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Geometry of the parallelism in polar spine spaces and their line reducts

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Abstract

The concept of the spine geometry over a polar Grassmann space belongs to a wide family of partial affine line spaces. It is known that the geometry of a spine space over a projective Grassmann space can be developed in terms of points, so called affine lines, and their parallelism (in this case the parallelism is not intrinsically definable as it is not Veblenian). This paper aims to prove an analogous result for the polar spine spaces. As a by-product we obtain several other results on primitive notions for the geometry of polar spine spaces.

Keywords: Grassmann space, projective space, polar space, spine space, coplanarity, pencil of lines.

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Introduction

Some properties of the polar spine spaces were already established in [8], where the class of such spaces was originally introduced. Its definition resembles the definition of a spine space defined within a (projective) Grassmann space (= the space $\mathbf{P}_k(\mathbb{V})$ of pencils of k-subspaces in a fixed vector space $\mathbb{V}$), cf. [12, 13]. In every case, a spine space is a fragment of a (projective) Grassmannian whose points are subspaces which intersect a fixed subspace W in a fixed dimension m. In case of *polar spine spaces* we consider a two-step construction, in fact: we consider the subspaces of $\mathbb{V}$ that are totally isotropic (self conjugate, singular) under a fixed nondegenerate reflexive bilinear form ξ on $\mathbb{V}$, and then we restrict this class to the subspaces which touch W in dimension m.

It is a picture which is seen from the view of $\mathbb{V}$. Clearly, W can be extended to a subspace M of $\mathbb{V}$ with codimension 1 and then M yields a hyperplane $\mathcal{M}$ of the polar space

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Q_0 determined by ξ in $\mathbb{V}$. In other words, the projective points on W that are points of Q_0 yield a subspace $\mathcal{W}$ of Q_0 extendable to a hyperplane. Recall that situation of this sort was already investigated in [10]. The isotropic k-subspaces of $\mathbb{V}$ are the $(k-1)$-dimensional linear subspaces of Q_0, and first-step restriction yields the so called polar Grassmann space $\mathrm{Q}_{k-1} = \mathbf{P}_{k-1}(\mathrm{Q})$. The points of Q_{k-1} which touch W in dimension m are – from view of Q – the elements of Q_{k-1} which touch $\mathcal{W}$ in dimension $m-1$. So, *a polar spine space* is also the fragment of a polar Grassmannian which consists of subspaces which touch a fixed subspace extendable to a hyperplane in a fixed dimension. The analogy seems full.

In particular, when W is a hyperplane of $\mathbb{V}$ i.e. $\mathcal{W}$ is a hyperplane of Q_0 then a k-subspace of Q either is contained in $\mathcal{W}$ or it touches it in dimension $k-1$. It is seen that in this case the only reasonable value of m is $m = k-1$ and the obtained structure is the Grassmannian of subspaces of the affine polar space obtained from Q_0 by deleting $\mathcal{W}$ (cf. [3, 11]). So, the class of polar spine spaces contain Grassmannians of k-subspaces of arbitrary polar slit space: of a polar space with a subspace (extendable to a hyperplane) removed, see [10]. An interesting case appears, in particular, when we assume that W is isotropic.

1 Generalities

This section is quoted after [8] with slight modifications.

1.1 Point-line spaces and their fragments

A point-line structure $\mathfrak{B} = \langle S, \mathcal{L}\rangle$, where the elements of S are called *points*, the elements of $\mathcal{L}$ are called *lines*, and where $\mathcal{L} \subset 2^S$, is said to be a *partial linear space*, or a *point-line space*, if two distinct lines share at most one point and every line is of size (cardinality) at least 2 (cf. [2]).

A *subspace* of $\mathfrak{B}$ is any set $X \subseteq S$ with the property that every line which shares with X two or more points is entirely contained in X. We say that a subspace X of $\mathfrak{B}$ is *strong* if any two points in X are collinear. If S is strong, then $\mathfrak{B}$ is said to be a *linear space*.

Let us fix a nonempty subset $\mathcal{H} \subset S$ and consider the set

$$\mathcal{L}|_{\mathcal{H}} := \big\{k \cap \mathcal{H} : k \in \mathcal{L} \text{ and } |k \cap \mathcal{H}| \geq 2\big\}. \tag{1.1}$$

The structure

$$\mathfrak{M} := \mathfrak{B} \restriction \mathcal{H} = \langle \mathcal{H}, \mathcal{L}|_{\mathcal{H}}\rangle$$

is a *fragment* of $\mathfrak{B}$ induced by $\mathcal{H}$ and itself it is a partial linear space. The incidence relation in $\mathfrak{M}$ is again $\in$, inherited from $\mathfrak{B}$, but limited to the new point set and line set. Following a standard convention we call the points of $\mathfrak{M}$ *proper*, and the points in $S \setminus \mathcal{H}$ *improper*. The set $S \setminus \mathcal{H}$ will be called the *horizon of* $\mathfrak{M}$. To every line $L \in \mathcal{L}|_{\mathcal{H}}$ we can assign uniquely the line $\overline{L} \in \mathcal{L}$, the *closure* of L, such that $L \subseteq \overline{L}$. For a subspace $X \subseteq \mathcal{H}$ the closure of X is the minimal subspace $\overline{X}$ of $\mathfrak{B}$ containing X. A line $L \in \mathcal{L}|_{\mathcal{H}}$ is said to be a *projective line* if $L = \overline{L}$, and it is said to be an *affine line* if $|\overline{L} \setminus L| = 1$. With every affine line L one can correlate the point $L^\infty \in S \setminus \mathcal{H}$ by the condition $L^\infty \in \overline{L} \setminus L$. We write $\mathcal{A}$ for the class of affine lines. In what follows we consider sets $\mathcal{H}$ which satisfy the following

$$|L \setminus \mathcal{H}| \leq 1 \text{ or } |L \cap \mathcal{H}| \leq 1 \text{ for all } L \in \mathcal{L}.$$

Note that the above holds when $\mathcal{H}$ or $S \setminus \mathcal{H}$ is a subspace of $\mathfrak{B}$, but the above does not force $\mathcal{H}$ or $S \setminus \mathcal{H}$ to be a subspace of $\mathfrak{B}$. In any case, under this assumption every line is either projective or affine. In case $\mathcal{L}|_{\mathcal{H}}$ contains projective or affine lines only, then $\mathfrak{M}$ is a *semiaffine* geometry (for details on terminology and axiom systems see [18]). In this approach an affine space is a particular case of a semiaffine space. For affine lines $L_1, L_2 \in \mathcal{L}|_{\mathcal{H}}$ we can define a parallelism in a natural way:

$$L_1, L_2 \text{ are } \textit{parallel } (L_1 \parallel L_2) \quad \text{iff} \quad \overline{L_1} \cap \overline{L_2} \cap (S \setminus \mathcal{H}) \neq \emptyset.$$

In what follows we assume that the notion of 'a plane' (= 2-dimensional strong subspace) is meaningful in $\mathfrak{B}$: e.g. $\mathfrak{B}$ is an exchange space, or a dimension function is defined on its strong subspaces. In the article in most parts we consider $\mathfrak{B}$ such that its planes are, up to an isomorphism, projective planes. We say that E is a plane in $\mathfrak{M}$ if $\overline{E}$ is a plane in $\mathfrak{B}$. Observe that there are two types of planes in $\mathfrak{M}$: projective and semiaffine. A semiaffine plane E arises from $\overline{E}$ by removing a point or a line. In result we get a punctured plane or an affine plane respectively. For lines $L_1, L_2 \in \mathcal{L}|_{\mathcal{H}}$ we say that they are *coplanar* and write

$$L_1 \boldsymbol{\pi} L_2 \quad \text{iff} \quad \text{there is a plane } E \text{ such that } L_1, L_2 \subset E. \tag{1.2}$$

Let E be a plane in $\mathfrak{M}$ and $U \in \overline{E}$. The set

$$\mathbf{p}(U, E) := \{L \in \mathcal{L}|_{\mathcal{H}} : U \in \overline{L} \subseteq \overline{E}\} \tag{1.3}$$

will be called a *pencil of lines* if U is a proper point, or a *parallel pencil* otherwise. The point U is said to be the *vertex* and the plane E is said to be the *base plane* of that pencil. We write

$$L_1 \boldsymbol{\rho} L_2 \quad \text{iff} \quad \text{there is a pencil } p \text{ such that } L_1, L_2 \in p. \tag{1.4}$$

1.2 Cliques

Let ϱ be a binary symmetric relation defined on a set $\mathcal{X}$. A subset of $\mathcal{X}$ is said to be a *ϱ-clique* iff every two elements of this set are ϱ-related.

For any $x_1, x_2, \ldots, x_s$ in $\mathcal{X}$ we introduce

$$\begin{aligned} \mathbf{\Delta}^s_\varrho(x_1, x_2, \ldots, x_s) \quad \text{iff} \quad & \neq (x_1, x_2, \ldots, x_s) \text{ and } x_i \ \varrho\ x_j \text{ for all } i, j = 1, \ldots, s \\ & \text{and for all } y_1, y_2 \in \mathcal{X} \text{ if } y_1, y_2 \ \varrho\ x_1, x_2, \ldots, x_s \text{ then } y_1 \ \varrho\ y_2, \end{aligned} \tag{1.5}$$

cf. analogous definition of $\mathbf{\Delta}^\varrho_s$ in [9]. For short we will frequently write $\mathbf{\Delta}_\varrho$ instead of $\mathbf{\Delta}^s_\varrho$. Next, we define

$$[x_1, x_2, \ldots, x_s]_\varrho := \{y \in \mathcal{X} : y \ \varrho\ x_1, x_2, \ldots x_s\}. \tag{1.6}$$

It is evident that if $\mathbf{\Delta}_\varrho(x_1, \ldots, x_s)$ holds (and ϱ is reflexive) then $[x_1, \ldots, x_s]_\varrho$ is the (unique) maximal ϱ-clique which contains $\{x_1, \ldots, x_s\}$. Finally, for an arbitrary integer $s \geq 3$ we put

$$\mathcal{K}^s_\varrho = \{[x_1, x_2, \ldots, x_s]_\varrho : x_1, x_2, \ldots, x_s \in \mathcal{X} \text{ and } \mathbf{\Delta}_\varrho(x_1, x_2, \ldots, x_s)\}. \tag{1.7}$$

Then we write

$$\mathcal{K}_\varrho := \bigcup_{s=3}^{\infty} \mathcal{K}^s_\varrho.$$

In most of the interesting situations there is an integer $s_{\max}$ such that $\mathcal{K}_\varrho = \bigcup_{s=3}^{s_{\max}} \mathcal{K}^s_\varrho = \mathcal{K}^*(\varrho)$, where

$$\mathcal{K}^*(\varrho) \text{ is the set of maximal } \varrho\text{-cliques.}$$

1.3 Grassmann spaces and spine spaces

We start with some constructions of a general character. Let X be a nonempty set and let $\mathcal{P}$ be a family of subsets of X. Assume that there is a dimension function $\dim\colon \mathcal{P} \to \{0,\dots,n\}$ such that $\mathfrak{B} = \langle \mathcal{P}, \subset, \dim\rangle$ is an incidence geometry, cf. e.g. [1]. Write $\mathcal{P}_k$ for the set of all $U \in \mathcal{P}$ with $\dim(U) = k$.

Given $H \in \mathcal{P}_{k-1}$ and $B \in \mathcal{P}_{k+1}$ with $H \subset B$, a k*-pencil* over $\mathfrak{B}$ is a set of the form

$$\mathbf{p}(H,B) = \{U \in \mathcal{P}_k : H \subset U \subset B\}.$$

The idea behind this concept is the same as in (1.3), though this definition is more general. The family of all such k-pencils over $\mathfrak{B}$ will be denoted by $\mathcal{P}_k$. Then, the structure

$$\mathbf{P}_k(\mathfrak{B}) = \langle \mathcal{P}_k, \mathcal{P}_k\rangle$$

will be called a *Grassmann space* over $\mathfrak{B}$ (cf. [5, Section 2.1.3]). It is a partial linear space for $0 < k < n$.

Let us fix $W \in \mathcal{P}$ and an integer m. We will write

$$\mathcal{F}_{k,m}(\mathfrak{B}, W) := \{U \in \mathcal{P}_k : \dim(U \cap W) = m\}.$$

The fragment

$$\mathbf{A}_{k,m}(\mathfrak{B}, W) := \mathbf{P}_k(\mathfrak{B}) \upharpoonright \mathcal{F}_{k,m}(\mathfrak{B}, W)$$

will be called a *spine space* over $\mathfrak{B}$ determined by W. It will be convenient to have an additional symbol for the line set of a spine space, which is

$$\mathcal{G}_{k,m}(\mathfrak{B}, W) := \mathcal{P}_k|_{\mathcal{F}_{k,m}(\mathfrak{B},W)}.$$

What follows are more specific examples of the above constructions that we actually investigate in our paper. Let $\mathbb{V}$ be a vector space and let $\mathrm{Sub}(\mathbb{V})$ be the set of all vector subspaces of $\mathbb{V}$. Then $\mathbf{P}_k(\mathbb{V})$ is a partial linear space called a *projective Grassmann space*. In particular $\mathbf{P}_1(\mathbb{V})$ is the projective space over $\mathbb{V}$. It is well known that $\mathbf{P}_k(\mathbb{V}) \cong \mathbf{P}_{k-1}(\mathbf{P}_1(\mathbb{V}))$.

Let $W \in \mathrm{Sub}(\mathbb{V})$. The spine space $\mathbf{A}_{k,m}(\mathbb{V}, W)$ was introduced in [12] and developed in [13, 14, 15, 16]. Note that $\mathbf{A}_{k,m}(\mathbb{V}, W) \cong \mathbf{A}_{k-1,m-1}(\mathbf{P}_1(\mathbb{V}), \mathrm{Sub}_1(W))$. The concept of a spine space makes a little sense without the assumption that

$$0, k-n+w \le m \le k, w, \tag{1.8}$$

where $w = \dim(W)$. It is a partial linear space when (1.8) is satisfied.

For possibly maximal values of m we get $\mathbf{A}_{k,k}(\mathbb{V}, W) = \mathbf{P}_k(W)$, where the points are basically vector subspaces of W, and $\mathbf{A}_{k,w}(\mathbb{V}, W) \cong \mathbf{P}_{k-w}(\mathbb{V}/W)$, where the points are those vector subspaces of $\mathbb{V}$ which contain W. Therefore, we assume that

$$m < k, w. \tag{1.9}$$

Now, let ξ be a nondegenerate reflexive bilinear form of index r on $\mathbb{V}$. For $U, W \in \mathrm{Sub}(\mathbb{V})$ we write $U \perp W$ iff $\xi(U, W) = 0$, meaning that $\xi(u, w) = 0$ for all $u \in U$, $w \in W$. Then the set of all totally isotropic subspaces of $\mathbb{V}$ w.r.t. ξ is

$$\mathrm{Q} := \{U \in \mathrm{Sub}(\mathbb{V}) : U \perp U\},$$

and $\mathrm{Q}_k := \mathrm{Q} \cap \mathrm{Sub}_k(\mathbb{V})$. The set Q_k is nonempty iff

$$k \leq r. \tag{1.10}$$

Provided that $2 \leq r$ the structure $\mathfrak{Q} = \mathbf{P}_1(\mathrm{Q})$ is a classical polar space embeddable into the projective space $\mathbf{P}_1(\mathbb{V})$. It is clear that $\mathfrak{Q} \cong \langle \mathrm{Q}_1, \mathrm{Q}_2, \subset \rangle$ and usually polar space is defined that way.

A *polar Grassmann space* is the structure $\mathbf{P}_k(\mathrm{Q})$. It is a partial linear space whenever

$$k < r. \tag{1.11}$$

Note that $\mathbf{P}_k(\mathrm{Q}) \cong \mathbf{P}_{k-1}(\mathbf{P}_1(\mathrm{Q}))$.

Finally,

$$\mathfrak{M} := \mathbf{A}_{k,m}(\mathrm{Q}, W),$$

a *polar spine space*, the main subject of our paper, arises. Note that we have $\mathfrak{M} \cong \mathbf{A}_{k-1,m-1}(\mathbf{P}_1(\mathrm{Q}), \mathrm{Sub}(W) \cap \mathrm{Q}_1)$.

Let $r_W = \mathrm{ind}(\xi \restriction W)$ be the index of the form ξ restricted to W. If $r_W < m$, then there is no totally isotropic subspace of $\mathbb{V}$, which meets W in some m-dimensional subspace. Every $U \in \mathrm{Q}$ can be extended to an $Y \in \mathrm{Q}_r$. Assume that $\dim(Y \cap W) > r - k + m$ for all $Y \in \mathrm{Q}_r$. This means that all totally isotropic subspaces of $\mathbb{V}$, which meet W in some m-dimensional subspace, are at most $(k-1)$-dimensional. On the other hand, this assumption implies $r_W > r - k + m$. Thus

$$m \leq r_W \leq r - k + m \tag{1.12}$$

is a sufficient condition for $\mathcal{F}_{k,m}(\mathrm{Q}, W) \neq \emptyset$.

Warning. The condition (1.12) is – in the context above – *only sufficient*. As we shall see there are sets W such that $r - k + m < r_W$ but $\mathcal{F}_{k,m}(\mathrm{Q}, W) \neq \emptyset$. Clearly, the condition $m \leq r_W$ *is necessary*.

Under (1.12) no point of $\mathfrak{M}$ is isolated and $\mathfrak{M}$ is a partial linear space. Now, let us have a look at the structure of strong subspaces of polar spine spaces. Following [13] they are called: α-stars, ω-stars, α-tops and ω-tops. For details see Table 2. Actually, this is an 'adaptation' of the classification of strong subspaces of $\mathbf{A}_{k,m}(\mathbb{V}, W)$ (consult [13]) to the case when we restrict $\mathbf{P}_k(\mathbb{V})$ to $\mathbf{P}_k(\mathrm{Q})$. With a slight abuse of language all sets of the type $\mathcal{T}^\alpha$ and $\mathcal{T}^\omega$ we call *tops*, and sets of the form $\mathcal{S}^\alpha$ and $\mathcal{S}^\omega$ *stars*. But note that due to some specific values of r, k, m and $\dim(Y \cap W)$ with $Y \in \mathrm{Q}_r$ families of some of these types may be empty. Moreover, stars and tops consist of strong subspaces of $\mathfrak{M}$, but stars or tops of some kind may be not maximal among strong. In general, $\mathcal{S}^\omega$ and $\mathcal{T}^\alpha$ consist of projective spaces, while the other consist of proper slit spaces (cf. [4, 18]), but if $\mathcal{F}_{k-1,m}(\mathrm{Q}, W) \ni H \subset Y \in \mathcal{F}_{r,m}(\mathrm{Q}, W)$ then $[H, Y]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W) = [H, Y]_k \in \mathcal{S}^\alpha$ is a projective space as well.

Generally, $H \in \mathcal{P}_{k-1}$ determines a star and $B \in \mathcal{P}_{k+1}$ determines a top as follows

$$\mathrm{S}(H) = \{U \in \mathcal{P}_k : H \subset U\}, \qquad \mathrm{T}(B) = \{U \in \mathcal{P}_k : U \subset B\}.$$

Here, we occasionally make use of this convention in the context of polar spine spaces, where $\mathcal{P}_k = \mathcal{F}_{k,m}(\mathrm{Q}, W)$.

2 Lines classification and existence problems

In analogy to [13, 17] the lines of $\mathfrak{M}$ can be of three sorts: affine (in $\mathcal{A}$), α-projective (in $\mathcal{L}^\alpha$), and ω-projective (in $\mathcal{L}^\omega$). To be more concrete, comp. Table 1, these are pencils $L = \mathbf{p}(H, B) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ such that (we consider parameters k, m, Q, W as fixed)

$\mathcal{A}$: $H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W),\ B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$; in this case $L^\infty = H + (W \cap B) = (H + W) \cap B$. Note that $L^\infty \subset B \in \mathrm{Q}$ and therefore $L^\infty \in \mathcal{F}_{k,m+1}(\mathrm{Q}, W)$. In other words, L^∞ is a point of $\mathbf{A}_{k,m+1}(\mathrm{Q}, W)$.

$\mathcal{L}^\alpha$: $H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W),\ B \in \mathcal{F}_{k+1,m}(\mathrm{Q}, W)$.

$\mathcal{L}^\omega$: $H \in \mathcal{F}_{k-1,m-1}(\mathrm{Q}, W),\ B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$

(cf. Table 1).

Note that if $r_W < m + 1$ (in view of the global assumption $r_W \geq m$ this means $r_W = m$) then $\mathcal{A} \cup \mathcal{L}^\omega = \emptyset$. Looking at [8, Lemma 1.6] we see that in this case $\mathfrak{M}$ is disconnected as well or $m = w$. In the latter case also $\mathcal{A} \cup \mathcal{L}^\omega = \emptyset$. Besides, this also contradicts (1.9). Consequently, for $r_W < m + 1$ the horizon $\mathbf{A}_{k,m+1}(\mathrm{Q}, W)$ of $\mathfrak{M}$ looses its sense.

The problem whether one of the three above classes of lines is nonempty reduces, in fact, to the problem whether the corresponding class of 'possible tops' of these lines is nonempty. More precisely, we have the following criterion.

Lemma 2.1.

(i) *Let* $B \in \mathcal{F}_{k+1,m}(\mathrm{Q}, W)$*; then* $\mathrm{T}(B) \neq \emptyset$.

(ii) *Let* $B \in \mathcal{F}_{k+1,m}(\mathrm{Q}, W)$ *and* $U \in \mathrm{T}(B)$*; then there is an* $L = [H, B]_k \in \mathcal{L}^\alpha$ *such that* $U \in L$.

So, if $\mathcal{F}_{k+1,m}(\mathrm{Q}, W) \neq \emptyset$ *then* $\mathcal{L}^\alpha \neq \emptyset$.

(iii) *Let* $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$*; then* $\mathrm{T}(B) \neq \emptyset$.

(iv) *Let* $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$ *and* $U \in \mathrm{T}(B)$*. Then there are:*

- *an* $L' = [H', B]_k \in \mathcal{L}^\omega$ *(provided that* $m > 0$*) such that* $U \in L'$ *and*
- *an* $L'' = [H'', B]_k \in \mathcal{A}$ *such that* $U \in L''$.

Consequently, if $\mathcal{F}_{k+1,m+1}(\mathrm{Q}, W) \neq \emptyset$ *then* $\mathcal{A} \neq \emptyset$*, and* $\mathcal{L}^\omega \neq \emptyset$ *when* $m > 0$.

Proof. To justify (i) present B in the form $B = (B \cap W) \oplus D$, where $D \cap W = \Theta$ and $\dim(D) = k + 1 - m$. Let D' be a $(k - m)$-dimensional subspace of D and put $H := (B \cap W) + D'$. To justify (ii) we simply use (i) with B replaced by U to obtain the subspace H.

To justify (iii) we present B in the form $B = (B \cap W) \oplus D$ (now, $\dim(D) = k - m$) and proceed analogously to (i): $U = D + Z$, where Z is an m-dimensional subspace of $B \cap W$. To justify (iv) to get H' we apply (iii) with B replaced by U, and to get H'' we apply (i) with B replaced by U. □

Note that, sufficient conditions for the existence of the corresponding subspaces B in Lemma 2.1, i.e. for $\mathcal{F}_{k+1,m}(\mathrm{Q},W), \mathcal{F}_{k+1,m+1}(\mathrm{Q},W) \neq \emptyset$ are

$$m \leq r_W \leq r-(k+1)+m \quad \text{and} \quad m+1 \leq r_W \leq r-k+m, \text{ respectively.}$$

We say that an $U \in \mathcal{F}_{k,m}(\mathrm{Q},W)$ is an *α-point* iff each top containing U is of type α, i.e. each line through U is of type α. Similarly, an $U \in \mathcal{F}_{k,m}(\mathrm{Q},W)$ is an *ω-point* iff each top containing U is of type ω, i.e. each line through U is either affine or of type ω.

Lemma 2.2. *Let $U \in \mathcal{F}_{k,m}(\mathrm{Q},W)$.*

(i) *There is B such that $U \subset B \in \mathcal{F}_{k+1,m}(\mathrm{Q},W) \cup \mathcal{F}_{k+1,m+1}(\mathrm{Q},W)$.*

(ii) *U is an α-point iff $U^\perp \cap W \subset U$. In this case*

$$w \leq k+m. \tag{2.1}$$

Otherwise, if $U^\perp \cap W \not\subset U$ then there is a $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q},W)$ such that $U \subset B$.

(iii) *U is an ω-point when $U^\perp \subset U+W$. In this case*

$$w \geq n+m-2k. \tag{2.2}$$

Otherwise, if $U^\perp \not\subset U+W$ then there is a $B \in \mathcal{F}_{k+1,m}(\mathrm{Q},W)$ such that $U \subset B$.

Proof. Clearly, U is not maximal isotropic, so there is a B such that $U \subset B \in \mathrm{Q}_{k+1}$. As in [12] we obtain $m \leq \dim(B \cap W) \leq m+1$. This justifies (i).

To justify (ii) note that every $B \in \mathrm{Q}_{k+1}$ containing U belongs to $\mathcal{F}_{k+1,m}(\mathrm{Q},W)$, and then $U \prec B \subset U^\perp$ and $U \cap W \subset B \cap W \subset U^\perp \cap W$. If we have $\dim(B \cap W) = m$ for all B, then $\dim(U^\perp \cap W) = m$ and $U \cap W = U^\perp \cap W$. As $U \subset U^\perp$ by definition of U, the obtained condition is equivalent to $U^\perp \cap W \subset U$.

In this case we have $W = (U \cap W) \oplus D$, where D is contained in a linear complement of $U^\perp$. D is at most $\operatorname{codim}(U^\perp) = k$-dimensional, so $\dim(W) \leq m+k$.

To justify (iii) note, first, that if $U \subset B \in \mathrm{Q}_{k+1}$ then $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q},W)$. So, if $U \prec B \in \mathrm{Q}$, then $B = U \oplus \langle y \rangle$ with $y \in U^\perp \setminus U$. If $U^\perp \subset U+W$ then $y = u+w$ for some $u \in U$ and $w \in W \setminus U$ and then $B = U \oplus \langle w \rangle$. So, $B \cap W = (U \cap W) \oplus \langle w \rangle$. If there is $y \in U^\perp \setminus (U+W)$, then $U + \langle y \rangle$ intersects W in $U \cap W$.

If U is as required above then $n-k = \dim(U^\perp) \leq \dim(U+W) = w+k-m$. This gives $w \geq m+n-2k$. □

From Lemmas 2.1 and 2.2(ii), 2.2(iii) we infer the following geometrical fact.

Corollary 2.3.

(i) *If $w > k+m$ then through* each *point of $\mathfrak{M}$ there passes an ω-line and an affine line.*

(ii) *If $w < n+m-2k$ then through* every *point of $\mathfrak{M}$ there passes an α-line.*

Combining Lemmas 2.2(i) with 2.1(ii) and 2.1(iv) we obtain the following Corollary, a weakening of Corollary 2.3 but with more general assumptions.

Corollary 2.4. *If $U \in \mathcal{F}_{k,m}(\mathrm{Q},W)$ then there is a line in $\mathcal{G}_{k,m}(\mathrm{Q},W)$ through U. Consequently, if $\mathcal{F}_{k,m}(\mathrm{Q},W) \neq \emptyset$, then $\mathcal{G}_{k,m}(\mathrm{Q},W) \neq \emptyset$.*

Comments to Lemma 2.2.

ad (ii) Condition (2.1) is a *necessary* condition for the existence of an α-point.

By (1.12) and (2.1) we get $r_W \leq r - k + m \leq r - w$ (this implies $r - r_W \geq w$). This condition is not inconsistent. So, it may happen that $\mathfrak{M}$ contains both α-points and ω-tops.

One can note (it is, practically, proved in the proof of Lemma 2.2(ii) that if (2.1) is satisfied and $U \in \mathrm{Q}_k$ then there is a subspace W such that U is an α-point in $\mathbf{A}_{k,m}(\mathrm{Q}, W)$ and $\dim(W) = w$.

ad (iii) Analogously, condition (2.2) is a *necessary* condition for the existence of an ω-point.

It is seen that (under suitable assumption, obtained by (1.12) and (2.2): $r - r_W \geq k - m \geq n - k - w$) the space $\mathfrak{M}$ may contain both ω-points and α-tops.

And there do exist W for which associated spine spaces contain an ω-point.

As an immediate consequence of Lemma 2.2(iii) we obtain the following.

Corollary 2.5. *Assume that* $w < n + m + 1 - 2k$. *Then, for every* $U \in \mathcal{F}_{k,m+1}(\mathrm{Q}, W)$ *there is* $L \in \mathcal{A}_{k,m}(\mathrm{Q}, W)$ *such that* $U = L^\infty$.

3 Examples, particular cases

Let us examine in some detail polar spine spaces of some, particularly natural classes.

3.1 Grassmannians of affine polar spaces

Assume that W is a hyperplane of $\mathfrak{P}$; in turn this is equivalent to say that $\mathrm{Sub}_1(W)$ is a hyperplane in $\mathfrak{Q}$. In this case we have

$$m = k - 1 \text{ and} \tag{3.1}$$

$$\dim(W \cap Y) = \begin{cases} r & \text{when } Y \subset W \\ r - 1 & \text{when } Y \not\subset W \end{cases} \quad \text{for every } Y \in \mathrm{Q}_r. \tag{3.2}$$

It is clear that in this case

$$\mathcal{F}_{k,m}(\mathrm{Q}, W) \neq \emptyset; \text{ in view of Corollary 2.4, } \mathbf{A}_{k,m}(\mathrm{Q}, W) \text{ is nontrivial}$$

simply, because it is impossible to have $\mathrm{Q}_k \subset \mathrm{Sub}_k(W)$. However, this case raises several degenerations concerning the structure of strong subspaces of $\mathfrak{M}$.

Lemma 3.1.

(i) *Let* $B \in \mathrm{Sub}_{k+1}(\mathbb{V})$. *Then either* $\dim(B \cap W) = k + 1 = m + 2$ *(and then* $B \subset W$*) or* $\dim(B \cap W) = k = m + 1$. *Therefore, there is no strong subspace in* $\mathcal{T}^\alpha$. *Moreover, by the same reasons,* $\mathcal{L}^\alpha = \emptyset$.

(ii) *If* $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$ *then* $\mathrm{T}(B) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W) \in \mathcal{T}^\omega$ *is a k-dimensional punctured projective space.*

(iii) *Let* $X = [H, Y]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$, $H \in \mathrm{Sub}_{k-1}(\mathbb{V})$, $Y \in \mathrm{Q}r$. *Assume that* $\dim(H \cap W) = m = k - 1$ *i.e.* $H \subset W$. *If* $Y \subset W$ *then, clearly,* $X = \emptyset$. *If* $Y \not\subset W$ *then* $X \in \mathcal{S}^\alpha$ *is a* $(r - k)$*-dimensional affine space.*

(iv) *Let* $X = [H, Y]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$, $H \in \mathrm{Sub}_{k-1}(\mathbb{V})$, $Y \in \mathrm{Q}r$. *Assume that* $\dim(H \cap W) = m - 1 = k - 2$ *i.e.* $H \not\subset W$. *Then* $Y \not\subset W$ *and, consequently,* $\dim(Y \cap W) = r - 1$. *In this case* $X \in \mathcal{S}^\omega$ *is a* $(r - k)$*-dimensional projective space.*

Corollary 3.2. *If* $4 \le k + 2 \le r$ *then every line of* $\mathfrak{M}$ *has at least two extensions to a maximal at least* 2*-dimensional strong subspace: one to a top, and one to a star.*

3.2 Spine spaces with isotropic 'holes'

Next, let us assume that $W \in \mathrm{Q}$ i.e. W is isotropic. In this case we have

$$r_W = w. \tag{3.3}$$

So, let $m < w, k$; let us take arbitrary $D \in \mathrm{Sub}_m(W)$ and $Y \in \mathrm{Q}_r$ with $W \subset Y$. Then there is $Y_0 \in \mathrm{Q}_r$ such that $Y \cap Y_0 = D$. Consider any U such that $\dim(U) = k$ and $D \subset U \subset Y_0$; then $U \in \mathcal{F}_{k,m}(\mathrm{Q}, W)$. Thus we have proved that

$$\mathcal{F}_{k,m}(\mathrm{Q}, W) \neq \emptyset;\ \text{in view of Corollary 2.4, } \mathbf{A}_{k,m}(\mathrm{Q}, W) \text{ is nontrivial.}$$

Note that if we assume (1.10) then $k + w - m \le r + r - m \le n - m \le n$ follows, so (1.8) holds as well.

Next, let us pay attention to the problem of extending lines. Namely, let $L = \mathbf{p}(H, B) \in \mathcal{L}^\omega$. So, $\dim(B \cap W) = m + 1$. Suppose that $r = m + 1$; then we obtain contradictory $m < k < r = m + 1$. As above, we extend W to a maximal isotropic Y and find maximal isotropic Y' with $Y \cap Y' = B$. This proves

Lemma 3.3. *If* $k < r-1$*, then every line in* $\mathcal{L}^\omega$ *can be extended to an at least* 2*-dimensional star.*

4 Binary collinearity

Let us start with a Chow'like result concerning binary collinearity $\boldsymbol{\lambda}$ of points in a polar spine space $\mathfrak{M} = \mathbf{A}_{k,m}(\mathrm{Q}, W)$ defined for some integers k, m and a fixed subspace W of a vector space V equipped with a suitable form ξ. To this aim standard reasoning similar to this of [6, 7, 17] can be used:

> a line through two distinct points is the intersection of all the maximal $\boldsymbol{\lambda}$-cliques which contain these points.

In the sequel we intensively analyse Table 2. Let $U_1 \ \boldsymbol{\lambda}\ U_2$, $U_1 \neq U_2$. Put $L = \overline{U_1, U_2}$. Evidently, every line $L = \mathbf{p}(H, B)$ can be extended to a top $T = \mathrm{T}(B) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$, which is a $(k - m)$-dimensional ($T \in \mathcal{T}^\alpha$) or a k-dimensional ($T \in \mathcal{T}^\omega$) slit space. We have assumed that $k > 1$. So, when $m < k - 1$ then T is greater than L. For any triangle $U_1, U_2, U_3 \in T$ we have $\boldsymbol{\Delta}_{\boldsymbol{\lambda}}(U_1, U_2, U_3)$ and $T = [U_1, U_2, U_3]_{\boldsymbol{\lambda}}$.

If L is an α-projective line or an affine line then it has at least one extension to a star S in $\mathcal{S}^\alpha$, which are $(r - k)$-dimensional slit spaces. Consequently, $L = T \cap S$. Assume that $k < r - 1$, so $L \subsetneq S$.

In this point we can choose one of the following two ways. Firstly, we notice that there is a finite system $U_1, U_2, \dots, U_t \in S$ such that $\boldsymbol{\Delta}_{\boldsymbol{\lambda}}(U_1, U_2, \dots, U_t)$, so $S' := S = [U_1, U_2, \dots, U_t]_{\boldsymbol{\lambda}}$. Secondly, we can extend U_1, U_2 to any triangle $U_1, U_2, U_3 \in S$ and note that $S' := [U_1, U_2, U_3]_{\boldsymbol{\lambda}}$ is the union of all the extensions of the plane spanned by U_1, U_2, U_3 to a maximal $\boldsymbol{\lambda}$-clique. In both cases $L = S' \cap T$ and thus L can be defined in terms of $\boldsymbol{\lambda}$.

A problem may arise when $L \in \mathcal{L}^\omega$. In this case each extension of L to a star S is contained in a segment $[H, Y]_k$ with a maximal totally isotropic extension Y of $B \supset H$ and it has dimension $\dim(W \cap Y) - m$. So, it may degenerate to the line L when $\dim(W \cap Y) = m + 1$. Is it possible that every such an extension Y intersects W in dimension $m + 1$? Recall that the condition $L \in \mathcal{L}^\omega$ yields $\dim(B \cap W) = m + 1$ and therefore we obtain $W \cap Y = W \cap B$, for every $\mathrm{Q}_r \ni Y \supset B$. So, our problematic case reduces to the question: for which $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$ there is no reasonable extension Y and: when each such a B has a required extension. Note that to find Y it suffices to find D such that $B \prec D \in \mathrm{Q}$ and $\dim(D \cap W) = m + 2$; then Y is an extension of D to a maximal totally isotropic subspace. On the other hand, the existence of D in question can be assured by a suitable substitution in Lemma 2.2(ii), which yields a sufficient condition for the existence of our Y:

$$w > k + m + 2. \tag{4.1}$$

As a consequence we can formulate the following result.

Theorem 4.1 (The Chow Theorem for $\mathfrak{M}$)**.** *If $m < k - 1$, $k < r - 1$, and each line in $\mathcal{L}^\omega$ can be extended to at least 2-dimensional star (which is assured, e.g. by* (4.1)*) then the structures $\mathfrak{M}$ and $\langle \mathcal{F}_{k,m}(\mathrm{Q}, W), \boldsymbol{\lambda} \rangle$ are definitionally equivalent.*

In particular, in view of Corollary 3.2 and Lemma 3.3, the Chow theorem holds in $\mathfrak{M}$ when W is an isotropic subspace and $k < r - 1$, and it holds in $\mathfrak{M}$ when W is a hyperplane and $4 \le k + 2 \le r$.

One can continue these investigations in the fashion of [17] considering graphs of collinearity with some sorts of lines distinguished ($\boldsymbol{\lambda}^\alpha$, $\boldsymbol{\lambda}^\omega$, $\boldsymbol{\lambda}^{\alpha\vee\omega}$ etc.). Observing criteria in Lemma 2.2 and Corollary 2.5 we see that it may be a hard work: α-points and ω-points may appear, 'deep' improper points may appear as well.

5 Maximal cliques of $\boldsymbol{\lambda}^\sigma$

Let σ be a one of the symbols

$$\alpha, \omega, \alpha \vee \omega, \alpha^+, \omega^+.$$

The classes $\mathcal{L}^\sigma$ with $\sigma \in \{\alpha, \omega\}$ are already defined (usually, the arguments like k, m, $\mathbb{V}$, Q, W will be omitted, if unnecessary or fixed). Next, $\mathcal{L}^{\sigma^+} := \mathcal{L}^\sigma \cup \mathcal{A}$, and, finally $\mathcal{L}^{\alpha\vee\omega} = \mathcal{L}^\alpha \cup \mathcal{L}^\omega$. It is evident that

$$\mathfrak{M}^\sigma := \langle \mathcal{F}_{k,m}(\mathrm{Q}, W), \mathcal{L}^\sigma_{k,m}(\mathrm{Q}, W) \rangle \tag{5.1}$$

is a partial linear space for every admissible symbol σ as above, but it may be trivial for particular values of k, m, r, w etc.: *it may have a void line set*. Let us write $\boldsymbol{\lambda}^\sigma$ for the binary collinearity of points of $\mathfrak{M}^\sigma$. Let $\boldsymbol{\lambda}^{\mathrm{af}}$ be the binary collinearity in

$$\mathfrak{A} = \langle \mathcal{F}_{k,m}(\mathrm{Q}, W), \mathcal{A}_{k,m}, \| \rangle.$$

In the first part of this section we shall determine (maximal) cliques of $\boldsymbol{\lambda}^\sigma$ for particular values of σ as above. Clearly, each such a clique is a $\boldsymbol{\lambda}$-clique. So, it is contained in an appropriate strong subspace of $\mathfrak{M}$.

We begin with some results which state, generally, that the affine lines in many cases can be 'eliminated': they are definable in terms of other projective lines.

Proposition 5.1. *Assume that $m > 0$ or $w < r - k$. Then for arbitrary triple $U_1, U_2, U_3 \in \mathcal{F}_{k,m}(\mathrm{Q}, W)$ we have*

$$\begin{aligned} &\textit{there is a line } L_0 \in \mathcal{A} \textit{ s.t. } U_1, U_2, U_3 \in L_0 \iff \\ &\qquad \textit{there is a triangle } L_1, L_2, L_3 \in \mathcal{L}^{\alpha\vee\omega}_{k,m} \textit{ s.t. } U_i \in L_i \textit{ for } i = 1,2,3 \\ &\qquad\qquad \textit{\& there is no } L \in \mathcal{L}^{\alpha\vee\omega} \textit{ s.t. } U_i, U_j \in L \textit{ for some } 1 \le i < j \le 3. \end{aligned} \tag{5.2}$$

Proof. Let $U_1, U_2, U_3 \in L_0 \in \mathcal{A}$; then we can write $L_0 = \mathbf{p}(H, B) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ for suitable H, B. As in the proof of Lemma 6.6 we examine extensions of L_0 to maximal strong subspaces of $\mathfrak{M}$. First, let us have a look at $\mathrm{T}(B) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$. It is an affine space only when $m = 0$; otherwise it contains a nonaffine semiaffine plane A which contains L_0. The lines on A are all in $\mathcal{L}^{\alpha\vee\omega}$ except the direction of L_0. It suffices to find adequate triangle on A to justify ($\Rightarrow$:) of (5.2).

Next, assume that $m = 0$ and take a look at extensions of L_0 of the form $[H, Y]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$, then $B \subset Y \in \mathrm{Q}_r$. This extension is an affine space when $\dim(W \cap Y) = r - k$. If there is no such Y, which is assured by the condition assumed, our extension contains a plane A as above and ($\Rightarrow$:) of (5.2) is justified.

To prove ($\Leftarrow$:) it suffices to note that a triangle spans a plane A in $\mathfrak{M}$. Since this plane contains projective lines it is not affine, and since there are non projectively joinable points on A it contains just one direction of affine lines. The rest is evident. □

Thus we have proved the following result.

Proposition 5.2. *Under assumptions made in Proposition 5.1 the class $\mathcal{A}_{k,m}(\mathrm{Q}, W)$ is definable in $\mathfrak{M}^{\alpha\vee\omega}$. That means: $\mathfrak{M}$ is definable in $\mathfrak{M}^{\alpha\vee\omega}$.*

Remark 5.3. Analysing the proof of Proposition 5.1 one can note an even more detailed result:

(i) If $m > 0$ then $\mathcal{A}$ is definable in $\mathfrak{M}^\omega$ and therefore then $\mathfrak{M}^{\omega^+}$ is definable in $\mathfrak{M}^\omega$.

(ii) If every affine line $L = \mathbf{p}(H, B)$ can be extended to a non-affine star ($\dim(W \cap Y) \ge r - k + m - 3$ for some maximal isotropic Y containing B) then $\mathcal{A}$ is definable in $\mathfrak{M}^\alpha$. So, $\mathfrak{M}^{\alpha^+}$ is definable in $\mathfrak{M}^\alpha$.

For an arbitrary set X of points we write

$$\mathrm{L}(X) = \{L \in \mathcal{G}_{k,m}(\mathrm{Q}, W) : L \subset X\}.$$

Let us remind well known and fundamental classification of lines in strong subspaces of $\mathfrak{M}$.

Fact 5.4. *Let X be a strong subspace of $\mathfrak{M}$ and $\mathcal{X} = \mathrm{L}(X)$.*

$$\begin{aligned} &\textit{If } X \in \mathcal{T}^\alpha \textit{ then } \mathcal{X} \subset \mathcal{L}^\alpha, && \textit{if } X \in \mathcal{S}^\alpha \textit{ then } \mathcal{X} \subset \mathcal{L}^{\alpha^+}, \\ &\textit{if } X \in \mathcal{T}^\omega \textit{ then } \mathcal{X} \subset \mathcal{L}^{\omega^+}, && \textit{if } X \in \mathcal{S}^\omega \textit{ then } \mathcal{X} \subset \mathcal{L}^\omega. \end{aligned}$$

Let us note an elementary

Fact 5.5. *Let $\mathfrak{S}$ be a n_0-dimensional slit space with a w_0-dimensional hole i.e. let $\mathfrak{S}$ result from a n_0-dimensional projective space by deleting a w_0-dimensional subspace $\mathcal{D}$. Let $\mathcal{L}_0$ be the class of projective lines of $\mathfrak{S}$ and $\boldsymbol{\lambda}_0$ be the binary collinearity determined by $\mathcal{L}_0$. Then*

(i) *The maximal affine subspaces of $\mathfrak{S}$ (i.e. maximal strong subspace w.r.t. to the family of affine lines of $\mathfrak{S}$) are w_0+1 dimensional affine spaces. Two such subspaces either coincide or are disjoint.*

(ii) *The maximal projective subspaces of $\mathfrak{S}$ are $(n_0 - w_0 - 1)$-dimensional projective spaces. These are linear complements of D and the elements of $\mathcal{K}^*(\boldsymbol{\lambda}_0)$.*

(iii) *Let X be a maximal projective subspace of $\mathfrak{S}$; then $X \in \mathcal{K}^{n_0-w_0}_{\boldsymbol{\lambda}_0}$.*

If $w_0 \le n_0 - 3$ (i.e. every projective line of $\mathfrak{S}$ has two distinct extensions to maximal projective subspaces) then the Chow Theorem holds:

The class $\mathcal{L}_0$ is definable in terms of $\boldsymbol{\lambda}_0$.

Observing Table 2 and Fact 5.5 we conclude with the following.

Corollary 5.6.

(i) *The maximal $\boldsymbol{\lambda}^\alpha$-cliques are $(k-m)$-dimensional projective tops: elements of $\mathcal{T}^\alpha$, and $(r+m-k-\dim(W\cap Y))$-dimensional projective spaces of the form*

$$[H,E]_k, \text{ where } H \subset E \subset Y,\ E \cap ((W\cap Y)+H) = H$$

contained in a suitable element $[H,Y]_k \cap \mathcal{F}_{k,m}(\mathrm{Q},W)$ of $\mathcal{S}^\alpha$.

(ii) *The maximal $\boldsymbol{\lambda}^\omega$-cliques are $(\dim(W\cap Y)-m)$-dimensional projective stars: elements of $\mathcal{S}^\omega$, and m-dimensional projective spaces of the form*

$$[G,B]_k, \text{ where } G \subset B,\ G \cap (B \cap W) = \Theta$$

contained in a suitable element $\mathrm{T}(B) \cap \mathcal{F}_{k,m}(\mathrm{Q},W)$ of $\mathcal{T}^\omega$.

(iii) *The maximal $\boldsymbol{\lambda}^{\alpha^+}$-cliques are elements of $\mathcal{T}^\alpha \cup \mathcal{S}^\alpha$, and the maximal $\boldsymbol{\lambda}^{\omega^+}$-cliques are elements of $\mathcal{T}^\omega \cup \mathcal{S}^\omega$.*

(iv) $\mathcal{K}^*(\boldsymbol{\lambda}^{\alpha\vee\omega}) = \mathcal{K}^*(\boldsymbol{\lambda}^\alpha) \cup \mathcal{K}^*(\boldsymbol{\lambda}^\omega)$, *so the maximal $\boldsymbol{\lambda}^{\alpha\vee\omega}$-cliques are of the form* (i) *and of the form* (ii) *above.*

Corollary 5.7. *The following variants of the Chow Theorem hold in projective reducts of $\mathfrak{M}$.*

(i) *If $m > 1$ then $\mathfrak{M}^\omega$ is definable in $\langle \mathcal{F}_{k,m}(\mathrm{Q},W), \boldsymbol{\lambda}^\omega\rangle$.*

(ii) *If every projective line $L = \mathbf{p}(H,B) \in \mathcal{L}^\alpha$ can be extended to a non-affine star ($\dim(W\cap Y) \le r-k+m-2$ for some maximal isotropic Y containing B) then $\mathfrak{M}^\alpha$ is definable in $\langle \mathcal{F}_{k,m}(\mathrm{Q},W), \boldsymbol{\lambda}^\alpha\rangle$.*

6 Parallelism, horizon, projective completion(s)

Let us summarize the following

(i) $\{L^\infty : L \in \mathcal{A}_{k,m}\} \subset \mathcal{F}_{k,m+1}(\mathrm{Q}, W)$.

(ii) by Lemma 2.2(ii) $\{L^\infty : L \in \mathcal{A}_{k,m}\} \supset \{U \in \mathcal{F}_{k,m+1}(\mathrm{Q}, W) : U^\perp \cap W \not\subset U\}$,

(iii) $\{L^\infty : L \in \mathcal{A}_{k,m}\} \supset \mathcal{F}_{k,m+1}(\mathrm{Q}, W)$, when $w < n + m + 1 - 2k$ by Corollary 2.5.

Note 6.1. The set $\{L^\infty : L \in \mathcal{A}_{k,m}\}$ will be frequently referred to as *the horizon of* $\mathfrak{M}$. We warn that, generally it does not coincide with the horizon $\mathrm{Q}_k \setminus \mathcal{F}_{k,m}(\mathrm{Q}, W)$ as defined in Section 1.

Note that the inequality in (iii) above is only sufficient. One can compute e.g.

Lemma 6.2. *Let $W \in \mathrm{Q}$. Then the claim of Corollary 2.5 holds i.e. for every $U \in \mathcal{F}_{k,m+1}(\mathrm{Q}, W)$ there is an $L \in \mathcal{A}_{k,m}(\mathrm{Q}, W)$ such that $U = L^\infty$. Consequently,*

$$\{L^\infty : L \in \mathcal{A}_{k,m}\} = \mathcal{F}_{k,m+1}(\mathrm{Q}, W).$$

Proof. By assumption, $\dim(U \cap W) = m + 1$. There are extensions $Y_1, Y_2 \in \mathrm{Q}_r$ such that $U \subset Y_1$, $W \subset Y_2$, and $Y_1 \cap Y_2 = U \cap W$. Take $B \in [U, Y_1]_{k+1}$; then $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$ and we are through. $\square$

For a subset X of $\mathcal{F}_{k,m}(\mathrm{Q}, W)$ we write

$$X^\infty := \{N^\infty : \mathcal{A}_{k,m} \ni N \subset X\}.$$

Lemma 6.3. *Let $L = \mathbf{p}(H, B) \in \mathcal{L}^\omega_{k,m+1} \cup \mathcal{L}^\alpha_{k,m+1}$.*

(i) *If $L \in \mathcal{L}^\alpha$ then there is in $\mathfrak{M}$ a plane $A = [G, B]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ with $G \in \mathcal{F}_{k-2,m}(\mathrm{Q}, W)$ such that $A^\infty = L$.*

(ii) *Assume that $w < n + m - 2k$. If $L \in \mathcal{L}^\omega$ then $A = [H, E]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ with some $E \in \mathcal{F}_{k+2,m+2}(\mathrm{Q}, W)$ is a plane in $\mathfrak{M}$ such that $A^\infty = L$.*

Proof. Ad (i): By assumption, $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$ and $H \in \mathcal{F}_{k-1,m+1}(\mathrm{Q}, W)$. There is a point $U \in L$, so $U \in \mathcal{F}_{k,m+1}(\mathrm{Q}, W)$. By Lemma 2.1(iii) there is an H_0 such that $U \succ H_0 \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W)$. Set $G = H_0 \cap H$; clearly, $\dim(G) = k - 2$, so $[G, B]_k$ is a plane in $\mathbf{P}_k(\mathrm{Q})$. Taking into account the fact that $H, H_0 \succ G$ we obtain $\dim(G \cap W) \in \{m+1, m\}$ and $\dim(G \cap W) \in \{m, m-1\}$. Thus $\dim(G \cap W) = m$. As $L \subset [G, B]_k$ and $[G, B]_k \supset [H_0, B]_k$ while $[H_0, B]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W) \in \mathcal{A}_{k,m}$ we get that $A \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ is a plane in $\mathfrak{M}$ with $A^\infty = L$.

Ad (ii): By assumption, $B \in \mathcal{F}_{k+1,m+2}(\mathrm{Q}, W)$ and $H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W)$. As above, we take any $U \in L$, so $U \in \mathcal{F}_{k,m+1}(\mathrm{Q}, W)$. By assumption of (ii) (they yield $w < n + (m+2) - 2(k+1)$) and Lemma 2.2(iii) there is an E such that $B \prec E \in \mathcal{F}_{k+2,m+2}(\mathrm{Q}, W)$. Next, there is $B_0 \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$ with $U \subset E$: $B = U + \langle b \rangle$ with a $b \in W$ and $E = B + \langle e \rangle$ with an $e \notin W$; we take $B_0 = U + \langle e \rangle$. Clearly, $E = B + B_0$ and $[H, B_0]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W) \in \mathcal{A}_{k,m}$. As above we argue that $A = [H, E]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ is a plane in $\mathfrak{M}$, and $L = A^\infty$. $\square$

Roughly speaking, Lemma 6.3 gives sufficient condition under which a (projective) line L of $\mathbf{A}_{k,m+1}(\mathrm{Q}, W)$ can be considered as a 'horizon' – the set of improper points of a plane in $\mathbf{A}_{k,m}(\mathrm{Q}, W)$. On the other hand, considering classification of planes in $\mathbf{A}_{k,m}(\mathbb{V}, W)$ presented in some details in [14] we easily conclude with the following

Lemma 6.4. *Let* $X \subset \mathrm{Sub}_k(\mathbb{V})$ *and* $A = X \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ *be a plane of* $\mathfrak{M}$ *such that* A^∞ *is a line of* $\mathbf{A}_{k,m+1}(\mathrm{Q}, W)$*. Then one of the following holds:*

(i) $X = [G, B]_k$ *for some* $G \in \mathcal{F}_{k-2,m}(\mathrm{Q}, W)$*,* $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$*.*

(ii) $X = [H, E]_k$ *for some* $H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W)$ *and* $E \in \mathcal{F}_{k+2,m+2}(\mathrm{Q}, W)$*.*

Conversely, if X *is defined by* (i) *then* $X \cap \mathcal{F}_{k,m+1}(\mathrm{Q}, W) = \big(X \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)\big)^\infty \in \mathcal{L}^\alpha_{k,m+1}$*, and if* (ii) *holds, then* $X \cap \mathcal{F}_{k,m+1}(\mathrm{Q}, W) \in \mathcal{L}^\omega_{k,m+1}$*.*

So, Lemma 6.4 states that the 'horizon' of any (affine) plane of $\mathcal{F}_{k,m}(\mathrm{Q}, W)$ is a (projective) line of $\mathcal{F}_{k,m+1}(\mathrm{Q}, W)$. As usually, the conditions of Lemma 6.3 are only sufficient. Dealing with concrete cases one should look for suitable extendability more or less 'by hand'. Let us quote an example:

Lemma 6.5. *Let* $W \in \mathrm{Q}$*. If* $L = \mathbf{p}(H, B) \in \mathcal{L}^\omega_{k,m+1}$ *then* $A = [H, E]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ *with some* $E \in \mathcal{F}_{k+2,m+2}(\mathrm{Q}, W)$ *is a plane in* $\mathfrak{M}$ *such that* $A^\infty = L$*.*

Hint. With the reasoning as in the proof of Lemma 6.3(ii) we look for an E such that $B \prec E \in \mathcal{F}_{k+2,m+2}(\mathrm{Q}, W)$. It suffices to find an E such that $E \cap W = B \cap W$ just considering suitable maximal isotropic extensions of B and W. □

To accomplish this part of investigations on the parallelism let us check if directions are 'isolated': when for an affine line L of $\mathfrak{M}$ there are other lines parallel to L and coplanar with L; with the plane in question being affine in $\mathfrak{M}$.

Lemma 6.6. *Let* $L = \mathbf{p}(H, B) \in \mathcal{A}_{k,m}$ *and* $U = L^\infty$*.*

(i) *Assume that* $k > m + 1$*. There is an* $L_0 = \mathbf{p}(H_0, B) \in \mathcal{L}^\alpha_{k,m+1}$ *such that* $U \in L_0$ *and* $A = [H_0 \cap H, B]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ *is a plane in* $\mathfrak{M}$ *such that* $A^\infty = L_0$*. We have* $\dim((H_0 \cap H) \cap W) = m - 1$*.*

(ii) *If* B *has an extension to a* $Y \in \mathrm{Q}_r$ *such that* $\dim(W \cap Y) \geq m + 2$ *(this yields, necessarily,* $m + 2 \leq r_W$*) then there exists an* $L_1 = \mathbf{p}(H, B_1) \in \mathcal{L}^\omega_{k,m+1}$ *such that* $U \in L_1$ *and* $A = [H, B + B_1]_k$ *is a plane in* $\mathfrak{M}$ *such that* $A^\infty = L_1$*. We have* $\dim((B_1 + B) \cap W) = m + 2$*.*

Proof. Let us begin with a reminder: $H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W)$, $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$. Have a look at the extension of L to a top $T = \mathrm{T}(B) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ (an ω-top in this case). Since $k > m + 1$, this is a semiaffine space, and its hole is at least 1-dimensional. Let L_0 be any line of $\mathbf{P}_k(\mathbb{V})$ contained in this hole and A be the plane spanned by $L \cup L_0$. That way we justify (i).

Next, let us look for appropriate extension of L to an α-star $S = [H, Y]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$. In general, it is a $(r-k)$-dimensional semiaffine space. Since $\mathrm{Q}_{k+1} \neq \emptyset$ we have $k+1 \leq r$. So, S is at least a line. To assure that the hole of S contains at least a line of $\mathbf{P}_k(\mathbb{V})$ we must assume that $\dim(W \cap Y) \geq m + 2$. That way we justify (ii). □

Let us remind that for distinct affine lines L_1, L_2 contained in a strong subspace of $\mathbf{A}_{k,m}(\mathbb{V}, W)$ their parallelism $\parallel$ can be characterized by the following formula (so called *Veblenian parallelism*).

$$L_1 \parallel_{\mathsf{V}} L_2 \iff \text{there are lines } L_1', L_2' \text{ s.t. } |L_1' \cap L_2'| = 1, \\ \text{and } L_1' \cap L_2' \cap L_i = \emptyset,\ |L_i' \cap L_j| = 1 \text{ for } i = 1,2, \quad (6.1)$$

and then $L_1 \parallel L_2$ iff $L_1 \parallel_{\mathsf{V}} L_2$. It is easy to note that the same formula (6.1) characterizes parallelism of affine lines contained in a common strong subspace of $\mathfrak{M}$.

Let us begin with a special form of connectedness of the space of lines over $\mathfrak{M}$:

Lemma 6.7. *Let $U \in \mathcal{F}_{k,m+1}(\mathrm{Q}, W)$ and $L_1^\infty = U = L_2^\infty$ for $L_1, L_2 \in \mathcal{A}_{k,m}$. Moreover, assume that $k \le r-2$. Then there are lines $M_1, \ldots, M_t \in \mathcal{A}_{k,m}$ ($t \le r+1$) such that $L_1 = M_1$, $L_2 = M_t$, and $M_i^\infty = U$, M_i, M_{i+1} are in a strong (semiaffine) subspace of $\mathfrak{M}$ or $M_i = M_{i+1}$, for $i = 1, \ldots, t-1$.*

Proof. Write $M_1 := L_1$. We have $H_1, H_2 \subset U \subset B_1, B_2, U \in \mathcal{F}_{k,m+1}(\mathrm{Q}, W)$ and $B_i \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$, $H_i \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W)$ for $i = 1,2$. Put $N_1 := [H_1, B_2]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$, $N_2 := [H_2, B_1]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$. Then $N_1, N_2 \in \mathcal{A}_{k,m}$, $N_2^\infty = U = N_1^\infty$.

If $L_1 = N_2$ we set $M_2 := L_1$. Assume that $L_1 \ne N_2$. Note that $L_1, N_2 \in \mathrm{T}(B_1) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W) \in \mathcal{T}^\omega$. So, we set $M_2 := N_2$.

Observe that $N_2, L_2 \subset [H_2, V]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$. So, the problem reduces to find a required sequence of lines in the projective star $\mathrm{S}(H_2)$. Let $B_1 \subset Y' \in \mathrm{Q}_r$, $B_2 \subset Y'' \in \mathrm{Q}_r$. There is a sequence $Y_2, \ldots, Y_t$ of elements of Q_r such that $Y' = Y_2$, $Y'' = Y_t$, and $U \subset Y_i$, $E_i := Y_i \cap Y_{i+1}$, $\dim(E_i) = r-1$ for $i = 2, \ldots, t-1$, $t \le r+1$. Then $\dim(E_i \cap W) \ge m+1$. From our assumption $k+1 \le r-1 = \dim(E_i)$. So, for every $i = 3, \ldots, t-1$ one can find D_i such that $U \prec D_i \subset E_{i-1}$ and $\dim(D_i \cap W) = m+1$. With $N_i = [H_2, D_i]_k$ we close our proof. $\square$

Corollary 6.8. *Under assumptions of Lemma 6.7 the parallelism $\parallel$ in $\mathfrak{M}$ coincides with the transitive closure of $\parallel_{\mathsf{V}}$. Actually, it is the $(r+1)$-th relational power $\underbrace{\parallel_{\mathsf{V}} \circ \cdots \circ \parallel_{\mathsf{V}}}_{(r+1)\ times}$ of $\parallel_{\mathsf{V}}$, defined by* (6.1)*, and therefore $\parallel$ is definable in the incidence structure $\mathfrak{M}$.*

As an immediate corollary we conclude with the following theorem.

Theorem 6.9. *Assume the following*

(1) *$w < n+m-2k$ to assure that every line in $\mathcal{L}^\omega_{k,m+1}$ can be extended to a nontrivial α-star of $\mathfrak{M}$ (cf. Lemma 6.3),*

(2) *$w < n+m+1-2k$ to assure extendability of each improper point to an affine line (cf. Corollary 2.5),*

(3) *$m+1 > 0$ or $w < r-k$ to assure definability of $\mathcal{A}_{k,m+1}$ in $\mathbf{A}_{k,m+1}(\mathrm{Q}, W)$ in terms of its projective lines (cf. Proposition 5.2),*

(4) *$k \le r-2$, to assure definability of parallelism in $\mathfrak{M}$ (cf. Corollary 6.8).*

Then $\mathbf{A}_{k,m+1}(\mathrm{Q}, W)$ is definable within $\mathbf{A}_{k,m}(\mathrm{Q}, W)$.

In analogy to [17] in the fragment of $\mathbf{P}_k(\mathrm{Q})$ determined by $\mathcal{R} := \mathcal{F}_{k,m}(\mathrm{Q}, W) \cup \mathcal{F}_{k,m+1}(\mathrm{Q}, W)$ (i.e. the points of $\mathfrak{M}$ and the points of the "affine horizon" of $\mathfrak{M}$) we distinguish two substructures corresponding to two possible sorts of lines. Let us set $\mathcal{L}^{\tau}_{k,m} := \{[H, B]_k : H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W),\ B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)\}$; it is seen that $\mathcal{L}^{\tau}_{k,m} = \{\bar{L} : L \in \mathcal{A}_{k,m}\}$. Note evident relation:

$$\{L : L \text{ is a line of } \mathbf{P}_k(\mathrm{Q}),\ L \subset \mathcal{R}\} = \mathcal{L}^{\alpha}_{k,m} \cup \mathcal{L}^{\omega}_{k,m} \cup \mathcal{L}^{\alpha}_{k,m+1} \cup \mathcal{L}^{\omega}_{k,m+1} \cup \mathcal{L}^{\tau}_{k,m}. \quad (6.2)$$

We define (write: $-\alpha = \omega,\ -\omega = \alpha$)

$$\mathfrak{N}^{\sigma} := \langle \mathcal{R}, \mathcal{L}^{\sigma}_{k,m} \cup \mathcal{L}^{\tau}_{k,m} \cup \mathcal{L}^{-\sigma}_{k,m+1} \rangle \text{ with } \sigma \in \{\alpha, \omega\}.$$

Evidently, $\mathfrak{M}^{\sigma}$ can be embedded into $\mathfrak{N}^{\sigma}$. Intuitively, while the structure

$$\langle \mathcal{R}, \{L : L \text{ is a line of } \mathbf{P}_k(\mathrm{Q}),\ L \subset \mathcal{R}\} \rangle$$

can be considered as a *projective completion of* $\mathfrak{M}$ and, under specific assumptions, it is definable in $\mathfrak{M}$, $\mathfrak{N}^{\sigma}$ is a projective completion of $\mathfrak{M}^{\sigma}$.

To close this part it is worth to note the following analogue of Remark 5.3 and, at the same time, an analogue of [17, Fact 3.1].

Remark 6.10. Assume (2) and (4) from Theorem 6.9.

(i) If $m > 0$ (cf. Remark 5.3) then the structure $\mathfrak{N}^{\omega}$ is definable in $\mathfrak{M}^{\omega}$.

(ii) If for each affine line $L = \mathbf{p}(H, B)$ there is a maximal isotropic Y such that $B \subset Y$ and $\dim(W \cap Y) \geq r - k + m - 3$ (cf. Remark 5.3) and $w < n + m - 2k$ (cf. Lemma 6.3), then the structure $\mathfrak{N}^{\alpha}$ is definable in $\mathfrak{M}^{\alpha}$.

According to Corollary 2.5 and Lemma 6.3, under condition $w < n + m - 2k$ each point of $\mathbf{A}_{k,m+1}(\mathrm{Q}, W)$ is a direction of a line in $\mathfrak{M}$ and each line of $\mathbf{A}_{k,m+1}(\mathrm{Q}, W)$ is a direction of a plane in $\mathfrak{M}$. This observation leads to the following.

Proposition 6.11. *If $w < n + m - 2k$, then the horizon $\mathbf{A}_{k,m+1}(\mathrm{Q}, W)$ of $\mathfrak{M}$ can be defined in terms of $\mathfrak{A}$.*

Finally, the question arises whether the adjacency of $\mathfrak{M}$ is definable purely in terms of the geometry of $\mathfrak{A}$? Unfortunately, the answer is not straightforward. The reasoning for spine spaces that justifies [17, Proposition 4.12], based on the fact that two distinct stars or tops of $\mathfrak{A}$ share no line on the horizon, cannot be adopted here without significant alterations. Note that if $\mathcal{L}^{\omega} \cup \mathcal{L}^{\alpha} = \emptyset$, then practically $\mathfrak{A} = \mathfrak{M}$. Therefore we assume that $\mathcal{L}^{\omega} \cup \mathcal{L}^{\alpha} \neq \emptyset$.

Theorem 6.12. *If the ground field of $\mathbb{V}$ is of odd characteristic, then the structure $\mathfrak{M}$ can be defined in terms of $\mathfrak{A}$.*

Proof. The proof is divided into several steps. For distinct points U_1, U_2 of $\mathfrak{M}$ we define

$$\begin{aligned} U_1 \sim^{+} U_2 &:\iff U_1, U_2 \subset B \text{ for some } B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W), \\ U_1 \sim_{-} U_2 &:\iff H \subset U_1, U_2 \text{ for some } H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W), \\ U_1 \sim U_2 &:\iff U_1 \sim^{+} U_2 \text{ or } U_1 \sim_{-} U_2. \end{aligned}$$

Note that $U_1 \sim^{+} U_2$ yields that either $U_1 \boldsymbol{\lambda}^{\mathrm{af}} U_2$ or $U_1 \boldsymbol{\lambda}^{\omega} U_2$, while $U_1 \sim_{-} U_2$ yields that either $U_1 \boldsymbol{\lambda}^{\mathrm{af}} U_2$, $U_1 \boldsymbol{\lambda}^{\alpha} U_2$, or U_1, U_2 are not collinear in $\mathfrak{M}$.

Step 1. The following conditions are equivalent.

(i) $U_1 \sim^+ U_2$ or $U_1 \sim_- U_2$.

(ii) There is a plane Π_1 through U_1 parallel to a plane Π_2 through U_2 in $\mathfrak{A}$.

Proof of Step 1. (i) $\Longrightarrow$ (ii): Assume that $U_1, U_2 \subset B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$, then $\mathrm{T}(B)$ is a semiaffine space (of the form $\mathcal{T}^\omega$) and one easily finds Π_1, Π_2 in it.

Next, assume that $U_1, U_2 \supset H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W)$. Set $B := U_1 + U_2$. If $B \in \mathrm{Q}$ then $L = \mathbf{p}(H, B)$ is a line of $\mathfrak{M}$. Applying analogous reasoning we find Π_1, Π_2 in an extension $[H, Y]_k$ of the type $\mathcal{S}^\alpha$. If $B \notin \mathrm{Q}$ then, in any case L is a line of the surrounding $\mathbf{A}_{k,m}(\mathbb{V}, W)$. Let us restrict to the subspaces around H; they form a spine space in the projective space $\mathbf{P}_1(\mathbb{V}/H)$ with the quadric $\mathrm{Q}(\xi/H)$ distinguished. Projective reasoning proves that required planes Π_1, Π_2 exist.

(ii) $\Longrightarrow$ (i): Let Π_i be parallel planes of $\mathfrak{A}$ with $U_i \in \Pi_i$, $i = 1, 2$. Let $L_0 = \Pi_1^\infty = \Pi_2^\infty$ be the improper line of Π_i. Then $L_0 \in \mathcal{L}^\alpha_{k,m+1}$ or $L_0 \in \mathcal{L}^\omega_{k,m+1}$. In the first case L_0, U_1, U_2 are contained in the (unique) extension to a top $\mathrm{T}(B)$ with $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$ and therefore $U_1 \sim^+ U_2$. In the second case extensions of Π_i to maximal strong subspaces have form $[H, Y_i]_k$ (they have L_0 in common), where $H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W)$. So, $U_1 \sim_- U_2$. $\Diamond$

Let us write
$$\mathfrak{M}_0 := \mathbf{A}_{k,m}(\mathbb{V}, W) \restriction \mathcal{F}_{k,m}(\mathrm{Q}, W)$$
for the surrounding spine space with point set restricted to totally isotropic subspaces. Note that the distinction between $\mathfrak{M}$ and $\mathfrak{M}_0$ consists in the range of their line sets. More precisely, for a line $L = \mathbf{p}(H, B)$ of $\mathfrak{M}_0$ its base B needs not to be totally isotropic and
$$L \text{ is a line of } \mathfrak{M} \quad \text{iff} \quad |L| \geq 3.$$

Step 2. Let $U_1, U_2 \in \mathcal{F}_{k,m}(\mathrm{Q}, W)$ and $U_1 \neq U_2$. The following conditions are equivalent.

(i) $U_1 \sim U_2$.

(ii) U_1, U_2 are collinear in $\mathfrak{M}_0$ with exception when the line L of $\mathfrak{M}_0$ which joins them has form $L = \mathbf{p}(H, B)$ where $H \in \mathcal{F}_{k-1,m-1}(\mathrm{Q}, W)$, $B \in \mathcal{F}_{k+1,m+1}(\mathbb{V}, W)$, and $B \notin \mathrm{Q}$ (i.e. L is an ω-line in $\mathbf{A}_{k,m}(\mathbb{V}, W)$).

Proof of Step 2. (i) $\Longrightarrow$ (ii): It is clear that U_1, U_2 are collinear in the surrounding Grassmann space. If $U_1 \sim^+ U_2$, then they lie on an affine or ω-line in $\mathfrak{M}$ by Table 1, while if $U_1 \sim_- U_2$, then they lie on an affine or α-line in $\mathfrak{M}_0$.

(ii) $\Longrightarrow$ (i): Now, let U_1, U_2 be collinear in $\mathfrak{M}_0$. Hence $U_1, U_2 \in \mathbf{p}(H, B)$ for suitable H, B. If $\dim(B \cap W) = m$, then $\dim(H \cap W) = m$ and thus $U_1 \sim_- U_2$. If $\dim(B \cap W) = m + 1$, then two cases arise: $\dim(H \cap W) = m, m - 1$. In the former we have $U_1 \sim_- U_2$. In the later $H \in \mathcal{F}_{k-1,m-1}(\mathrm{Q}, W)$ and $B \in \mathcal{F}_{k+1,m+1}(\mathbb{V}, W)$. If $B \in \mathrm{Q}$, then $U_1 \sim^+ U_2$, otherwise we get the excluded case. $\Diamond$

Step 3. A set X of points of $\mathfrak{A}$ is a maximal at least 3-element $\sim$-clique iff X has one of the following forms:

(a) $X = \mathrm{T}(B)$ for some $B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q}, W)$,

(b) $X = \mathrm{T}(B)$ for some $B \in \mathcal{F}_{k+1,m}(\mathrm{Q}, W)$,

(c) $X = \mathrm{S}(H)$ for some $H \in \mathcal{F}_{k-1,m}(\mathrm{Q}, W)$, or

(d) $X = [H, Y] \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ for some $H \in \mathcal{F}_{k-1,m-1}(\mathrm{Q}, W)$ and $H \subset Y \in \mathrm{Q}_r$.

Proof of Step 3. It is easy to verify that sets defined in (a) – (d) are maximal $\sim$-cliques. Now, let X be a maximal at least 3-element $\sim$-clique. In view of Step 2, X is a subset of a clique in $\mathfrak{M}_0$. So, we need general tops $\mathrm{T}_0(B) = [\Theta, B]_k$ for $B \in \mathrm{Sub}_{k+1}(\mathbb{V})$ and stars $\mathrm{S}_0(H) = [H, V]_k$ for $H \in \mathrm{Sub}_{k-1}(\mathbb{V})$. Let us examine the following four cases:

$X \subseteq \mathrm{T}_0(B)$, $B \in \mathcal{F}_{k+1,m+1}(\mathbb{V}, W)$
If $B \in \mathrm{Q}$, then any two points of $\mathfrak{M}_0$ in $\mathrm{T}(B)$ are $\sim^+$-adjacent and thus $X = \mathrm{T}(B) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ is a $\sim$-clique as in (a). If $B \notin \mathrm{Q}$, then $|X| \leq 2$ by [19, Proposition 4.4], a contradiction.

$X \subseteq \mathrm{T}_0(B)$, $B \in \mathcal{F}_{k+1,m}(\mathbb{V}, W)$
Since $|X| \geq 3$ we have $B \in \mathrm{Q}$ by [19, Proposition 4.4]. Any two points of $\mathfrak{M}_0$ in $\mathrm{T}(B)$ are $\sim_-$-adjacent, so $X = \mathrm{T}(B) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ has form (b).

$X \subseteq \mathrm{S}_0(H)$, $H \in \mathcal{F}_{k-1,m}(\mathbb{V}, W)$
Note that $H \in \mathrm{Q}$ as X is nonempty. This implies that any two points of $\mathfrak{M}_0$ in $\mathrm{S}(H)$ are $\sim_-$-adjacent. Consequently, $X = \mathrm{S}(H) \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ has form (c).

$X \subseteq \mathrm{S}_0(H)$, $H \in \mathcal{F}_{k-1,m-1}(\mathbb{V}, W)$
As above $H \in \mathrm{Q}$. The points of $\mathfrak{M}_0$ in $\mathrm{S}_0(H)$ are $\sim$-adjacent iff they are $\sim^+$-adjacent i.e. they are collinear in the surrounding polar Grassmann space where the appropriate clique has form $[H, Y]_k$ for some $Y \in \mathrm{Q}_r$ (cf. [7, Section 3]). Hence $X = [H, Y]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ has form (d).

That way we obtain the desired list (a) – (d). $\Diamond$

Note that the $\boldsymbol{\lambda}^{\mathsf{af}}$-cliques are essentially smaller than $\sim$-cliques.

Step 4. At least 3-element minimal intersections of the maximal $\sim$-cliques are lines of $\mathfrak{M}$.

Proof of Step 4. Let $\mathcal{K}_{\mathrm{x}}$ be the family of cliques of the form (x) defined in Step 3. Let X_1, X_2 be two distinct $\sim$-cliques and $Z = X_1 \cap X_2$. If $X_1, X_2 \in \mathcal{K}_{\mathrm{(a)}} \cup \mathcal{K}_{\mathrm{(b)}}$, $X_1, X_2 \in \mathcal{K}_{\mathrm{(c)}}$, or $X_1 \in \mathcal{K}_{\mathrm{(b)}} \cup \mathcal{K}_{\mathrm{(c)}}$ and $X_2 \in \mathcal{K}_{\mathrm{(d)}}$, then Z contains at most a single point. If $X_1 \in \mathcal{K}_{\mathrm{(a)}}$ and $X_2 \in \mathcal{K}_{\mathrm{(c)}}$, then Z is an affine line of $\mathfrak{M}$. If $X_1 \in \mathcal{K}_{\mathrm{(b)}}$ and $X_2 \in \mathcal{K}_{\mathrm{(c)}}$, then Z is an α-line of $\mathfrak{M}$. If either $X_1 \in \mathcal{K}_{\mathrm{(a)}}$ and $X_2 \in \mathcal{K}_{\mathrm{(d)}}$ or $X_1, X_2 \in \mathcal{K}_{\mathrm{(d)}}$, then at least 3-element minimal Z is an ω-line of $\mathfrak{M}$. $\Diamond$

It is evident that every projective line of $\mathfrak{M}$ can be presented as the intersection of cliques enumerated in Step 3. So, applying Step 4 we get the line set of $\mathfrak{M}$ recovered which makes the proof of Theorem 6.12 complete. $\square$

Remark 6.13. The horizon of a star in $\mathfrak{M}$ may have strange properties. Assume that $W \in \mathrm{Q}$ and let $H \in \mathrm{Q}_{k-1}$, $H \subset W^{\perp}$. Set $m := \dim(H \cap W)$. This means that $k - 1 + w - m < r$. Then there is an $Y_0 \in \mathrm{Q}_r$ such that $H \cup W \subset Y_0$. So, $Y_0 \cap W = W$. Write $S_0 = [H, Y_0]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$. Then $S_0^{\infty} = [H, H + W]_k$. Take any $S = [H, Y]_k \cap \mathcal{F}_{k,m}(\mathrm{Q}, W)$ contained in $\mathrm{S}(H)$. Then $S^{\infty} = [H, H + (W \cap Y)]_k \subset [H, H + W]_k = S_0^{\infty}$. So, in this case

$$\mathrm{S}(H)^{\infty} \text{ is the projective space } [H, H + W]_k \text{ contained in } \mathfrak{M}^{\infty}.$$

Nevertheless, $\mathrm{S}(H)$ contains affine subspaces of different dimensions.

Note that in this case $k-m-1 = \dim(\mathrm{T}(B)^\infty) = \dim(\mathrm{S}(H)^\infty) = w-m-1$ yields $w=k$, so horizons of stars and tops may have equal dimensions only when $\mathfrak{M}$ consists of points in Q_k that are at the fixed distance $k-m$ from the fixed point W.

Remark 6.14. Theorem 6.12 for polar spine spaces and its counterpart [17, Proposition 4.12] for spine spaces both say that the respective geometry depends only on affine lines together with parallelism, that is, projective lines can be recovered using affine line structure. However, the idea of the proof presented in this paper is more general than that in [17] because it does not rely on specific horizons and intersections of stars which are completely different in $\mathfrak{M}$ and in $\mathbf{A}_{k,m}(\mathbb{V}, W)$. As such it can be applied for spine spaces and is expected to give less complex reasonings.

7 Classifications

Table 1: The classification of lines in a polar spine space $\mathbf{A}_{k,m}(\mathrm{Q}, W)$.

Class	Representative line $g = \mathbf{p}(H,B) \cap \mathcal{F}_{k,m}(\mathrm{Q},W)$	g^∞
$\mathcal{A}_{k,m}(\mathrm{Q},W)$	$H \in \mathcal{F}_{k-1,m}(\mathrm{Q},W),\ B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q},W)$	$H + (B \cap W)$
$\mathcal{L}^\alpha_{k,m}(\mathrm{Q},W)$	$H \in \mathcal{F}_{k-1,m}(\mathrm{Q},W),\ B \in \mathcal{F}_{k+1,m}(\mathrm{Q},W)$	–
$\mathcal{L}^\omega_{k,m}(\mathrm{Q},W)$	$H \in \mathcal{F}_{k-1,m-1}(\mathrm{Q},W),\ B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q},W)$	–

Each strong subspace X of a polar spine space is a slit space, that is a projective space $\mathbf{P}$ with a subspace $\mathcal{D}$ removed. In the extremes $\mathcal{D}$ can be void, then X is basically a projective space, or a hyperplane, then X is an affine space.

Table 2: The classification of stars and tops in a polar spine space $\mathbf{A}_{k,m}(\mathrm{Q}, W)$.

Class	Representative subspace		
	$\dim(\mathbf{P})$	$\mathcal{D}$	$\dim(\mathcal{D})$
$\mathcal{S}^\omega_{k,m}(\mathrm{Q},W)$	$[H,(H+W)\cap Y]_k : H \in \mathcal{F}_{k-1,m-1}(\mathrm{Q},W), Y \in \mathrm{Q}_r, H \subset Y$		
	$\dim(W\cap Y) - m$	$\emptyset$	-1
$\mathcal{S}^\alpha_{k,m}(\mathrm{Q},W)$	$[H,Y]_k \cap \mathcal{F}_{k,m}(\mathrm{Q},W) : H \in \mathcal{F}_{k-1,m}(\mathrm{Q},W), Y \in \mathrm{Q}_r, H \subset Y$		
	$r-k$	$[H,(H+W)\cap Y]_k$	$\dim(W\cap Y) - m - 1$
$\mathcal{T}^\alpha_{k,m}(\mathrm{Q},W)$	$[B\cap W, B]_k : B \in \mathcal{F}_{k+1,m}(\mathrm{Q},W)$		
	$k-m$	$\emptyset$	-1
$\mathcal{T}^\omega_{k,m}(\mathrm{Q},W)$	$[\Theta, B]_k \cap \mathcal{F}_{k,m}(\mathrm{Q},W) : B \in \mathcal{F}_{k+1,m+1}(\mathrm{Q},W)$		
	k	$[B\cap W, B]_k$	$k-m-1$

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