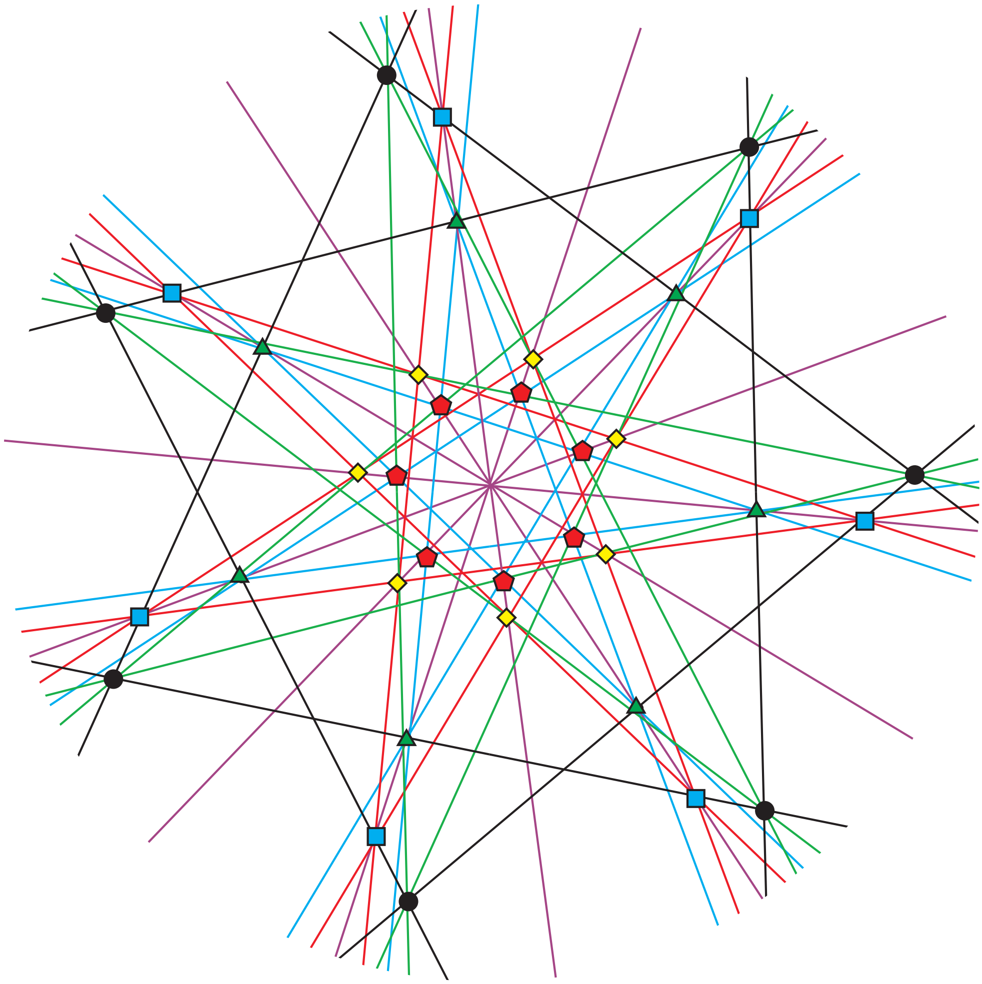

Figure 3: A diagram showing the elements of the construction of the new movable configurations. Points in $\mathcal{M}$ are shown as black circles, points in $\mathcal{A}$ are shown as blue squares, points in $\mathcal{B}$ are shown as green triangles, points in $\mathcal{D}$ are shown as yellow diamonds, and points in $\mathcal{E}$ are red pentagons. The lines of span a connecting points in $\mathcal{A}$ are black. The lines $L_{\mathcal{M}}$ are green, $L_{\mathcal{A}}$ are red, and $L_{\mathcal{B}}$ are blue. The diameters are purple. In this diagram, $m = 7$, $a = 2$ and $b = 3$.

this point B_0 around to form a second regular m-gon, $\mathcal{B}$; we will label these points cyclically as $B_0, B_1, \ldots, B_{m-1}$ (the green triangles in Figure 3).

Now construct all diagonals of span b using the m-gons $\mathcal{M}$, $\mathcal{A}$, and $\mathcal{B}$, and label these sets of lines as $L_{\mathcal{M}}$, $L_{\mathcal{A}}$ and $L_{\mathcal{B}}$, respectively; in Figure 3, lines in $L_{\mathcal{M}}$ are green, lines in $L_{\mathcal{A}}$ are red, and lines in $L_{\mathcal{B}}$ are blue.

By Theorem 1, two diagonals from $L_{\mathcal{A}}$ and one diagonal from $L_{\mathcal{M}}$ intersect in a single point. Call the m-gon of points formed $\mathcal{D}$ (the yellow diamonds in Figure 3); similarly, call $\mathcal{E}$ the m-gon of points formed by the intersection of two diagonals from $L_{\mathcal{B}}$ with one diagonal of $L_{\mathcal{M}}$ (the red pentagons in Figure 3).

We now have five sets of points. The points of $\mathcal{M}$ have four lines passing through them, but the points in $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$, and $\mathcal{E}$ only have three lines passing through them. To be a (n_4) configuration, every point must have four lines passing through them.

The final set of lines will be diameters—that is, lines that pass through the origin. However, which diameters need to be used and which points lie on them depends on whether m is odd or even and on how the points of the various polygons are aligned; the construction is slightly different depending on the parity of m.

To determine how the points in $\mathcal{D}$ and $\mathcal{E}$ are aligned with respect to the points of $\mathcal{A}$ and $\mathcal{B}$, consider the points of $\mathcal{A}$ and all the diagonals of span b using those points (the lines $L_{\mathcal{A}}$). Given a point A_i in $\mathcal{A}$ and a point P in $\mathcal{D}$ which lies on a line of span b using the points A_i and has label $[[b, a]]$, the angle $A_i\mathcal{O}P$ is $\frac{\pi j}{m}$ for some j, where j is odd if a is odd and j is even if a is even. That is, a ray which intersects A_i also passes through a point in $\mathcal{D}$, which has label $[[b, a]]$, if b and a have the same parity. Note that a line through the origin that passes through a point A_i is composed of one ray through A_i and one ray with angle π with respect to $\overrightarrow{\mathcal{O}A_i}$; if m is odd, this second ray does not pass through a second point in $\mathcal{A}$, while if m is even, it does. Therefore, if m is even and b and a are of the same parity, a line through the origin passing through A_i contains two points from $\mathcal{A}$, two points from $\mathcal{B}$ and two points from $\mathcal{D}$; this is too many points for an (n_4) configuration, which is why a and b were required to be of opposite parity when m is even.

What remains is to show how to add appropriate diameters, so that a configuration is constructed in which every line passes through precisely four points and every point lies on the intersection of four lines.

Case 1: m is even

Suppose in this case that $m = 2k$. Given a point A_i in $\mathcal{A}$, the point B_i in $\mathcal{B}$ lies on the line $\overline{\mathcal{O}A_i}$. However, since m is even, $\mathcal{M}$ has $180°$ rotational symmetry, so the points A_{i+k} and B_{i+k} also lie on $\overline{\mathcal{O}A_i}$. Thus, if we add to the set of lines in the configuration the set of all diameters of the form $\overline{\mathcal{O}A_i}$ for A_i in $\mathcal{A}$, all of the points in $\mathcal{A}$ and $\mathcal{B}$ have a diameter passing through them, and each of these diameters has four points lying on it, two from $\mathcal{A}$ and two from $\mathcal{B}$.

Now consider the points in $\mathcal{D}$ and $\mathcal{E}$. Recall that both D_i and E_i were chosen to be points with label $[[b, a]]$ based on the span b lines $L_{\mathcal{A}}$ and $L_{\mathcal{B}}$; that is, point D_i is the a-th intersection on a particular line in $L_{\mathcal{A}}$ with other lines in $L_{\mathcal{A}}$, counted from the center. Moreover, since m is even, we chose b and a to be of opposite parity. Therefore, points D_i and E_i lie on a single line through the origin, and because of the parity constraint, this line does not pass through A_i and B_i. However, because $\mathcal{D}$ and $\mathcal{E}$ also have $180°$ rotational symmetry, a line $\overline{\mathcal{O}D_i}$ also passes through the points D_{i+k} and E_{i+k}. Therefore, if we include these diameters as lines

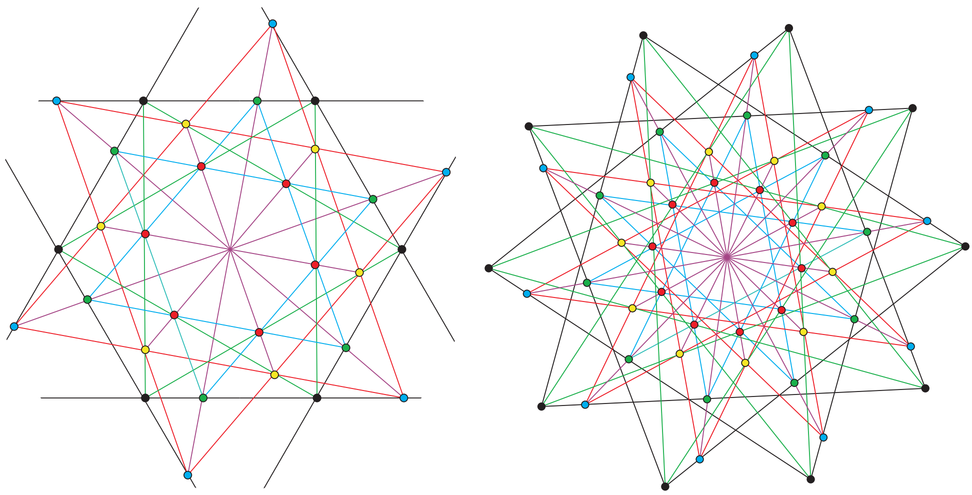

Figure 4: Movable configurations where m is even. Left hand side: a (30_4) configuration with $m = 6$, $a = 1$, $b = 2$; right hand side: a (50_4) configuration with $m = 10$, $a = 3$, $b = 4$.

of the configuration, we have five classes of points (polygons $\mathcal{M}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$, and $\mathcal{E}$) and six classes of lines (lines of span a through $\mathcal{M}$, lines of span b through $\mathcal{M}$, $\mathcal{A}$, and $\mathcal{B}$ (that is, $L_\mathcal{M}, L_\mathcal{A}$ and $L_\mathcal{B}$), diameters through A_i, which contain two points from $\mathcal{A}$ and two points from $\mathcal{B}$, and diameters through D_i, which contain two points from $\mathcal{D}$ and two points from $\mathcal{E}$). Two examples of such configurations are shown in Figure 4.

Case 2: m is odd.

By construction, given a point A_i in $\mathcal{A}$, the point B_i in $\mathcal{B}$ lies on the line $\overline{\mathcal{O}A_i}$, since that was how the B_i's were chosen. Because $\mathcal{M}$ does not have $180°$ rotational symmetry (since m is odd), no other points of $\mathcal{A}$ and $\mathcal{B}$ lie on that line.

The intersection points D_i and E_i also lie on $\overline{\mathcal{O}A_i}$. To see this, note that points in $\mathcal{D}$ are of label $[[b, a]]$ with respect to $\mathcal{A}$, points in $\mathcal{E}$ are of label $[[b, a]]$ with respect to $\mathcal{B}$, and $\mathcal{A}$ and $\mathcal{B}$ are aligned. If a and b are of the same parity, then points of label $[[b, a]]$ lie on *rays* $\overrightarrow{\mathcal{O}A_i}$, so points D_i and E_i lie on the same side of the origin as A_i and B_i (see the right hand side of Figure 5). However, if a and b are of opposite parity points D_i and E_i lie on a ray that makes an angle of π with respect to $\overrightarrow{\mathcal{O}A_i}$; since m is odd, this ray combines with $\overrightarrow{\mathcal{O}A_i}$ to make a line through the origin, with D_i and E_i lying on the opposite side of the origin from A_i and B_i (see the left hand side of Figure 5). In either case, four points of the configuration lie on each line $\overline{\mathcal{O}A_i}$.

Thus, we use as points of the configuration the points in the m-gons $\mathcal{M}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$, and $\mathcal{E}$, and the lines of the configuration are the diagonals of span a using $\mathcal{M}$, the diagonals of span b using $\mathcal{M}$, $\mathcal{A}$, $\mathcal{B}$ (that is, $L_\mathcal{M}, L_\mathcal{A}$ and $L_\mathcal{B}$), and the diameters $\overline{\mathcal{O}A_i}$ for $i = 0, 1, \ldots, m$. Each diameter contains one point from $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$, and $\mathcal{E}$.

Examples of such configurations are shown in Figure 5.

Finally, note that A_0 was chosen arbitrarily; its position along the line of span a forms a

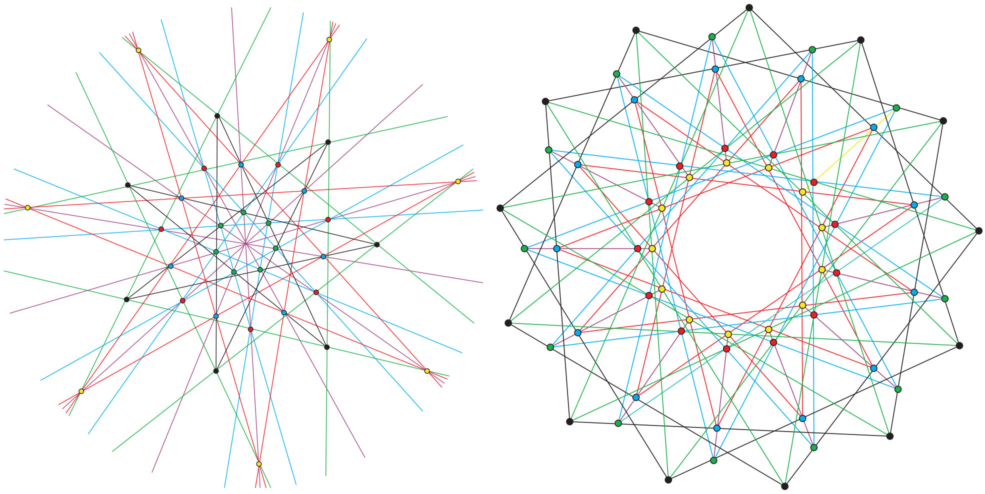

Figure 5: Movable configurations where m is odd. Left hand side: a (35_4) configuration with $m = 7$, $a = 3$, $b = 2$; right hand side: a (65_4) configuration with $m = 13$, $a = 3$, $b = 5$.

continuous parameter. Figure 1 shows three versions of a (35_4) configuration, with various choices for the position of A_0. Also, if $b > a$, as in the left hand side of Figure 5, then the polygons $\mathcal{D}$ and $\mathcal{E}$ will be outside the polygons $\mathcal{A}$ and $\mathcal{B}$.

References

[1] L. W. Berman, Movable (n_4) configurations, *Electron. J. Combin.* **13** (2006), #R104.

[2] L. W. Berman, A characterization of astral (n_4) configurations, *Discrete Comput. Geom.* **26** (2001), no. 4, 603–612.

[3] M. Boben and T. Pisanski, Polycyclic configurations, *European J. Combin.* **24** (2003), 431–457.

[4] J. Bokowski, B. Grünbaum and L. Schewe, Topological configurations (n_4) exist for all $n \geq 17$, *European J. Combin.*, to appear.

[5] B. Grünbaum, Configurations of points and lines. In *The Coxeter Legacy: Reflections and Projections* (C. Davis and E. W. Ellers, eds), American Mathematical Society (2006), 179–225.

[6] B. Grünbaum, Astral (n_4) configurations, *Geombinatorics* **9** (2000), 127–134.

[7] B. Grünbaum, Which (n_4) configurations exist?, *Geombinatorics* **9** (2000), 164–169.