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PETRA ŠPARL (1975–2016)

Petra Šparl passed away on 21st August 2016 after an unfair battle against severe cancer. She fought the cancer with incredible courage for over a year, and those who knew her sincerely hoped that she would be among the few who might overcome the disease. She left two children, Alja and Žiga, aged 14 and 12. Petra studied mathematics at the University of Maribor, where she received her bachelor's degree in 1998, her MASc in 2001, and her PhD degree in 2005. In her thesis, Petra developed an algorithm for multicolouring on a special class of graphs, called hexagonal graphs, and this is still achieving the best approximation bound among 2-local algorithms. During her PhD studies she taught at the Faculty of Civil Engineering, and while she was writing her PhD thesis, she was also involved in renovating the mathematical curricula for civil engineering students, and introducing some fresh topics in discrete mathematics. Soon after completing her PhD, Petra joined the Faculty of Organisational Sciences, where she was immediately asked (with some urgency) to develop the curricula for mathematical subjects. At the same time, she started a successful collaboration with colleagues in the new Faculty, which resulted in several publications on several different topics. Graph theory remained one of her major research interests. For example, in December 2015 she was working on the final version of her last paper, on matching in hexagonal graphs [1]. This year Petra co-authored a paper in *Ars Mathematica Contemporanea,* which initiated the study of multicolourings of 3D-analogues of planar hexagonal graphs [2]. The motivation for the studying multicolourings of hexagonal graphs is derived from the recently very popular problems of channel assignment, which have appeared in wireless networking. Petra loved to see the successful application of serious mathematics. She also had the necessary energy and skills to bring mathematics closer to engineering students. Petra was at the peak of her potential when she had to start a fight for her life. Who knows what more she would have achieved if she had not left us so young. I am very proud that Petra was my PhD student.

Janez Žerovnik

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8ECM

The Eighth European Congress of Mathematics (8ECM) will take place in Portorož, Slovenia, the week 5–11 July 2020 (see `http://www.8ecm.si/`). The ECM is the quadrennial congress of the European Mathematical Society.

The 8ECM will be not only a great opportunity for Slovenia (and for the University of Primorska in particular) to showcase its mathematical achievements, but also a wonderful chance for authors, referees and editors of our journal to present work at its best. This can be done through posters, contributed talks, mini symposia and satellite conferences.

Here is the chronology of the European Congresses of Mathematics: Paris (1992), Budapest (1996), Barcelona (2000), Stockholm (2004), Amsterdam (2008), Krakow (2012), Berlin (2016) … and now Portorož (2020). The list is impressive, and we are quite honoured to be in such good company. Also we are grateful to everyone who supported our bid to host the 8th ECM.

There is only one problem we foresee, namely persuading members of the AMC community to take part in the Congress. Of course many of us prefer to attend more specialised conferences and workshops, where one can enjoy some really good and interesting talks and the company of many mathematicians with similar interests. But the 8ECM offers something special, in terms of a wider programme, plenary lectures by leading and upcoming mathematicians across a range of fields, and a spectacular Adriatic venue!

An important task of those of us involved with the organisation of the 8ECM is to make the Congress friendly and welcoming for mathematics communities like that of AMC. We are confident that our experience in organising numerous mathematical conferences and workshops is giving us the necessary skills to perform this task. Even so, it will be a great challenge.

Klavdija Kutnar
Associate Editor

Dragan Marušič and Tomaž Pisanski
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The Cartesian product of graphs with loops

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Abstract

We extend the definition of the Cartesian product to graphs with loops and show that the Sabidussi–Vizing unique factorization theorem for connected finite simple graphs still holds in this context for all connected finite graphs with at least one unlooped vertex. We also prove that this factorization can be computed in $O(m)$ time, where m is the number of edges of the given graph.

Keywords: Graphs, monoids, factorizations, algorithms.

Math. Subj. Class.: 05C70, 13A05, 20M13, 05C85.

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1 Introduction

This paper considers finite undirected graphs that may contain loops, or, put differently, symmetric binary relations on finite sets. One may define several binary operations on such graphs; these are explored in the recently revised monograph [1]. The well-known *Cartesian product* of finite undirected graphs is usually defined only for *simple* graphs, that is, for graphs that do not contain multiple edges between the same pair of vertices and, more importantly for us, do not contain loops. Here we extend this definition.

Before doing so, let us fix the notation. For us, a *graph* $G = (V, E)$ will always be a finite undirected graph without multiple edges. The edge set E is taken to be a set of ordered pairs of vertices; thus, a loop on the vertex $v \in V$ corresponds to the edge $(v, v) \in E$, and as all graphs are undirected, we have $(v, w) \in E$ if and only if $(w, v) \in E$. We will occasionally call a loop a 1-*edge* and an edge that is not a loop a 2-*edge*. Moreover, given a graph G, we will refer to its vertex set as $V(G)$ and to its edge set as $E(G)$.

Definition 1.1. Let $G_1, \ldots, G_k$ be graphs. The *Cartesian product* $G = G_1 \,\square\, \cdots \,\square\, G_k$ is a graph with vertex set $V(G) = V(G_1) \times \cdots \times V(G_k)$, and edge set $E(G)$ defined as follows: two vertices $(v_1, \ldots, v_k) \in V(G)$ and $(w_1, \ldots, w_k) \in V(G)$ are adjacent if there exists an index i such that $(v_i, w_i) \in E(G_i)$, and $v_j = w_j$ for all $j \neq i$.

Note that this definition extends the classical one for simple graphs. The product graph has a loop on a vertex $(v_1, \ldots, v_k) \in V(G)$ if and only if there is a loop on at least one of the constituents $v_i \in V(G_i)$. Thus, the distribution of loops (or 1-edges) on the product graph is independent from the distribution of the 2-edges.

Definition 1.2. Let $G_1, \ldots, G_k$ be graphs, and $G = G_1 \,\square\, \cdots \,\square\, G_k$. The *ith projection* $p_i : V(G) \to V(G_i)$ is given by $(v_1, \ldots, v_k) \mapsto v_i$.

Using Definition 1.1, we observe the property that the projections $p_i : V(G) \to V(G_i)$ are weak homomorphisms from G to G_i. Recall that a *weak homomorphism* between graphs G and H is a map $\varphi : V(G) \to V(H)$ such that, whenever $(v, w) \in E(G)$, either $(\varphi(v), \varphi(w)) \in E(H)$ or $\varphi(v) = \varphi(w)$. In particular, the presence of loops in G or H does not impose any restriction on a weak homomorphism from G to H.

Definition 1.3. Let $G_1, \ldots, G_k$ be graphs, and $G = G_1 \,\square\, \cdots \,\square\, G_k$. For every vertex $a = (a_1, \ldots, a_k) \in V(G)$, the *$G_i$-layer through a* is the induced subgraph

$$\begin{aligned} G_i^a &= \langle \{x \in V(G) \mid p_j(x) = a_j \text{ for } j \neq i\} \rangle \\ &= \langle \{(a_1, a_2, \ldots, x_i, \ldots, a_k) \mid x_i \in V(G_i)\} \rangle. \end{aligned}$$

Note that $G_i^a = G_i^b$ if and only if $p_j(a) = p_j(b)$ for each index $j \neq i$. With the usual Cartesian product, the restrictions $p_i|V(G_i^a) : V(G_i^a) \to V(G_i)$ are isomorphisms between G_i^a and G_i [1, Section 4.3]. Under Definition 1.1, we obtain a dichotomy, as follows.

Lemma 1.4. *Let $G_1, \ldots, G_k$ be graphs, and $G = G_1 \,\square\, \cdots \,\square\, G_k$. Then, the following two conditions hold for every vertex $a = (a_1, \ldots, a_k) \in V(G)$ and every $i \in \{1, \ldots, k\}$:*

(i) If $a_j \in V(G_j)$ is unlooped for every $j \neq i$, then $p_i|V(G_i^a) : V(G_i^a) \to V(G_i)$ is an isomorphism between G_i^a and G_i.

(ii) Otherwise, G_i^a is isomorphic to G_i with a loop attached to every vertex.

Proof. Easy from the definitions. □

2 Matrix and semiring properties

From the definition of the Cartesian product we infer that it is commutative and distributive over the disjoint union. Moreover, the trivial graph K_1, that is, a vertex without edges, is a unit. As the Cartesian product is also associative, see below, the set Γ_0 of isomorphism classes of finite undirected graphs with loops is a commutative semiring.

To prove associativity we could adapt the proof of [1, Proposition 4.1] for associativity of the Cartesian product of graphs without loops, or we could modify the multiplication table method of [1, Exercise 4.15], which was introduced for the classification of associative products. However, we follow a different path and use the fact that the adjacency matrix $A(G \,\square\, H)$ of the Cartesian product of two simple graphs is the Kronecker sum of the adjacency matrices $A(G)$ and $A(H)$ of the factors, see [1, Section 33.3].

Let us first recall that the Kronecker sum $A \oplus B$ of an $n \times n$ matrix A by an $m \times m$ matrix B is defined as $I_n \otimes B + A \otimes I_m$. Here, I_n and I_m denote the identity matrices of size n and m, respectively, and $P \otimes Q$ denotes the Kronecker product. In our situation, the first factor $P = (p_{ij})$ is always an $n \times n$ matrix and the Kronecker product is defined by

$$P \otimes Q = \begin{bmatrix} p_{11}Q & \cdots & p_{1n}Q \\ \vdots & \ddots & \vdots \\ p_{n1}Q & \cdots & p_{nn}Q \end{bmatrix}.$$

Notice that both the Kronecker sum and the Kronecker product are associative but not commutative.

For simple graphs G and H we have $A(G \,\square\, H) = A(G) \oplus A(H)$. For graphs with loops we find that the diagonal entries take positive integer values that are not restricted to $\{0, 1\}$. If we agree on the convention that a positive diagonal entry in the adjacency matrix means a loop, whereas a 0 means no loop, then the product given in Definition 1.1 still corresponds to the Kronecker sum. It follows that, up to isomorphism of graphs, this product is associative.

We note in passing that the fact that the Kronecker sum is not commutative does not contradict the commutativity of the Cartesian product: $A(G) \oplus A(H)$ and $A(H) \oplus A(G)$ represent adjacency matrices of $G \,\square\, H$ for different vertex numberings.

Finally, we briefly call a graph *entirely looped* if every vertex has a loop. For any graph G, we let $\mathcal{N}(G)$ be G with its loops removed.

Lemma 2.1. *Let G, H, H_1, H_2 be graphs. Assume that G is entirely looped. Then $G \,\square\, H$ is entirely looped as well. Moreover, if $\mathcal{N}(H_1) \cong \mathcal{N}(H_2)$, then $G \,\square\, H_1 \cong G \,\square\, H_2$.*

Proof. The first statement follows directly from Definition 1.1. As remarked earlier, the 2-edges of the products $G \,\square\, H_i$ do not depend on the loops of either factor. Thus

$$\begin{aligned} \mathcal{N}(G \,\square\, H_1) &= \mathcal{N}(G) \,\square\, \mathcal{N}(H_1) \\ &\cong \mathcal{N}(G) \,\square\, \mathcal{N}(H_2) \\ &= \mathcal{N}(G \,\square\, H_2)\,. \end{aligned}$$

Next, we insert the loops on the product; but, as every vertex of G has a loop, it follows that every vertex of either product $G \,\square\, H_i$ has a loop as well, and the two products are obviously isomorphic. □

It follows that the subset Γ_{00} of Γ_0 given by the isomorphism classes of entirely looped graphs constitutes an *ideal* of the semiring Γ_0. It is obviously closed under the disjoint union and the Cartesian product, and, since the loop K_1^* is a unit for the Cartesian product inside Γ_{00}, it is a semiring itself. The loop-removing map $\mathcal{N}$ constitutes an isomorphism of semirings between Γ_{00} and the set of simple graphs Γ.

3 Unique factorization

One fundamental property of the Cartesian product, proved independently by Sabidussi [5] and Vizing [6] in the 1960s, is the unique factorization of connected simple graphs into irreducibles with respect to this product. We will extend this result to graphs with loops, where we will have to exclude the set of entirely looped graphs (Lemma 2.1 suggests why). Algebraically speaking, we might want to form the *quotient semiring* Γ_0/Γ_{00}, so that also any fully looped components in disconnected graphs are annulled. However, since we will only consider connected graphs in what follows, this is not of great consequence.

Definition 3.1. A nontrivial, connected graph G with at least one unlooped vertex is called *irreducible* with respect to the Cartesian product if, for every factorization $G = H \,\Box\, L$, either H or L is trivial.

Recall that a graph is called *trivial* if it is a vertex without edges. Consider a nontrivial, connected graph G with at least one unlooped vertex. One can easily check that, if G is not irreducible, it can be expressed as Cartesian product of two factors each of which is, again, a nontrivial, connected graph with at least one unlooped vertex. Iteration of this procedure yields a representation of G as a product of irreducible graphs. It is occasionally called a *prime factorization*.

Another way to prove the existence of a prime factorization is the following: Any factorization of G with a maximum number of nontrivial factors must be a product of irreducible graphs. If G has n vertices, this maximum number is at most $\log_2(n)$.

Our main results are the following.

Theorem 3.2. *Every nontrivial, connected graph with at least one unlooped vertex has a representation as a product of irreducible graphs with respect to the Cartesian product. The representation is unique up to isomorphisms and the order of the factors.*

Theorem 3.3. *The unique prime factorization with respect to the Cartesian product of a nontrivial, connected graph G with at least one unlooped vertex can be computed in $O(m)$ time, where m is the number of edges of G.*

To prove Theorem 3.2, we follow the method of [1, Section 6.1], for Theorem 3.3 we extend the ideas of [4]. First, let us define convex subgraphs and boxes.

Definition 3.4. A subgraph H of a graph G is *convex* in G if every shortest path in G that connects two vertices of H is completely contained in H. A subgraph H of a Cartesian product $G = G_1 \,\Box\, \cdots \,\Box\, G_k$ is called a *box* or *subproduct* if there are subgraphs $H_i \subseteq G_i$ such that

$$H = H_1 \,\Box\, \cdots \,\Box\, H_k\,.$$

In order to determine whether a subgraph is convex or not, only the 2-edges need to be concerned. In particular, a subgraph H is convex in G if and only if the subgraph $\mathcal{N}(H)$ is convex in $\mathcal{N}(G)$.

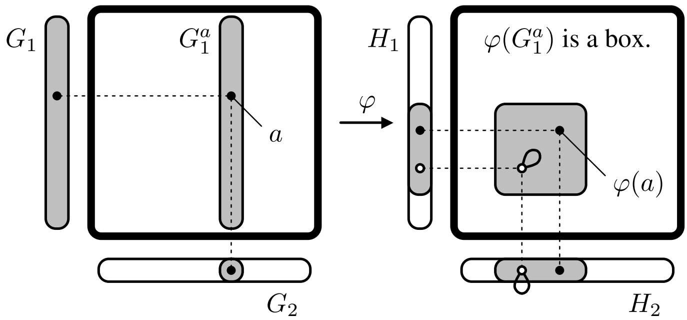

Figure 1: An isomorphism between factored graphs with loops.

Lemma 3.5. *Let H be a subgraph of a Cartesian product $G = G_1 \,\Box\, \cdots \,\Box\, G_k$. Then the following are equivalent:*

(i) H is an induced and convex subgraph of G;

(ii) There are induced and convex subgraphs $H_i \subseteq G_i$ such that $H = H_1 \,\Box\, \cdots \,\Box\, H_k$. In other words, H is a box whose factors are induced and convex.

Proof. As far as only the 2-edges are concerned, all convex subgraphs are induced and the assertion is Lemma 6.5 of [1]. This means that $p_1(V(H)) \times \cdots \times p_k(V(H)) = V(H)$. Now, let H_i be the subgraph of G_i induced by $p_i(V(H))$, where $i \in \{1, \ldots, k\}$. Then the lemma follows by the definition of the Cartesian product. □

As remarked after Definition 3.1 every finite graph has a factorization into irreducibles. Thus we only have to show that it is unique in order to prove Theorem 3.2. The next lemma and its corollary makes this precise; the situation is illustrated in Figure 1.

Lemma 3.6. *Let φ be an isomorphism between nontrivial, connected graphs G and H with at least one unlooped vertex. Assume that G and H are representable as products $G = G_1 \,\Box\, \cdots \,\Box\, G_k$ and $H = H_1 \,\Box\, \cdots \,\Box\, H_\ell$ of irreducible graphs. Then $k = \ell$ and, for every unlooped vertex $a \in V(G)$, there is a permutation π of $\{1, \ldots, k\}$ such that*

$$\varphi(G_i^a) = H_{\pi(i)}^{\varphi(a)} \text{ for every } i \in \{1, \ldots, k\}\,.$$

Formally, φ is a bijection between the vertex sets $V(G)$ and $V(H)$. But since φ is a homomorphism of graphs, it induces a well-defined mapping between the edge sets $E(G)$ and $E(H)$. In the above theorem, we slightly abuse notation and denote the image of the subgraph G_i^a, including vertices and edges, by $\varphi(G_i^a)$.

Proof. Fix an unlooped vertex $a = (a_1, \ldots, a_k) \in V(G)$, and set $(b_1, \ldots, b_\ell) := \varphi(a)$. By Lemma 1.4 we infer that $G_i^a \cong G_i$ and $H_j^{\varphi(a)} \cong H_j$ for every i and j. Every layer G_i^a is induced and, as a consequence of Lemma 3.5, convex in G. So, its image $\varphi(G_i^a)$ is induced and convex in H. Again, as a consequence of Lemma 3.5, $\varphi(G_i^a) = U_1 \,\Box\, \cdots \,\Box\, U_\ell$, where every U_j is induced and convex in H_j. But $\varphi(G_i^a) \cong G_i^a \cong G_i$ is irreducible. Since

$(b_1, \dots, b_\ell) = \varphi(a) \in \varphi(G_i^a)$, we conclude that $V(U_j) = \{b_j\}$ for all indices but one, say $\pi(i)$. In other words, $\varphi(G_i^a) \subseteq H_{\pi(i)}^{\varphi(a)}$. But then

$$G_i^a \subseteq \varphi^{-1}\left(H_{\pi(i)}^{\varphi(a)}\right) .$$

Because the latter graph is induced and convex, it is a box; and because it is irreducible, it must be contained in G_i^a. Therefore, $\varphi(G_i^a) = H_{\pi(i)}^{\varphi(a)}$.

We claim that the map $\pi : \{1, \dots, k\} \to \{1, \dots, \ell\}$ is injective. If $\pi(i) = \pi(j)$, then

$$\varphi(G_i^a) = H_{\pi(i)}^{\varphi(a)} = H_{\pi(j)}^{\varphi(a)} = \varphi(G_j^a) .$$

But φ is an isomorphism, and therefore the above equation implies $G_i^a = G_j^a$. Since every layer contains at least two vertices, we obtain $i = j$. So, π is injective, and $k \leq \ell$. Repetition of the above argument for φ^{-1} yields $\ell \leq k$. So, $k = \ell$ and π is a permutation. □

Corollary 3.7. *$G_i \cong H_{\pi(i)}$ for every $i \in \{1, \dots, k\}$.*

Proof. Since a is unlooped, $G_i \cong G_i^a$ and $H_j \cong H_j^{\varphi(a)}$ for every i and j. By Lemma 3.6 the corollary follows. □

Clearly Lemma 3.6 and Corollary 3.7 prove the validity of Theorem 3.2.

A remark about automorphisms

In Lemma 3.6 the permutation π of $\{1, \dots, k\}$ is constructed to a fixed unlooped vertex $a \in V(G)$. Actually π is independent of the choice of a, and one can extend Lemma 3.6 to the following description of the automorphisms of G.

Theorem 3.8. *Suppose φ is an automorphism of a nontrivial, connected graph G with at least one unlooped vertex and prime factorization $G = G_1 \,\Box\, \cdots \,\Box\, G_k$. Then there are a permutation π of $\{1, \dots, k\}$ and isomorphisms $\varphi_i : G_{\pi(i)} \to G_i$ for which*

$$\varphi(x_1, \dots, x_k) = (\varphi_1(x_{\pi(1)}), \dots, \varphi_k(x_{\pi(k)})) .$$

The proof of this theorem can be led on the same lines as that of [1, Theorem 6.10]. Among other consequences this implies that the automorphism group of G is isomorphic to the automorphism group of the disjoint union of the prime factors $G_1, \dots, G_k$.

4 Algorithms

In this section we present two algorithms for the decomposition of a nontrivial, connected graph G with at least one unlooped vertex into its prime factors. One is straightforward and has complexity $O(mn)$, where m is the number of edges and n the number of vertices of G. The other one is linear in the number of edges of G and depends on the algorithm of Imrich and Peterin [4] for the prime factorization of graphs without loops.

Let $G = G_1 \,\Box\, \cdots \,\Box\, G_k$ be the prime factorization of a nontrivial, connected graph G with at least one unlooped vertex. Then also $\mathcal{N}(G) = \mathcal{N}(G_1) \,\Box\, \cdots \,\Box\, \mathcal{N}(G_k)$. Clearly the graphs $\mathcal{N}(G_i)$, $i \in \{1, \ldots, k\}$, need not be irreducible with respect to the Cartesian product. Let $\mathcal{N}(G_i) = H_{i,1} \,\Box\, \cdots \,\Box\, H_{i,\ell(i)}$ be their prime factorizations. Thus

$$\mathcal{N}(G) = \prod_{i=1}^{k} \prod_{j=1}^{\ell(i)} H_{i,j}$$

is a representation of $\mathcal{N}(G)$ as a Cartesian product of irreducible graphs. Because the prime factorization is unique, it is the prime factorization of $\mathcal{N}(G)$, up to the order and isomorphisms of the factors. In other words, if $\prod_{j \in J} Z_j$ is a prime factorization of $\mathcal{N}(G)$, then there is a partition $J = J_1 \cup \cdots \cup J_k$ such that $\mathcal{N}(G_i) = \prod_{j \in J_i} Z_j$. Our task is to find this partition. We begin with a straightforward approach and prove the following lemma.

Lemma 4.1. *Let G be a nontrivial, connected graph with at least one unlooped vertex. Then its prime factorization can be found in $O(mn)$ time.*

Proof. If G has n vertices, then this is also true for $\mathcal{N}(G)$, and so the number of factors of $\mathcal{N}(G)$, say r, is at most $\log_2(n)$. This also bounds the size of J and implies that the number s of subsets of J is at most $2^{\log_2(n)}$, i. e. $s \leq n$. Notice that the factors of $\mathcal{N}(G)$ can be found in $O(m)$ time by [4].

Let $J_1, J_2, \ldots, J_s$ be all subsets of J, ordered in such a way that $|J_i| \leq |J_j|$ whenever $1 \leq i \leq j \leq s$. For every $i \in \{1, \ldots, s\}$ set $Y_i := \prod_{j \in J_i} Z_j$ and $Y_i^* := \prod_{j \in J \setminus J_i} Z_j$. Let $\langle Y_i^a \rangle_G$ denote the subgraph of G induced by the layer Y_i^a of Y_i through a, and define $\langle (Y_i^*)^a \rangle_G$ analogously. If the partition $J_i \cup (J \setminus J_i)$ of J leads to a factorization of G, then $\langle Y_i^a \rangle_G$ is isomorphic to a factor of G.

We begin the algorithm by scanning the J_i in the given order. For every J_i and every vertex $v \in V(G)$ we consider the projections $p_{Y_i}(v)$ and $p_{Y_i^*}(v)$ into $\langle Y_i^a \rangle_G$ and $\langle (Y_i^*)^a \rangle_G$. If $v = (v_1, \ldots, v_r)$, then $p_{Y_i}(v) = (w_1, \ldots, w_r)$, where $w_j = v_j$ if $j \in J_i$, and $w_j = a_j$ otherwise. Notice that $p_{Y_i}(v)$ is the vertex of shortest distance from v in $\langle Y_i^a \rangle_G$. The other projection $p_{Y_i^*}(v)$ is defined analogously. Again, $p_{Y_i^*}(v)$ is the vertex of shortest distance from v in $\langle (Y_i^*)^a \rangle_G$. Clearly $G = \langle Y_i^a \rangle_G \,\Box\, \langle (Y_i^*)^a \rangle_G$ if and only if for every vertex $v \in V(G)$ the following two conditions are satisfied:

1. If v is unlooped, then both $p_{Y_i}(v)$ and $p_{Y_i^*}(v)$ are unlooped.
2. If v has a loop then at least one of the vertices $p_{Y_i}(v)$, $p_{Y_i^*}(v)$ has a loop.

The time necessary to compute $p_{Y_i}(v)$ and $p_{Y_i^*}(v)$ for a given v is proportional to r. As one can check in constant time whether $p_{Y_i}(v)$ or $p_{Y_i^*}(v)$ has a loop, one can check in $O(nr)$ time whether $G = \langle Y_i^a \rangle_G \,\Box\, \langle (Y_i^*)^a \rangle_G$.

Notice that r is the number of factors of $\mathcal{N}(G)$, which is also bounded by the minimum degree δ of $\mathcal{N}(G)$. This is easily seen, since every vertex meets every layer and, in a connected graph, is incident with at least one edge of that layer. Hence the number of factors cannot exceed the degree of any vertex, and $nr \leq n\delta \leq m$.

For a given J_i one can thus check in $O(m)$ time whether $\langle Y_i^a \rangle_G$ is a factor of G. If it is, and if J_i is minimal with respect to inclusion, then it clearly is an irreducible factor. Hence, this is true for the first factor that we encounter, because of having ordered the J_i by size. We now continue the scan, omitting the J_j that are not disjoint from J_i, to find the next factor. Clearly it will also be irreducible. We continue until we have found all irreducible factors. Since there are no more than n subsets of J, we can find them in $O(nm)$ time. □

In order to reduce the complexity to $O(m)$, we need some more preparation. So let a be an unlooped vertex of G and L_i be the levels of a BFS-ordering of the vertices of G with respect to the root a. That is, L_i consists of all vertices of distance i from a. Furthermore, we enumerate the vertices of G by giving them so-called BFS-numbers that satisfy $\mathrm{BFS}(v) > \mathrm{BFS}(u)$ if the distance from a to v is larger than the one from a to u.

It is important to observe that the projection $p_{Y_i}(v)$ is a vertex of $\langle Y_i^a \rangle_G$ and always closer to a than v, unless v already is a vertex of $\langle Y_i^a \rangle_G$, because then $p_{Y_i}(v) = v$.

Proof of Theorem 3.3. Let $\prod_{j \in J} Z_j$ be a prime factorization of $\mathcal{N}(G)$. We begin with the trivial partition of J and wish to check, whether it already leads to a factorization of G. We scan the vertices v of G in BFS-order and, given v, check the validity of Conditions

(i) If v is unlooped, then all $p_{Y_i}(v)$ are unlooped.
(ii) If v has a loop, then at least one of the projections $p_{Y_i}(v)$ has a loop.

If one of these conditions is not satisfied, then the partition of J is obviously inconsistent with the loop structure. In either case we have too many factors and have to make the partition of J coarser. Before we go on, notice that in L_1 these conditions are trivially satisfied for any partition of J, because all projections $p_{Y_i}(v)$ are a, except one, which is v.

Suppose we arrive at a vertex v where one of the conditions (i) or (ii) is violated for the first time. Assume first that Condition (i) is violated, that is, v is unlooped, but $p_{Y_i}(v)$ has a loop for an index i. In the end, all projections have to be unlooped. We must combine the set J_i with one or more other sets of the partition. Using the fact that we proceed in BFS-order, it is easy to see that we have to make v a unit layer vertex, that is, we combine all those sets J_j for which $p_{Y_j}(v) \neq a$. Assume now that Condition (ii) is violated, that is, v has a loop, but no $p_{Y_i}(v)$ does. In the end, at least one of the projections has to have a loop. As above, the only way to achieve this is to make v a unit layer vertex, that is, we combine all factors J_j for which $p_{Y_j}(v) \neq a$.

In both cases we arrive at a coarser partition of J than the one we started out with. By associativity of the Cartesian product with loops, we need not recheck the vertices we have already considered and continue in BFS-order.

Notice that this process yields a factorization, because both (i) and (ii) are satisfied. For every finer partition of J one of these conditions is violated, hence the factorization is the unique prime factorization we are looking for.

Considering the computational cost of these operations, we observe that all projections that we need for the n vertices can be computed, in $O(n|J|)$ time. Since we can check in constant time whether a vertex has a loop or not, the checks for conditions (i) and (ii) can also be done in $O(n|J|)$ time. As $|J| \leq \delta$, we have $O(n|J|) = O(n\delta) = O(m)$.

Finally, recomputing the partition needs at most $O(|J|)$ time, and this has to be done at most $|J|$ times, so the cost is $O(\delta^2)$. □

5 Remarks

In [2] it was shown that connected set systems, or hypergraphs, as they are called now, also have unique prime factorizations with respect to the Cartesian product if one-element sets, or loops in our terminology, are excluded. Our result also extends to hypergraphs with loops: Connected hypergraphs have unique prime factorization with respect to the Cartesian product, if there is a least one vertex without a loop. Furthermore, the same arguments yield unique prime factorization for connected infinite graphs or hypergraphs with respect to the weak Cartesian product; compare [3].

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Spectral centrality measures in temporal networks

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Abstract

In our previous article we defined temporal quantities used for the description of temporal networks with zero latency and we showed that some centrality measures (e.g. degree, betweenness, closeness) can be extended to the case of temporal networks. In this article we broaden the scope of centrality measures in temporal networks to centrality measures derived from the eigenvectors of network matrices, namely the eigenvector in-centrality, the eigenvector out-centrality, the Katz centrality, the Bonacich α and $(\alpha, \beta)-$centrality, the HITS algorithm (also known as Hubs and Authorities) introduced by Kleinberg, and the PageRank algorithm defined by Page and Brin.

We extended our Python library TQ (Temporal Quantities) to include the algorithms from our research. The library is available online. The procedures will also be added to the user friendly program called Ianus. We tested the proposed algorithms on Franzosi's violence network and on Corman's Reuter terror news network and show the results.

Keywords: Temporal network, semiring, algorithm, network measures, Python library, violence.

Math. Subj. Class.: 91D30,16Y60,90B10,68R10.

1 Introduction

Many real-life problems can be represented as networks in which the actors are represented with nodes (or vertices) and interactions between the actors are represented with links – arcs or edges, according to the nature of interactions (whether the interactions are directed or

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not). Networks have been widely studied in mathematics, computer science, biology, social sciences and other disciplines. There are many examples of data that have an underlying network structure, such as the Internet, the phone-calls data, the co-authorship graphs, the email graphs, the biological and the chemical networks, the transaction networks, the trade networks, etc. These networks are tipically generated by human activity and often exhibit similar structure. The network analysis has seen an ever increasing research activity in the past years due to the amount of data available and to the global interest in data analysis. See for example [7] and [22]. In the last two decades, there has been an increased interest in temporal network analysis where a time dimension is also considered.

The node centrality has been a fundamental tool in the study of social networks since the late 1940s, beginning with the Group Networks Laboratory at MIT directed by Alex Bavelas (see [3], [4], [21]). The node degree is probably the oldest measure of a node's importance in a network. In a network, every node has some measure of influence or importance within the rest of the network and the importance of a node is determined by the structure of the network it belongs to. Centrality measures are designed to rank nodes based on their structural position inside the network and different centrality measures aim to quantitatively measure the importance of a node in the network. Various measures of centrality were employed in different contexts. There is no consensus on what centrality is and there is little agreement on the best way to measure it. It still falls to the network analysts to decide which centrality measure is the most appropriate for the given network and context and to define exactly what the purpose of the computation is. The usual questions that are approximately answered using network centrality measures are - Who are the influential people in a social network? Which roads are most often used? Which web pages are important?

In this article, we make a step towards connecting two of the most frequent questions arising in contemporary network analysis: we consider the temporal changes of the centralities of nodes.

The paper is organized as follows: in Sections 2 and 3 we review some basic ideas and notations on centrality measures and temporal networks. In Section 4 we present the algorithms for computing the spectral centrality measures, and we give examples on real-life data in Section 5. We conclude with possible directions of future research in Section 6.

2 Centrality Measures and Graph Matrices

Let $G = (V, L)$ be a graph with a node set $V = \{v_1, v_2, \dots, v_n\}$ and a link set $L \subseteq V \times V$. An adjacency matrix $\mathbf{A}(G) = [a_{uv}]$ of the graph G is a binary $n \times n$ matrix with elements

$$a_{uv} = \begin{cases} 1, & (u, v) \in L, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the undirected graphs have a symmetric adjacency matrix and the graphs with no loops have adjacency matrices with zero diagonal elements. If the network has an arc without its opposite arc, the adjacency matrix is not symmetric. In directed networks, we have two types of links adjacent to a node – links pointing to the node (incoming) and links pointing away from the node (outgoing). The number of incoming links is the node indegree, the number of outgoing links is the node outdegree.

If G has weights on the arcs we let a_{uv} be the weight of the arc (u, v). Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be the corresponding matrix. An eigenvector of $\mathbf{A}$ is a non-zero vector x such that $\mathbf{A}x = \lambda x$ for some complex λ, which is called an eigenvalue of $\mathbf{A}$ belonging to the vector x. The eigenvalues of a graph G are defined as the eigenvalues of its adjacency matrix $\mathbf{A}(G)$. The set of the eigenvalues of G is called the spectrum of G. There are many connections between the eigenvalues of a graph and its combinatorial properties. These include eigenvalues that are not dominant and are beyond the scope of this article.

It is well known that if $\mathbf{A}$ is a real symmetric matrix then its eigenvalues are real. Even more is true for nonnegative matrices ([5]).

Theorem 2.1. *(The Perron-Frobenius Theorem) If an $n \times n$ matrix has nonnegative entries then it has a nonnegative real eigenvalue λ which has a maximum absolute value among all eigenvalues. This eigenvalue λ has a nonnegative real eigenvector. If, in addition, the matrix has no block – triangular decomposition (i.e., it does not contain a $k \times (n-k)$ block of zeros disjoint from the diagonal), then λ has a multiplicity of 1 and the corresponding eigenvector is positive. If the matrix is positive, λ has the strictly largest absolute value.*

Theorem 2.1 implies that if a graph G is strongly connected (and the link weights are nonnegative in case of weighted graphs), then the strongly largest eigenvalue $\lambda_{\max}$ of $\mathbf{A}(G)$ has a multiplicity of 1 and the corresponding eigenvector is positive. If the graph is not strongly connected, the uniqueness of the largest eigenvalue is not guaranteed.

All centrality measures are real valued functions on the nodes of the network. Spectral centrality measures are based on the (left) dominant eigenvector of a network adjacency matrix or some other matrix derived from it. Existence and uniqueness of the spectral measures follow from the theory of nonnegative matrices.

The rationalization behind using an eigenvector as a centrality measure is that important nodes have many connections, but the nodes with the highest degree are not necessarily the most important. It is not just the number of neighbors that counts, but also the importance of these neighbors.

3 Temporal Quantities in Networks

In [2], we proposed a definition of temporal networks with zero latency that is based on temporal quantities. Here, we repeat some of the definitions and describe our approach to the temporal networks which we will use in the rest of the paper.

A temporal network $\mathcal{N}_T = (\mathcal{V}, \mathcal{L}, \mathcal{T}, \mathcal{P}, \mathcal{W})$ is obtained by attaching the time $\mathcal{T}$ to an ordinary network of nodes $\mathcal{V}$ and links $\mathcal{L}$. The sets $\mathcal{P}$ and $\mathcal{W}$ represent the node properties and the link properties or weights, respectively. The time $\mathcal{T}$ is a set of time points, $t \in \mathcal{T}$, and is usually a subset of positive integers, $\mathcal{T} \subseteq \mathbb{N}$.

In a temporal network, nodes $v \in \mathcal{V}$ and links $\ell \in \mathcal{L}$ are not necessarily present or active at all time points. We denote the activity sets of time points for the nodes v with $T(v), T \in \mathcal{P}$, and for the links ℓ with $T(\ell), T \in \mathcal{W}$. The activity set $T(e)$ of a node/link e is described as a sequence of time intervals $([s_i, f_i))_{i=1}^{k}$, where s_i is the starting time and f_i is the finishing time of the activity.

Besides the presence/absence of nodes and links also their properties can change through time. Let a describe the temporal property of a node/link. The activity set of the corresponding node/link is denoted with T_a. To describe the changes we introduce a notion of a temporal quantity

$$\tilde{a}(t) = \begin{cases} a(t) & t \in T_a \\ \maltese & t \in \mathcal{T} \setminus T_a \end{cases}$$

where $a(t)$ is the value of a at an instant t, and $\maltese$ denotes the value undefined. In the following we are talking about temporal quantities and we write simply a instead of $\tilde{a}$. We assume that the values of temporal properties belong to a set A which is a semiring $(A, \oplus, \odot, 0, 1)$. We can extend both operations to the set $A_\maltese = A \cup \{\maltese\}$ by requiring that for all $a \in A_\maltese$ it holds

$$a \oplus \maltese = \maltese \oplus a = a \quad \text{and} \quad a \odot \maltese = \maltese \odot a = \maltese.$$

The structure $(A_\maltese, \oplus, \odot, \maltese, 1)$ is also a semiring. For more about semirings see [1] or [16].

Let $A_\maltese(\mathcal{T})$ denote the set of all temporal quantities over $A_\maltese$ in the time $\mathcal{T}$. To extend the operations to networks and their matrices we define the sum of temporal quantities (corresponds to parallel links)

$$a \oplus b = s$$

as

$$s(t) = \begin{cases} a(t) \oplus b(t) & t \in T_a \cap T_b \\ a(t) & t \in T_a \setminus T_b \\ b(t) & t \in T_b \setminus T_a \\ \maltese & \text{otherwise} \end{cases}$$

and $T_s = T_a \cup T_b$; and the product of temporal quantities (corresponds to sequential links)

$$a \odot b = p$$

as

$$p(t) = \begin{cases} a(t) \odot b(t) & t \in T_a \cap T_b \\ \maltese & \text{otherwise} \end{cases}$$

and $T_p = T_a \cap T_b$. This definition of product is restricted to temporal networks with zero latency – the time needed to traverse the link is equal to zero and there is no waiting in nodes for the next transition.

We define the temporal quantities $\mathbf{0}$ and $\mathbf{1}$ with $\mathbf{0}(t) = \maltese$ and $\mathbf{1}(t) = 1$ for all $t \in \mathcal{T}$. The structure $(A_\maltese(\mathcal{T}), \oplus, \odot, \mathbf{0}, \mathbf{1})$ is also a semiring, and therefore so is the set of square matrices of order n over it for the addition $\mathbf{A} \oplus \mathbf{B} = \mathbf{S}$

$$s_{ij} = a_{ij} \oplus b_{ij}$$

and multiplication $\mathbf{A} \odot \mathbf{B} = \mathbf{P}$

$$p_{ij} = \bigoplus_{k=1}^{n} a_{ik} \odot b_{kj}.$$

The operations $\oplus$ and $\odot$ on the left hand side operate on matrices and on the right hand side in the semiring of temporal quantities.

The static network consisting of links and nodes that are present in the temporal network $\mathcal{N}$ at the time $t \in \mathcal{T}$ is denoted by $\mathcal{N}(t)$ and is called a *time slice* of the network $\mathcal{N}$. The addition and the multiplication of temporal quantities operate inside the chosen time slice. They are defined as pointwise operations on functions. The operations in the matrix semiring also operate on the network time slices. Using these operations on a temporal network is equivalent to using the usual operations on a sequence of static networks that represent the time slices of the temporal network and then combining them into one result.

Using our algebraic approach avoids creating the network time slices and the problem of choosing the time intervals for which the time slices should be computed. The appropriate intervals are chosen by the operations on temporal quantities.

The procedures we have developed for the analysis of temporal networks using temporal quantities are available as a Python library TQ (Temporal Quantities) at `http://vladowiki.fmf.uni-lj.si/doku.php?id=tq`. In the TQ library the temporal quantities are represented with the sequences of ordered triples $[(s_i, f_i, v_i)]_{i \in I}, I \subseteq \mathbb{N}$, where $[s_i, f_i)$ is the time interval in which the temporal quantity has the value $v_i \in A$. Note that this means that the temporal quantities are constant functions on time intervals. The procedures that will be used for the computation of spectral centrality measures use and extend this library. A user friendly program Ianus is also being developed.

4 Algorithms

4.1 Eigenvector Centrality

The most intuitive notion of centrality is the degree centrality which says that the most important nodes in the network are the ones with the highest degree. In many applications, the degree centrality is flawed as it measures the exposure (the number of arcs) and not the actual influence of the node. Wasserman and Faust [23] discuss what they call prestige measures of centrality where the centralities of nodes in a network are recursively related to the centralities of the nodes to which they are linked, the idea being "It is better to have less friends who are powerful than to have a lot of non-powerful friends." This measure has the following form.

Let $\mathbf{A}$ be the (weighted) adjacency matrix of the network in which $a_{vu} \neq 0$ implies that there exists an arc $\ell = (v, u)$ and let x be the in-centrality vector. The form of the prestige measure is

$$x_v = a_{1v}x_1 + a_{2v}x_2 + \cdots + a_{nv}x_n = \sum_{u:u \to v} a_{uv}x_u,$$

where we denote "u links to v" with $u \to v$. The in-centrality of the node v is a combination of the in-centralities of the in-neighbors of v. This set of equations has a matrix representation

$$\mathbf{A}^T x = x. \tag{4.1}$$

In the equation (4.1), x is an eigenvector of $\mathbf{A}^T$ corresponding to the eigenvalue of 1. It has no non-zero solutions unless $\mathbf{A}^T$ has an eigenvalue of 1. One way to solve this problem is to normalize the rows so that each row sums to 1. Then the normalized matrix has an eigenvalue 1 and there is a solution to the equation (4.1).

Eigenvector centrality, first proposed in [8], generalizes the equation to the general eigenvector equation for $\mathbf{A} \in \mathbb{R}^{n \times n}$. The underlying assumption is that the node's in-centrality is proportional to the weighted sum of the in-centralities of its neighbors

$$\lambda x_v = a_{1v}x_1 + a_{2v}x_2 + \cdots + a_{nv}x_n = \sum_{u:u \to v} a_{uv}x_u$$

which (in the matrix notation) is equivalent to

$$\mathbf{A}^T x = \lambda x. \tag{4.2}$$

This equation always has a non-zero solution.

In some cases, it is more appropriate to define the out-centrality of the node v as a combination of the centralities of the out-neighbors of v. In this case, with the same reasoning, the eigenvector out-centrality is the solution to the equation

$$\mathbf{A}x = \lambda x. \tag{4.3}$$

When computing both (in- and out-) eigenvector centralities, we are looking for the dominant eigenvalue and the corresponding eigenvector of the network adjacency matrix $\mathbf{A}$. The simplest numerical method to compute them is the power iteration, see [14] for a more detailed description and for the convergence conditions.

```
function power(A, x_0)                 ▷ x_0 is the initial approximation for the eigenvector
    i = 0
    while no convergence do
        y_{i+1} = A x_i
        x_{i+1} = y_{i+1} / ||y_{i+1}||_2          ▷ Approximate eigenvector.
        i = i + 1
    λ̄ = x_i^T A x_i                                ▷ Approximate eigenvalue.
```

Our implementation of the power iteration algorithm for temporal networks with zero latency as a function *eigTemp* is described in Algorithm 1. The algorithm returns the approximate eigenvector x, the approximate eigenvalue ev and the parameter *convergence,* which tells us whether the algorithm ended when the required tolerance was achieved (its value is **True**) or not (its value is **False**). The function *MatVecRight*(A, x) computes the product $\mathbf{A}x$ for a temporal matrix $\mathbf{A}$ and a temporal vector x, the function *normalize*(x) implements the temporal version of $x/\|x\|_2$. The function *test_dif*(x, y) finds the maximal value (over time) of $\|x - y\|_2$, which we compare to the desired tolerance in line 7. If we achieved the desired tolerance *tol,* we exit the loop. We compute the approximate eigenvalue after the algorithm exits the loop to avoid numerous matrix multiplications. The function *scalProd*(x, y) computes the scalar product of two temporal vectors x and y. Line 11 is the temporal version of $\lambda = x^T\mathbf{A}x$.

Algorithm 1 Temporal power iteration.

```
1:  function eigTemp(A, x, tol = 10^-6, maxIter = 100)
2:      i = 0
3:      convergence = False
4:      while i < maxIter do
5:          x_old = x
6:          x = normalize(MatVecRight(A, x))
7:          if test_dif(x, x_old) < tol then
8:              convergence = True
9:              break
10:         i = i + 1
11:     ev = scalProd(x, MatVecRight(A, x))
12:     return (x, ev, convergence)          ▷ x is an approximate eigenvector
                                               and ev is an approximate eigenvalue
```

Theorem 4.1. *The temporal power iteration algorithm converges if and only if the non-temporal power iteration converges for every time slice matrix* $\mathbf{A}(t),\ t \in \mathcal{T}$.

Proof. The addition and the multiplication of temporal quantities correspond to pointwise operations with functions $a : A_{⌘} \to A_{⌘}$ as we noted on page 4 after the definition of the operations. For every $t \in \mathcal{T}$ the values of the temporal quantities describe a static network – the time slice of the temporal network at the time t. Therefore the conditions of convergence for static matrices translate pointwise to temporal matrices. In the algorithm, the function *test_dif* checks the maximum difference over all times. Since the lifetime is a finite set, the pointwise convergence implies that this maximum converges to 0. □

Corollary 4.2. *Let $\lambda_1(t)$ and $\lambda_2(t)$ be the eigenvalues with the greatest absolute values of a network time slice matrix $\mathbf{A}(t)$. The rate of convergence is $\mu = \max\{|\lambda_2(t)/\lambda_1(t)|,\ t \in \mathcal{T}\}$. The temporal power iteration algorithm converges for $\mu < 1$ and the convergence is slower when the value of μ is near to* 1.

Proof. The temporal power iteration algorithm converges at the time point t when the quotient $|\lambda_2(t)/\lambda_1(t)| < 1$. The proof of convergence for static matrices can be found in [14]. The rate is calculated pointwise and the maximum over a finite set of time points is computed. □

Note that by the proof of Theorem 4.1 the temporal power iteration algorithm can converge for some times $t \in \mathcal{T}$ and not converge for others. We give two stopping conditions for the loop: The first condition is the desired tolerance *tol* which has a default value of 10^{-6} and the second condition is the number of iterations *maxIter* with a default value of 100. In our implementation we set the convergence parameter to **True** when convergence of all the time slice matrices is achieved. It could easily be altered to require convergence of at least one of the time slices.

We use the temporal power iteration algorithm to compute the eigenvector in-centrality (function *inEig*) and the eigenvector out-centrality (function *outEig*). Both algorithms are written in the Algorithm 2. The function *MatTrans*(A) computes the transpose of a temporal matrix $\mathbf{A}$. The function *VecConst*(n) creates a temporal vector of the dimension n, which has components equal to $\mathbf{1}$. The function *numInv*(a) replaces the value of the temporal quantity with its inverse value, leaving the time component intact. The function *numVecProd*(a, x) computes the product of a temporal quantity a and a temporal vector x. In line 2 (or 6), we compute the approximate eigenvalue and eigenvector for $\mathbf{A}^T$ (or $\mathbf{A}$) with an initial vector of (temporal) ones. In line 3 (or 7), we scale the vector according to the eigenvalue.

Algorithm 2 Temporal eigenvalue centrality.

1: **function** *inEig*(A)
2: $\quad (x, ev, conv) =$ *eigTemp*(*MatTrans*(A), *VecConst*$(len(A))$)
3: $\quad x =$ *numVecProd*(*numInv*(ev), x)
4: $\quad$ **return** $(x, conv)$
5: **function** *outEig*(A)
6: $\quad (x, ev, conv) =$ *eigTemp*$(A$, *VecConst*$(len(A)))$
7: $\quad x =$ *numVecProd*(*numInv*(ev), x)
8: $\quad$ **return** $(x, conv)$

If the network is not strongly connected, the network matrix (with the right renumeration of nodes) has a block form

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}.$$

In this case, the corresponding dominant eigenvector is not necessarily unique and there is some debate on how to interpret the result. A lot of the times, the right eigenvector has the form $[\tilde{x}, 0]^T$ which means that we get no information about a lot of the nodes. When the given network is not connected, the matrix has a block diagonal form. Let

$$\mathbf{A} = \begin{bmatrix} 0.8000 & 0.7500 & 0 & 0 \\ 0.2000 & 0.2500 & 0 & 0 \\ 0 & 0 & 0.4000 & 0.5455 \\ 0 & 0 & 0.6000 & 0.4545 \end{bmatrix}$$

be the matrix of a disconnected network. It has the eigenvalues $\lambda_{1,2} = 1, \lambda_3 = -0.1455$ and $\lambda_4 = 0.0500$. The dominant eigenvectors (corresponding to the eigenvalues of 1) are $v_1 = [0.9662, 0.2577, 0, 0]$ and $v_2 = [0, 0, -0.6727, -0.7399]$. Because they correspond to the same eigenvalue, also their sum $v_1 + v_2 = [0.9662, 0.2577, -0.6727, -0.7399]$ or any linear combination of v_1 and v_2 is also an eigenvector. How do we choose the right one? No definite answer to this question has been given. The first two eigenvectors correspond to the centralities of nodes in each component, which makes sense, but there is no good way to compare the two scores.

Another problem with disconnected networks is that the node scores in the largest component do not neccessarily get non-zero values, and the highest scores are often those, that correspond to dyads (strongly connected components with two nodes), which are usually not of high interest. The nodes in the largest strongly connected component (that are of greatest interest most of the time) are not likely to have scores higher than those of the dyadic component. This is usually solved by introducing some normalization factor, which we have not implemented in our algorithms.

The problem of finding the strongly / weakly connected components in temporal networks with zero latency and no waiting in nodes has been addressed in our article [2]. If the network is not strongly connected, the user can choose how to proceed – one can either extract the strongly connected components and compute the eigenvector centralities separately for each component, or use one of the other, more elaborate measures that are described in the later sections of this article and have no such limitation to their use.

4.2 Katz Centrality

In his article [18] Katz describes the centrality index which computes the centrality of a node v by taking into account the centralities of all the nodes from which the node v is reachable. In the proposed approach a weight α is used to dampen the effects of more distant nodes. The weight α could depend on the group and the context and could also vary through time. We only consider the case when it is constant through time. We assume that it is known or we compute it in a way that guarantees the convergence of the algorithm. The constant α can be viewed as the probability of success of the link: the value $\alpha = 0$ means that even the neighboring nodes have no impact on the node and the value $\alpha = 1$ means that the distant nodes are as important as the neighbors.

This idea is modelled with powers of the binary adjacency matrix $\mathbf{A}$ of the network, as the element a_{vu} from $\mathbf{A}^r$ equals to the number of walks of length r from the node v to the node u through other nodes. The column sums of $\mathbf{A}$ give the indegrees of nodes (walks of length 1) and the column sums of $\mathbf{A}^r$ give the number of walks of length r from other nodes.

The idea is to find the column sums of the matrix

$$\mathbf{T} = \alpha\mathbf{A} + \alpha^2\mathbf{A}^2 + \cdots + \alpha^k\mathbf{A}^k + \cdots = (\mathbf{I} - \alpha\mathbf{A})^{-1} - \mathbf{I}.$$

It has been shown in [18] that this is equivalent to solving the system of linear equations

$$\left(\frac{1}{\alpha}\mathbf{I} - \mathbf{A}^T\right) t = d, \tag{4.4}$$

where d is a vector of indegrees. The vector t has elements t_v which are the column sums of the matrix $\mathbf{T}$, i.e. the answers to the original question. This means that for a given network with the binary adjacency matrix $\mathbf{A}$ and for a given α we only need to solve the system of linear equations (4.4). In his article, Katz states that reasonable values of $1/\alpha$ are those between the largest eigenvalue of $\mathbf{A}$ and about twice that value. For smaller values of $1/\alpha$ the effect of distant nodes is greater.

The usual centrality indices are normalized – in case of degree, for example, by $n-1$, the number of possible choices. Using the same notion, Katz [18] defined the divisor of t_v by

$$m = \alpha(n-1) + \alpha^2(n-1)^{(2)} + \alpha^3(n-1)^{(3)} + \ldots,$$

where $(n-1)^{(k)} = (n-1)(n-2)\cdots(n-k)$. A good approximation for m is

$$m \doteq (n-1)!\alpha^{n-1}e^{1/\alpha},$$

which improves with increasing n.

The Katz centrality vector is given by $\frac{1}{m}t$, where t is the solution to the equation (4.4).

We used the Jacobi's method (see [14]) to compute the solution to the linear system of equations. It is an iterative method for solving linear systems of the form $\mathbf{A}x = b$. The idea of the Jacobi's method is to rewrite the original system in the form $\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U}$, where $\mathbf{D} = \text{diag}(\mathbf{A})$ and $\mathbf{L}$ and $\mathbf{U}$ are the lower and upper triangles of $\mathbf{A}$, respectively. Then iterate

$$\mathbf{D}x_{m+1} = -(\mathbf{L} + \mathbf{U})x_m + b.$$

The impletation of the Jacobi's method for solving a system $\mathbf{A}x = b$, where $\mathbf{A}$ and b have elements that are temporal quantities, is written in the Algorithm 3 as a function *jacobi*. We give two conditions for exiting the loop: when we reach the desired precision *tol* of the solution or when we compute a predetermined number of steps *maxIter*. In line 2, we compute the inverse of the diagonal matrix $\mathbf{D}$, and in line 3, we compute the matrix $\mathbf{B} = -(\mathbf{L} + \mathbf{U})$, by setting the diagonal of $\mathbf{A}$ to undefined (semiring neutral element) and negating the values. Line 8 computes the next approximation to the solution as a temporal version of $x_n = invD(Bx + b)$. Lines 9-11 test whether the desired tolerance has been achieved and end the computation if that is the case.

Theorem 4.3. *The temporal Jacobi iteration algorithm converges if and only if the non-temporal Jacobi iteration converges for all the time slice network matrices* $\mathbf{A}(t),\ t \in \mathcal{T}$.

Proof. The reasoning is the same as in the proof of Theorem 4.1. The addition and the multiplication of temporal quantities correspond to pointwise operations on functions. The operations on temporal matrices can therefore be viewed as if we were operating on sequences of static matrices and the convergence conditions for static matrices translate to temporal matrices. □

Definition 4.4. The static matrix $\mathbf{A}$ is *strictly diagonally dominant* if it holds

$$|a_{jj}| > \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}|, \quad j = 1, 2, \ldots, n.$$

Corollary 4.5. *If all the time slice matrices are strictly diagonally dominant the temporal Jacobi iteration converges with any temporal vector as the initial aproximation to the solution of the linear system* $\mathbf{A}x = b$

Proof. The proof of convergence for static matrices can be found in [14]. □

Similarly to the power iteration, the Jacobi iteration algorithm can converge for some times $t \in \mathcal{T}$ and not converge for others. We set the convergence parameter to **True** if it converges in all time points for which the values of temporal quantities are defined.

Algorithm 3 Temporal Jacobi iteration.

function *jacobi*$(A, b, x, tol = 10^{-6}, maxIter = 100)$ ▷ x is the initial approximation for the solution
    $invD = MatSetDiagVec(vecInv(diag(A)))$
    $B = MatMinus(MatSetDiagZero(A))$
    $i = 0$
    convergence = **False**
    while $i < maxIter$ **do**
        $i = i + 1$
        $xn = MatVecRight(invD, VecSum(MatVecRight(B, x), b))$
        if *test_dif*$(x, xn) < tol$ **then**
            convergence = **True**
            break
        $x = xn$
    return $(xn, convergence)$

The algorithm for computing the Katz centrality for temporal networks is written as Algorithm 4. In the algorithm for computing the Katz centrality, the input parameter a, corresponding to α, can be left out.

Corollary 4.6. *The Algorithm 4 computes the parameter* a *in a way that insures that the Jacobi's algorithm converges when* a *is not given as an input parameter.*

Proof. In lines 4-9 of the algorithm we compute a from the maximum of all the column sums (indegrees), so that a is a little bigger than this maximum and every time slice matrix in the equation (4.4) is strictly diagonally dominant. Therefore the algorithm converges by the Corollary 4.5. □

Lines 10-13 compute $\mathbf{B} = \frac{1}{a}\mathbf{I} - \mathbf{A}^T$, and line 14 computes the solution to the linear equation $\mathbf{B}t = d$ with the initial approximation equal to the temporal vector with all elements equal to $\mathbf{1}$. Lines 15-17 normalize the solution with an appropriate m.

Note that the algorithm also works for weighted adjacency matrices. In this case, the powers of the adjacency matrix are the weighted sums of the walks and the above explanation is not that straightforward.

Algorithm 4 Temporal Katz centrality.

```
1: function katz(A, a = Null)
2:     n = len(A)
3:     d = MatVecLeft(A, VecConst(n))          ▷ Column sums – temporal indegrees.
4:     if a = Null then                        ▷ Compute a if it is not given.
5:         max = 0
6:         for i = 1 : len(d) do
7:             if VecMax(d[i]) > max then
8:                 max = VecMax(d[i])
9:         a = 0.999/max
10:    B = n × n temporal matrix
11:    for i = 1 : n do
12:        B[i][i] = [(1, ∞, 1/a)]
13:    B = MatDiff(B, MatTrans(A))
14:    (t, conv) = jacobi(B, d, VecConst(n))
15:    m = math.factorial(n − 1) ∗ (a ∗ ∗(n − 1)) ∗ math.exp(1/a)
16:    m = [(1, ∞, 1/m)]
17:    return (numVecProd(m, t), conv)
```

4.3 Bonacich α and (α, β) Centrality

The dominant eigenvector from Section 4.1 is one of the standard measures of network centrality but it also has its flaws. The nodes with zero indegree also have a zero centrality. Nodes pointed at by nodes with zero centrality also have a zero centrality and the effect propagates to other nodes. In many cases the eigenvector centrality gives no information about a lot of nodes. Some solutions to this problem were given, see for example [10], [9] and [22].

We can assign each node v some status s_v that is independent of the connections. It is possible for the vector s to reflect the effects of external status but it is often assumed to be a vector of ones. The new equation is

$$x = \alpha(\mathbf{A}^T x) + s.$$

The parameter α weighs the relative importance of the network sources versus the outside sources. This measure is called $\alpha-$centrality. It has a matrix solution

$$x = (\mathbf{I} - \alpha\mathbf{A}^T)^{-1} s$$

and is almost identical to the measure proposed by Katz in [18] which we study in Section 4.2. The temporal version of $\alpha-$centrality is written in Algorithm 5. The parameter a in the algorithm corresponds to the parameter α from the definition. If the status vector s is not given, we set it to be a temporal vector of ones in line 3. The solution to the linear system is computed with the temporal version of Jacobi's iteration (Algorithm 3).

Algorithm 5 Temporal Bonacich $\alpha-$centrality.

1: **function** *alpha*$(A, a, s = None)$
2: **if** $s =$ **None then**
3: $s =$ *VecConst*$(len(A))$
4: **return** *jacobi*(*MatSum*(*MatEye*$(len(A))$, *numMatProd*$([(1, \infty, -a)]$, *MatTrans*$(A)))$, s, *VecConst*$(len(A)))$

Another proposed solution from [9], written in Algorithm 6, is also very similar to Katz's centrality measure. It depends on two parameters α and β. The parameter β affects how much of the node's influence is due to the node's neighborhood. If β is positive the status of the node is increasing with its connections. This would be the case in a communication network, for example, where the amount of information available to the individual is increasing with the amount of information available to its contacts. A positive β is chosen in situations in which the node's status (power, influence) increases with connections to influential nodes.

In some situations it is advantageous to have connections to people who have few other options (e.g. in bargaining). In this case power comes with connections to powerless nodes and the node's power reduces with connections to powerful nodes. In such cases a negative β is chosen. The main difference between this measure and Katz's is that we allow $\beta < 0$.

The magnitude of β affects the influence of more distant nodes. When $\beta = 0$, the $(\alpha, \beta)-$centrality measure is proportional to the degree. With increasing $|\beta|$ the distant (reachable) nodes influence the node's centrality in a greater proportion.

The $(\alpha, \beta)-$centrality of a node v is defined as

$$c_v(\alpha, \beta) = \sum_u (\alpha + \beta c_u) a_{uv},$$

which we write in matrix notation as

$$c(\alpha, \beta) = \alpha(\mathbf{I} - \beta\mathbf{A})^{-1}\mathbf{A}\mathbf{e}, \tag{4.5}$$

where $\mathbf{e}$ is a column vector of ones.

From (4.5) we see that α only affects the length of the solution vector. If α is not given, we normalize the solution in such a way that $\|c(\alpha, \beta)\|_2^2 = n$. Using this normalization, $c_v(\alpha, \beta) = 1$ means that the node v has no special standing in the network.

Our temporal version of Bonacich $(\alpha, \beta)-$centrality is given as a function *bonacich* and is described in Algorithm 6. The parameters a and b in the algorithm correspond to the parameters α and β from the definition, respectively. We introduce an auxiliary variable $normB$ that tells whether the solution is normalized in a way we described above (we do that in line 10) or not. We compute the temporal version of the statements $b_1 = a\mathbf{A}\mathbf{e}$ in line 6 and $\mathbf{B} = \mathbf{I} - b\mathbf{A}$ in line 7. We use Jacobi's iteration with the initial approximation of all (temporal) ones to compute the solution to the equation.

Algorithm 6 Temporal Bonacich $(\alpha, \beta)-$centrality.

function *bonacich*$(A, b, a = None)$
  $normB =$ **False**
  if $a =$ **None then**
    $a = 1$
    $normB =$ **True**
  $b1 =$ *numVecProd*$([(1, \infty, a)],$ *MatVecRight*$(A,$ *VecConst*$(len(A))))$
  $B =$ *MatSum*$($*MatEye*$(len(A)),$ *numMatProd*$([(1, \infty, -b)], A))$
  $(x, conv) =$ *jacobi*$(B, b1,$ *VecConst*$(len(A)))$
  if $normB$ **then**
    $x =$ *numVecProd*$([(1, \infty, \sqrt{len(A)})],$ *normalize*$(x))$
  return $(x, conv)$

4.4 Hubs and Authorities

This centrality measure is motivated by the problem of searching the Web but its use is not limitted to text search networks. It is useful in arbitrary networks, especially those that present data with some duality of actor roles (for example, agressors and victims, bidders and recipients, providers and consumers, etc.). At the time when it first appeared, search engines relied on indexing the Web and creating a structured collection of the indexed pages. The problem was the fast growth of the Internet. Because of the enormous size of the network, text-based searching became slow and inefficient. The idea was to use the structure of the hyperlink network to infer the importance of the page from its connections to other pages on the Internet – more relevant pages will be pointed at by many other pages. But the simple indegree measure does not discriminate between the relevant pages for the query and the universally popular pages. Human judgement of relevance is in some way underlying the network structure. The creator of the page v inferred some authority on the page u when he included the link to u on his page. Kleinberg [19] defined two roles of Web pages – hubs and authorities. The idea behind the HITS algorithm for computing hubs and authorities is that inlinks endorse the importance of a page – the page referred to by many other pages is preferred by many (such pages are authorities for a given query). But also, there exist pages that compile lists of relevant resources (these are hubs for a given query). If a page lists a high number of relevant sources it should score high. Good hubs point to good authorities and good authorities are pointed at by good hubs. A page gets authority ranking from the hub rankings of the pages pointing to it, and gets a hub ranking from the authority rankings of the pages it points to. Kleinberg defined the authority update rule and the hub update rule. Both scores are applied iteratively. For an overview, see also [22].

The algorithm operates on focused subnetworks of the Web that are constructed from the output of a text-based search engine. We will not deal with the construction of such a subnetwork and will assume that it is given. We denote its adjacency matrix by $\mathbf{A}$. To each node v of a network (the node represents a Web page) two scores are assigned: the hub score x_v and the authority score y_v. The scores are stored in two distinct vectors. We get coupled relations

$$\lambda y_v = \sum_{u:u \to v} x_u = \sum_u a_{uv} x_u = (\mathbf{A}^T x)_v,$$
$$\mu x_v = \sum_{u:v \to u} y_u = \sum_u a_{vu} y_u = (\mathbf{A} y)_v,$$

which can be rewritten in matrix notation as

$$\lambda\mu x = \mathbf{A}\mathbf{A}^T x, \quad \lambda\mu y = \mathbf{A}^T\mathbf{A} y.$$

This means that the hub and authority scores are just the elements of the dominant eigenvectors of the matrices $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$, respectively. Our version of the HITS algorithm for temporal networks is given in Algorithm 8.

For the computation of the eigensystem of $\mathbf{A}^T\mathbf{A}$ we implemented a more efficient algorithm that computes the eigensystem directly, without computing the product $\mathbf{A}^T\mathbf{A}$. It is implemented as a function *singTemp* and is written in Algorithm 7. The algorithm is similar to Algorithm 1, the difference is in lines 6 and 11, where we multiply with $\mathbf{A}^T$, using the function *MatTransVecRight*.

Algorithm 7 Power iteration for computing the eigenvalues of $\mathbf{A}^T\mathbf{A}$.

```
1:  function singTemp(A, x, tol = 10^-6, maxIter = 100)
2:      i = 0
3:      convergence = False
4:      while i < maxIter do
5:          x_old = x
6:          x = normalize(MatTransVecRight(A, MatVecRight(A, x)))
7:          if test_dif(x, x_old) < tol then
8:              convergence = True
9:              break
10:         i = i + 1
11:     ev = scalProd(x, MatTransVecRight(A, MatVecRight(A, x)))
12:     return (x, ev, convergence)
```

We compute the hubs and authorities scores in Algorithm 8 by first computing the eigensystem of the matrix $\mathbf{A}^T\mathbf{A}$ in line 2, using the initial approximation of ones, from which we get the authority scores vector y. In line 3, we compute the hub scores vector x from y. In lines 4-6 we scale them according to the eigenvalue.

4.5 PageRank

PageRank is the centrality measure used by Google to rank Web pages. Because of the success of Google there is a lot of literature on PageRank, see for example [11], [12], [6], [17], [20] and [22]. Brin and Page first described the calculation of PageRank in their original paper [11].

The PageRank algorithm can be viewed in two different ways – as a random walk on a graph and as an eigenvector of a network matrix. We briefly explain both and compute the

Algorithm 8 Hubs and authorities (HITS algorithm).

function *hits*(A)
  $(y, evy, conv) =$ *singTemp*$(A,$ *VecConst*$(len(A)))$
  $x =$ *normalize*$($*MatVecRight*$(A, y))$
  $evInv =$ *numInv*(evy)
  $y =$ *numVecProd*$(evInv, y)$
  $x =$ *numVecProd*$(evInv, x)$
  return $(x, y, conv)$ ▷ x is the hub scores vector and y is the authority scores vector

PageRank using the eigenvector. Due to the size of the Internet, the random walk version is used in practice.

A random walk is a stationary process on any undirected graph. The centrality of a node derived from a random walk is defined as the number of times that the walker stops at the node in the random process. In directed graphs, the process may not be stationary as the nodes with zero outdegree (dangling ends) act as sinks for the process. Once we get to a node with a zero outdegree we cannot leave it. To make the process stationary, the random walker is given the opportunity to leave a dangling end.

The random walker of PageRank simulates the behaviour of a user browsing the Internet. Most of the time, the user is clicking links on the pages (is surfing), but sometimes he types an URL (jumps). These jumps are added to the random walk in the model. They occur with a probability q and take the simulated user to a random page. The process is described by a simple set of relations

$$p_v = \frac{q}{n} + (1-q) \sum_{u:u\to v} \frac{p_u}{\text{outdeg}(u)}, \; v = 1, 2, \ldots, n, \tag{4.6}$$

where n is the number of nodes, p_v is the PageRank value of the node v, and outdeg(u) is the outdegree of the node u. The sum runs over all the nodes incoming to v.

Typically, the probability of jumps is chosen as $q = 0.15$. Small values of q preserve the information about the network connections better. When $q = 0$ the process may not be stationary and PageRank is ill-defined. When $q = 1$ the jumps dominate and all the nodes have the same PageRank value equal to $\frac{1}{n}$.

For the (equivalent) matrix version of PageRank: Let $\mathbf{A}$ be the adjacency matrix of the network and let $\mathbf{D}$ be the diagonal matrix of outdegrees so that the scaled matrix $\mathbf{S} = \mathbf{D}^{-1}\mathbf{A}$ has row sums equal to 1. When a page v has no outgoing links the row sum corresponding to v in $\mathbf{A}$ is equal to zero and we cannot compute the corresponding row of the matrix $\mathbf{S}$. In this case, we take $\mathbf{S}_{vu} = \frac{1}{n}$ for all u. We construct the matrix $\mathbf{M}$ as

$$\mathbf{M} = \frac{q}{n}\,\mathbf{1} + (1-q)\,\mathbf{S}, \tag{4.7}$$

where $\mathbf{1}$ is a temporal matrix of all (temporal) ones. This matrix is positive and has a unique normed positive left eigenvector x, so that $x\mathbf{M} = x$. The PageRank of a node v is the value of x_v.

The version of PageRank for temporal networks is given as Algorithm 9. In line 4 we compute the vector of outdegrees and in lines 5-7 we compute the matrix $\mathbf{S}$. In line 5, we use the function *vecInvPR* that returns a vector of the inverses of the degrees or, when the degree is undefined (zero), the vector with the value $[(1, \infty, \frac{1}{n})]$ and the matrix, which has elements $[(1, \infty, 1)]$ in the rows that correspond to the nodes with zero outdegree. Line 6 basically changes the original matrix so that the rows corresponding to the nodes with zero outdegree contain all ones. Line 7 scales the matrix according to outdegrees. We do this using a special function *DiagMatProd*(x, A) instead of full matrix multiplication to make the algorithm more efficient. This function computes the product of a matrix that has the vector x on the diagonal and the matrix A. In line 8 we compute $\mathbf{M}$, creating a matrix with all values equal to $[(1, \infty, \frac{1}{n})]$ using the function *constantMat*. We compute the left eigenvector of $\mathbf{M}$ as a right eigenvector of $\mathbf{M}^T$. Finally, we normalize the result. The function *norm1* normalizes the vector using the first norm, meaning that the sum of the vector components is equal to 1 at all times.

Algorithm 9 The temporal PageRank algorithm.

1: **function** *pageRank*$(A, q = 0.15)$
2: $n = len(A)$
3: $S = n \times n$ temporal matrix
4: $s =$ *MatVecRight*$(A,$ *VecConst*$(n))$ ▷ vector of outdegrees
5: $(S, s) =$ *vecInvPR*(S, s)
6: $S =$ *MatSum*(A, S)
7: $S =$ *DiagMatProd*(s, S)
8: $M =$ *MatSum*(*numMatProd*$([(1, \infty, 1 - q)], S)$, *numMatProd*$([(1, \infty, q)],$ *constantMat*$(n, [(1, \infty, 1/n)])))$
9: $(x, ev, conv) =$ *eigTemp*(*MatTrans*(M), *VecConst*$(n))$
10: $x =$ *norm1*(x)
11: **return** $(x, conv)$

Corollary 4.7. *The temporal pageRank algorithm always converges.*

Proof. The matrix $\mathbf{M}$ from the equation (4.7) is positive and has a unique eigenvalue that has the strictly largest absolute value by the Theorem 2.1. Therefore the temporal power iteration converges by the Theorem 4.1. □

4.6 A Note on the Time Complexity of the Algorithms

We use n for the number of nodes of the given network, m for the number of arcs, and k for the number of iterations of the iterative algorithms (Algorithms 1, 3and 7). Because of the assumption that $\mathcal{T} \subseteq \mathbb{N}$, the length of the temporal quantities describing the network vectors and matrices is bounded with the lifetime of the network. We denote the lifetime with L. The underlying semiring is plain floating point numbers field so the time complexity of the operations is $\mathcal{O}(1)$.

We showed in [2], that the addition and the multiplication of temporal quantities have the time complexity of $\mathcal{O}(L)$. Therefore the complexity of the multiplication of two temporal vectors is $\mathcal{O}(nL)$, the complexity of the multiplication of a temporal matrix with a temporal vector is $\mathcal{O}(n^2L)$ and the complexity of the multiplication of two temporal matrices is $\mathcal{O}(n^3L)$.

From this, it follows that all the algorithms we proposed have a time complexity of $\mathcal{O}(kn^2L)$. The time complexity of Algorithm 1 follows from the complexities of the operations in the temporal quantities semiring. The functions of Algorithm 2 have the same complexity, as *eigTemp* is the major part of them. This is also true for Algorithm 9. (Note that the matrix product in line 7 would have the time complexity of $\mathcal{O}(n^3L)$ if we computed the full matrices.) Algorithm 3 also has a time complexity of $\mathcal{O}(kn^2L)$, the major part is line 8. We use the function *jacobi* in line 14 of Algorithm 4, in line 4 of Algorithm 5 and in line 8 of Algorithm 6. It is the major part of the computation in all cases, so these algorithms have the same complexity. The computation of the singular values in Algorithm 7 also has this complexity with our implementation (note that if we were to compute the matrix product and compute its the eigenvalues, the complexity would be $\mathcal{O}(kn^3L)$). We use the results as a major part of Algorithm 8, again of the same complexity.

5 Examples of Spectral Centralities in Temporal Networks

5.1 Spectral centrality measures – test case

We will test our algorithms on the temporal network from Figure 1. The network changes are outlined with the weights on the arcs and with dotted arcs as follows:

The full arcs are present through all of the network lifetime, that is in the time interval $[1, 9)$. In the time intervals $[1, 3) \cup [7, 9)$ the weight of these arcs is equal to one, on the interval $[3, 7)$, the weight is equal to the number written on the arc (note that some of the weights remain 1). The dashed arcs are present only in the time interval $[5, 9)$. In the interval $[5, 7)$ the weight on the arc is equal to the number on the arc in the figure, in the interval $[7, 9)$ all the weights are equal to 1.

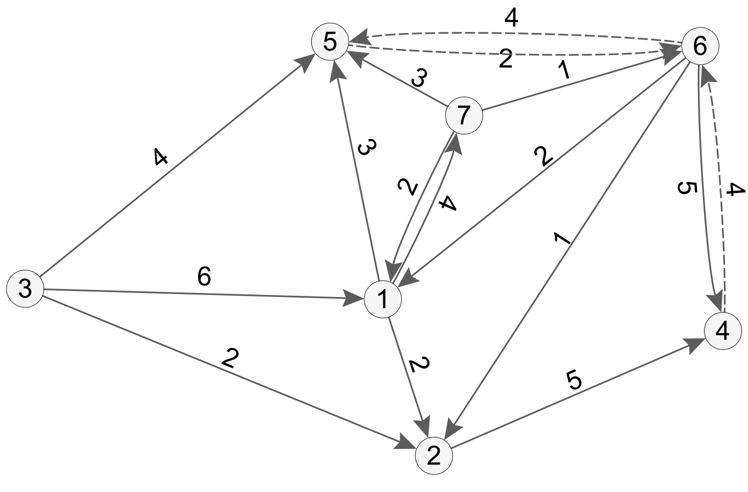

Figure 1: Test temporal graph.

The temporal vectors describing the centrality measures from Section 4 for the test graph are too long to be written in full. The changes in the standings of the nodes that are usually what interests us are written in Table 1. From the Table, we can see that some centrality measures remain undefined for certain nodes in some time intervals. For example, the nodes 2,4,5 in the time interval $[1, 3)$ are missing in the row, corresponding to out-eig. That can also be seen from the Figure, as the nodes 4 and 5 have outdegree equal to zero in

	time 1–3	time 3–5	5–7	7–9
in-eig	4,5,2,1,7,6	5,4,7,1,2,6	4,6,5,1,2,7	6,5,4,2,1,7
out-eig	7,1,3,6	3,1,7,6	6,3,1,4,7,2,5	7,6,1,3,4,5,2
Katz $a = 0.15$	2,5,1,4,7,6	5,4,7,1,2,6	5,6,4,1,7,2	5,6,2,1,4,7
Bonacich $\alpha = 0.85$	1,2,5,4,6,7,3	1,4,5,2,7,6,3	5,1,4,6,2,7,3	5,1,2,6,4,7,3
Bonacich $\beta = 0.15$	7,1,3,6,2	3,1,6,7,2	3,6,1,4,7,2,5	6,7,1,3,4,5,2
hub	3,6,1,7,2	3,7,6,1,2	3,6,7,1,2,4,5	6,3,7,1,2,4,5
authority	1,2,5,4,6,7	1,5,2,4,7,6	5,1,4,2,7,6	5,1,2,4,6,7
pageRank $q = 0.15$	4,2,5,1,7,6,3	4,5,7,1,2,6,3	6,4,5,1,7,2,3	6,4,5,2,1,7,3

Table 1: The order of the nodes of the test graph by their centralities through time.

this interval and the node 2 only points to 4, which has zero centrality.

The second interesting thing is that all the centrality measures return similar results, if we put them into two groups: One group chooses nodes that are central as the ones that have “more inlinks” (in-eig, Katz, α centrality, authority score), the other group chooses the nodes that have “more outlinks” (out-eig, $(\alpha, \beta)-$ centrality, hub score, pageRank).

5.2 Franzosi’s violence network

We applied our algorithms to compute the centrality scores of the nodes in Franzosi’s violence temporal network [15]. From the newspapers in the period from January 1919 to December 1922, Roberto Franzosi collected data about the reported violent actions – interactions between different political groups and other groups of people in Italy. The network nodes represent the involved groups of people (for example, “people”, “police”, “fascists”, “communists”, “socialists”, “workers”) and the arc weights correspond to the number of interactions between two groups (the arc (u, v) with a weight 3 would mean that the group u committed 3 violent actions on the group v). The temporal network contains data about violent activities for each month in the given time period – the temporal quantities corresponding to an arc tell the information about the violent activities for the whole 4 years.

We get the clearest results with the hub and authority scores for the nodes, which is expected because of the nature of the network – the underlying duality of the actors. The actors can be seen as the aggressors and as the groups at which the aggression was directed. For the sake of clarity, we created the timeline of changes in the highest scores. The hub scores can be seen in Figure 2. The time points are months and the heights of the symbols correspond to the value of the normalized authority score. From Figure 2 it is clearly seen that at one time the violent actions of police were replaced with that of the fascists. That happens at the time point 23 which corresponds to November 1920. From the Figure on the right, we see that for some time, the police retained some control and was second by the violent activities, but later it dissapeared altogether.

The authority scores are outlined in Figure 3. There is no clear trend and it seems that the violent activities were not limited to one particular group through time which is in

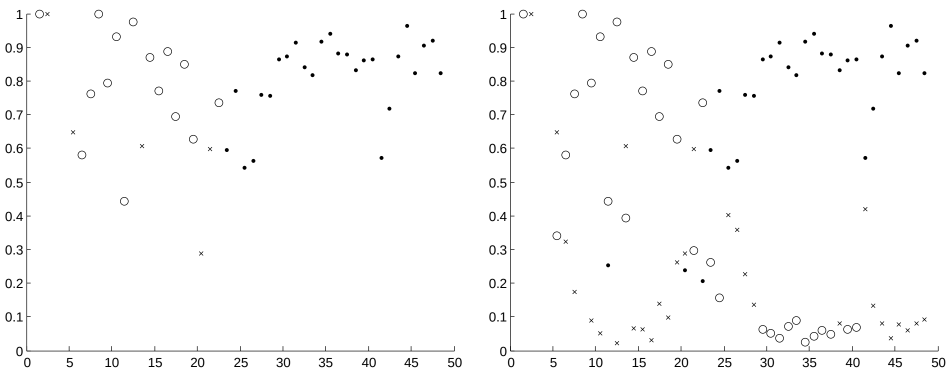

Figure 2: The highest hub score of the Franzosi's network through time (left) and the two highest scores (right). The "fascists" are the black dots, the "police" are circles, others are crosses.

accordance with our intuition.

With the other centrality scores the results are similar, but the boundary is not that obvious. For eigenvector in-centrality we get 24 counts of "workers," "workers (agricultural)" or "socialists," and 14 counts of"undefined," "people" or "protesters." There are three others (once "police" and two times "fascists").

The eigenvector out-centrality returns a mix of "police" (4), "protesters" (3), "?" (3), "undefined" (2), "workers" (2), "workers (agricultural)" (1) and "republicans" (1). The first appearance of "fascists" is at the time point 23 (November 1920). The fascists have the highest centrality score until the end od the timeline, except for 4 instances ("the right", "?", "workers", "police").

The pageRank centrality for $q = 0.15$ gives us 18 counts of "fascists", starting from the time point 22 (October 1920), which is then interrupted with "workers" (3), "people" (3), "undefined" (2) and "police." Until that time, we have a mix of "police" (6), "undefined" (5), "people" (4), "socialists" (3), "war affected" and "the right."

As it seems that the aggressor is more distinct than the groups that were targeted, we computed the Katz and the $\alpha-$centrality measure on the transpose of the original matrix.

The Bonacich $\alpha-$centrality for $\alpha = 0.9$ returns 17 counts of "police" until the time point 23 (others with the maximal centrality score until this time are "thugs," "undefined," and twice "workers"). After the time point 23, we have 23 counts of "fascists," others are "thugs," "police," and twice "workers."

The Katz centrality measure has 16 counts of "police," and one appearance of "thugs," "undefined," "protesters" and "?". After the time point 23 the "fascists" are the only ones with the highest centrality score. The Bonacich $(\alpha, \beta)-$centrality returns the same score.

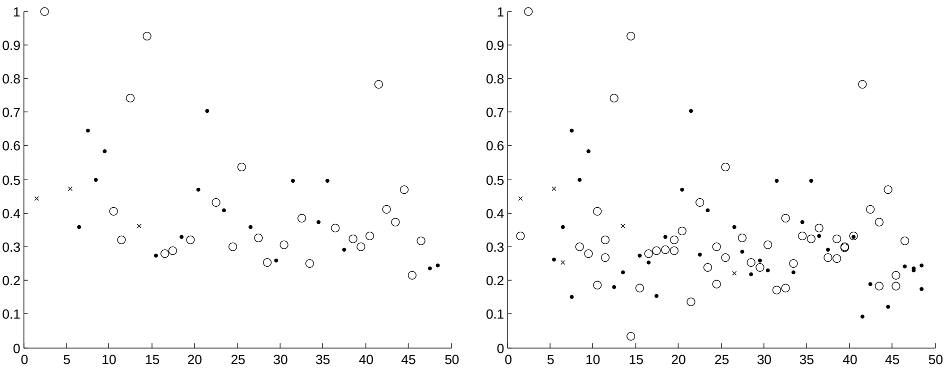

Figure 3: The highest authority score of the Franzosi's network through time (left) and the two highest scores (right). The "workers," "workers (agricultural)" and "socialists" are the black dots, the "undefined," "people" and "protesters" are circles, and other groups are crosses.

These results are summarized in Table 2 in which we have written the count of "police," "fascists," and others with the maximum value of centrality for different centrality measures, divided into two columns – the first for the count before November 1920, the second after that. We only do this for the centrality measures that correspond to the aggressor. From the table, we can see that the fascist aggression was central in the studied news after November 1920 in all cases. Because of the undefined values on some intervals the number of data in the columns varies.

group of people	hub score		pageRank		out-eig		α		α, β		Katz	
police	15	0	6	1	4	1	17	1	16	0	16	0
fascists	0	26	1	18	0	21	0	23	0	25	0	25
other	5	0	14	8	12	3	4	3	4	0	4	0

Table 2: The summary of the maximum centrality scores before and after November 1920 for the Franzosi's violence network.

5.3 9/11/2001 Reuters terror news network

The Reuters terror news network about the 9/11 attack on the United States was obtained from the CRA (Centering Resonance Analysis) networks created by Steve Corman and Kevin Dooley at Arizona State University [13] and was used as a case network for the

Viszards visualization session on the Sunbelt XXII International Sunbelt Social Network Conference, New Orleans, USA, 13-17. February 2002.

The network is based on the September 11 attack news that were released by the news agency Reuters during the 66 consecutive days after the attack. The nodes of the network are words and the edges tell whether the two words appear in the same news sentence. The weight of the edge is the frequency of these common appearances. The network has $n = 13332$ nodes (different words) and $m = 243447$ edges, of which 50859 have weights larger than 1. There are no loops in the network. We extracted a subnetwork of the 50 most active nodes as in [2]. We tested our algorithms on this smaller network.

The methods *inEig* and *outEig* do not converge with the initial approximation vector of all temporal ones. Also, the PageRank ranking of nodes tells almost nothing about the importance of nodes as it jumps around – in value as well as in the node with the highest value of centrality.

The other methods are twofold: The first group corresponds to the question "Which words are the news pointing at the most? What's the end-game?" All methods return the terms "attack," "afghanistan," and "anthrax" as the most frequent terms with the highest value of centrality. The methods belonging to this group are the Katz centrality index computed on the transposed adjacency matrix, Kleinberg's hub score, Bonacich α centrality on the transposed matrix, and Bonacich $(\alpha, \beta)-$centrality. The value of the maximal centrality is getting smaller as the time increases.

The second group answers to the question "From which words do the news spread? What started it all?" and all the centrality measures have the most frequent term "united_states," except for the first week after the attack during which the term with the highest centrality is "world_trade_ctr." The methods belonging to this group are the Katz centrality index, Kleinberg's authority score, Bonacich α centrality, and Bonacich $(\alpha, \beta)-$centrality computed on the transposed adjacency matrix.

We list the count of the terms with the highest α centrality (for the transposed matrix) through time as an example of the first group: 50 times "attack," 10 times "afghanistan," 4 times "anthrax" and once "leader."

As an example of the second group, we list the count of the terms that have the highest Katz centrality measure through time: 49 times "united_states," 7 times "world_trade_ctr," 4 times "washington," 2 times "taliban" and "war," once "world" and "wednesday."

6 Conclusions and Future Work

In the article, we show that spectral centrality measures can be extended to the analysis of temporal networks with zero latency described with temporal quantities. In the application we are using only the combinatorial semiring, but the underlying linear algebra could be extended to other semirings in the future, providing some reasonable motivation is found. Also, the meaning of non-dominant eigenvalues and/or eigenvectors could be explored. With the theory of perturbations of eigenvectors, we feel that it would be possible to continue this research to predict the changes in the standing of the nodes in the network for the near future.

Algorithms for the efficient computation of eigenvalues and for solving linear systems in other semirings could be developed. The problem is that, in semirings, the inverse is not necessarily available. There has been some research on this topic which we have not approached yet.

Methods for the visualisation of temporal networks and for the visualisation of the changes in the node importance through time should be developed.

Our current representation is based on the network matrix, which means that it is not very efficient for large sparse networks. In the future, data structures for the representation of sparse temporal networks could be studied and implemented.

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Z_3-connectivity of $K_{1,3}$-free graphs without induced cycle of length at least 5

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Abstract

Jaeger *et al.* conjectured that every 5-edge-connected graph is Z_3-connected. In this paper, we prove that every 4-edge-connected $K_{1,3}$-free graph without any induced cycle of length at least 5 is Z_3-connected, which partially generalizes the earlier results of Lai [Graphs and Combin. 16 (2000) 165–176] and Fukunaga [Graphs and Combin. 27 (2011) 647–659].

Keywords: Z_3-connectivity, $K_{1,3}$-free, nowhere-zero 3-flow.

Math. Subj. Class.: 05C40

1 Introduction

Graphs in this paper are finite, loopless, and may have multiple edges. Terminology and notations not defined here are from [1].

For a graph G and $v \in V(G)$, denote by $N_G(v)$ (or shortly $N(v)$) the set of neighbors of v in G. Let $d_G(v) = |N_G(v)|$ and $N[v] = N(v) \cup \{v\}$. For $A \subset V(G)$, let $N(A) = \cup_{v \in A} N(v) \setminus A$. A graph G is *trivial* if $|V(G)| = 1$, and *non-trivial* otherwise. An *n-cycle* is a cycle of length n. A path P_n is a path on n vertices. The complete graph on n vertices is denoted by K_n, and K_n^- is obtained from K_n by deleting an edge. For two vertex-disjoint subgraphs H_1 and H_2 of G, denote by $e_G(H_1, H_2)$ (or simply $e(H_1, H_2)$) the number of edges with one end vertex in H_1 and the other one in H_2. If $V(H_1) = \{a\}$, we use $e_G(a, H_2)$(or simply $e(a, H_2)$) instead of $e_G(H_1, H_2)$. For simplicity, if V_1, V_2 are two disjoint subsets of $V(G)$, we use $e_G(V_1, V_2)$ for $e_G(G[V_1], G[V_2])$. Similarly, we define $e(V_1, V_2)$ and $e(a, V_2)$. For graphs $H_1, \ldots, H_s$, a graph G is *$\{H_1, \ldots, H_s\}$-free* if for each $i \in \{1, 2, \ldots, s\}$, G has no induced subgraph H_i.

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Let G be a graph and let D be an orientation of G. If an edge $e = uv \in E(G)$ is directed from a vertex u to a vertex v, then u is a tail of e, v is a head of e. For a vertex $v \in V(G)$, let $E^+(v)$={$e \in E(D)$: v is a tail of e }, and $E^-(v)$={$e \in E(D)$: v is a head of e }. Let A be an abelian group with identity 0 and $A^* = A - \{0\}$. Define $F(G, A) = \{f : E(G) \to A\}$ and $F^*(G, A) = \{f : E(G) \to A^*\}$. For each $f \in F(G, A)$, the boundary of f is a function $\partial f : V(G) \to A$ given by,

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

where "$\sum$" refers to the addition in A.

A function $b : V(G) \to A$ is called an *A-valued zero-sum function* on G if $\sum_{v \in V(G)} b(v) = 0$. The set of all A-valued zero-sum functions on G is denoted by $Z(G, A)$. A graph G is *A-connected* if G has an orientation D such that for any $b \in Z(G, A)$, there is a function $f \in F(G, A^*)$ such that $\partial f(v) = b$. In particular, if $\partial f(v) = 0$ for each vertex $v \in V(G)$, then f is called a nowhere-zero A-flow of G. More specifically, a nowhere-zero k-flow is a nowhere-zero Z_k-flow, where Z_k is the cyclic group of order k. Tutte [16] proved that G admits a nowhere-zero A-flow with $|A| = k$ if and only if G admits a nowhere-zero k-flow.

Integer flow problems were introduced by Tutte in [16]. Group connectivity was introduced by Jaeger *et al.* in [7] as a generalization of nowhere-zero flows. The following longstanding conjecture is due to Jaeger *et al.* and is still open.

Conjecture 1.1. *(Jaeger* et al. *[7]) Every 5-edge-connected graph is Z_3-connected.*

Conjecture 1.1 was extensively studied over thirty years. For the literature, some results can be seen in [3, 4, 10, 13, 17, 18] and so on. Recently, Thomassen [15] proved that every 8-edge-connected graph is Z_3-connected, which improved by Lovász, Thomassen, Wu and Zhang [12] as follows.

Theorem 1.2. *Every 6-edge-connected graph is Z_3-connected.*

However, Conjectures 1.1 is still open. A graph is *chordal* if every cycle of length at least 4 has a chord. A graph G is *bridged* if every cycle C of length at least 4 has two vertices x, y such that $d_G(x, y) < d_C(x, y)$. A graph is *HHD-free* if any k-cycle for $k \geq 5$ in the graph has at least two chords. Lai [9] characterized Z_3-connectivity of 3-edge-connected chordal graphs. Li *et al.* [11] and Fukunaga [6] generalized this result to bridged graphs and 4-edge-connected HHD-free graphs.

Theorem 1.3. *(Fukunaga[6]) Every 4-edge-connected HHD-free graph is Z_3-connected.*

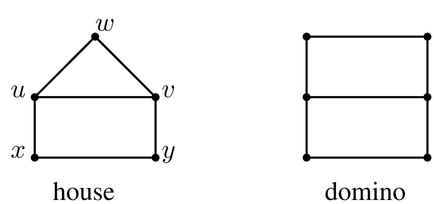

Figure 1: 2 forbidden graphs

On the other hand, it is easy to see that a graph G is HHD-free if and only if G contains no induced subgraph isomorphic to house, domino and k-cycle where $k \geq 5$. Note that a domino contains a $K_{1,3}$ as a subgraph. One naturally ask whether both house and domino may be replaced by a $K_{1,3}$. On the other hand, Xu [14] proved that Conjecture 1.1 is true if and only if every 5-edge-connected $K_{1,3}$-free graph is Z_3-connected. Thus, we consider Z_3-connectivity of $K_{1,3}$-free graphs without induced cycle of length at least 5 and prove the following theorem in this paper.

Theorem 1.4. *Let G be a 4-edge-connected, $K_{1,3}$-free simple graph. If G does not contain any induced cycle of length at least 5, then G is Z_3-connected.*

Theorem 1.4 cannot be implied by Theorem 1.2 in the sense that there are infinite graphs which is Z_3-connected by Theorem 1.4 but not by Theorem 1.2 as follows. Let H_1 be a copy of K_5 and H_2 be a copy of K_m where $m \geq 5$. Pick a vertex u of H_1 and a vertex v of H_2. Define G_m to be the graph obtained from H_1 and H_2 by identifying u and v. It is easy to see that for each $m \geq 5$, G_m is a 4-edge-connected $K_{1,3}$-free graph without any induced cycle of length at least 5. Thus, G_m is Z_3-connected by Theorem 1.4. Clearly, G_m has an edge cut of size 4 which implies Theorem 1.2 does not show that G_m is Z_3-connected.

Theorem 1.3 cannot imply Theorem 1.4 in the sense that there are infinite graphs which is Z_3-connected by Theorem 1.4 but not by Theorem 1.3 as follows. Let H_i be a copy of K_{n_i} where $1 \leq i \leq 4$ and $n_i \geq 5$ for $i \in \{1, 2, 3, 4\}$. Pick two distinct vertices u_i and v_i of H_i. Denote by Γ_n the graph obtained from H_1, H_2, H_3, H_4 by identifying v_i with u_{i+1} for $i = 1, 2, 3$, and v_4 with u_1. It is easy to verify that Γ_n contains a house and so Theorem 1.3 cannot guarantee that Γ_n is Z_3-connected but Theorem 1.4 does.

The paper is organized as follows: In Section 2, the former related results are presented, and some lemmas are established. In Section 3, the main theorem is proved.

2 Lemmas

For a subset $X \subseteq E(G)$, the contraction G/X denotes the graph obtained from G by identifying the two ends of each edge in X and then deleting all the resulting loops. Note that even if G is simple, G/X may have multiple edges. For convenience, we write G/e for $G/\{e\}$, where $e \in E(G)$. If H is a subgraph of G, then we write G/H for $G/E(H)$.

For $k \geq 2$, a *wheel* W_k is the graph obtained from a k-cycle by adding a new vertex, called the *center* of the wheel, which is adjacent to every vertex of the k-cycle. A wheel W_k is *odd* (*even*) if k is odd (or even). For technical reasons, we refer the wheel W_1 to a 3-cycle.

In order to prove Theorem 1.4, we need some lemmas. Some results [2, 5, 7, 8, 9, 10] on group connectivity are summarized as follows.

Lemma 2.1. Let A be an abelian group and G a simple graph. Then each of the following holds:
(1) K_1 is Z_3-connected.
(2) If $e \in E(G)$ and if G is A-connected, then G/e is A-connected.
(3) If H is a subgraph of G and if both H and G/H are A-connected, then G is A-connected.
(4) For $n \geq 5$, K_n^- and K_n are Z_3-connected;
(5) An n-cycle is A-connected if and only if $|A| \geq n + 1$;

(6) For every positive integer k, W_{2k} is Z_3-connected and W_{2k+1} is not Z_3-connected.
(7) Let H be a Z_3-connected subgraph of G. If $e(v, V(H)) \geq 2$ for $v \in V(G-H)$, then the subgraph induced by $V(H) \cup \{v\}$ is Z_3-connected.
(8) Let H_1, H_2 be subgraphs of G such that H_1 and H_2 are A-connected, If $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2$ is A-connected.

For a graph G with $u, v, w \in V(G)$ such that $uv, uw \in E(G)$, let $G_{[uv,uw]}$ denote the graph obtained from G by deleting two edges uv and uw, and then adding a new edge vw, that is, $G_{[uv,uw]} = G \cup \{vw\} - \{uv, uw\}$.

Lemma 2.2. (Chen *et al.* and Lai, [2, 9]) Let A be an abelian group, let G be a graph and u, v, w be three vertices of G such that $d(u) \geq 4$ and $v, w \in N(u)$. If $G_{[uv,uw]}$ is A-connected, then so is G.

A graph G satisfies the *Ore-condition* if $d_G(u) + d_G(v) \geq n$ for every pair of nonadjacent vertices u and v of G.

Theorem 2.3. (Luo *et al.*[13]) *Let G be a simple graph on n vertices, where $n \geq 3$. If G satisfies the Ore-condition, then G is not Z_3-connected if and only if G is one of $\{G_1, G_2, \ldots, G_{12}\}$ shown in Figure 2.*

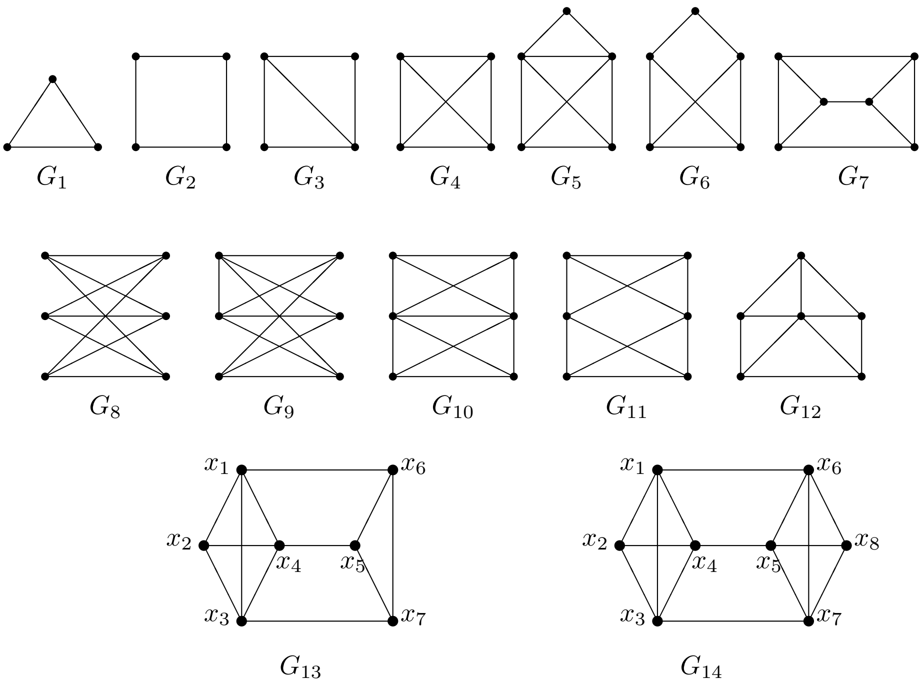

Figure 2: 14 specified graphs

Lemma 2.4. Suppose that H is one graph of $\{G_7, G_{13}, G_{14}\}$. Denote by G the graph obtained from H by adding an edge $e = xy$ which is neither of H nor parallel to any existing edge of H. Then G is Z_3-connected.

Proof. We use the same notation of G_{13}, G_{14} shown in Figure 2. Let $H = G_7$, then G satisfies the Ore-condition. By Theorem 2.3, G is Z_3-connected.

Let $H = G_{13}$. If $x_2 \in \{x, y\}$, then G satisfies the Ore-condition. By Theorem 2.3, G is Z_3-connected. Thus, assume that $x_2 \notin \{x, y\}$. By symmetry, let $e = x_1x_5$. Contracting 2-cycle in $G_{[x_1x_2,x_1x_3]}$ and contracting all 2-cycles generated in the process, we get an even wheel W_4 with the center at x_5, which is Z_3-connected by Lemma 2.1 (6) and so G is Z_3-connected by Lemma 2.2.

Let $H = G_{14}$. If $e = x_2x_8$, then G satisfies the Ore-condition. Since $|V(H)| = 8$, by Lemma 2.3, G is Z_3-connected. Thus, assume that $e \neq x_2x_8$. By symmetry, assume that $e = x_1x_5$ or $e = x_2x_6$. In the former case, contracting 2-cycle in $G_{[x_1x_2,x_1x_3]}$ and contracting all 2-cycles generated in the process, we obtain an even wheel W_4 induced by $\{x_1, x_4, x_5, x_6, x_7\}$ with the center at x_5. Contracting this W_4 into one vertex and contracting 2-cycle generated in the process, finally we get a K_1 which is Z_3-connected. By Lemmas 2.1 (7) and 2.2, G is Z_3-connected. In the latter case, contracting 2-cycle in $G_{[x_1x_2,x_1x_3]}$ and contracting all 2-cycles generated in the process, we obtain an even wheel W_4 induced by $\{x_4, x_5, x_6, x_7, x_8\}$ with the center at x_5, which is Z_3-connected by Lemma 2.1. Note that x_1 has two neighbors in this even wheel. By Lemma 2.1(7), $G_{[x_1x_2,x_1x_3]}$ is Z_3-connected. By Lemma 2.2, G is Z_3-connected. □

3 Proof of Theorem 1.4

Throughout this section, we assume that $\kappa'(G) \geq 4$, $K_{1,3}$-free simple graph and G does not contain any induced cycle of length at least 5. We argue our proof by contradiction, assume that G is a counterexample to Theorem 1.4 with $|V(G)|$ minimized.

Lemma 3.1. Suppose that H is a maximal Z_3-connected subgraph of G and H_i is a component of $G - V(H)$. Let $x_1 \in V(H)$ such that $x_1y_1, \ldots, x_1y_k$, where $y_1, \ldots, y_k \in V(H_i)$ and $2 \leq k \leq 3$. Then each of $y_1, \ldots, y_k$ is not a cut vertex of H_i.

Proof. We only prove the case that $k = 3$. The proof for that $k = 2$ is similar. Without loss of generality, we will prove that y_3 is a cut vertex of H_i. Suppose otherwise that y_3 is not a cut vertex of H_i. Since the maximality of H, $e(y_i, H) = 1$ by Lemma 2.1 (7). Since G is $K_{1,3}$-free, $y_1y_2, y_1y_3, y_2y_3 \in E(G)$. Since $\kappa'(G) \geq 4$, let $x_4 \in V(H)$ and $y_4 \in V(H_i)$ such that $x_4y_4 \in E(G)$, and y_4 is not in the component of $H_i - y_3$ containing y_1 and y_2.

Consider the neighbors of y_1 and y_2. Let $N(y_1) \setminus \{x_1, y_2, y_3\} = \{u_1, u_2, \ldots, u_a\}$ and $N(y_2) \setminus \{x_1, y_1, y_3\} = \{v_1, v_2, \ldots, v_b\}$. Since G is $K_{1,3}$-free, both subgraphs induced by $\{u_1, \ldots, u_a\}$ and by $\{v_1, \ldots, v_b\}$ are complete graphs. We assume, without loss of generality, that $a \geq b$. Since G is 4-edge-connected, $a \geq 1$ and $b \geq 1$. Note that y_3 is a cut vertex of H_i and G is $K_{1,3}$-free. The following claim is straightforward.

Claim. All neighbors of y_3 are y_1, y_2 in the component of $H_i - y_3$ containing $\{y_1, y_2\}$.

Case 1. $\{u_1, \ldots, u_a\} \cap \{v_1, v_2, \ldots, v_b\} \neq \emptyset$.

If $a \geq 4$, then the subgraph induced by $\{y_1, u_1, u_2, \ldots, u_a\}$ is a complete graph K_{a+1}, which is Z_3-connected by Lemma 2.1 (4). By Lemma 2.1 (7), G contains a Z_3-connected subgraph induced by $V(H) \cup \{y_1, y_2, y_3, u_1, u_2, \ldots, u_a\}$, contrary to the maximality of H. Thus, $a \leq 3$.

Assume that $a = 3$. If $|\{u_1, u_2, \ldots, u_a\} \cap \{v_1, v_2, \ldots, v_b\}| \geq 2$, then the subgraph induced by $\{y_1, y_2, u_1, \ldots, u_a\}$ is K_5 or K_5^-, which is Z_3-connected by Lemma 2.1 (4). By Lemma 2.1 (7), G contains a Z_3-connected subgraph induced by $V(H) \cup \{y_1, y_2, y_3, u_1, u_2, \ldots, u_a\}$ which is larger than H, contrary to the choice of H. Thus, $|\{u_1, u_2, \ldots, u_a\} \cap \{v_1, v_2, \ldots, v_b\}| = 1$ and let $u_1 = v_1$. Assume that $3 \geq b \geq 2$. Since $\kappa'(G) \geq 4$, there is a path from $\{u_2, u_3\}$ to v_2 avoiding each vertex of $\{y_1, y_2, u_1\}$. Since G contains no induced cycle of length at least 5, $u_i v_2 \in E(G)$ where $i \in \{2, 3\}$. In this case, G contains an even wheel W_4 induced by $\{y_1, y_2, u_1, u_i, v_2\}$ with the center at u_1, which is Z_3-connected by Lemma 2.1 (6). By Lemma 2.1 (7), G contains a Z_3-connected subgraph induced by $V(H) \cup \{y_1, y_2, y_3, u_1, u_2, \ldots, u_a\}$, contrary to the maximality of H. Thus, $b = 1$. In this case, since $\kappa'(G) \geq 4$, let $u_2 p_1, u_3 q_1 \in E(G)$ where $p_1 \notin \{u_1, u_3, y_1\}$ and $q_1 \notin \{u_1, u_2, y_1\}$. Since $\kappa'(G) \geq 4$ and G contains no cycle of length at least 5, $p_1 q_1, p_1 u_3, q_1 u_2 \in E(G)$. We replace p_1 with u_2 and replace q_1 with u_3. By argument above, we obtain p_2, q_2 such that $p_2 q_2, p_2 p_1, q_2 q_1, p_2 q_1, q_2 p_1 \in E(G)$. Repeating such a way, we can obtain two infinite sequences of $p_1, p_2, \ldots$ and $q - 1, q_2 \ldots$ such that $p_i p_{i+1}, q_i q_{i+1}, p_i q_i, p_i q_{i+1}, q_i, q_{i+1} \in E(G)$ for $i = 1, 2, \ldots$. This contradicts that G is finite.

We are left to consider that $a \leq 2$. In this case, since G is 4-edge-connected, $a = b = 2$ and $\{u_1, u_2\} = \{v_1, v_2\}$. As the proof above, we also obtain a contradiction.

Case 2. $\{u_1, \ldots, u_a\} \cap \{v_1, v_2, \ldots, v_b\} = \emptyset$.

We claim that $a + b \geq 4$. Suppose otherwise that $a + b \leq 3$. It follows that either $a = 2, b = 1$ or $a = b = 1$. We only prove the case when $a = 2$ and $b = 1$. The proof is similar for the case that $a = b = 1$. Since $a = 2$ and $b = 1$, $y_1 u_1, y_1 u_2, y_2 v_1 \in E(G)$. By the Claim, y_3 is not adjacent to one of u_1, u_2 and v_1. Thus, $\{y_1 u_1, y_1 u_2, y_2 v_1\}$ is an edge cut of size 3, contrary to that $\kappa'(G) \geq 4$.

Assume that $a \geq 4$. If $b \geq 4$, then G contains a path from $\{u_1, \ldots, u_a\}$ to $\{v_1, \ldots, v_b\}$. Note that $\kappa'(G) \geq 4$ and G has no cycle of length at least 5. If $2 \leq b \leq 3$, then each vertex of $\{v_1, v_2 \ldots, v_b\}$ has a neighbor in $\{u_1, u_2, \ldots, u_a\}$. If $b = 1$, then v_1 has three neighbors in $\{u_1, \ldots, u_a\}$. By Lemma 2.1 (4), G contains a Z_3-connected subgraph K_{a+1}. By Lemma 2.1 (7), G contains a Z_3-connected subgraph induced by $V(H) \cup \{y_1, y_2, y_3, u_1, \ldots, u_a, v_1, \ldots, v_b\}$, contrary to the maximality of H.

Assume that $a = 3$. If $b = 3$, denote by F the subgraph induced by $\{u_1, u_2, u_3, v_1, v_2, v_3, y_1, y_2\}$. Since $\kappa'(G) \geq 4$ and G contains no cycle of length at least 5, each vertex of $\{u_1, u_2, u_3\}$ is adjacent to one of $\{v_1, v_2, v_3\}$ and each vertex of $\{v_1, v_2, v_3\}$ is adjacent to each vertex of $\{u_1, u_2, u_3\}$. Since $\kappa'(G) \geq 4$, $e(\{u_1, u_2, u_3\}, \{v_1, v_2, v_3\}) \geq 3$ and each vertex of F is of degree 4 and this subgraph satisfies the Ore-condition. By Theorem 2.3, F is Z_3-connected. By Lemma 2.1 (7), G contains a Z_3-connected subgraph induced by $V(H) \cup V(F)$, contrary to the maximality of H.

Let $b = 2$. Since $\kappa'(G) \geq 4$ and G contains no cycle of length at least 5, each vertex of $\{u_1, u_2, u_3\}$ is adjacent to one of $\{v_1, v_2\}$ and each vertex of $\{v_1, v_2\}$ is adjacent to two vertices of $\{u_1, u_2, u_3\}$. It follows that one, say u_3, of $\{u_1, u_2, u_3\}$ has two neighbors in $\{v_1, v_2\}$. It implies that the subgraph induced by $\{u_1, u_2, u_3, v_1, v_2\}$ is an even wheel W_4 with the center at u_3, which is Z_3-connected by Lemma 2.1 (6). By Lemma 2.1 (7), G contains a Z_3-connected subgraph induced by $V(H) \cup \{y_1, y_2, y_3, u_1, u_2, u_3, v_1, v_2\}$, contrary to the maximality of H.

Let $b = 1$. Since $\kappa'(G) \geq 4$ and G contains no cycle of length at least 5, v_1 is adjacent to each vertex of $\{u_1, u_2, u_3\}$. The subgraph induced by $\{u_1, u_2, u_3, v_1, y_1\}$ is K_5^-, which is Z_3-connected by Lemma 2.1 (4). By Lemma 2.1 (7), G contains a Z_3-connected subgraph induced by $V(H) \cup \{y_1, y_2, y_3, u_1, u_2, u_3, v_1\}$, contrary to the maximality of H.

Next, assume that $a = 2$. Let $b = 2$. Since $\kappa'(G) \geq 4$ and G contains no cycle of length at least 5, each vertex of $\{u_1, u_2\}$ is adjacent to two of $\{v_1, v_2\}$ and each vertex of $\{v_1, v_2\}$ is adjacent to two vertices of $\{u_1, u_2\}$. Denote by F the subgraph induced by $\{y_1, y_2, u_1, u_2, v_1, v_2\}$. It follows that F satisfies the Ore-condition and each of 4 vertices of F is of degree 4. By Theorem 2.3, F is Z_3-connected. By Lemma 2.1 (7), G contains a Z_3-connected subgraph induced by $V(H) \cup V(F)$, contrary to the maximality of H. □

Lemma 3.2. G does not contain a nontrivial Z_3-connected subgraph H.

Proof. Suppose that our lemma fails and H is a maximal Z_3-connected subgraph of G. Suppose that $H_1, H_2, \ldots, H_k$ are components of $G - V(H)$, where $k \geq 1$. Let $G' = G/H$ and v' be the vertex into which H is contracted.

Observe H_i, where $i \in \{1, 2, \ldots, k\}$. Let $E(H, H_i) = \{x_1y_1, x_2y_2, \ldots, x_ty_t\}$, where $x_i \in V(H)$ and $y_j \in V(H_i)$ for $i, j \in \{1, 2, \ldots, t\}$. Since G is 4-edge-connected, $t \geq 4$. By the maximality and by Lemma 2.1 (7), $y_1, \ldots, y_t$ are distinct t vertices of H_i. Let $e_i = x_iy_i$ for $i \in \{1, 2, \ldots, t\}$.

Claim 1. $E(H, H_i)$ does not contain 4 edges having a common end-vertex.

Proof of Claim 1. Suppose otherwise that without loss of generality, that e_1, e_2, e_3, e_4 have a common vertex x_1, that is, $x_1 = x_2 = \ldots = x_4$. Then the subgraph induced by $\{x_1, y_1, \ldots, y_4\}$ is a complete graph K_5 since G is $K_{1,3}$-free. By Lemma 2.1 (4), K_5 is Z_3-connected. By Lemma 2.1 (8), G contains a Z_3-connected subgraph induced by $V(H) \cup \{x_1, y_1, \ldots, y_4\}$, contrary to the choice of H. Thus, $E(H, H_i)$ contains at most three edges having a common vertex. This proves Claim 1.

Claim 2. $E(H, H_i)$ does not contain 4 independent edges.

Proof of Claim 2. Suppose otherwise that $E(H, H_i)$ contains 4 independent edges. We assume,without loss of generality, that e_1, e_2, e_3, e_4 are independent edges. Since G has no induced cycle of length at least 5, as the argument above, $y_iy_j \in E(G)$ for $1 \leq i < j \leq 4$. This means that the subgraph the subgraph induced by $\{y_1, y_2, y_3, y_4\}$ is a K_4. In the graph G', the subgraph induced by $\{v', y_1, y_2, y_3, y_4\}$ is a K_5 which is Z_3-connected by Lemma 2.1 (4). By Lemma 2.1 (3), the subgraph induced by $V(H) \cup \{y_1, y_2, y_3, y_4\}$ is Z_3-connected, contrary the maximality of H. This proves Claim 2.

Claim 3. $E(H, H_i)$ does not contain 2 edges having a common end-vertex.

Proof of Claim 3. By Claim 2, we assume that $t = 4$ and e_1, e_2, e_3, e_4 have at least a pair of two edges sharing a vertex in H. Suppose otherwise that we assume, without loss of generality, that e_1, e_2 have a common vertex x_1, that is, $x_1 = x_2$. Since $t = 4$, we need to consider e_3 and e_4 do not share a common end-vertex or e_3 and e_4 share a common end-vertex.

In the former case, the subgraph induced by $\{x_1, y_1, y_2\}$ is a K_3 since G is $K_{1,3}$-free. Since G has no induced cycle of length at least 5, $y_3y_4 \in E(G)$, $y_3y_i, y_4y_j \in E(G)$ where $i, j \in \{1, 2\}$. By Lemma 3.1, the subgraph induced by $\{y_1, y_2, y_3, y_4\}$ is a K_4 since G has no induced cycle of length at least 5. In the graph G', the subgraph induced by $\{v', y_1, y_2, y_3, y_4\}$ is a K_5 which is Z_3-connected by Lemma 2.1 (4). By Lemma 2.1 (3), the subgraph induced by $V(H) \cup \{y_1, y_2, y_3, y_4\}$ is Z_3-connected, contrary the choice of H.

In the latter case, we assume, without loss of generality, that e_3 and e_4 share a common end-vertex x_3. Since G is $K_{1,3}$-free, the subgraph induced by $\{x_1, y_1, y_2\}$ is a complete graph and so is the subgraph induced by $\{x_3, y_3, y_4\}$. Since G has no induced cycle of length at least 5, as the argument above, $y_iy_j \in E(G)$ for some $i \in \{1, 2\}$ and some $j \in \{3, 4\}$. We assume, without loss of generality, that $i = 2, j = 3$. By Lemma 3.1, each vertex of $\{y_1, y_2, y_3, y_4\}$ is not a cut vertex. Since G has no induced cycle of length at least 5 and G is 4-edge-connected, y_2 is adjacent to y_4, and y_3 is adjacent to y_1. In the graph G', the subgraph induced by $\{v', y_1, y_2, y_3, y_4\}$ is a K_5^- which is Z_3-connected by Lemma 2.1 (4). By Lemma 2.1 (3), the subgraph induced by $V(H) \cup \{y_1, y_2, y_3, y_4\}$ is Z_3-connected, contrary the maximality of H. This proves Claim 3.

By Claims 1, 2, and 3, we assume, without loss of generality, that e_1, e_2, e_3 have a common vertex x_1, that is, $x_1 = x_2 = x_3$. Thus, $t = 4$ and $x_4 \neq x_1$. It follows that the subgraph induced by $\{x_1, y_1, y_2, y_3\}$ is a complete graph K_4. Consider the cycle $x_1Px_4y_4Qy_j$, where $V(P) \subset V(H)$, $V(Q) \subset V(H_i)$ and $j \in \{1, 2, 3\}$. Since G contains no any induced cycle of length at least 5, $V(P) = V(Q) = \emptyset$ and $x_1x_4, y_4y_j \in E(G)$. We assume, without loss of generality, that $j = 3$, that is, $y_3y_4 \in E(G)$. By Lemma 3.1, each of $\{y_1, y_2, y_3\}$ is not cut vertex. Since G contains no any induced cycle of length at least 5 and $\kappa'(G) \geq 4$, $y_1y_4, y_2y_4 \in E(G)$. This, in the graph G', the subgraph induced by $\{v', y_1, y_2, y_3, y_5\}$ is a K_5, which is Z_3-connected by Lemma 2.1 (4). By Lemma 2.1 (3), the subgraph induced by $V(H) \cup \{y_1, y_2, y_3, y_4\}$ is Z_3-connected, contrary the maximality of H. □

Proof of Theorem 1.4

Since domino contains an induced $K_{1,3}$ and G contains no induced $K_{1,3}$, G contains no induced domino. By Theorem 1.3 and the choice of G, G contains an induced house. We use the same notations depicted in Figure 2. By symmetry, assume that $d(u) \leq d(v)$.

Claim 1. $|N(u) \cap N(v) \setminus \{w\}| \leq 1$.

Proof of Claim 1. Suppose otherwise that $|N(u) \cap N(v) \setminus \{w\}| \geq 2$. Let $u_1, v_1 \in N(u) \cap N(v) \setminus \{w\}$. Denote by F the subgraph induced by $\{u_1, v_1, w\}$. Since G is $K_{1,3}$-free, F contains at least one edge. If F contains two edges, then the subgraph induced by $\{u_1, v_1, w, u, v\}$ contains an even wheel W_4, which is Z_3-connected by Lemma 2.1 (6), contrary to Lemma 3.2. Thus, F contains only one edge e. By symmetry, assume that $e = wu_1$ or $e = u_1v_1$. In each case, since G is $K_{1,3}$-free, $xv_1, yv_1 \in E(G)$. This means that the subgraph induced by $\{v_1, u, v, x, y\}$ is an even wheel W_4 with the center at v_1, which is Z_3-connected by Lemma 2.1 (6), contrary to Lemma 3.2. This proves Claim 1.

Claim 2. $|N(u) \cap N(v) \setminus \{w\}| \neq 0$.

Proof of Claim 2. Suppose otherwise that $|N(u) \cap N(v) \setminus \{w\}| = 0$. Since $\kappa'(G) \geq 4$, $\delta(G) \geq 4$. First, we claim that $\max\{d(u), d(v)\} \leq 5$. Suppose otherwise that $d(u) \geq 6$. Let $u_1, u_2, u_3 \in N(u) \setminus \{w, v, x\}$. Since G is $K_{1,3}$-free, either $G[\{u, x, u_1, u_2, u_3\}]$ or $G[\{u, w, u_1, u_2, u_3\}]$ is a complete subgraph K_5 which is Z_3-connected by Lemma 2.1, contrary to Lemma 3.2. Thus, $4 \leq d(u), d(v) \leq 5$.

Assume first that $d(u) = d(v) = 4$. Let $N(u) \setminus \{w, v, x\} = \{u_1\}$ and $N(v) \setminus \{w, u, y\} = \{v_1\}$. Since G is $K_{1,3}$-free and $u_1v, v_1u \notin E(G))$, $u_1x, v_1y \in E(G)$.

Since G contains no induced cycle of length at least 5 and $\kappa'(G) \geq 4$, $u_1v_1 \in E(G)$. If $u_1y \in E(G)$ or $xv_1 \in E(G)$, then $G[\{u, v, u_1, v_1, x, y\}]$ contains a subgraph isomorphic to $G_7 + e$ which is Z_3-connected by Lemma 2.4, contrary to Lemma 3.2. Thus, assume that $u_1y, xv_1 \notin E(G)$. Since G contains no induced cycle of length at least 5, $wv_1, wu_1 \notin E(G)$. Since $\kappa'(G) \geq 4$, there exists a shortest (u_1, w)-path P such that $N_P(u_1) \notin \{u, x, v_1\}$. Since $wu_1 \notin E(G)$, $u_2 \in V(P)$ such that $u_1u_2, u_2w \in E(G)$ since G contains no induced cycle of length at least 5. Consider the cycle wu_2u_1xyvw. Since G contains no induced cycle of length at least 5, $u_2y, u_2x \in E(G)$. Since $|N(u) \cap N(v) \setminus \{w\}| = 0$, $u_2v \notin E(G)$. This implies that G contains a $K_{1,3}$ induced by $\{u_2, u_1, w, y\}$, a contradiction.

Next, assume that $d(u) = 4$ and $d(v) = 5$. Let $N(u) \setminus \{w, v, x\} = \{u_1\}$ and $N(v) \setminus \{w, v, y\} = \{v_1, v_2\}$. Since G is $K_{1,3}$-free and $|N(u) \cap N(v) \setminus \{w\}| = 0$, $u_1x, v_1y, v_2y, v_1v_2 \in E(G)$. If $wv_1, wv_2 \in E(G)$, then G contains a K_5^- induced by $\{w, v, v_1, v_2, y\}$ which is Z_3-connected by Lemma 2.1 (4), contrary to Lemma 3.2. Thus, assume that $wv_1 \notin E(G)$. Since G contains no induced cycle of length at least 5 and $\kappa'(G) \geq 4$, $u_1v_1 \in E(G)$. If $u_1y \in E(G)$ or $u_1v_2 \in E(G)$, then $G[\{u, v, x, y, u_1, v_1, v_2\}]$ contains a subgraph isomorphic to $G_{13} + e$ which is Z_3-connected by Lemma 2.4, contrary to Lemma 3.2. Thus, assume that $u_1y, u_1v_2 \notin E(G)$. As the proof above, there is u_2 such that such that $u_1u_2, u_2w \in E(G)$ and $u_2y, u_2x \in E(G)$. It follows that G contains a $K_{1,3}$ induced by $\{u_2, u_1, w, y\}$, a contradiction.

Finally, assume that $d(u) = d(v) = 5$. Let $N(u) \setminus \{w, v, x\} = \{u_1, u_2\}$ and $N(v) \setminus \{w, u, y\} = \{v_1, v_2\}$. Since G is $K_{1,3}$-free and $|N(u) \cap N(v) \setminus \{w\}| = 0$, $u_1x, u_2x, u_1u_2, v_1y, v_2y, v_1v_2, u_1v_1 \in E(G)$. If $\{u_2y, u_2v_2, u_2v_1\} \cap E(G) \neq \emptyset$, then $G[\{u, v, x, y, u_1, u_2, v_1, v_2\}]$ contains a subgraph isomorphic to $G_{14} + e$ which is Z_3-connected by Lemma 2.4, contrary to Lemma 3.2. Thus, assume that $u_2y, u_2v_2, u_2v_1 \notin E(G)$. Since G contains no induced cycle of length at least 5, $u_2w \notin E(G)$. Since $\kappa'(G) \geq 4$, as the proof above, there exists a vertex $u_3 \in V(P)$ such that $u_2u_3, u_3w \in E(G)$ and $u_3x, u_3y \in E(G)$. In this case, G contains a $K_{1,3}$ induced by $\{u_3, u_2, y, w\}$, a contradiction. This proves Claim 2.

By Claims 1 and 2, assume that $N(u) \cap N(v) \setminus \{w\} = \{z\}$. If $xz, yz \in E(G)$, then $G[\{u, v, x, y, z\}]$ is a Z_3-connected subgraph W_4, contrary to Lemma 3.2. Thus, $xz \notin E(G)$ or $yz \notin E(G)$. Recall that $d(u) \leq d(v)$. We claim that $d(v) \leq 6$. Otherwise, since G is $K_{1,3}$-free, $G[N[v] \setminus \{w, u, z\}]$ contains a complete subgraph K_m, where $m \geq 5$, which is Z_3-connected by Lemma 2.1, contrary to Lemma 3.2. Thus, $4 \leq d(u), d(v) \leq 6$.

Case 1. $xz, yz \notin E(G)$.

Since $G[\{u, w, x, z\}]$ is not an induced $K_{1,3}$, $wz \in E(G)$. We first establish a claim.

Claim 3. If $d(u) = 4$, then $d(x) = 4$; if $d(v) = 4$, then $d(y) = 4$.

Proof of Claim 3. Suppose otherwise that $d(x) \geq 5$. Since $d(u) = 4$, each $s \in N(x) \setminus \{u\}$ is not adjacent to u. Thus, $G[N[x] \setminus \{u\}]$ is a Z_3-connected K_m, where $m \geq 5$, since G is $K_{1,3}$-free, contrary to Lemma 3.2. Since G is 4-edge-connected, $d(x) \geq 4$. Thus, $d(x) = 4$. The proof for the case that $d(y) = 4$ is similar. This proves Claim 3.

Assume that $d(u) = d(v) = 4$. By Claim 3, $d(x) = 4$. Let $N(x) \setminus \{u, y\} = \{x_1, x_2\}$. Since G is $K_{1,3}$-free, $yx_1, yx_2, x_1x_2 \in E(G)$. Since $\kappa'(G) \geq 4$, G contains a path from x_1 to w which does not contains any vertex of $\{x_2, x, y, u, v\}$. Since G contains no induced cycle of length at least 5, this path is an edge, that is, $x_1w \in E(G)$ or $x_1z \in E(G)$. Similarly, we can prove that $x_2z \in E(G)$ or $x_2w \in E(G)$. In each case, $H =$

$G[\{u, v, x, y, x_1, x_2, w, z\}]$ satisfies the Ore-condition. By Lemma 2.3, H is Z_3-connected, contrary to Lemma 3.2.

Assume that $d(u) = 4$ and $d(v) = 5$. Let $N(v) \setminus \{u, w, z, y\} = \{v_1\}$. Since G is $K_{1,3}$-free, $yv_1 \in E(G)$. By the Claim, $d(x) = 4$. Assume that $xv_1 \in E(G)$. Let $xx_1 \in E(G)$. Since G is $K_{1,3}$-free, $x_1y, x_1v_1 \in E(G)$. Let $H = G[\{u, v, x, y, x_1, v_1, w, z\}]$. If $wv_1 \in E(G)$, contract the 2-cycle (v, v_1) in $H_{[wv,wv_1,]}$ and repeatedly contact the 2-cycles generated in the process, eventually, we get a K_1 which is Z_3-connected. By Lemmas 2.1 and 2.2, H is Z_3-connected, contrary to Lemma 3.2. Thus, $wv_1 \notin E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, $x_1w \in E(G)$. As the proof above, we can get $H_{[x_1y,x_1v_1]}$ is Z_3-connected. By Lemma 2.2, H is Z_3-connected, contrary to Lemma 3.2.

Thus, $xv_1 \notin E(G)$. Let $xx_1, xx_2 \in E(G)$. Since G is $K_{1,3}$-free, $yx_1, yx_2, x_1x_2 \in E(G)$. Since G contains no induced cycle of length at least 5, $x_1v_1, x_2v_1, wv_1, zv_1 \notin E(G)$. Since G contains no induced cycle of length at least 5 and $\kappa'(G) \geq 4$, $x_1w, x_2z \in E(G)$ or $x_1z, x_2w \in E(G)$. In each case, $L = G[\{u, v, x, y, x_1, x_2, w, z\}]$ satisfies the Ore-condition. By Lemma 2.3, L is Z_3-connected, contrary to Lemma 3.2.

If $d(u) = 4$ and $d(v) = 6$, let $N(v) \setminus \{u, w, z, y\} = \{v_1, v_2\}$. Since G is $K_{1,3}$-free, $v_1y, v_2y, v_1v_2 \in E(G)$. By the Claim, $d(x) = 4$. First assume that $xv_1, xv_2 \in E(G)$. In this case, G contains a Z_3-connected subgraph K_5^- induced by $\{x, y, v, v_1, v_2\}$, contrary to Lemma 3.2. Next, assume that $xv_1 \in E(G)$ and $xv_2 \notin E(G)$. Let $xx_1 \in E(G)$. Since G is $K_{1,3}$-free, $x_1y, x_1v_1 \in E(G)$. Let $H = G[\{w, u, v, x, y, x_1, v_1, v_2\}]$. If $wv_1 \in E(G)$ or $wv_2 \in E(G)$ or $x_1z \in E(G)$, we can prove that $H_{[wv,wv_1]}$ or $H_{[wv,wv_2]}$ or $H_{[x_1y,x_1v_1]}$ is Z_3-connected. By Lemma 2.2, H is Z_3-connected, contrary to Lemma 3.2. If $x_1v_2 \in E(G)$, then G contains a Z_3-connected subgraph K_5^- induced by $\{x_1, y, v, v_1, v_2\}$, a contradiction. Thus, $wv_1, wv_2, x_1z, x_1v_2 \notin E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, $wx_1 \in E(G)$. As the argument above, $H_{[x_1y,x_1v_1]}$ is Z_3-connected. By Lemma 2.2, H is Z_3-connected, contrary to Lemma 3.2. Finally, assume that $xv_1, xv_2 \notin E(G)$. Let $xx_1, xx_2 \in E(G)$. Since G is $K_{1,3}$-free, $x_1x_2, yx_1, yx_2 \in E(G)$. Since G contains no induced cycle of length at least 5, $wv_1, wv_2, zv_1, zv_2 \notin E(G)$ and $e(\{x_1, x_2\}, \{v_1, v_2\}) = 0$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, $wx_1, zx_2 \in E(G)$ or $wx_2, zx_1 \in E(G)$. In each case, $L = G[\{w, u, v, x, y, x_1, x_2, z\}]$ satisfies the Ore-condition, by Lemma 2.3, L is Z_3-connected, contrary to Lemma 3.2.

If $d(u) = 5$ and $d(v) = 5$, let $N(u) \setminus \{v, w, z, x\} = \{u_1\}$ and $N(v) \setminus \{u, w, z, y\} = \{v_1\}$. Since G is $K_{1,3}$-free, $u_1x, v_1y \in E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, $u_1v_1 \in E(G)$. If $u_1y \in E(G)$ or $v_1x \in E(G)$, then $G[\{u, v, x, y, u_1, v_1\}]$ contains a subgraph isomorphic to $G_7 + e$ which is Z_3-connected by Lemma 2.4, contrary to Lemma 3.2. Thus, $u_1y, v_1x \notin E(G)$. Assume that $u_1z \in E(G)$. Since G is $K_{1,3}$-free, $v_1z \in E(G)$. It follows that G contains a Z_3-connected subgraph W_4 induced by $\{u, v, u_1, v_1, z\}$ with the center at z, contrary to Lemma 3.2. Thus, by symmetry, we assume that $u_1z, v_1z \notin E(G)$ and $wu_1, wv_1 \notin E(G)$. As $\kappa'(G) \geq 4$, there is w_1 such that $u_1w_1, w_1w \in E(G)$. Observe cycle ww_1u_1xyvw. Since G contains no induced cycle of length at least 5, $w_1y \in E(G)$. It follows that G contains a $K_{1,3}$ induced by $\{w_1, u_1, w, y\}$, a contradiction.

If $d(u) = 5$ and $d(v) = 6$, let $N(u) \setminus \{v, w, z, x\} = \{u_1\}$ and $N(v) \setminus \{u, w, z, y\} = \{v_1, v_2\}$. Since G is $K_{1,3}$-free, $u_1x, v_1y, v_2y, v_1v_2 \in E(G)$. Since G contains no induced cycle of length at least 5, $wv_1, wv_2, zv_1, zv_2 \notin E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, by symmetry, we assume that $u_1v_1 \in E(G)$. If $\{u_1y, u_1v_2, v_1x, v_2x\} \cap E(G) \neq \emptyset$, then $G[\{u, v, x, y, u_1, v_1, v_2\}]$ contains a subgraph isomorphic to $G_{13} + e$ which is Z_3-connected by Lemma 2.4, contrary to Lemma 3.2. Thus, assume that $u_1y, u_1v_2, v_1x, v_2x \notin E(G)$. Since G has no induced cycle of length at least 5, $u_1z, wu_1 \notin E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, there is w_1 such that such that $u_1w_1, w_1w \in E(G)$. Since G is $K_{1,3}$-free, $w_1y, w_1x \in E(G)$. This implies that $G[\{w_1, u_1, w, y\}]$ is an induced $K_{1,3}$, a contradiction.

If $d(u) = 6$ and $d(v) = 6$, let $N(u)\setminus\{v, w, z, x\} = \{u_1, u_2\}$ and $N(v)\setminus\{u, w, z, y\} = \{v_1, v_2\}$. since G is $K_{1,3}$-free, $u_1x, u_2x, u_1u_2, v_1y, v_2y, v_1v_2 \in E(G)$. If either $e(\{u_1, u_2\}, \{v_1, v_2\}) \geq 2$ or $e(\{u_1, u_2\}, \{v_1, v_2\}) = 1$ and $\{u_1y, u_1v_2, u_2y, u_2v_2, u_2v_1\} \cap E(G) \neq \emptyset$, then $G[\{u, v, x, y, u_1, u_2, v_1, v_2\}]$ contains a subgraph isomorphic to $G_{14}+e$ which is Z_3-connected by Lemma 2.4, contrary to Lemma 3.2. Thus, $e(\{u_1, u_2\}, \{v_1, v_2\}) \leq 1$. Moreover, if $e(\{u_1, u_2\}, \{v_1, v_2\} = 1$ and $u_1y, u_1v_2, u_2y, u_2v_2, u_2v_1 \notin E(G)$. In this case, let $u_1v_1 \in E(G)$. Since G contains no induced cycle of length at least 5, $wu_1, wu_2, wv_1, wv_2, u_2z \notin E(G)$. Consider the case that $e(\{u_1, u_2\}, \{v_1, v_2\}) = 0$. By Lemmas 2.4 and 3.2, $e(x, \{v_1, v_2\}) \leq 1$ and $e(y, \{u_1, u_2\}) \leq 1$. Since G contains no induced cycle of length at least 5, $wu_2, u_2z \notin E(G)$. In each case, since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, there is w_1 such that such that $u_2w_1, w_1w \in E(G)$ and $w_1y, w_1x \in E(G)$. In this case, G contains a $K_{1,3}$ induced by $\{w_1, u_2, w, y\}$, a contradiction.

Case 2. one edge of $\{xz, yz\}$ is not in $E(G)$.

We assume, without loss of generality, that $xz \in E(G)$ and $yz \notin E(G)$. Since G is $K_{1,3}$-free, $wz \in E(G)$. Consider that $d(u) = d(v) = 4$. Since $\delta(G) \geq 4$ and G is $K_{1,3}$-free, $d(y) = 4$. Let $\{y_1, y_2\} \subseteq N(y) \setminus \{x, v\}$. Assume that one edge of y_1z, y_2z is in G, without loss of generality, assume that $y_1z \in E(G)$. Since G is $K_{1,3}$-free, $y_1x, y_2x, y_1y_2 \in E(G)$. Let $H = G[\{u, v, w, x, y, z, y_1, y_2\}]$. Contracting the 2-cycle (y_1, y_2) in $H_{[yy_1,yy_2]}$ and repeatedly contacting the 2-cycles generated in the process, eventually, we get a K_1 which is Z_3-connected. By Lemmas 2.1 and 2.2, H is Z_3-connected, contrary to Lemma 3.2. Thus, $y_1z, y_2z \notin E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, $wy_1 \in E(G)$ or $wy_2 \in E(G)$. In each case, Contracting 2-cycle (u, w) and contracting all 2-cycle generated in the process in $H_{[wu,wz]}$, we obtain a K_5^- which is Z_3-connected by Lemma 2.1 (1). By Lemma 2.2, H is Z_3-connected, contrary to Lemma 3.2.

If $d(u) = 4$ and $d(v) = 5$, let $v_1 \in N(v) \setminus \{w, u, y, z\}$. Since G is $K_{1,3}$-free, $v_1y \in E(G)$. Since $\kappa'(G) \geq 4$, let $yy_1 \in E(G)$. Let H be the subgraph induced by $\{u, v, x, y, w, z, y_1, v_1\}$. Since G is $K_{1,3}$-free, $xy_1 \in E(G)$. Since G contains no induced cycle of length at least 5, $v_1w \notin E(G)$. We claim that $v_1x \notin E(G)$ for otherwise, assume that $v_1x \in E(G)$. Since G is $K_{1,3}$-free, $y_1v_1 \in E(G)$. Contracting 2-cycle (y_1, v_1) and contracting all 2-cycles generated in the process in $H_{[xy_1,xv_1]}$, we get a K_5^- which is Z_3-connected by Lemma 2.1 (4). By Lemma 2.2, H is Z_3-connected, contrary to Lemma 3.2. If $v_1z \in E(G)$, by Lemma 3.2, $v_1u, v_1w \notin E(G)$. In this case, the subgraph induced by $\{z, x, w, v_1\}$ is a $K_{1,3}$, a contradiction. Thus, $v_1z \notin E(G)$. If $wy_1 \in E(G)$, then $H_{[wu,wz]}$ contains a 2-cycle (u, z). Contracting this 2-cycle and contracting all 2-cycles generated in the process, finally we obtain a K_1. By Lemma 2.1 (1) (3) (5), and by Lemma 2.2, H is Z_3-connected, contrary to Lemma 3.2. Thus, $wy_1 \notin E(G)$. Recall that $wx \notin E(G)$. Since $\kappa'(G) \geq 4$, there is a vertex w_1 such that $ww_1, w_1v_1 \in E(G)$. Since $d(u) = 4$

and $d(v) = 5$, $w_1u, w_1v \notin E(G)$. Since G has no induced cycle of length at least 5, $w_1x \in E(G)$. In this case, the subgraph induced by $\{w, w_1, x, v_1\}$ is a $K_{1,3}$, a contradiction.

If $d(u) = 4$ and $d(v) = 6$, let $N(v) \setminus \{w, u, x, z\} = \{v_1, v_2\}$. Since G is $K_{1,3}$-free, $yv_1, yv_2.v_1v_2 \in E(G)$. Assume that $v_1z \in E(G)$. Observe the subgraph $G[\{z, x, w, v_1\}]$. Since G is $K_{1,3}$-free, $xv_1 \in E(G)$ or $wv_1 \in E(G)$. In the former case, G contains a Z_3-connected subgraph W_4 induced by $\{z, u, x, v_1, v\}$ with the center at z, contrary to Lemma 3.2. In the latter case, G contains a Z_3-connected subgraph W_4 induced by $\{w, u, z, v_1, v\}$ with the center at v, contrary to Lemma 3.2. Thus, $v_1z \notin E(G)$. Similarly, $v_2z \notin E(G)$. If $v_1x, v_2x \in E(G)$, then G contains a Z_3-connected subgraph K_5^- induced by $\{y, x, v_1, v, v_2\}$, contrary to Lemma 3.2. Thus,$|\{v_1x, v_2x\} \cap E(G)| \leq 1$. Assume that $v_1x \notin E(G)$. Since G contains no induced cycle of length at least 5, $wv_1, wv_2 \notin E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, there exists a vertex w_1 such that $ww_1, w_1v_1 \in E(G)$ and $w_1x \in E(G)$. In this case, G contains a $K_{1,3}$ induced by $\{w_1, w, x, v_1\}$, a contradiction.

If $d(u) = d(v) = 5$, let $N(u) \setminus \{w, v, x, z\} = \{u_1\}$ and $N(v) \setminus \{w, u, y, z\} = \{v_1\}$. Since G is $K_{1,3}$-free, $u_1x, v_1y \in E(G)$. Since G is $K_{1,3}$-free, $zu_1 \in E(G)$. Since G has no induced cycle of length at least 5, $wv_1 \notin E(G)$. We claim that $zv_1 \notin E(G)$. To the contrary, assume that $zv_1 \in E(G)$. Since G is $K_{1,3}$-free, $u_1v_1, xv_1 \in E(G)$. Let $H = G[\{u, v, w, x, y, z, u_1, v_1\}]$. Contracting the 2-cycle (u, x) in $H_{[u_1u, u_1x]}$ and repeatedly contacting the all 2-cycles generated in the process, eventually, we get a K_1 which is Z_3-connected. By Lemmas 2.1 and 2.2, H is Z_3-connected, contrary to Lemma 3.2. Thus, $v_1z \notin E(G)$. In this case, since $\kappa'(G) \geq 4$, there is a path Q from u_1 to v_1 avoiding any vertex in $\{z, w, u, v\}$. Since G has no induced cycle of length at least 5, $|E(Q)| = 1$, that is, $v_1u_1 \in E(G)$. If $u_1y \in E(G)$ or $v_1x \in E(G)$, then $G[\{u, v, x, y, u_1, v_1\}]$ contains a subgraph isomorphic to $G_7 + e$ which is Z_3-connected by Lemma 2.4, contrary to Lemma 3.2. Thus, $u_1y, v_1x \notin E(G)$. Since G has no induced cycle of length at least 5, $wu_1 \notin E(G)$. As $\kappa'(G) \geq 4$, there is a path P from w to v_1. Since $wv_1 \notin E(G)$, there is $w_1 \in V(G)$ such that $w_1w, w_1v_1 \in E(G)$. Since G has no induced cycle of length at least 5, $w_1x, w_1y \in E(G)$. Since G is $K_{1,3}$-free, $xv_1 \in E(G)$. This is a contradiction, as we have proved $xv_1 \notin E(G)$.

If $d(u) = 5$ and $d(v) = 6$, let $N(u) \setminus \{w, v, x, z\} = \{u_1\}$ and $N(v) \setminus \{w, u, y, z\} = \{v_1, v_2\}$. Since G is $K_{1,3}$-free, $u_1x, v_1y, v_2y, v_1v_2, zu_1 \in E(G)$. Since G has no induced cycle of length at least 5, $wv_1, wv_2 \notin E(G)$. We claim that none of $\{zv_1, zv_2\}$ is in $E(G)$. Suppose otherwise that assume that $zv_1 \in E(G)$. Since G is $K_{1,3}$-free, $u_1v_1, xv_1 \in E(G)$. Let $H = G[\{u, v, w, x, y, z, u_1, v_1, v_2\}]$. Then H is isomorphic to $G_{14} + e$, which is Z_3-connected by Lemma 2.4, contrary to Lemma 3.2. Thus, $zv_1, zv_2 \notin E(G)$. As $\kappa'(G) \geq 4$, there is a path P from u_1 to v_1 avoiding any vertex in $\{z, w, u, v, x, y\}$. Since G has no induced cycle of length at least 5, $u_1v_1 \in E(G)$. In this case, the subgraph induced by $\{u, v, x, y, z, u_1, v_1, v_2\}$ is also isomorphic to $G_{14} + e$, which is Z_3-connected by Lemma 2.4, contrary to Lemma 3.2.

If $d(u) = d(v) = 6$, let $N(u) \setminus \{w, v, x, z\} = \{u_1, u_2\}$ and $N(v) \setminus \{w, u, y, z\} = \{v_1, v_2\}$. Since G is $K_{1,3}$-free, $u_1x, u_2x, u_1u_2, v_1y, v_2y, v_1v_2, zu_1, zu_2 \in E(G)$. This means that the subgraph induced by $\{z, u, u_1, u_2, x\}$ is a K_5, which is Z_3-connected by Lemma 2.1, contrary to Lemma 3.2.

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Finite two-distance-transitive graphs of valency 6

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Abstract

A non-complete graph Γ is said to be $(G, 2)$-*distance-transitive* if, for $i = 1, 2$ and for any two vertex pairs (u_1, v_1) and (u_2, v_2) with $d_\Gamma(u_1, v_1) = d_\Gamma(u_2, v_2) = i$, there exists $g \in G$ such that $(u_1, v_1)^g = (u_2, v_2)$. This paper classifies the family of $(G, 2)$-distance-transitive graphs of valency 6 which are not $(G, 2)$-arc-transitive.

Keywords: 2-Distance-transitive graph, 2-arc-transitive graph, permutation group.

Math. Subj. Class.: 05E18, 05B25

1 Introduction

In this paper, all graphs are finite, simple, connected and undirected. For a graph Γ, we use $V(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its *vertex set* and *automorphism group*, respectively. For the group theoretic terminology not defined here we refer the reader to [4, 8, 26]. Let $u, v \in V(\Gamma)$. Then the distance between u, v in Γ is denoted by $d_\Gamma(u, v)$. A non-complete graph Γ is said to be $(G, 2)$-*distance-transitive*, if for $i = 1, 2$ and for any two vertex pairs (u_1, v_1) and (u_2, v_2) with $d_\Gamma(u_1, v_1) = d_\Gamma(u_2, v_2) = i$, there exists $g \in G$ such that $(u_1, v_1)^g = (u_2, v_2)$. An *arc* is an ordered pair of adjacent vertices. A vertex triple (u, v, w) with v adjacent to both u and w is called a 2-*arc* if $u \neq w$. The graph Γ is said to be $(G, 2)$-*arc-transitive* if G is transitive on both the set of arcs and the set of 2-arcs.

The first remarkable result about $(G, 2)$-arc-transitive graphs comes from Tutte [20, 21], and since then, this family of graphs has been studied extensively, see [1, 12, 15, 16, 17, 23, 24]. By definition, every non-complete $(G, 2)$-arc-transitive graph is $(G, 2)$-distance-transitive. The converse is not necessarily true. If a $(G, 2)$-distance-transitive graph has

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girth 3 (length of the shortest cycle is 3), then this graph is not $(G, 2)$-arc-transitive. Thus, the family of non-complete $(G, 2)$-arc-transitive graphs is properly contained in the family of $(G, 2)$-distance-transitive graphs. The graph in Figure 1 is the Kneser graph $KG_{6,2}$ which is $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive of valency 6 for $G = \mathrm{Aut}(KG_{6,2})$. Therefore the following problem naturally arises: characterize the family of $(G, 2)$-distance-transitive graphs. At the moment, Corr, Schneider and the first author are investigating such graphs, and they classified the family of $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive graphs of valency at most 5 in [6]. Hence 6 is the next smallest valency for $(G, 2)$-distance-transitive graphs to investigate. Our main theorem gives a classification of such graphs.

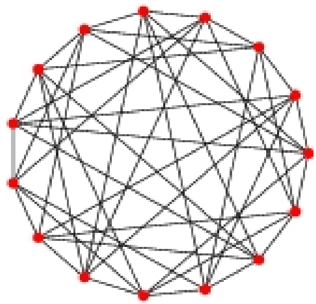

Figure 1: Kneser graph $KG_{6,2}$

Remark 1.1. Let Γ be a connected $(G, 2)$-distance-transitive graph. If Γ has girth at least 5, then for any two vertices u, v with $d_\Gamma(u, v) = 2$, there exists a unique 2-arc between u and v. Hence Γ is $(G, 2)$-distance-transitive implies that it is $(G, 2)$-arc-transitive. If Γ has girth 4, then Γ can be $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive. There are infinitely many such graphs. For instance, let Γ be the complement of the $(2 \times p^k) -$ grid where p is a prime, and let $M = \mathbb{Z}_p^k : \mathbb{Z}_{p^k-1}$, $G = \mathbb{Z}_2 \times M$. Then Γ is $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive of valency $p^k - 1$ and girth 4. There are also infinitely many $(G, 2)$-distance-transitive graphs of girth 4 that are $(G, 2)$-arc-transitive, for example the complete bipartite graphs $\mathrm{K}_{m,m}$. If Γ has girth 3, then since Γ is non-complete, it follows that G_u is not 2-transitive on $\Gamma(u)$, hence it is not $(G, 2)$-arc-transitive.

The *line graph* $L(\Gamma)$ of a graph Γ has the set of edges of Γ as its vertex set, and two edges are adjacent in $L(\Gamma)$ if and only if they have a common vertex in Γ. The line graph of a complete bipartite graph $\mathrm{K}_{m,n}$ is called an $(m \times n)-$grid. Let Γ be a connected graph. The *complement graph* $\overline{\Gamma}$ of Γ, is the graph with vertex $V(\Gamma)$, and two vertices are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in Γ. The *Hamming graph* $\mathrm{H}(d, n)$ has vertex set $\mathbb{Z}_n^d = \mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n$, and two vertices are adjacent if and only if they have exactly one different coordinate. We denote by $\mathrm{K}_{m[b]}$ the *complete multipartite graph* with m parts, and each part has b vertices where $m \geq 3, b \geq 2$. Let p be a prime such that $p \equiv 1 \pmod 4$. Then, the *Paley graph* $P(p)$ is the Cayley graph $\mathrm{Cay}(T, S)$ for the additive group $T = F_p^+$ with $S = \{w^2, w^4, \ldots, w^{p-1} = 1\}$ and $\Gamma_2(1) = \{w, w^3, \ldots, w^{p-2}\}$, where w is a primitive element of F_p, and $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_p : \mathbb{Z}_{\frac{p-1}{2}}$. In particular, Hamming graphs and Paley graphs are $(G, 2)$-distance-transitive for $G = \mathrm{Aut}(\Gamma)$, see [3, 13].

The *diameter* $\mathrm{diam}(\Gamma)$ of a graph Γ is the maximum distance occurring over all pairs of vertices. Let $u \in V(\Gamma)$ and $i = 1, 2, \ldots, \mathrm{diam}(\Gamma)$. We use $\Gamma_i(u)$ to denote the set of vertices at distance i with vertex u in Γ. Sometimes, $\Gamma_1(u)$ is also denoted by $\Gamma(u)$. Let Ω be a set of cardinality n. Then the *Kneser graph* $KG_{n,k}$ is the graph with vertex set all k-subsets of Ω, and two k-subsets are adjacent if and only if they are disjoint. The *triangular graph* $T(n)$ is the graph with vertex set all 2-subsets of Ω, and two 2-subsets are adjacent if and only if they share one common element. Thus $KG_{n,2} = \overline{T(n)}$. A subgraph X of Γ is an *induced subgraph* if two vertices of X are adjacent in X if and only if they are adjacent in Γ. When $U \subseteq V(\Gamma)$, we use $[U]$ to denote the subgraph of Γ induced by U.

Since complete graphs have diameter 1, they do not provide interesting examples. Our main theorem determines the family of non-complete $(G, 2)$-distance-transitive graphs of valency 6 which are not $(G, 2)$-arc-transitive.

Theorem 1.2. *Let Γ be a connected non-complete $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive graph of valency 6. Let $u \in V(\Gamma)$. Then one of the following holds.*

(1) Γ *has girth* 4*, and* $(\Gamma, G) = (\overline{(2 \times 7)-\mathsf{grid}}, S_2 \times M)$ *where M is a 2-transitive but not 3-transitive subgroup of S_7.*

(2) $[\Gamma(u)]$ *is connected, and Γ is isomorphic to one of: $T(5)$, Paley graph $P(13)$, $\mathrm{K}_{3[3]}$ or $\mathrm{K}_{4[2]}$.*

(3) $[\Gamma(u)]$ *is disconnected, and either*

(3.1) $[\Gamma(u)] \cong 2\mathrm{K}_3$*,* $\Gamma \cong \mathrm{H}(2, 4)$*, or* $|\Gamma_2(u)| = 18$ *and Γ is a line graph; or*

(3.2) $[\Gamma(u)] \cong 3\mathrm{K}_2$*,* $\Gamma \cong KG_{6,2}$*, or* $|\Gamma_2(u)| = 12$*,* 24*.*

Remark 1.3. (1) There exist graphs Γ in Theorem 1.2 (3.1) such that $|\Gamma_2(u)| = 18$. For instance the generalized hexagon of order $(3, 1)$ and the generalized dodecagon of order $(3, 1)$. These two graphs are locally isomorphic to $2\mathrm{K}_3$ and $|\Gamma_2(u)| = 18$. By [3, p.223], they are $(G, 2)$-distance-transitive for $G = \mathrm{Aut}(\Gamma)$, since they are non-complete and have girth 3, they are not $(G, 2)$-arc-transitive.

(2) There exist graphs Γ in Theorem 1.2 (3.2) such that $|\Gamma_2(u)| = 12$ and also exist graphs such that $|\Gamma_2(u)| = 24$. For instance $\mathrm{H}(3, 3)$ has valency 6, $[\Gamma(u)] \cong 3\mathrm{K}_2$ and $|\Gamma_2(u)| = 12$; the halved foster graph has valency 6, $[\Gamma(u)] \cong 3\mathrm{K}_2$ and $|\Gamma_2(u)| = 24$. By [3, p.223], these two graphs are $(G, 2)$-distance-transitive for $G = \mathrm{Aut}(\Gamma)$, since they are non-complete and have girth 3, they are not $(G, 2)$-arc-transitive.

2 Proof of Theorem 1.2

In this section, we will prove our main theorem by a series of lemmas. All graphs are non-complete graphs.

A graph Γ is said to be *G-distance-transitive* if G is transitive on the ordered pairs of vertices at any given distance. The study of finite G-distance-transitive graphs goes back to Higman's paper [10] in which "groups of maximal diameter" were introduced. These are permutation groups G which act distance-transitively on some graph. Then G-distance-transitive graphs have been studied extensively and a classification is almost done, see [2, 9, 11, 18, 19, 22, 25]. By definition, every non-complete G-distance-transitive graph is $(G, 2)$-distance-transitive.

The following remark gives an useful observation.

Remark 2.1. Let Γ be a $(G, 2)$-distance-transitive graph. Let u, w be two vertices such that $d_\Gamma(u, w) = 2$.

Suppose that $|\Gamma_3(u) \cap \Gamma(w)| = 0$. Then since Γ is $(G,2)$-distance-transitive, Γ has diameter 2 and so it is G-distance-transitive.

Suppose that $|\Gamma_3(u) \cap \Gamma(w)| = 1$. Let $(u_0, \ldots, u_i)$ be a path with $d_\Gamma(u_0, u_i) = i$ where $i = \mathrm{diam}(\Gamma)$. Then for each $j \leq \mathrm{diam}(\Gamma) - 2$, $|\Gamma_3(u_j) \cap \Gamma(u_{j+2})| = 1$. Note that, $\Gamma_{j+3}(u_0) \cap \Gamma(u_{j+2}) \subseteq \Gamma_3(u_j) \cap \Gamma(u_{j+2})$, and so $|\Gamma_{j+3}(u_0) \cap \Gamma(u_{j+2})| = 1$, hence Γ is also G-distance-transitive.

We use $G_u^{[1]}$ to denote the kernel of the G_u-action on $\Gamma(u)$.

Lemma 2.2. *Let Γ be a $(G,2)$-distance-transitive graph. Let $u, w \in V(\Gamma)$ be such that $d_\Gamma(u,w) = 2$. Let $g \in G_u^{[1]}$ be with order a prime p. Suppose that $|\Gamma_3(u) \cap \Gamma(w)| < p$. Then g is not trivial on $\Gamma_2(u)$.*

Proof. Suppose that g is trivial on $\Gamma_2(u)$. Let $w_i \in \Gamma_2(u)$. Since $g \in G_u^{[1]}$ and g is trivial on $\Gamma_2(u)$, g fixes all the vertices in $(\Gamma(u) \cup \Gamma_2(u)) \cap \Gamma(w_i)$ and $g \in G_{w_i}$. In particular, g fixes $\Gamma_3(u) \cap \Gamma(w_i)$ setwise.

Since Γ is $(G,2)$-distance-transitive and $|\Gamma_3(u) \cap \Gamma(w)| < p$, $|\Gamma_3(u) \cap \Gamma(w_i)| < p$. Since the order of g is prime p and g fixes $\Gamma_3(u) \cap \Gamma(w_i)$ setwise, it follows that g fixes all the vertices in $\Gamma_3(u) \cap \Gamma(w_i)$. Thus $g \in G_{w_i}^{[1]}$. Since w_i is any vertex of $\Gamma_2(u)$, g fixes all the vertices of $\Gamma_3(u)$. For any $v \in \Gamma(u)$, $\Gamma_2(v) \subseteq \Gamma(u) \cup \Gamma_2(u) \cup \Gamma_3(u)$. Thus $g \in G_v^{[1]}$ and fixes all the vertices of $\Gamma_2(v)$.

Since Γ is $(G,2)$-distance-transitive, for any $z \in \Gamma_2(v)$, $|\Gamma_3(v) \cap \Gamma(z)| < p$. Since g fixes all the vertices in $(\Gamma(v) \cup \Gamma_2(v)) \cap \Gamma(z)$, g fixes all the vertices in $\Gamma_3(v) \cap \Gamma(z)$. Thus $g \in G_z^{[1]}$. In particular, g fixes all the vertices of $\Gamma_4(u)$. Since Γ is connected, by induction, g fixes all the vertices of Γ, so $g = 1$, which is a contradiction. Thus g is not trivial on $\Gamma_2(u)$. □

Lemma 2.3. *Let Γ be a $(G,2)$-distance-transitive graph of valency 6. Let $u, w \in V(\Gamma)$ be such that $d_\Gamma(u,w) = 2$. If Γ has girth 4 and $|\Gamma(u) \cap \Gamma(w)| = 3$, then Γ is $(G,2)$-arc-transitive.*

Proof. Suppose that Γ has girth 4 and $|\Gamma(u) \cap \Gamma(w)| = 3$. Let (u, v, w) be a 2-arc. Then $d_\Gamma(u,w) = 2$ and $|\Gamma_2(u) \cap \Gamma(v)| = 5$. Since Γ is $(G,2)$-distance-transitive, there are 30 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Since $|\Gamma(u) \cap \Gamma(w)| = 3$ and $|\Gamma(u) \cap \Gamma(w)| \cdot |\Gamma_2(u)| = 30$, it follows that $|\Gamma_2(u)| = 10$. Again since Γ is $(G,2)$-distance-transitive, G_u is transitive on both $\Gamma(u)$ and $\Gamma_2(u)$, so both $|\Gamma(u)|$ and $|\Gamma_2(u)|$ divide $|G_u|$, hence 30 divides $|G_u|$. Thus 5 divides $|G_{u,v}|$, so $G_{u,v}$ has an element g of order 5. Therefore either $\langle g \rangle$ is regular on $\Gamma(u) \setminus \{v\}$ or is trivial on $\Gamma(u) \setminus \{v\}$. If $\langle g \rangle$ is regular on $\Gamma(u) \setminus \{v\}$, then $G_{u,v}$ is transitive on $\Gamma(u) \setminus \{v\}$, so G_u is 2-transitive on $\Gamma(u)$. Thus Γ is $(G,2)$-arc-transitive.

Now suppose that g is trivial on $\Gamma(u) \setminus \{v\}$. Then $g \in G_u^{[1]}$. Since $|\Gamma(u) \cap \Gamma(w)| = 3$, it follows that $|\Gamma_3(u) \cap \Gamma(w)| \leq 3 < 5$. Thus by Lemma 2.2, g is not trivial on $\Gamma_2(u)$. Hence $\langle g \rangle$ has orbits of size 5 on $\Gamma_2(u)$. Since g fixes $\Gamma_2(u) \cap \Gamma(v_i)$ setwise and $|\Gamma_2(u) \cap \Gamma(v_i)| = 5$, it follows that $\langle g \rangle$ is transitive on $\Gamma_2(u) \cap \Gamma(v_i)$. Thus G_{u,v_i} is transitive on $\Gamma_2(u) \cap \Gamma(v_i)$, so Γ is $(G,2)$-arc-transitive. □

Lemma 2.4. ([6]) *Let $\Gamma \cong \mathrm{K}_{m,m}$ with $m \geq 2$. Then Γ is $(G,2)$-distance-transitive if and only if it is $(G,2)$-arc-transitive.*

A permutation group G on a set Ω is said to be 2-*homogeneous*, if G is transitive on the set of 2-subsets of Ω.

Lemma 2.5. ([8, Theorem 9.4B]) *Let G be a 2-homogeneous permutation group which is not 2-transitive of degree n. Then $n = p^e \equiv 3 \pmod 4$ where p is a prime.*

Lemma 2.6. *Let Γ be a $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive graph of valency 6. If Γ has girth 4, then $(\Gamma, G) = (\overline{(2 \times 7)-\mathsf{grid}}, S_2 \times M)$ where M is a 2-transitive but not 3-transitive subgroup of S_7.*

Proof. Suppose that Γ has girth 4. Let (u, v, w) be a 2-arc. Then $d_\Gamma(u, w) = 2$, $|\Gamma_2(u) \cap \Gamma(v)| = 5$ and $|\Gamma(u) \cap \Gamma(w)| \geq 2$. Further there are 30 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Since Γ is $(G, 2)$-distance-transitive, $|\Gamma(u) \cap \Gamma(w)|$ divides 30. Since $2 \leq |\Gamma(u) \cap \Gamma(w)| \leq 6$, we have $|\Gamma(u) \cap \Gamma(w)| = 2$, 3, 5 or 6.

Suppose first that $|\Gamma(u) \cap \Gamma(w)| = 2$. Then since Γ has girth 4, each 2-arc of Γ lies in a unique 4-cycle. Thus, there is a 1-1 mapping between the unordered vertex pairs in $\Gamma(u)$ and vertices in $\Gamma_2(u)$. Since G_u is transitive on $\Gamma_2(u)$, it follows that G_u is transitive on the set of unordered vertex pairs in $\Gamma(u)$. Hence $G_u^{\Gamma(u)}$ is 2-homogeneous on $\Gamma(u)$. Further, since Γ is not $(G, 2)$-arc-transitive, $G_u^{\Gamma(u)}$ is not 2-transitive on $\Gamma(u)$. Thus by Lemma 2.5, the valency of Γ is $p^e \equiv 3 \pmod 4$ where p is a prime, contradicting the fact that Γ has valency 6.

Next, if $|\Gamma(u) \cap \Gamma(w)| = 3$, then by Lemma 2.3, Γ is $(G, 2)$-arc-transitive, which is a contradiction.

Thirdly, suppose that $|\Gamma(u) \cap \Gamma(w)| = 5$. Then $|\Gamma_3(u) \cap \Gamma(w)| \leq 1$. It follows from Remark 2.1 that Γ is G-distance-transitive. By inspecting the graphs in [3, p. 222-223], Γ is isomorphic to $\overline{(2 \times 7)-\mathsf{grid}}$. Noting that $\overline{(2 \times 7)-\mathsf{grid}}$ is $(\mathrm{Aut}(\Gamma), 2)$-arc-transitive. Thus $S_2 < G < \mathrm{Aut}(\Gamma) \cong S_2 \times S_7$. Let $G = S_2 \times M$ where $M < S_7$. Then $G_u = M_u$. Since Γ is $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive, M_u is transitive but not 2-transitive on $\Gamma(u)$. Thus M is a 2-transitive but not 3-transitive subgroup of S_7.

Finally, if $|\Gamma(u) \cap \Gamma(w)| = 6$, then $\Gamma \cong \mathrm{K}_{6,6}$, and by Lemma 2.4, Γ is $(G, 2)$-distance-transitive implies that it is $(G, 2)$-arc-transitive, which is a contradiction. □

In a non-complete graph Γ, a 2-*geodesic* of Γ is a 2-arc (u_0, u_1, u_2) such that $d_\Gamma(u_0, u_2) = 2$. The graph Γ is said to be $(G, 2)$-*geodesic-transitive*, if G is transitive on both the set of arcs and the set of 2-geodesics. Hence, a non-complete G-arc-transitive graph is $(G, 2)$-geodesic-transitive if, for any arc (u, v), $G_{u,v}$ is transitive on $\Gamma_2(u) \cap \Gamma(v)$. By definition, every $(G, 2)$-geodesic-transitive graph is $(G, 2)$-distance-transitive.

Suppose that Γ is a G-distance-transitive graph of valency k and diameter d. Then the cells of the distance partition with respect to vertex u are orbits of G_u, every vertex in $\Gamma_i(u)$ is adjacent to the same number of other vertices in $\Gamma_{i-1}(u)$, say c_i. Similarly, every vertex in $\Gamma_i(u)$ is adjacent to the same number of other vertices in $\Gamma_{i+1}(u)$, say b_i. The notation $(k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d)$ is called the *intersection array* of Γ.

Lemma 2.7. *Let Γ be a $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive graph of valency 6. Let $u \in V(\Gamma)$. If $[\Gamma(u)]$ is connected, then Γ is isomorphic to one of: $T(5)$, Paley graph $P(13)$, $\mathrm{K}_{3[3]}$ or $\mathrm{K}_{4[2]}$.*

Proof. Suppose that $[\Gamma(u)]$ is connected. Let (u, v, w) be a 2-arc such that $d_\Gamma(u, w) = 2$. Since Γ is $(G, 2)$-distance-transitive, G_u is transitive on $\Gamma(u)$, so $[\Gamma(u)]$ is a vertex-transitive graph. Let k be the valency of $[\Gamma(u)]$. Since $[\Gamma(u)]$ is connected and $|\Gamma(u)| = 6$, it follows that $k = 2, 3, 4, 5$. Let $\Gamma(u) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$.

If $k = 5$, then $[\Gamma(u)] \cong \mathrm{K}_6$, and so $\Gamma \cong \mathrm{K}_7$, contradicting the fact that Γ is non-complete.

Suppose that $k = 4$. Then $|\Gamma(u) \cap \Gamma(v_1)| = 4$, say $\Gamma(u) \cap \Gamma(v_1) = \{v_2, v_3, v_4, v_5\}$. Since $|\Gamma(u) \cap \Gamma(v_6)| = 4$ and v_1, v_6 are non-adjacent, it follows that $\Gamma(u) \cap \Gamma(v_6) = \{v_2, v_3, v_4, v_5\}$. Thus $[\Gamma(u)]$ has diameter 2, and $\{v_1, v_6\}$ is a block. Since $[\Gamma(u)]$ is vertex-transitive, $[\Gamma(u)] \cong \mathrm{K}_{3[2]}$, and by [3, p.5] or [5], $\Gamma \cong \mathrm{K}_{4[2]}$.

Suppose that $k = 3$. Then $|\Gamma(u) \cap \Gamma(v_1)| = 3$, say $\Gamma(u) \cap \Gamma(v_1) = \{v_2, v_3, v_4\}$. Assume first that $[\Gamma(u)]$ does not have triangles. Then every vertex of $\{v_2, v_3, v_4\}$ is adjacent to both v_5 and v_6. Thus $[\Gamma(u)] \cong \mathrm{K}_{3,3}$. Then by [3, p.5] or [5], $\Gamma \cong \mathrm{K}_{3[3]}$. Next, assume that $[\Gamma(u)]$ has a triangle. Since $[\Gamma(u)]$ is vertex-transitive, every vertex of $\Gamma(u)$ lies in a triangle. Let (v_1, v_2, v_3) be a triangle. Since $[\Gamma(u)]$ is connected, v_4 is adjacent to neither v_2 nor v_3. Thus v_4 is adjacent to both v_5 and v_6. Since v_4 lies in a triangle and $\{v_5, v_6\} \subset \Gamma_2(v_1)$, it follows that v_5, v_6 are adjacent. Further, v_2 is adjacent to one of $\{v_5, v_6\}$, say v_5, and v_3 is adjacent to the remaining vertex v_6. Thus $[\Gamma(u)]$ is isomorphic to the 3-prism, (v_1, v_2, v_3) and (v_4, v_5, v_6) are the two triangles, and $\{v_1, v_4\}$, $\{v_2, v_5\}$ and $\{v_3, v_6\}$ are edges. Since $k = 3$, it follows that $|\Gamma_2(u) \cap \Gamma(v_1)| = 2$. Set $\Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2\}$. Then $\Gamma(v_1) = \{u, v_2, v_3, v_4, w_1, w_2\}$. Since $[\Gamma(v_1)]$ is isomorphic to the 3-prism, it follows that v_4 is adjacent to both w_1 and w_2, v_2 is adjacent to one of $\{w_1, w_2\}$, say w_1, and v_3 is adjacent to w_2. Thus $\Gamma(v_4) = \{u, v_1, v_5, v_6, w_1, w_2\}$. Since $[\Gamma(v_4)]$ is isomorphic to the 3-prism, it follows that w_1 is adjacent to one of $\{v_5, v_6\}$, say v_5. Thus $\{v_1, v_2, v_4, v_5\} \subseteq \Gamma(u) \cap \Gamma(w_1)$. Since $w_2 \in \Gamma(w_1)$, it follows that $|\Gamma_3(u) \cap \Gamma(w_1)| \leq 1$. Thus by Remark 2.1, Γ is G-distance-transitive.

Since $\{v_1, v_2, v_4, v_5\} \subseteq \Gamma(u) \cap \Gamma(w_1)$ and $\{w_1\} \subseteq \Gamma_2(u) \cap \Gamma(w_1)$, it follows that $|\Gamma(u) \cap \Gamma(w_1)| = 4$ or 5. Since Γ is $(G, 2)$-distance-transitive and $|\Gamma_2(u) \cap \Gamma(v_1)| = 2$, there are 12 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Thus $|\Gamma(u) \cap \Gamma(w_1)|$ divides 12, so $|\Gamma(u) \cap \Gamma(w_1)| = 4$. Hence $|\Gamma_2(u)| = 3$. Since G_u is transitive on $\Gamma_2(u)$, $[\Gamma_2(u)]$ is a vertex-transitive regular graph. Since w_1, w_2 are adjacent, $[\Gamma_2(u)] \cong C_3$. Therefore, $|\Gamma_3(u) \cap \Gamma(w_1)| = 0$, Γ has diameter 2 and has 10 vertices. In particular, the intersection array of Γ is $(6, 2; 1, 4)$. By inspecting the graphs in [3, p.222-223], Γ is $T(5)$ (also known as the Johnson graph $J(5, 2)$).

If $k = 2$, then $[\Gamma(u)] \cong C_6$. Let $(v_1, \dots, v_6)$ be a 6-cycle. Then $|\Gamma_2(u) \cap \Gamma(v_1)| = 3$, and set $\Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2, w_3\}$. Then $\Gamma(v_1) = \{u, v_2, v_5, w_1, w_2, w_3\}$. Since $[\Gamma(v_1)] \cong C_6$ and (v_2, u, v_6) is a 2-arc, it follows that v_2 is adjacent to one of $\{w_1, w_2, w_3\}$, say w_1; v_6 is adjacent to one of $\{w_2, w_3\}$, say w_3; and w_2 is adjacent to both w_1 and w_3. In particular, v_2 is not adjacent to any of $\{w_2, w_3\}$, and v_6 is not adjacent to any of $\{w_1, w_2\}$. Since $|\Gamma_2(u) \cap \Gamma(v_2)| = 3$, there exist w_4, w_5 in $\Gamma_2(u)$ that are adjacent to v_2, and so $\Gamma(v_2) = \{u, v_1, v_3, w_1, w_4, w_5\}$. Noting that $[\Gamma(v_2)] \cong C_6$ and (w_1, v_1, u, v_3) is a 3-arc, so v_3 is adjacent to one of $\{w_4, w_5\}$, say w_5, w_1 is adjacent to w_4, and w_4, w_5 are adjacent. Thus, $\{v_1, v_2, w_2, w_4\} \subseteq (\Gamma(u) \cup \Gamma_2(u)) \cap \Gamma(w_1)$. Hence $2 \leq |\Gamma(u) \cap \Gamma(w_1)| \leq 4$ and $|\Gamma_2(u) \cap \Gamma(w_1)| \geq 2$. Since Γ is $(G, 2)$-distance-transitive and $|\Gamma_2(u) \cap \Gamma(v_1)| = 3$, there are 18 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Since $|\Gamma(u) \cap \Gamma(w_1)|$ divides 18, $|\Gamma(u) \cap \Gamma(w_1)| = 2$ or 3.

Suppose that $|\Gamma(u) \cap \Gamma(w_1)| = 2$. Then $|\Gamma_2(u)| = 9$. Since $|\Gamma_2(u) \cap \Gamma(w_1)| \geq 2$, $|\Gamma_3(u) \cap \Gamma(w_1)| \leq 2$. If $|\Gamma_3(u) \cap \Gamma(w_1)| \leq 1$, then by Remark 2.1, Γ is G-distance-transitive. Inspecting the graphs in [3, p. 222-223], such a Γ does not exist. Hence $|\Gamma_3(u) \cap \Gamma(w_1)| = 2$. Since Γ is $(G, 2)$-distance-transitive, both $|\Gamma(u)|$ and $|\Gamma_2(u)|$ divide $|G_u|$, hence 18 divides $|G_u|$. Thus 3 divides $|G_{u,v}|$. Therefore $G_{u,v}$ has an element g of order 3. Since $|\Gamma(u) \setminus \{v\}| = 5$, it follows that g is trivial on $\Gamma(u) \setminus \{v\}$, so $g \in G_u^{[1]}$. Hence g fixes $\Gamma_2(u) \cap \Gamma(v_i)$ setwise. By Lemma 2.2, g is not trivial on $\Gamma_2(u)$. Hence $\langle g \rangle$ has orbits of size 3 on $\Gamma_2(u)$. Since g fixes $\Gamma_2(u)\cap\Gamma(v_i)$ setwise and $|\Gamma_2(u)\cap\Gamma(v_i)| = 3$, it follows that $\langle g\rangle$ is transitive on $\Gamma_2(u)\cap\Gamma(v_i)$. Thus G_{u,v_i} is transitive on $\Gamma_2(u)\cap\Gamma(v_i)$. Therefore Γ is $(G, 2)$-geodesic-transitive. Then by [7, Corollary 1.4], Γ is either the Octahedron or the Icosahedron. However, these two graphs do not have valency 6, which is a contradiction.

Finally, suppose that $|\Gamma(u)\cap\Gamma(w_1)| = 3$. Since there are 18 edges between $\Gamma(u)$ and $\Gamma_2(u)$, and $|\Gamma_2(u)|\cdot|\Gamma(u)\cap\Gamma(w_1)| = 18$, $|\Gamma_2(u)| = 6$. Since $|\Gamma_2(u)\cap\Gamma(w_1)| \geq 2$, $|\Gamma_3(u)\cap\Gamma(w_1)| \leq 1$. Thus by Remark 2.1, Γ is G-distance-transitive. Inspecting the graphs in [3, p. 222-223], Γ is the Paley graph $P(13)$. □

Lemma 2.8. *Let Γ be a $(G, 2)$-distance-transitive graph of valency 6. Let u be a vertex of Γ. If $[\Gamma(u)] \cong 2\mathrm{K}_3$, then $|\Gamma_2(u)| = 9$ or 18.*

Proof. Suppose that $[\Gamma(u)] \cong 2\mathrm{K}_3$. Then each arc lies in a unique K_4. Let $\Gamma(u) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ such that (v_1, v_2, v_3) and (v_4, v_5, v_6) are two triangles. Then for each v_i, $|\Gamma_2(u)\cap\Gamma(v_i)| = 3$. Since $[\Gamma(v_1)] \cong 2\mathrm{K}_3$, it follows that $\Gamma_2(u)\cap\Gamma(v_i)\cap\Gamma(v_j) = \emptyset$ for $i, j \in \{1, 2, 3\}$. Thus $|\Gamma_2(u)| \geq 9$.

On the other hand, since Γ is $(G, 2)$-distance-transitive and $|\Gamma_2(u)\cap\Gamma(v_1)| = 3$, there are 18 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Thus $|\Gamma_2(u)|$ divides 18, and so $|\Gamma_2(u)| = 9$ or 18. □

If further $|\Gamma_2(u)| = 9$, then such a graph is unique.

Lemma 2.9. *Let Γ be a $(G, 2)$-distance-transitive graph of valency 6. Let u be a vertex of Γ. Suppose that $[\Gamma(u)] \cong 2\mathrm{K}_3$ and $|\Gamma_2(u)| = 9$. Then $\Gamma \cong \mathrm{H}(2, 4)$*

Proof. Since $[\Gamma(u)] \cong 2\mathrm{K}_3$, each arc lies in a unique K_4. Let $\Gamma(u) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Let (v_1, v_2, v_3) and (v_4, v_5, v_6) be the two triangles of $[\Gamma(u)]$. Then for each v_i, $|\Gamma_2(u)\cap\Gamma(v_i)| = 3$. Since $[\Gamma(v_1)] \cong 2\mathrm{K}_3$, it follows that $\Gamma_2(u)\cap\Gamma(v_i)\cap\Gamma(v_j) = \emptyset$ for $i \neq j \in \{1, 2, 3\}$. Since $|\Gamma_2(u)| = 9$, $\Gamma_2(u) = (\Gamma_2(u)\cap\Gamma(v_1)) \cup (\Gamma_2(u)\cap\Gamma(v_2)) \cup (\Gamma_2(u)\cap\Gamma(v_3))$. Set $\Gamma_2(u)\cap\Gamma(v_1) = \{w_1, w_2, w_3\}$, $\Gamma_2(u)\cap\Gamma(v_2) = \{w_4, w_5, w_6\}$, and $\Gamma_2(u)\cap\Gamma(v_3) = \{w_7, w_8, w_9\}$. Since $[\Gamma(v_1)] \cong [\Gamma(v_2)] \cong [\Gamma(v_3)] \cong 2\mathrm{K}_3$, it follows that (w_1, w_2, w_3), (w_4, w_5, w_6) and (w_7, w_8, w_9) are three triangles.

Since Γ is $(G, 2)$-distance-transitive and $|\Gamma_2(u)\cap\Gamma(v_1)| = 3$, there are 18 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Since $|\Gamma_2(u)| = 9$, it follows that for each w_i, $|\Gamma(u)\cap\Gamma(w_i)| = 2$. By the previous argument, w_1 is not adjacent to any of $\{v_2, v_3\}$, so w_1 is adjacent to one of $\{v_4, v_5, v_6\}$, say v_4. Then $\Gamma(u)\cap\Gamma(w_1) = \{v_1, v_4\}$. As each arc lies in a unique K_4 and (v_1, w_1, w_2, w_3) is a K_4, it follows that v_4 is not adjacent to any of $\{w_2, w_3\}$. Since $|\Gamma_2(u)\cap\Gamma(v_4)| = 3$ and $|\Gamma(v_i)\cap\Gamma(v_4)| = 2$ for $i = 1, 2, 3$, v_4 is adjacent to one of $\{w_4, w_5, w_6\}$, say w_4, and is adjacent to one of $\{w_7, w_8, w_9\}$, say w_7. Then $\Gamma(v_4) = \{u, v_5, v_6, w_1, w_4, w_7\}$. Since $[\Gamma(v_4)] \cong 2\mathrm{K}_3$ and (u, v_5, v_6) is a triangle, it follows that (w_1, w_4, w_7) is a triangle. Thus, $\Gamma(w_1) = \{v_1, v_4, w_2, w_3, w_4, w_7\}$, and so $\Gamma_3(u)\cap\Gamma(w_1) = \emptyset$. Since Γ is $(G, 2)$-distance-transitive, it follows that Γ is G-distance-transitive with diameter 2 and has 16 vertices. Thus by inspecting the graphs in [3, p. 222-223], $\Gamma \cong \mathrm{H}(2, 4)$. □

Lemma 2.10. *Let Γ be a $(G, 2)$-distance-transitive graph of valency 6. Let u be a vertex of Γ. If $[\Gamma(u)] \cong 3\mathrm{K}_2$, then $|\Gamma_2(u)| = 8, 12$, or 24.*

Proof. Suppose that $[\Gamma(u)] \cong 3\mathrm{K}_2$. Then each arc lies in a unique triangle. Let $\Gamma(u) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ be such that (v_1, v_2), (v_3, v_4), and (v_5, v_6) are three arcs. Then for each v_i, $|\Gamma_2(u) \cap \Gamma(v_i)| = 4$. Since $[\Gamma(v_1)] \cong 3\mathrm{K}_2$, it follows that $\Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_2) = \emptyset$. Thus $|\Gamma_2(u)| \geq 8$.

Since Γ is $(G, 2)$-distance-transitive and $|\Gamma_2(u) \cap \Gamma(v_1)| = 4$, there are 24 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Since $|\Gamma_2(u)|$ divides 24, it follows that $|\Gamma_2(u)| = 8$, 12, or 24. □

If further $|\Gamma_2(u)| = 8$, then Γ is known.

Lemma 2.11. *Let Γ be a $(G, 2)$-distance-transitive graph of valency 6. Let u be a vertex of Γ. Suppose that $[\Gamma(u)] \cong 3\mathrm{K}_2$ and $|\Gamma_2(u)| = 8$. Then $\Gamma \cong KG_{6,2}$*

Proof. Since Γ is symmetric and $[\Gamma(u)] \cong 3\mathrm{K}_2$, each arc lies in a unique triangle. Set $\Gamma(u) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Let (v_1, v_2), (v_3, v_4) and (v_5, v_6) be three arcs. Then for each v_i, $|\Gamma_2(u) \cap \Gamma(v_i)| = 4$. Since $[\Gamma(v_1)] \cong 3\mathrm{K}_2$, it follows that $\Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_2) = \emptyset$. Since $|\Gamma_2(u)| = 8$, $\Gamma_2(u) = (\Gamma_2(u) \cap \Gamma(v_1)) \cup (\Gamma_2(u) \cap \Gamma(v_2))$. Set $\Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2, w_3, w_4\}$, and $\Gamma_2(u) \cap \Gamma(v_2) = \{w_5, w_6, w_7, w_8\}$. Since $[\Gamma(v_1)] \cong [\Gamma(v_2)] \cong 3\mathrm{K}_2$, it follows that (w_1, w_2), (w_3, w_4), (w_5, w_6) and (w_7, w_8) are arcs.

Since Γ is $(G, 2)$-distance-transitive and $|\Gamma_2(u) \cap \Gamma(v_1)| = 4$, there are 24 edges between $\Gamma(u)$ and $\Gamma_2(u)$. As $|\Gamma_2(u)| = 8$, it follows that for each w_i, $|\Gamma(u) \cap \Gamma(w_i)| = 3$. By the previous argument, w_1 is not adjacent to v_2. Noting that $\Gamma_2(u) \cap \Gamma(v_i) \cap \Gamma(v_j) = \emptyset$ for $(i, j) = (1, 2), (3, 4), (5, 6)$. Thus w_1 is adjacent to one of $\{v_3, v_4\}$, say v_3, and is also adjacent to one of $\{v_5, v_6\}$, say v_5. Then $\Gamma(u) \cap \Gamma(w_1) = \{v_1, v_3, v_5\}$. Since each arc lies in a unique triangle and (v_1, w_1, w_2) is a triangle, it follows that v_3 is not adjacent to w_2. By $|\Gamma_2(u) \cap \Gamma(v_3)| = 4$ and $|\Gamma(v_i) \cap \Gamma(v_3)| = 3$ for $i = 1, 2$, v_3 is adjacent to one of $\{w_3, w_4\}$, say w_3, and is also adjacent to two vertices of $\{w_5, w_6, w_7, w_8\}$, say w_5, w_7.

Then $\Gamma(v_3) = \{u, v_4, w_1, w_3, w_5, w_7\}$. Since $[\Gamma(v_3)] \cong 3\mathrm{K}_2$ and (u, v_4) is an arc, it follows that (w_1, w_5) and (w_3, w_7) are two arcs. Thus, $\{v_1, v_3, v_5\} \cup \{w_2, w_5\} \subseteq \Gamma(w_1)$, and so $|\Gamma_3(u) \cap \Gamma(w_1)| \leq 1$. Since Γ is $(G, 2)$-distance-transitive, it follows from Remark 2.1 that Γ is G-distance-transitive. One part of the intersection array of Γ is $(6, 4, \ldots; 1, 3, \ldots)$. By inspecting the graphs in [3, p.221], $\Gamma \cong KG_{6,2}$. □

Lemma 2.12. *Let Γ be an arc-transitive graph and let u be a vertex of Γ. Suppose that $\Gamma(u) = U \cup W$, where $|U| = |W| = n$ and $U \cap W = \emptyset$. Assume further that $[U] \cong [W] \cong \mathrm{K}_n$. Let $v_1 \in U$. If $|\Gamma(u) \cap \Gamma(v_1) \cap W| \leq n - 2$, then Γ is a line graph.*

Proof. Suppose that $|\Gamma(u) \cap \Gamma(v_1) \cap W| \leq n - 2$. Then $[U]$ and $[W]$ are the only two n-cliques of $\Gamma(u)$. It follows from [14, Proposition 2.1] that Γ is a line graph. □

Proof of Theorem 1.2. Let Γ be a connected non-complete $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive graph of valency 6. If Γ has girth at least 5, then for any two vertices with distance 2, there exists a unique 2-arc between these two vertices. Thus Γ is $(G, 2)$-arc-transitive, which is a contradiction. Hence Γ has girth 3 or 4. If Γ has girth 4, then it follows from Lemma 2.6 that $(\Gamma, G) = ((2 \times 7)\text{−grid}, S_2 \times M)$ where M is a 2-transitive but not 3-transitive subgroup of S_7, so that (1) holds.

Suppose that Γ has girth 3. Let (u, v, w) be a 2-arc such that $d_\Gamma(u, w) = 2$. If $[\Gamma(u)]$ is connected, then by Lemma 2.7, Γ is isomorphic to one of: $T(5)$, Paley graph $P(13)$, $\mathrm{K}_{3[3]}$ or $\mathrm{K}_{4[2]}$, (2) holds. If $[\Gamma(u)]$ is disconnected, then G_u has blocks in $\Gamma(u)$, and each block has cardinality 2 or 3. If each block has cardinality 3, then $[\Gamma(u)] \cong 2\mathrm{K}_3$; if each block has cardinality 2, then $[\Gamma(u)] \cong 3\mathrm{K}_2$. Suppose that $[\Gamma(u)] \cong 2\mathrm{K}_3$. Then by Lemma 2.8, $|\Gamma_2(u)| = 9$ or 18. If $|\Gamma_2(u)| = 9$, then by Lemma 2.9, $\Gamma \cong \mathrm{H}(2, 4)$. If $|\Gamma_2(u)| = 18$, then by Lemma 2.12, Γ is a line graph, (3.1) holds.

Finally, if $[\Gamma(u)] \cong 3\mathrm{K}_2$, then by Lemma 2.10, $|\Gamma_2(u)| = 8, 12$, or 24. In particular, if $|\Gamma_2(u)| = 8$, then by Lemma 2.11, $\Gamma \cong KG_{6,2}$, so that (3.2) holds. □

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Spherical folding tessellations by kites and isosceles triangles IV

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Abstract

The classification of the dihedral folding tessellations of the sphere and the plane whose prototiles are a kite and an equilateral triangle were obtained in [1]. Recently, this classification was extended to isosceles triangles so that the classification of spherical folding tesselations by kites and isosceles triangles in three cases of adjacency was presented in [2, 3, 4]. In this paper we finalize this classification presenting all the dihedral folding tessellations of the sphere by kites and isosceles triangles in the remaining three cases of adjacency, that consists of five sporadic isolated tilings. A list containing these tilings including its combinatorial structure is presented in Table 1.

Keywords: Dihedral f-tilings, combinatorial properties, spherical trigonometry, symmetry groups.

Math. Subj. Class.: 52C20, 05B45, 52B05

1 Introduction

By a *folding tessellation* or *folding tiling* of the Euclidean sphere S^2 we mean an edge-to-edge pattern of spherical geodesic polygons that fills the whole sphere with no gaps and no overlaps, and such that the “underlying graph” has even valency at any vertex and the sums of alternate angles around each vertex are π.

Folding tilings (*f-tiling*, for short) are strongly related to the theory of isometric foldings on Riemannian manifolds. In fact, the set of singularities of any isometric folding corresponds to a folding tiling, see [13] for the foundations of this subject.

The study of this special class of tessellations was initiated in [5] with a complete classification of all spherical monohedral folding tilings. Ten years latter Ueno and Agaoka

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[14] have established the complete classification of all triangular spherical monohedral tilings (without any restriction on angles).

Dawson has also been interested in special classes of spherical tilings, see [10], [11] and [12], for instance.

The complete classification of all spherical folding tilings by rhombi and triangles was obtained in 2005 [8]. A detailed study of the triangular spherical folding tilings by equilateral and isosceles triangles is presented in [9].

Spherical f-filings by two noncongruent classes of isosceles triangles in a particular case of adjacency were recently obtained [6].

Here we discuss dihedral folding tessellations by spherical kites and isosceles spherical triangles.

A spherical kite K (Figure 1(a)) is a spherical quadrangle with two congruent pairs of adjacent sides, but distinct from each other. Let us denote by $(\alpha_1, \alpha_2, \alpha_1, \alpha_3)$, $\alpha_2 > \alpha_3$, the internal angles of K in cyclic order. The length sides are denoted by a and b, with $a < b$. From now on T denotes a spherical isosceles triangle with internal angles β and γ ($\gamma \neq \beta$), and length sides c and d, see Figure 1(b).

We shall denote by $\Omega(K, T)$ the set, up to isomorphism, of all dihedral folding tilings of S^2 whose prototiles are K and T.

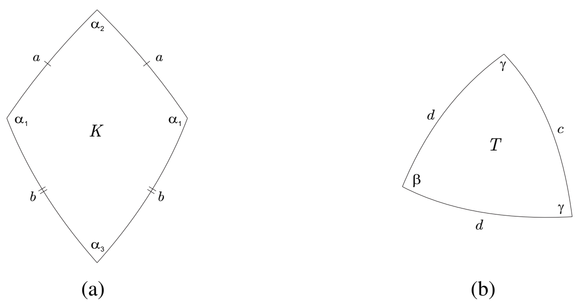


Figure 1: A spherical kite and a spherical isosceles triangle

Taking into account the area of the prototiles K and T we have

$$2\alpha_1 + \alpha_2 + \alpha_3 > 2\pi \quad \text{and} \quad \beta + 2\gamma > \pi.$$

As $\alpha_2 > \alpha_3$ we also have

$$\alpha_1 + \alpha_2 > \pi.$$

We begin by pointing out that any element of $\Omega\,(K, T)$ has at least two cells congruent to K and T, respectively, such that they are in adjacent positions and in one and only one of the situations illustrated in Figure 2.

After certain initial assumptions are made, it is usually possible to deduce sequentially the nature and orientation of most of the other tiles. Eventually, either a complete tiling or an impossible configuration proving that the hypothetical tiling fails to exist is reached. In the diagrams that follow, the order in which these deductions can be made is indicated by the numbering of the tiles. For $j \geq 2$, the location of tiling j can be deduced directly from the configurations of tiles $(1, 2, \ldots, j-1)$ and from the hypothesis that the configuration is part of a complete tiling, except where otherwise indicated.

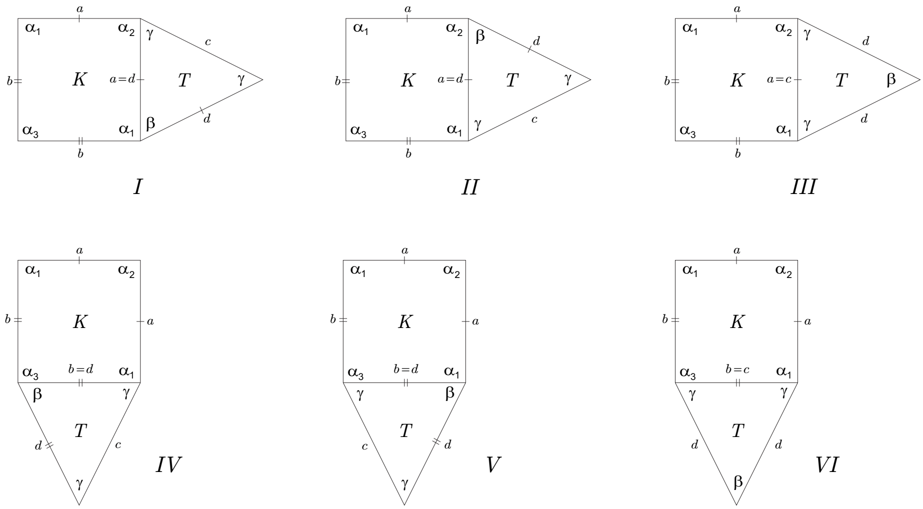


Figure 2: Distinct cases of adjacency of K and T

The cases of adjacency I, II and III have already been analyzed in [2, 3, 4]. In this paper we consider the remaining cases of adjacency IV, V and VI.

2 Case of Adjacency IV

Suppose that any element of $\Omega\,(K,T)$ has at least two cells congruent, respectively, to K and T, such that they are in adjacent positions as illustrated in Figure 2–IV. As $b = d$, using trigonometric formulas, we obtain

$$\frac{\cos\gamma(1+\cos\beta)}{\sin\gamma\sin\beta} = \frac{\cos\frac{\alpha_2}{2} + \cos\alpha_1\,\cos\frac{\alpha_3}{2}}{\sin\alpha_1\,\sin\frac{\alpha_3}{2}}. \tag{2.1}$$

Concerning the angles of the triangle T we have necessarily one of the following situations:

$$\gamma > \beta \quad \text{or} \quad \gamma < \beta.$$

In the following subsections we consider separately each one of these cases.

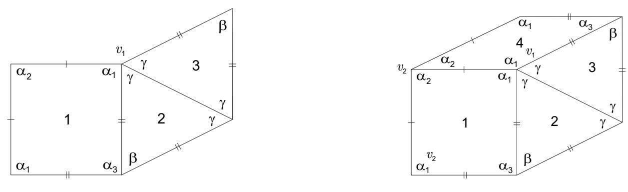
2.1 $\gamma > \beta$

In this case we have $\gamma > \frac{\pi}{3}$ and $a, c < b = d$.

Proposition 2.1. *Under the conditions assumed in this section, there is a single folding tiling, $\mathcal{L}$, such that $\alpha_2 = \frac{\pi}{2}$, $\alpha_1 + \gamma = \pi$ and $\alpha_3 = \beta = \frac{\pi}{3}$. Planar and 3D representations of $\mathcal{L}$ are given in Figure 9.*

Proof. Suppose that any element of $\Omega\,(K,T)$ has at least two cells congruent, respectively, to K and T, such that they are in adjacent positions as illustrated in Figure 2$-IV$.

Observing Figure 3(a), tile 3 cannot be a kite – this case was already analyzed in [4] (case 2.1) and, under the current conditions, give rise to no f-tilings. Consequently, a side of length c of each triangle must be adjacent to a side of length c of another triangle. Moreover, we have $\alpha_1 \geq \alpha_2 > \alpha_3$. In fact, if $\alpha_2 > \alpha_1$ and

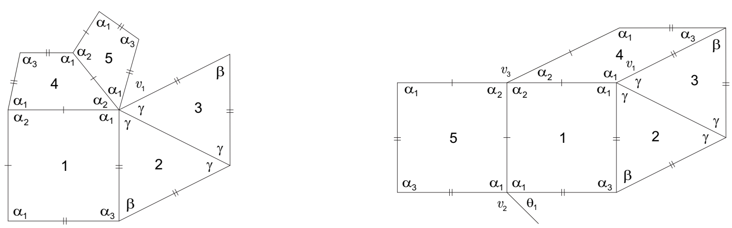


(a) (b)

Figure 3: Local configurations

(i) $\alpha_1 + \gamma = \pi$ (Figure 3(b)), we reach a contradiction at vertex v_2, as $\alpha_2 + \rho > \pi$, for all $\rho \in \{\alpha_1, \alpha_2\}$;

(ii) $\alpha_1 + \gamma < \pi$ (Figure 4(a)), at vertex v_1 we have necessarily $\alpha_2 + \gamma + k\alpha_3 = \pi$, $k \geq 1$. But $(\alpha_1 + \alpha_1 + \gamma) + (\alpha_2 + \gamma + \alpha_3) > (2\alpha_1 + \alpha_2 + \alpha_3) > 2\pi$, which is impossible.



(a) (b)

Figure 4: Local configurations

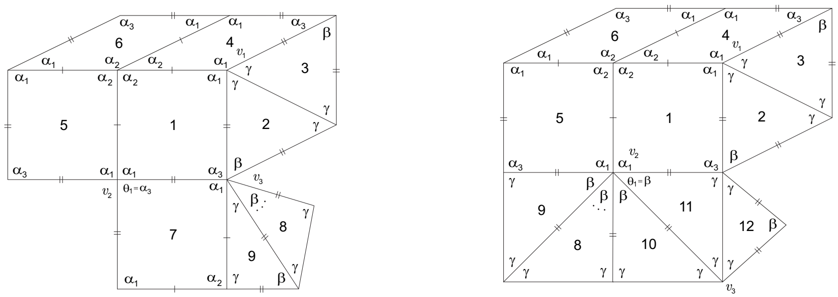
At vertex v_1 we have $\alpha_1 + \gamma = \pi$ or $\alpha_1 + \gamma < \pi$.

1. Suppose firstly that $\alpha_1 + \gamma = \pi$ (Figure 4(b)). At vertex v_3 we have $k\alpha_2 = \pi$, with $k \geq 2$. As $(\alpha_1 + \gamma) + (\alpha_2 + \alpha_2 + \alpha_2) > (\alpha_1 + \alpha_2) + (\gamma + \beta + \beta) > 2\pi$, it follows that $k = 2$ $\left(\alpha_2 = \frac{\pi}{2}\right)$. With the labeling of Figure 4(b), if

(i) $\theta_1 = \alpha_3$ (Figure 5(a)), then at vertex v_3 we have necessarily $\alpha_1 + k\beta = \pi$, $k \geq 2$, and so $\alpha_1 > \frac{\pi}{2} = \alpha_2 > \gamma \geq \alpha_3 > \beta$ (note that $\alpha_1 + \beta + \alpha_3 > \pi$). But then tile 9 must be a triangle, which is impossible;

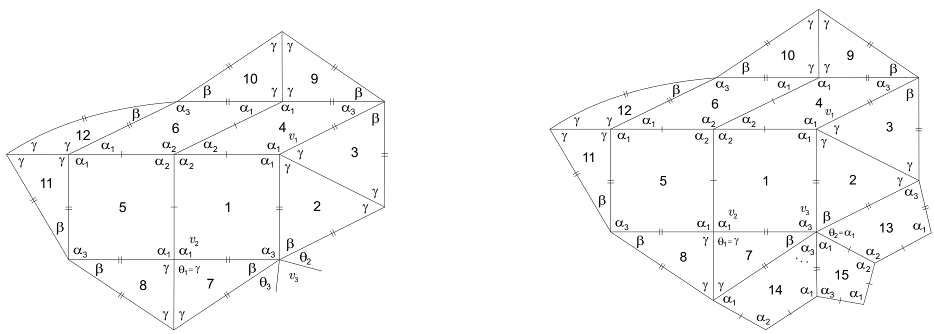
(ii) $\theta_1 = \beta$ (Figure 5(b)), then at vertex v_2 it follows that $\alpha_1 + k\beta = \pi$, $k \geq 2$ (note that we must have $\alpha_3 > \beta$). But at vertex v_3 we have $\gamma + \gamma < \pi$ and $\gamma + \gamma + \rho > \pi$, for all ρ.

(iii) $\theta_1 = \gamma$, we get the configuration illustrated in Figure 6(a). Now, if



(a) case 1(i) (b) case 1 (ii)

Figure 5: Local configurations



(a) case 1(iii) (b) case 1(iii)(a)

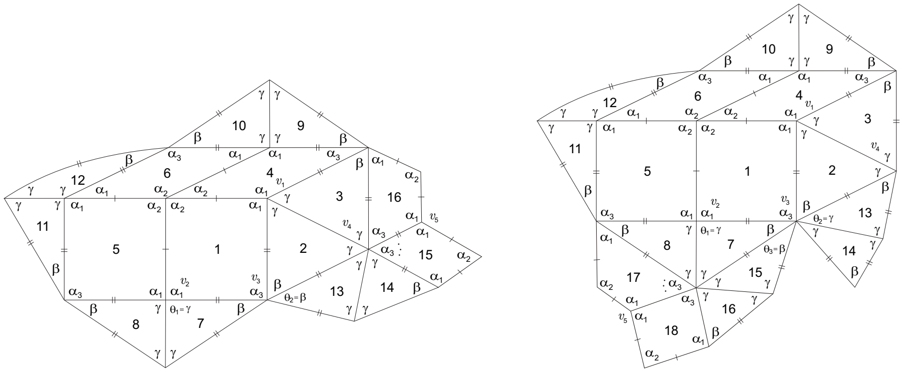
Figure 6: Local configurations

(a) $\theta_2 = \alpha_1$ (Figure 6(b)), we have necessarily $\alpha_1 + k\alpha_3 = \pi$, with $k \geq 2$, and $\alpha_1 > \frac{\pi}{2} = \alpha_2 > \gamma > \beta > \alpha_3$ $(\alpha_1 + \beta + \alpha_3 > \pi)$. But then, the other sum of alternate angles at vertex v_3 must be greater or equal than $\alpha_1 + \beta + \beta > \pi$, which is a contradiction $(3\pi \geq (\alpha_1 + \gamma) + (\alpha_2 + \alpha_2) + (\alpha_1 + \beta + \beta) > (2\alpha_1 + \alpha_2 + \alpha_3) + (\beta + \gamma + \gamma) > 3\pi)$.

(b) $\theta_2 = \beta$ (Figure 7(a)), at vertex v_3 we have $\gamma + \gamma + k\alpha_3 = \pi$, $k \geq 1$, and a contradiction arises at vertex v_4 as $\alpha_1 + \rho > \pi$, for all $\rho \in \{\alpha_1, \alpha_2\}$.

(c) $\theta_2 = \gamma$, at vertex v_3 we have $\theta_3 \in \{\beta, \alpha_3\}$. In the first case, illustrated in Figure 7(b), we reach a contradiction at vertex v_5. On the other hand, if $\theta_3 = \alpha_3$, due to the angles involved in the sums of alternate angles at vertex v_3, we must have $\alpha_3 = \beta$. Taking into account the triangle and the kite's areas, it follows that $\gamma + \beta + \beta = \gamma + \alpha_3 + \alpha_3 = \pi$ (Figure 8). At vertex v_6 we have $\alpha_1 + \beta < \pi$ and $\alpha_1 + \beta + \rho > \pi$, for all $\rho \in \{\alpha_1, \alpha_2, \alpha_3, \beta, \gamma\}$.

(d) $\theta_2 = \alpha_3$, taking into account the analysis of the previous cases, at vertex v_3 we have $k\alpha_3 = k\beta = \pi$, $k \geq 3$. Due to the kite's area, it follows that $\gamma - \frac{\pi}{4} < \frac{\beta}{2}$ and consequently $\cos\frac{\beta}{2} < \cos\left(\gamma - \frac{\pi}{4}\right)$. Using equation (2.1), we conclude that $\beta > \frac{\pi}{4}$, and so $k = 3$. The last configuration is then extended to the one illustrated in Figure 9(a). We shall denote this f-tiling by $\mathcal{L}$. Its 3D



(a) case 1(iii)(b) (b) case 1(iii)(c)

Figure 7: Local configurations

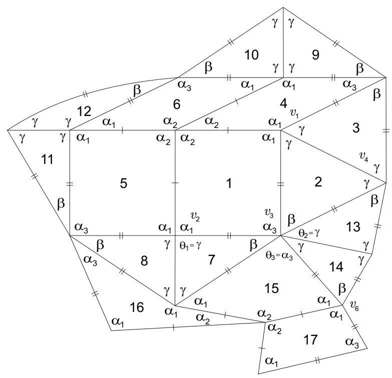


Figure 8: Local configuration occurring in case 1(iii)(c)

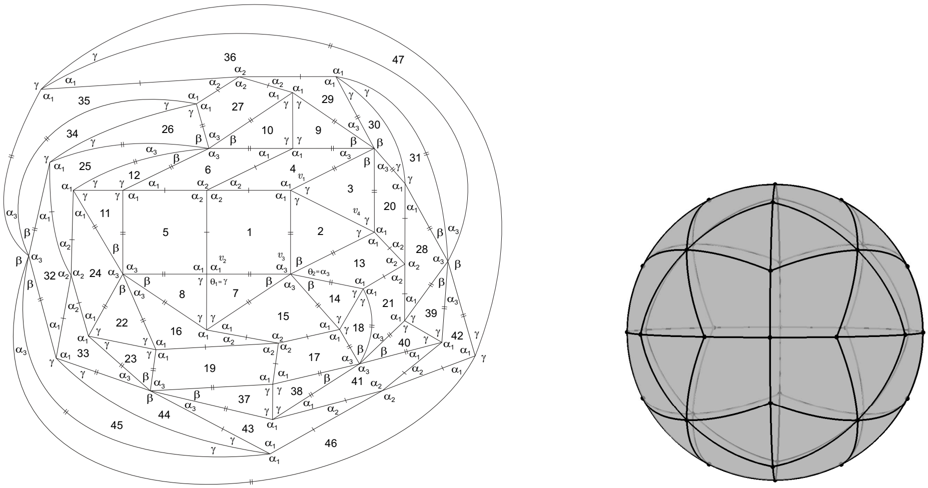
representation is given in Figure 9(b).

2. Suppose now that $\alpha_1 + \gamma < \pi$ (Figure 3(a)). Again, due to the analysis made in [4] (case 2.1), we use the fact that a side of length c of each triangle must be adjacent to a side of length c of other triangle. At vertex v_1 we must have $\alpha_1 + \gamma + k\alpha_3 = \pi$, with $k \geq 1$. Nevertheless, we reach a contradiction at vertex v_2 (Figure 10) since there is no way to satisfy the angle-folding relation around this vertex. □

2.2 $\gamma < \beta$

In this case we have $\beta > \frac{\pi}{3}$ and $a < b = d < c$.

Proposition 2.2. *Under the conditions assumed in this section, there is a single folding tiling, $\mathcal{J}$, such that $\alpha_2 = \frac{\pi}{2}$, $\alpha_1 + \gamma = \pi$, $\gamma = \frac{\pi}{3}$ and $\beta + \beta + \alpha_3 = \pi$. Planar and 3D representations of $\mathcal{J}$ are given in Figure 12.*



(a) planar representation

(b) 3D representation

Figure 9: f-tiling $\mathcal{L}$

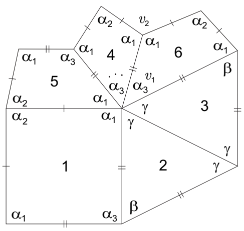
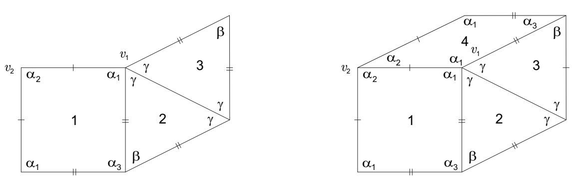


Figure 10: Local configuration occurring in case 2

Proof. Suppose that any element of $\Omega(K,T)$ has at least two cells congruent, respectively, to K and T, such that they are in adjacent positions as illustrated in Figure $2\!-\!IV$. As $a \neq c$, we get the configuration illustrated in Figure 11(a), and, at vertex v_1, we have



(a)

(b) case 1

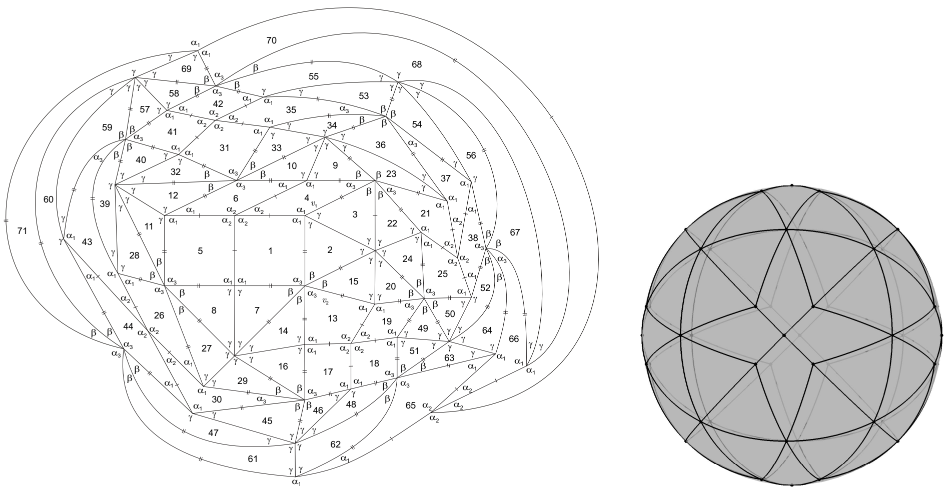
Figure 11: Local configurations

$\alpha_1 + \gamma = \pi$ or $\alpha_1 + \gamma < \pi$.

1. Suppose firstly that $\alpha_1 + \gamma = \pi$ (Figure 11(b)).

Note that the conditions $\alpha_2 > \alpha_1 \geq \alpha_3$ and $\alpha_2 > \alpha_3 > \alpha_1$ lead to a contradiction at vertex v_2, as $\alpha_2 + \rho > \pi$, for all $\rho \in \{\alpha_1, \alpha_2\}$. Therefore $\alpha_1 \geq \alpha_2 > \alpha_3$. Now, if

(i) $\alpha_2 + \alpha_2 = \pi$, then $\beta + \beta + k\alpha_3 = \pi$, $k \geq 1$, and so $\alpha_1 > \alpha_2 = \frac{\pi}{2} > \beta > \gamma > \alpha_3$. Consequently, $\gamma = \frac{\pi}{3}$ (as $\beta < \frac{\pi}{2}$, we have $\gamma > \frac{\pi}{4}$). Then, the last configuration is extended to the one illustrated in Figure 12(a). We shall denote this f-tiling by $\mathcal{J}$. Its 3D representation is given in Figure 12(b).



(a) planar representation (b) 3D representation

Figure 12: f-tiling $\mathcal{J}$

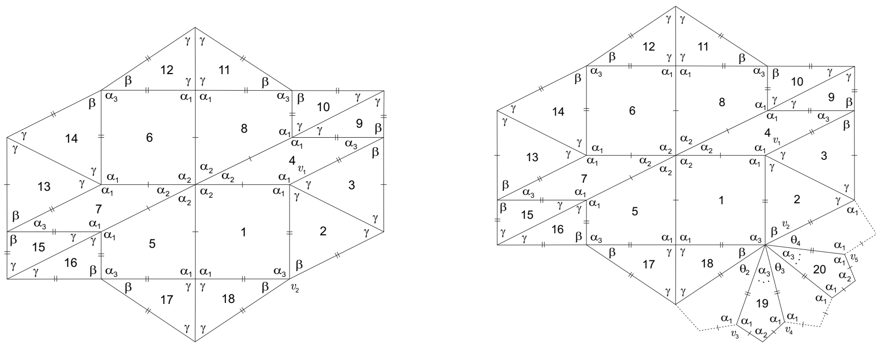
(ii) $\alpha_2 + \alpha_2 < \pi$, then $k\alpha_2 = \pi$, $k \geq 3$, $\beta + \beta + \overline{k}\alpha_3 = \pi$, $\overline{k} \geq 1$, and so $\alpha_1 > \frac{\pi}{2} > \beta > \alpha_2 > \gamma > \alpha_3$. As $\gamma > \frac{\pi}{4}$, we have necessarily $k = 3$ (Figure 13(a)). Now, if at vertex v_2 we have $\overline{k} > 1$ (Figure 13(b)), one of the angles θ_2, θ_3 or θ_4 must be α_3. But then we reach a contradiction at vertex v_3, v_4 or v_5, respectively, as $\alpha_1 + \rho > \pi$, for all $\rho \in \{\alpha_1, \alpha_2\}$. On the other hand, if $\overline{k} = 1$, we get the configuration illustrated in Figure 14(a) (note that at vertex v_3 we cannot have $\gamma+\gamma+\gamma = \pi$, as $\frac{\pi}{3} = \alpha_2 > \gamma$). At vertex v_4 we reach a similar contradiction as in the case $\overline{k} > 1$.

2. Suppose now that $\alpha_1 + \gamma < \pi$ (Figure 11(a)).

If $\alpha_2 > \alpha_1 \geq \alpha_3$ or $\alpha_2 > \alpha_3 > \alpha_1$, at vertex v_1 we must have $\alpha_2 + k\gamma = \pi$, with $k \geq 2$, and consequently at vertex v_2 it follows that $\alpha_1 + \alpha_1 \leq \pi$, and so $\alpha_1 \leq \frac{\pi}{2}$ and $\alpha_2 + \alpha_3 > \pi$. But then an incompatibility on the sides arises at vertex v_1.

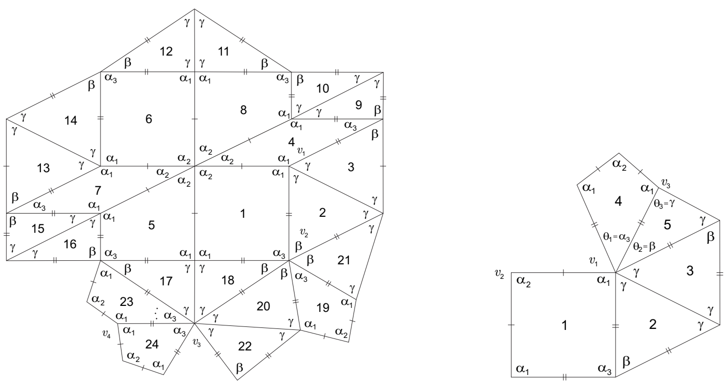
If $\alpha_1 \geq \alpha_2 > \alpha_3$, and

(i) $\theta_1 = \alpha_3$ (Figure 14(b)), then θ_2 must be β, otherwise we get, at vertex v_3, $\theta_3 = \alpha_1$ and $\alpha_1 + \rho > \pi$, for all $\rho \in \{\alpha_1, \alpha_2\}$. Nevertheless, an impossibility cannot be avoided at vertex v_1 since we obtain $\beta + \gamma + \rho > \pi$, for all $\rho \in \{\alpha_1, \alpha_2\}$.



(a) (b)

Figure 13: Local configurations occurring in case 1(ii)



(a) case 1(ii) (b) case 2(i)

Figure 14: Local configurations

(ii) $\theta_1 = \gamma$ and

(a) $\theta_2 = \beta$ (Figure 15(a)), then $\gamma < \frac{\pi}{4}$ and $\beta > \frac{\pi}{2}$. At vertex v_4 we must have $\beta + \alpha_2 \leq \pi$, however $2\pi \geq (\alpha_1+\gamma+\gamma)+(\beta+\alpha_2) = (\beta+\gamma+\gamma)+(\alpha_1+\alpha_2) > 2\pi$ is impossible.

(b) $\theta_2 = \gamma$, it follows that $\alpha_1 + k\gamma = \pi$, $k \geq 2$, as illustrated in Figure 15(b). But any choices for θ_3 and θ_4 lead to a contradiction.

□

3 Case of Adjacency *V*

Proposition 3.1. *$\Omega(K,T)$ is composed by a single folding tiling, $\mathcal{M}$, such that $\alpha_2 = \frac{\pi}{2}$, $\alpha_1 + \beta = \pi$ and $\gamma = \alpha_3 = \frac{\pi}{3}$. For a planar representation see Figure 20(b). Its 3D representation is given in Figure 21.*

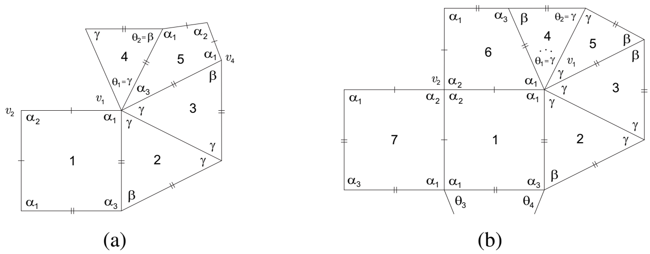


Figure 15: Local configurations occurring in case 2(ii)

Proof. Suppose that any element of $\Omega\,(K,T)$ has at least two cells congruent, respectively, to K and T, such that they are in adjacent positions as illustrated in Figure 2–V.

The case analyzed in [4] (case 2.1) give rise to no f-tilings including two cells in these adjacent positions, and so a side of length c of each triangle must be adjacent to a side of length c of another triangle.

1. If $\alpha_2 > \alpha_1$, then $\alpha_2 > \frac{\pi}{2}$ and we get the configuration of Figure 16(a).

If $\alpha_2 + \beta = \pi$ (Figure 16(b)), we have $\alpha_1 = \frac{\pi}{2}$ (vertex v_1), and so $\alpha_2 + \alpha_3 > \pi$, justifying the choice for θ_1. But at vertex v_2 we obtain a contradiction as $\alpha_3 + \gamma + \gamma > \pi$ $((\alpha_1 + \alpha_1) + (\alpha_2 + \beta) + (\alpha_3 + \gamma + \gamma) = (2\alpha_1 + \alpha_2 + \alpha_3) + (\beta + \gamma + \gamma) > 3\pi)$.

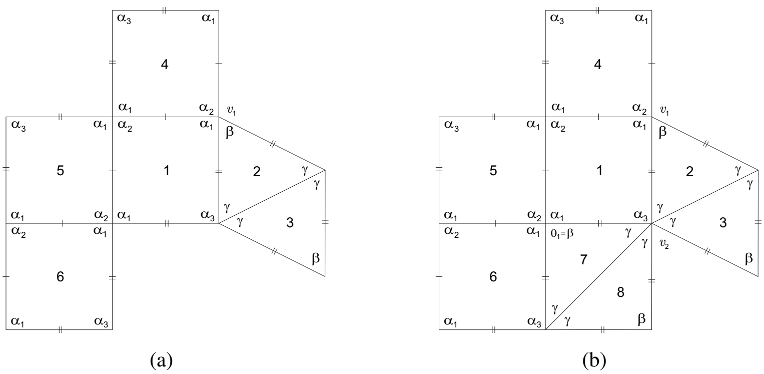


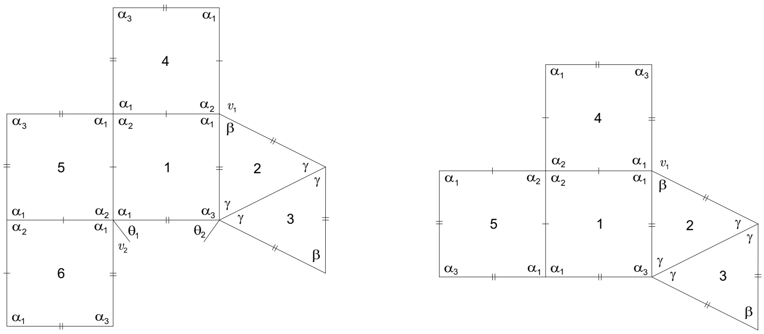
Figure 16: Local configurations occurring in case 1

If $\alpha_2 + \beta < \pi$, then $\alpha_2 + k\beta = \pi$, with $k \geq 2$ (note that $\alpha_2 + \alpha_3 > \pi$). Consequently, $\gamma > \beta$ and $\alpha_3 > \beta$. Observing Figure 17(a), we conclude that there is no way to satisfy the angle-folding relation around vertex v_2 ($\alpha_2 + \alpha_2 > \alpha_2 + \alpha_1 > \pi$, $\alpha_2 + \alpha_3 > \pi$, $\alpha_2 + \gamma + \rho > \pi$, for all ρ, and $\theta_1 = \beta$ implies $\theta_2 = \gamma$ and $\gamma + \gamma + \rho > \pi$, for all ρ).

2. Suppose now that $\alpha_1 \geq \alpha_2$ (Figure 17(b)). It follows that $\alpha_1 > \frac{\pi}{2} \geq \alpha_2 > \beta$ and $\gamma > \frac{\pi}{4}$.

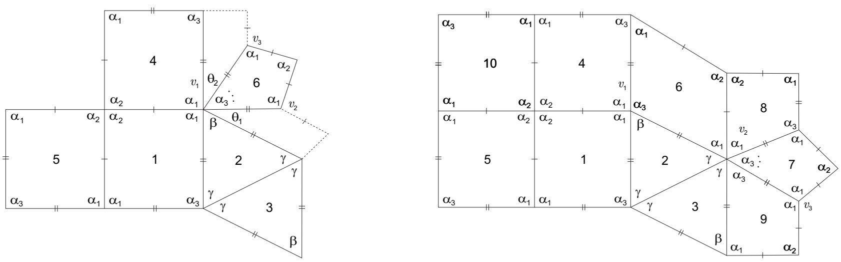
2.1 If $\beta > \gamma$, then at vertex v_1 we must have $\alpha_1 + \beta + k\alpha_3 = \pi$, with $k \geq 1$, or $\alpha_1 + \beta = \pi$.

In the first case we have $\alpha_1 > \frac{\pi}{2} \geq \alpha_2 > \beta > \gamma > \alpha_3$ (Figure 18(a)). As θ_1 or θ_2 must be α_3, we get an impossibility at vertex v_2 or v_3, respectively.



(a) case 1 (b) case 2

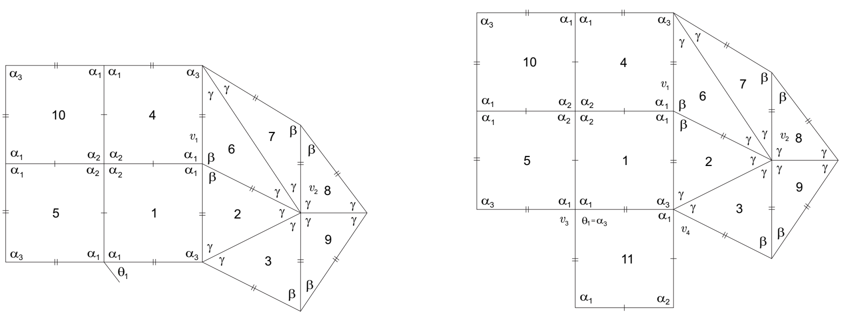
Figure 17: Local configurations



(a) (b)

Figure 18: Local configurations occurring in case 2.1

Therefore $\alpha_1+\beta=\pi$. At vertex v_1 we cannot have $\alpha_1+\beta=\pi=\alpha_1+\alpha_3$, as illustrated in Figure 18(b), otherwise at vertex v_2 we get $\alpha_1+\gamma+k\alpha_3=\pi$, $k\geq 1$, and a contradiction arises at vertex v_3. Consequently, we get the configuration illustrated in Figure 19(a). Note



(a) (b)

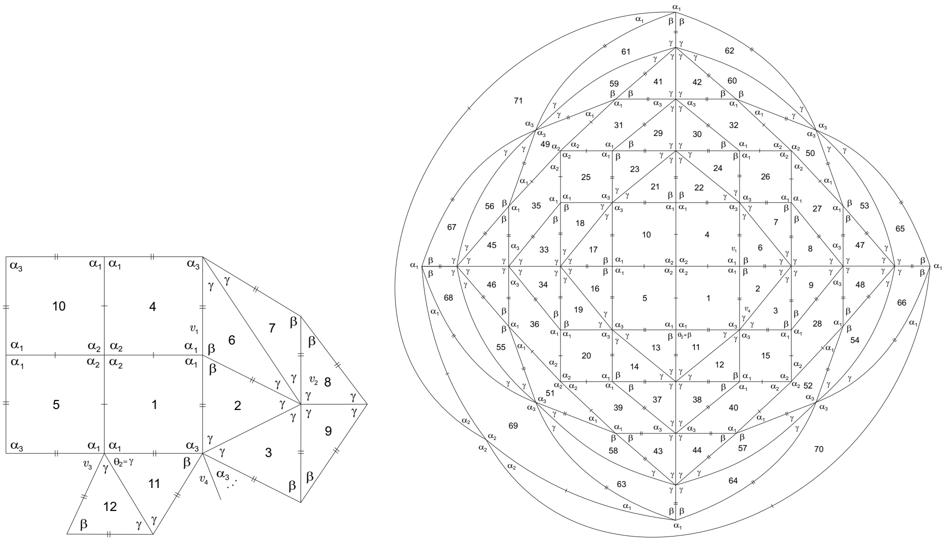
Figure 19: Local configurations occurring in case 2.1

that at vertex v_2 we cannot have $\gamma+\gamma+k\alpha_3=\pi$, $k\geq 1$, nor $\gamma+\gamma+\gamma+k\alpha_3=\pi$, $k\geq 1$, otherwise we obtain a similar contradiction as before (in fact we cannot have two angles α_3 adjacent). Observe also that we have necessarily $\alpha_2 + \alpha_2 = \pi$, as $\alpha_2 + \alpha_2 + \alpha_2 > \pi$.

Now, $\theta_1 = \alpha_3$, $\theta_1 = \gamma$ or $\theta_1 = \beta$.

2.1.1 If $\theta_1 = \alpha_3$ (Figure 19(b)), at vertex v_3 we must have $\alpha_1 + \alpha_3 = \pi$, which implies $\alpha_3 = \beta$. Nevertheless, a contradiction arises at vertex v_4 since we get $\alpha_1 + \gamma + k\alpha_3 > \pi$, for all $k \geq 1$.

2.1.2 If $\theta_1 = \gamma$ (Figure 20(a)), at vertex v_4 we obtain $\beta + \gamma + k\alpha_3 = \pi$. But at vertex v_3 we get $\alpha_1 + \gamma + \overline{k}\alpha_3 = \pi$, which is not possible as $3\pi \geq (\alpha_1 + \gamma + \alpha_3) + (\alpha_1 + \beta) + (\alpha_2 + \alpha_2) > (2\alpha_1 + \alpha_2 + \alpha_3) + (\beta + \gamma + \gamma) > 3\pi$.



(a) (b) f-tiling $\mathcal{M}$

Figure 20: Local configurations occurring in case 2.1

2.1.3 If $\theta_1 = \beta$, the last configuration is extended to the one illustrated in Figure 20(b). We shall denote this f-tiling by $\mathcal{M}$. Its 3D representation is given in Figure 21.

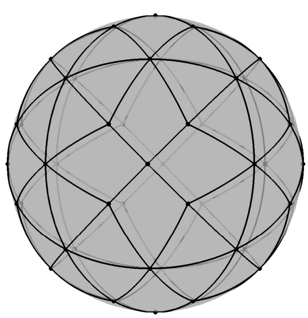
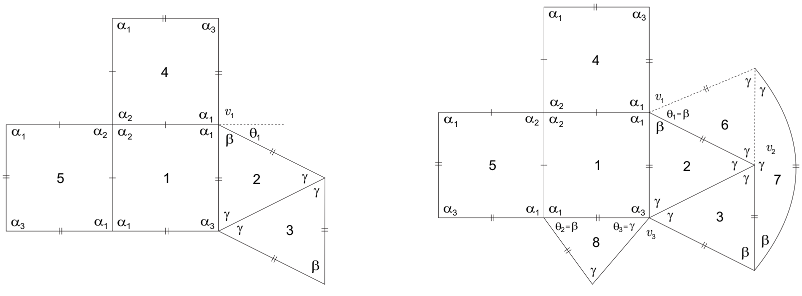


Figure 21: f-tiling $\mathcal{M}$

2.2 Suppose now that $\beta < \gamma$ (Figure 22(a)). In this case we have $\gamma > \frac{\pi}{3}$ and $\theta_1 = \beta$ or $\theta_1 = \alpha_3$.



(a) (b)

Figure 22: Local configurations occurring in case 2.2

If $\theta_1 = \beta$ ($\alpha_1 + \beta \leq \pi$, see Figure 22(b)), then at vertex we must have $\gamma + \gamma + k\alpha_3 = \pi$, with $k \geq 0$. As we seen before, as two angles α_3 in adjacent positions lead to a contradiction, we must have $\gamma + \gamma = \pi$. Moreover, θ_2 cannot be α_3, otherwise we would obtain $\theta_3 = \alpha_1$ and, at vertex v_3, $\alpha_1 + \gamma > \pi$. The case $\theta_2 = \beta$ also leads to a contradiction as $\gamma + \gamma = \pi$ and vertex v_3 cannot have valency four.

Finally, if $\theta_1 = \alpha_3$, we obtain the configuration illustrated in Figure 23. At vertex v_1 we reach a contradiction as $(\alpha_1 + \beta + \alpha_3) + (\alpha_1 + \gamma) + (\alpha_2 + \alpha_2) > (2\alpha_1 + \alpha_2 + \alpha_3) + (\beta + \gamma + \gamma) > 3\pi$. □

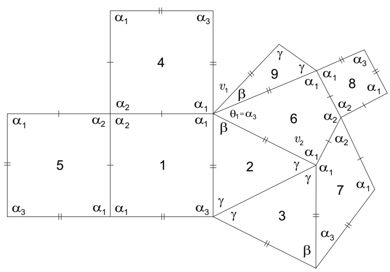


Figure 23: Local configuration occurring in case 2.2

4 Case of Adjacency VI

Suppose that any element of $\Omega\,(K, T)$ has at least two cells congruent, respectively, to K and T, such that they are in adjacent positions as illustrated in Figure 2–VI. As $b = c$, using trigonometric formulas, we obtain

$$\frac{\cos\beta + \cos^2\gamma}{\sin^2\gamma} = \frac{\cos\frac{\alpha_2}{2} + \cos\alpha_1\,\cos\frac{\alpha_3}{2}}{\sin\alpha_1\,\sin\frac{\alpha_3}{2}}. \tag{4.1}$$

Remark 4.1. The cases analyzed in [2] and [3] give rise to no f-tilings including two cells in these adjacent positions, and so a side of length c of each triangle must be adjacent to a side of length c of another triangle.

Proposition 4.2. *$\Omega(K,T) \neq \emptyset$ iff*

(i) $\alpha_1 + \gamma = \pi$, $\alpha_2 = \frac{\pi}{2}$, $\gamma + \gamma + \alpha_3 = \pi$ *and* $\beta = \frac{\pi}{3}$, *or*

(ii) $\alpha_1 + \gamma = \pi$, $\alpha_2 = \beta = \frac{\pi}{2}$ *and* $\gamma + \gamma + \alpha_3 = \pi$.

In the first case, there is a single f-tiling denoted by $\mathcal{N}$. A planar representation is given in Figure 26(b) and a 3D representation is given in Figure 27.

In the second case, there is a single f-tiling, $\mathcal{P}$. The corresponding planar and 3D representations are given in Figure 29(b) and Figure 30, respectively.

Proof. Concerning the angles of the triangle T we have necessarily one of the following situations:

$$\gamma > \beta \quad \text{or} \quad \gamma < \beta.$$

We consider separately each one of these cases.

1. Suppose firstly that $\gamma > \beta$.

If $\alpha_2 > \alpha_1$, then $\alpha_2 > \frac{\pi}{2}$ and we get the configuration of Figure 24(a). Due to the edge

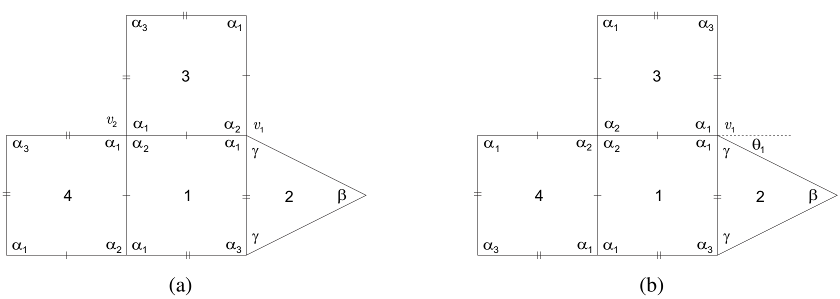


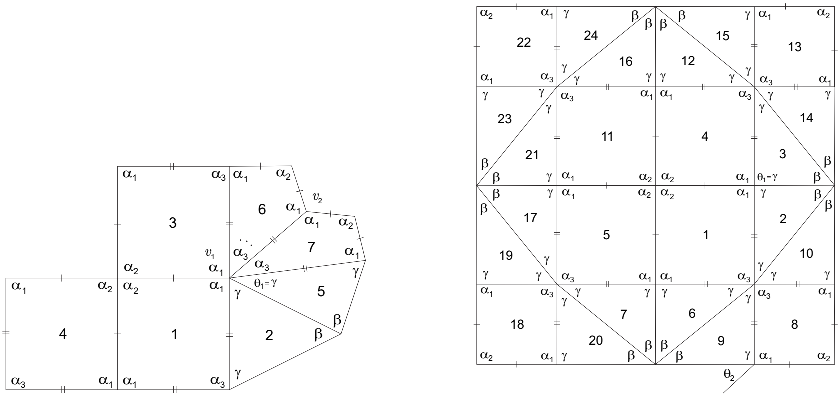
Figure 24: Local configurations occurring in case 1

lengths and also Remark 4.1, v_1 cannot have valency four and so $\alpha_2 + \gamma + k\alpha_3 = \pi$, $k \geq 1$. Therefore, analyzing vertices v_1 and v_2 we conclude that $\alpha_2 + \alpha_3 < \pi$ and $\alpha_1 \leq \frac{\pi}{2}$, which is impossible taking into account the kite's area.

Thus, $\alpha_1 \geq \alpha_2 > \alpha_3$ (Figure 24(b)) and $\theta_1 = \beta$ or $\theta_1 = \gamma$. In the first case, v_1 cannot have valency four and there is no way to satisfy the angle-folding relation around this vertex. Consequently, $\theta_1 = \gamma$ and

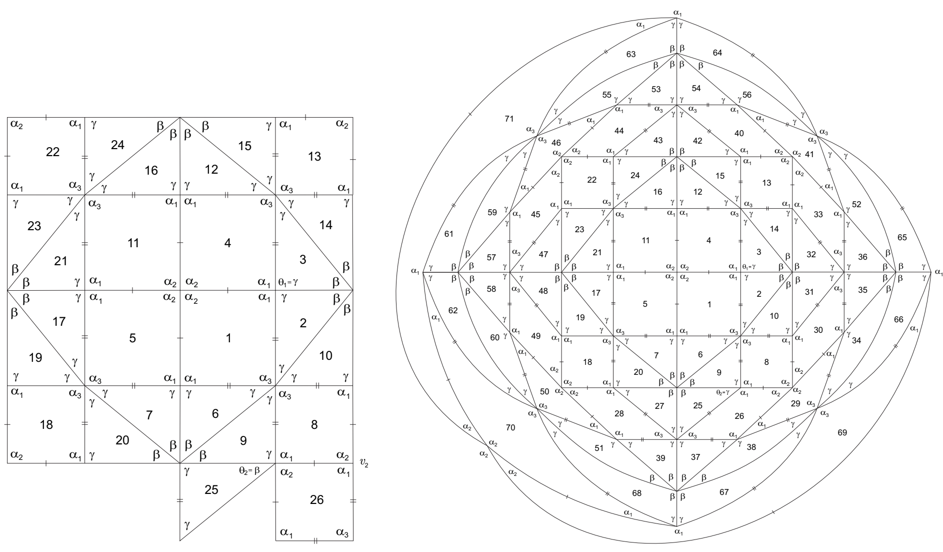
(i) if $\alpha_1 + \gamma < \pi$, then $\alpha_1 + \gamma + k\alpha_3 = \pi$, $k \geq 1$ (Figure 25(a)). At vertex v_2 we reach a contradiction as $\alpha_1 + \rho > \pi$, for all $\rho \in \{\alpha_1, \alpha_2\}$.

(ii) if $\alpha_1 + \gamma = \pi$, then the last configuration extends to the one illustrated in Figure 25(b). Now, if $\theta_2 = \beta$ (Figure 26(a)), we obtain a contradiction at vertex v_2. On the other hand, if $\theta_2 = \gamma$ a global planar representation is achieved as illustrated in Figure 26(b). We denote such f-tiling by $\mathcal{N}$. The corresponding 3D representation is given in Figure 27.



(a) case 1(i) (b) case 1(ii)

Figure 25: Local configurations occurring in case 1



(a) (b) f-tiling $\mathcal{N}$

Figure 26: Local configurations occurring in case 1(ii)

2. Suppose now that $\gamma < \beta$.

If $\alpha_2 > \alpha_1$, then $\alpha_2 > \frac{\pi}{2}$ and we get the configuration of Figure 28(a). Due to the edge lengths and also Remark 4.1, v_1 cannot have valency four and so $\alpha_1 + \alpha_1 + k\gamma = \pi$, $k \geq 1$. But then the other sum of alternate angles must contain $\alpha_2 + \alpha_3 > \pi$, which is not possible.

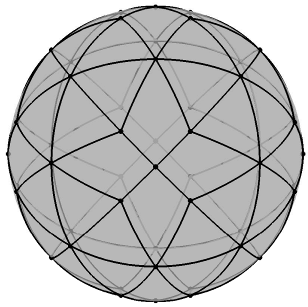
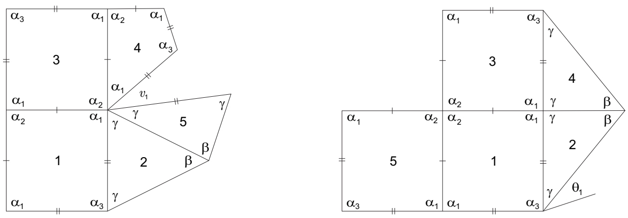


Figure 27: f-tiling $\mathcal{N}$



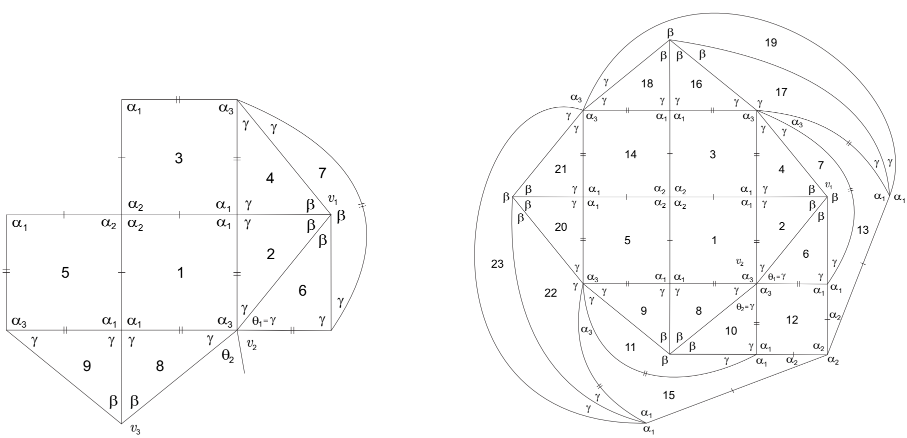
(a) (b)

Figure 28: Local configurations occurring in case 2

Therefore, $\alpha_1 \geq \alpha_2 > \alpha_3$ and $k\alpha_2 = \pi$, $k \geq 2$, and we have $\alpha_1 + \gamma = \pi$ or $\alpha_1 + \gamma < \pi$.

2.1 If $\alpha_1 + \gamma = \pi$, with the labeling of Figure 28(b), we have $\theta_1 = \gamma$ or $\theta_1 = \beta$.

2.1.1 If $\theta_1 = \gamma$, the last configuration is extended to the one illustrated in Figure 29(a).



(a) (b) f-tiling $\mathcal{P}$

Figure 29: Local configurations occurring in case 2.1

2.1.1.1 If $\theta_2 = \gamma$, at vertex v_2 we have $\alpha_3 + \gamma + \gamma = \pi$ or $\alpha_3 + \gamma + \gamma + \gamma = \pi$. Note that we cannot have more angles α_3 around v_2, as two angles of this type in adjacent positions lead to an impossibility, as seen before.

The condition $\alpha_3 + \gamma + \gamma = \pi$ implies $\alpha_2 + \alpha_2 = \frac{\pi}{2}$, and we get the configuration illustrated in Figure 29(b). We denote this f-tiling by $\mathcal{P}$, whose 3D representation is given in Figure 30.

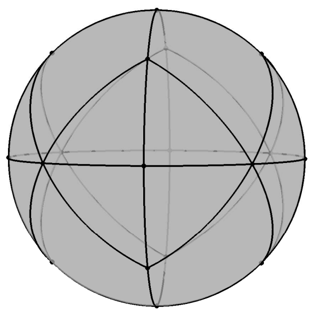
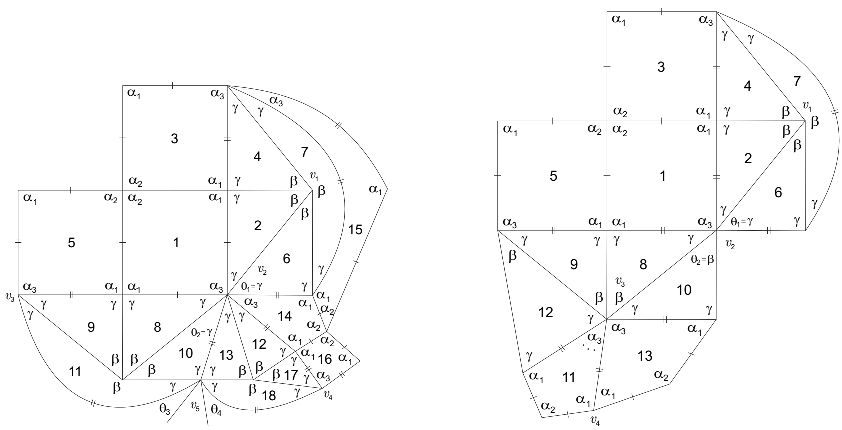


Figure 30: f-tiling $\mathcal{P}$

On the other hand, if $\alpha_3+\gamma+\gamma+\gamma = \pi$ (Figure 31(a)), the angles θ_3 and θ_4 cannot be α_3 otherwise we reach a contradiction at vertices v_3 and v_4, respectively. But this implies that at vertex v_5 we have two angles α_3 in adjacent positions, which is not also possible.



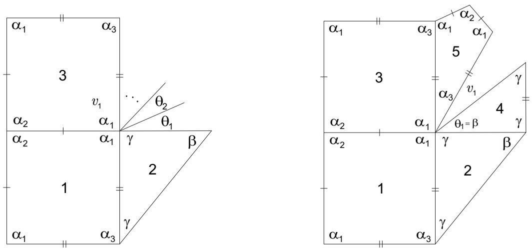
(a) (b)

Figure 31: Local configurations occurring in case 2.1

2.1.1.2 If $\theta_2 = \beta$, then at vertex v_3 we have $\beta + \gamma + k\alpha_3 = \pi$, $k \geq 1$, which leads to a contradiction as illustrated in Figure 31(b) (see vertex v_4).

2.1.2 If $\theta_1 = \beta$, we obtain a similar impossibility as in the previous case.

2.2 If $\alpha_1 + \gamma < \pi$ (Figure 32(a)), then $\theta_1 \in \{\beta, \gamma\}$.



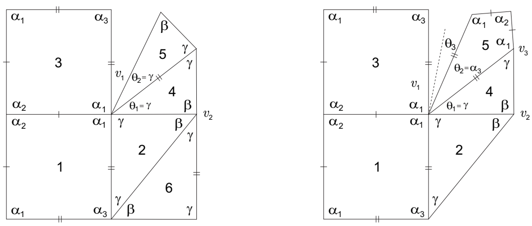
(a) (b)

Figure 32: Local configurations occurring in case 2.2

If $\theta_1 = \beta$ (Figure 32(b)), then $\alpha_1 + \beta + k\alpha_3 = \pi$, $k \geq 1$. It follows that the other sum of alternate angles at vertex v_1 must be greater or equal to $\alpha_1 + \gamma + \gamma > \pi$, which is an impossibility.

If $\theta_1 = \gamma$ and

(i) $\theta_2 = \gamma$ (Figure 33(a)), then $\beta > \alpha_1 > \frac{\pi}{2}$, which implies tile 6. At vertex v_2 we obtain $\beta + \gamma + k\alpha_3 = \pi$, $k \geq 1$, giving rise to two angles α_3 in adjacent positions, which leads to a contradiction, as seen previously.



(a) (b)

Figure 33: Local configurations occurring in case 2.2

(ii) $\theta_2 = \alpha_3$ (Figure 33(b)), vertex v_1 has valency six or greater than six. In the first case, we obtain two angles α_3 in adjacent positions, which is not possible. In the last case, we have necessarily $\theta_3 = \gamma$, and so $\beta > \alpha_1 > \frac{\pi}{2}$. Consequently, a contradiction arises at vertex v_2 or v_3.

□

Concerning to the combinatorial structure of each tiling obtained, we have that

(i) the symmetries of the f-tilings $\mathcal{L}$, $\mathcal{J}$, $\mathcal{M}$ and $\mathcal{N}$ that fix a vertex v of valency four and surrounded by $(\alpha_2, \alpha_2, \alpha_2, \alpha_2)$ are generated by a reflection and by the rotation through an angle $\frac{\pi}{2}$ around the axis by $\pm v$. On the other hand, for any vertices v_1 and v_2 of this type, there is a symmetry sending v_1 into v_2. It follows that the symmetry group has exactly $48 = 6 \times 8$ elements and it forms the group of all symmetries of the cube - the octahedral group, sometimes referred as $C_2 \times S_4$.

(ii) the f-tiling $\mathcal{P}$ has only two vertices surrounded by $(\alpha_2, \alpha_2, \alpha_2, \alpha_2)$, say the north and south poles. The symmetries of $\mathcal{P}$ that fix the north pole are generated by a reflection and by the rotation through an angle $\frac{\pi}{2}$ around the zz axis, giving rise to a subgroup isomorphic to D_4 (the dihedral group of order 8). Now, the reflection on the equator is also a symmetry of $\mathcal{P}$, and so it follows that the symmetry group of $\mathcal{P}$ is isomorphic to $C_2 \times D_4$.

5 Summary

In Table 1 is shown a list of the spherical dihedral f-tilings whose prototiles are a spherical kite and an isosceles spherical triangle, K and T, of internal angles $(\alpha_1, \alpha_2, \alpha_1, \alpha_3)$, and (β, γ, γ), respectively, in cases of adjacency IV, V and VI. Our notation is as follows:

- γ_1 is the solution of equation (2.1), with $\alpha_2 = \frac{\pi}{2}$, $\alpha_1 = \pi - \gamma_1$ and $\alpha_3 = \beta = \frac{\pi}{3}$; β_1 is the solution of equation (2.1), with $\alpha_2 = \frac{\pi}{2}$, $\alpha_1 = \pi - \gamma$ and $\alpha_3 = \pi - 2\beta_1$; β_2 is the solution of equation (2.1), with $\alpha_2 = \frac{\pi}{2}$, $\alpha_1 = \pi - \beta_2$ and $\alpha_3 = \gamma = \frac{\pi}{3}$; γ_2 is the solution of equation (4.1), with $\alpha_2 = \frac{\pi}{2}$, $\beta = \frac{\pi}{3}$, $\alpha_1 = \pi - \gamma_2$ and $\alpha_3 = \pi - 2\gamma_2$; γ_3 is the solution of equation (4.1), with $\alpha_2 = \beta = \frac{\pi}{2}$, $\alpha_1 = \pi - \gamma_3$ and $\alpha_3 = \pi - 2\gamma_3$.
- $|V|$ is the number of distinct classes of congruent vertices;
- N_1 is the number of triangles congruent T and N_2 is the number of kites congruent to K (used in the dihedral f-tilings);
- $G(\tau)$ is the symmetry group of each tiling $\tau \in \Omega(K, T)$.

f-tiling	α_1	α_2	α_3	β	γ	$\lvert V \rvert$	N_1	N_2	$G(\tau)$
$\mathcal{L}$	$\pi - \gamma_1$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	γ_1	3	24	24	$C_2 \times S_4$
$\mathcal{J}$	$\frac{2\pi}{3}$	$\frac{\pi}{2}$	$\pi - 2\beta$	β_1	$\frac{\pi}{3}$	4	48	24	$C_2 \times S_4$
$\mathcal{M}$	$\pi - \beta_2$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	β_2	$\frac{\pi}{3}$	4	48	24	$C_2 \times S_4$
$\mathcal{N}$	$\pi - \gamma_2$	$\frac{\pi}{2}$	$\pi - 2\gamma_2$	$\frac{\pi}{3}$	γ_2	3	48	24	$C_2 \times S_4$
$\mathcal{P}$	$\pi - \gamma_3$	$\frac{\pi}{2}$	$\pi - 2\gamma_3$	$\frac{\pi}{2}$	γ_3	3	16	8	$C_2 \times D_4$

Table 1: Combinatorial structure of dihedral f–tilings of S^2 by spherical kites and isosceles triangles in cases of adjacency IV, V and VI

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Distinguishing graphs by total colourings*

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Abstract

We introduce the *total distinguishing number* $D''(G)$ of a graph G as the least number d such that G has a total colouring (not necessarily proper) with d colours that is only preserved by the trivial automorphism. This is an analog to the notion of the distinguishing number $D(G)$, and the distinguishing index $D'(G)$, which are defined for colourings of vertices and edges, respectively. We obtain a general sharp upper bound: $D''(G) \leq \lceil \sqrt{\Delta(G)} \rceil$ for every connected graph G.

We also introduce the *total distinguishing chromatic number* $\chi''_D(G)$ similarly defined for proper total colourings of a graph G. We prove that $\chi''_D(G) \leq \chi''(G) + 1$ for every connected graph G with the total chromatic number $\chi''(G)$. Moreover, if $\chi''(G) \geq \Delta(G)+2$, then $\chi''_D(G) = \chi''(G)$. We prove these results by setting sharp upper bounds for the minimal number of colours in a proper total colouring such that each vertex has a distinct set of colour walks emanating from it.

Keywords: Total colourings of graphs, symmetry breaking in graphs, total distinguishing number, total distinguishing chromatic number.

Math. Subj. Class.: 05C15, 05E18

1 Introduction and definitions

In 1996, Albertson and Collins [1] introduced the *distinguishing number* $D(G)$ of a graph G as the least number d such that G admits a vertex colouring with d colours that is only preserved by the trivial automorphism of G. Ten years later Collins and Trenk [3] defined the *distinguishing chromatic number* $\chi_D(G)$ of a graph G for proper vertex colourings, so $\chi_D(G)$ is the least number d such that G has a proper colouring with d colours that is

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only preserved by the trivial automorphism. These concepts have already spawned tens of papers. For endomorphisms instead of automorphisms this approach was investigated in [4].

Obviously, $D(G) = 1$ for all asymmetric graphs. On the other hand, for a complete graph K_n and a complete bipartite graph $K_{p,p}$ we have $D(K_n) = n$, and $D(K_{p,p}) = p+1$. The distinguishing number of cycles C_3, C_4, C_5 equals three, while cycles C_n of length $n \geq 6$ have distinguishing number two.

This compares with a more general result of Collins and Trenk [3], and of Klavžar, Wong and Zhu [7].

Theorem 1.1. *[3],[7] If G is a connected graph with maximum degree* Δ*, then* $D(G) \leq \Delta + 1$*. Furthermore, equality holds if and only if G is a* K_n*,* $K_{p,p}$ *or* C_5*.*

In the same paper [3], Collins and Trenk obtained a general bound for the distinguishing chromatic number.

Theorem 1.2. *[3] If G is a connected graph with maximum degree* Δ*, then* $\chi_D(G) \leq 2\Delta$*. Furthermore, equality is achieved if and only if G is a* $K_{p,p}$ *or* C_6*.*

Edge colourings breaking automorphisms were investigated by the first two authors in [5]. If a graph G does not contain K_2 as a connected component, then the *distinguishing index* $D'(G)$ of a graph G as the least number d such that G admits an edge colouring with d colours that is only preserved by the trivial automorphism. And the *distinguishing chromatic index* $\chi'_D(G)$ of a graph G is the least number d such that G has a proper edge colouring with d colours that is not preserved by any nontrivial automorphism of G. A general upper bound for the distinguishing index was proved therin.

Theorem 1.3. *[5] If G is a connected graph of order* $n \geq 3$ *with maximum degree* Δ*, then* $D'(G) \leq \Delta$ *unless G is* C_3*,* C_4 *or* C_5*.*

It was also proved in [5] that $D'(G) \leq D(G) + 1$ for any connected graph of order $n \geq 3$, and this bound is sharp for each n. Actually, quite frequently $D'(G) < D(G)$. For a complete graph $D'(K_n) = 2$ for any $n \geq 6$, and also for a complete bipartite graph $D'(K_{p,p}) = 2$ for $p \geq 4$, whereas $D(K_n)$ and $D(K_{p,p})$ are equal to $\Delta + 1$.

The following theorem gives a sharp upper bound for the distinguishing chromatic index of connected graphs.

Theorem 1.4. *[5] If G is a connected graph of order* $n \geq 3$*, then*

$$\chi'_D(G) \leq \Delta(G) + 1$$

except for four graphs of small order C_4*,* K_4*,* C_6*,* $K_{3,3}$*.*

This theorem immediately implies the following interesting fact.

Corollary 1.5. *[5] Every connected Class 2 graph G admits an edge colouring with* $\chi'(G)$ *colours that is not preserved by any nontrivial automorphism of G.*

It has to be noted that Theorem 1.4 was a consequence of Theorem 1.6, the main result of [6]. To formulate it we need some definitions.

Let $f : E \to K$ be a proper edge colouring of a graph $G = (V, E)$. For a given vertex $x \in V$, each walk emanating from x defines a sequence of colors (α_i). We then say that the sequence (α_i) is *realizable* at a vertex x. The set of all sequences realizable at x is denoted by $W(x)$. We say that two vertices x and y of a graph G are *similar* if $W(x) = W(y)$, and the coloring f *personalizes the vertices* of G if no two vertices are similar. The minimum number of colours we need to obtain this property is denoted by $\mu(G)$ and called the *vertex distinguishing index by colour walks* of a graph G.

Theorem 1.6. *Let G be a connected graph of order* $n \geq 3$. *Then*

$$\mu(G) \leq \Delta(G) + 1$$

except for four graphs of small orders: C_4, K_4, C_6 *and* $K_{3,3}$.

The aim of this paper is to present analogous results for total colourings. We give general bounds, and an interesting relationship between the total distinguishing chromatic number and the total chromatic number.

Definition 1.7. The total distinguishing number $D''(G)$ of a graph G is the least number d such that G has a total colouring with d colours that is preserved only by the identity automorphism of G.

Observe that $D''(G) \leq \min\{D(G), D'(G)\}$. Clearly the equality holds for asymmetric graphs. And also for graphs with $\min\{D(G), D'(G)\} = 2$. The following observation can easily be verified.

Proposition 1.8. $D''(P_n) = D''(C_n) = D''(K_n) = 2$ *for* $n \geq 3$. $D''(K_{p,p}) = 2$ *for* $p \geq 1$.

However, quite frequently $D''(G) < \min\{D(G), D'(G)\}$. For instance, for a star $K_{1,n}$ of size $n \geq 3$, we shall show in the next section that $D''(K_{1,n}) = \lceil\sqrt{n}\rceil$, while $D(K_{1,n}) = D'(K_{1,n}) = n$.

We shall also investigate this concept for proper total colourings. A *proper total colouring* f of a graph G is an assignment of colours to the vertices and edges of G such that no two adjacent edges, no two adjacent vertices and no incident edges and vertices are assigned the same colour. The least number of colours among all such colourings is called the *total chromatic number* denoted by $\chi''(G)$.

Definition 1.9. The total distinguishing chromatic number $\chi''_D(G)$ of a graph G is the least number d such that G has a proper total colouring with d colours that is preserved only by the identity automorphism of G.

The total chromatic number of some simple classes of graphs was investigated first by Rosenfeld in [11]. He showed that $\Delta(G) + 2$ colours are enough for cliques, for complete bipartite and tripartite graphs, for balanced complete k-partite graphs and for graphs with maximum degree at most three. Next Kostochka proved the same bound for graphs with maximum degree at most four and five (see [8] and [9]). In the general case the following famous Behzad-Vizing conjecture is still open.

Conjecture 1.10. *[2] For every graph G, the total chromatic number satisfies the inequality*

$$\chi''(G) \leq \Delta(G) + 2.$$

So far, the best result in this direction was proved by Molloy and Reed in 1998.

Theorem 1.11. *[10] For every graph $G = (V, E)$, the total chromatic number satisfies the inequality*

$$\chi''(G) \leq \Delta(G) + 10^{26}.$$

In the next section we investigate total colourings, not necessarily proper. We prove a sharp upper bound $D''(G) \leq \lceil \sqrt{\Delta(G)} \rceil$ for all connected graphs.

In Section 3 we investigate total proper colourings. We show how one can personalize vertices of a graph by colour walks in total colourings. This approach is analogous to that of [6] for edge colourings.

In the last section we show that $\chi''(G)+1$ colours suffice to find a total proper colouring preserved only by the trivial automorphism. We shall infer this from the results of Section 3. However, it can also be easily shown using another argument. Namely, given a proper edge colouring of a graph G, the subgroup of $\mathrm{Aut}(G)$ preserving the colouring acts freely on vertices, i.e., the only element fixing a vertex is the identity. This follows since all paths beginning at a given vertex v are uniquely determined by the sequence of edge colours (which in effect give directions of where to go next at each vertex in the path). Thus any color preserving automorphism fixing v must fix all vertices. This immediately implies that $\chi''_D(G) \leq \chi''(G) + 1$ (just colour one vertex by an additional extra colour in a total colouring of G).

A much more intricate result of Section 4 states that $\chi''_D(G) = \chi''(G)$ whenever $\chi''(G) \geq \Delta(G) + 2$ (recall that if the Behzad-Vizing conjecture is true, then every graph has a total colouring with $\Delta(G) + 1$ or $\Delta(G) + 2$ colours). This will be proved using the main result of Section 3 concerning personalizing vertices by colour walks in proper total colorings.

2 Total distinguishing number

Every finite tree T has either a central vertex or a central edge which is fixed by every automorphism of T. For $k \geq 0$, let $S_k(x)$ denote a sphere of radius k with a center x, i.e., the set of all vertices at distance k from x.

Theorem 2.1. *If T is a tree of order $n \geq 3$, then $D''(T) \leq \lceil \sqrt{\Delta(T)} \rceil$.*

Proof. If T has a central vertex v_0, then the colour of v_0 can be arbitrary. Having $\lceil \sqrt{\Delta(T)} \rceil$ colours, we have at least $\Delta(T)$ different pairs (c_1, c_2) of colours, as the colouring need not be proper. Every edge incident to v_0 and its end vertex in the first sphere $S_1(v_0)$ obtain a distinct pair of colours (c_1, c_2). Hence, all vertices adjacent to v_0 are fixed by every automorphism of T preserving this colouring. Next, we colour edges going to subsequent spheres of T by pairs of colours in the same way as for the first sphere. By induction on the distance from v_0, all vertices of T are fixed.

If T has a central edge e_0, let T_1, T_2 be subtrees obtained by deleting the edge e_0. If we put distinct colours on the end vertices of e_0, then these verices are fixed by every automorphism. Next, for $i = 1, 2$, we colour the tree T_i using the same method as in the previous case. □

To see that the bound in Theorem 2.1 is sharp, observe that for any star $K_{1,n}$ we have $D''(K_{1,n}) = \lceil \sqrt{\Delta(K_{1,n})} \rceil = \lceil \sqrt{n} \rceil$. Indeed, if we used less than $\lceil \sqrt{n} \rceil$ colours then we

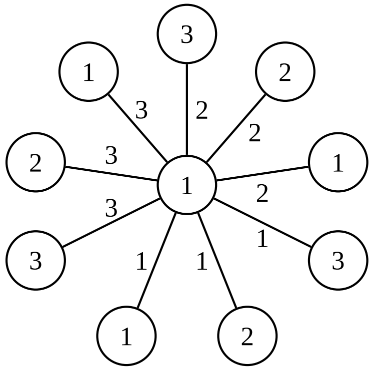

Figure 1: A total colouring of the star $K_{1,9}$ with three colours preserved only by the identity.

have less than n pairs of colours, so there would exist at least two edges coloured identically (together with their end vertices), thus a transposition of them would be a nontrivial automorphism preserving such a colouring.

Theorem 2.2. *If G is a connected graph of order $n \geq 3$, then $D''(G) \leq \lceil \sqrt{\Delta(G)} \rceil$.*

Proof. Denote $\Delta = \Delta(G)$. Clearly, $\Delta \geq 2$ and we have at least two colours. If G is a tree then the claim is true by Theorem 2.1. Suppose that G contains a cycle. If G is just a cycle or a complete graph, then the claim follows from Proposition 1.8.

Otherwise, we can always choose a vertex v_0 lying on a cycle such that the sphere $S_2(v_0)$ is nonempty. We colour v_0 with 2 and consider a BFS tree T of G rooted at v_0. We will first colour the tree T. For a given vertex v, denote

$$N_t(v) = \{(vu, u) \ : \ vu \in E(G)\}.$$

Let $S_1(v_0) = \{v_1, v_2, \ldots, v_p\}$. Without loss of generality we can assume that v_1 has a neighbour in $S_2(v_0)$. We colour both pairs (v_0v_1, v_1) and (v_0v_2, v_2) with a pair $(1, 1)$. Then we colour each pair of $N_t(v_0) \setminus \{(v_0v_1, v_1), (v_0v_1, v_2)\}$ with a distinct pair of colours different from $(1, 1)$. Thus $(1, 1)$ appears twice as a pair of colours in $N_t(v_0)$. We will then colour the graph G in such a way that v_0 will be the only vertex of G coloured with 2 such that the pair $(1, 1)$ appears twice in the neighbourhood $N_t(v_0)$. Hence v_0 will be fixed by every automorphism preserving the colouring. Therefore all vertices in $S_1(v_0)$ will also be fixed, except, possibly v_1 and v_2. To distinguish v_1 and v_2, we colour the sets $\{(v_1u, u) \in N_t(v_1) : u \in S_2(v_0)\}$ and $\{(v_2u, u) \in N_t(v_2) : v_2u \in E(T), u \in S_2(v_0)\}$ with two distinct sets of pairs of colours (this is possible since each of these sets contains at most $\Delta - 1$ elements, and we have Δ distinct pairs of colours). Therefore, every vertex adjacent to v_0, v_1 or v_2 will be fixed by every automorphism preserving our colouring. For each $i = 3, \ldots, p$, we then colour all elements of $\{(v_iu, u) : v_iu \in E(T), u \in S_2(v_0)\}$ with distinct pairs of colours different from the pair $(1, 1)$. This is again possible. Thus, all other vertices in $S_2(v_0)$ will be also fixed.

Then we proceed recursively with respect to the radius k of subsequent spheres $S_k(v_0)$ according to the ordering of vertices of the BFS tree T. Suppose all vertices of $S_i(v_0) = \{u_1, \ldots, u_{l_i}\}$, $i = 0, \ldots, k$, are fixed by every automorphism preserving colours. For each subsequent vertex $u_j, j = 1, \ldots, l_k$, we colour every pair (uju, u), where u is a descendent of u_j in T, with a distinct pair of colours except for $(1, 1)$. This is again possible since the number of pairs to be coloured is not greater than the number of admissible pairs of colours. Thus all neighbours of u_j in $S_{k+1}(v_0)$ will be also fixed.

Finally, we colour all remaining edges in $E(G) \setminus E(T)$ with 2. It is easily seen that if v is a vertex coloured with 2 such that the pair of colours $(1, 1)$ appears twice in $N_t(v)$, then $v = v_0$. Hence, all vertices of G are fixed by any automorphism preserving this colouring. □

Theorem 2.2 does not hold for disconnected graphs. For instance, consider a graph G of order n being the sum of r pairwise disjoint copies of K_2, i.e., $G = rK_2$ with $n = 2r$. It is easy to see that $D''(rK_2) = \min\{k : k^2(k-1) \geq r\}$. Hence, $D''(rK_2) \geq \sqrt[3]{\frac{n}{2}}$ while $\Delta(rK_2) = 1$.

3 Personalizing vertices by total colour walks

3.1 Total colour walks

In this section, we consider only proper colourings. Let f be a proper total colouring of a graph $G = (V, E)$. The *total palette of a vertex* v is the set

$$S(v) = \{f(u)\} \cup \{(f(vu), f(u)) : uv \in E\}.$$

For a given vertex $x \in V$, each walk emanating from x, say $xe_1x_1e_2x_2 \ldots e_px_p$, where $e_i = x_{i-1}x_i$ is an edge of G, $i = 1, 2, \ldots, p$, defines a sequence of colours $(f(x), f(e_1), f(x_1), f(e_2), f(x_2), \ldots, f(e_p), f(x_p))$. We then say that this sequence of colours is *realizable* at the vertex x. The set of all sequences realizable at x is denoted by $W(x)$.

We say that two vertices x and y of a graph G are *similar* with respect to f if $W(x) = W(y)$, and the colouring f *personalizes* the vertices of G if no two vertices are similar. The minimum number of colours we need to obtain this property is denoted by $\tau(G)$, and called the *vertex distinguishing index by total colour walks* of a graph G.

Denote by $W_k(x)$ all sequences of $W(x)$ of length $2k + 1$, i.e., generated by all walks of length k. We see that the total palette of a vertex v can be identified with $W_1(v)$.

For a given $(\alpha_i) \in W(x)$, denote by $m(x, (\alpha_i))$ the last vertex on a walk emanating from x and defining the sequence (α_i). The following observation will be used several times in the proof of our main result.

Proposition 3.1. *Two vertices x and y of G are similar if and only if for each $(\alpha_i) \in W(x)$, we have $(\alpha_i) \in W(y)$ and the vertices $m(x, (\alpha_i))$, $m(y, (\alpha_i))$ have the same total palettes.*

An analogous notion for edge colouring has been introduced in [6]. The corresponding parameter was denoted by $\mu(G)$. The main result of [6] was Theorem 1.6. In particular it follows that $\mu(G) = \chi'(G)$ if $\chi'(G) = \Delta(G) + 1$.

The aim of this section is to prove an analogous result for total colourings. More precisely we shall prove the following theorem.

Theorem 3.2. *Let G be a connected graph. Then*

$$\tau(G) \leq \chi''(G) + 1.$$

Moreover, if $\chi''(G) \geq \Delta(G) + 2$ *then* $\tau(G) = \chi''(G)$.

The proof of this theorem is divided into two parts. First, in the subsection below, we prove that $\tau(G) \leq \chi''(G) + 1$. In the next subsection, we prove the second part of the theorem for graphs with $\chi''(G) \geq \Delta(G) + 2$.

The above inequalities concerning $\tau(G)$ need not be true for disconnected graphs. For instance, consider again a graph $G = rK_2$ with $n = 2k$. It is easy to see that $\tau(rK_2) = \min\{k : 3\binom{k}{3} \geq r\}$. Hence, $\tau(rK_2) \geq \sqrt[3]{n}$ while $\Delta(rK_2) = 1$ and $\chi''(rK_2) = 3$.

3.2 Graphs with $\chi''(G) = \Delta(G) + 1$

In this subsection we prove Theorem 3.2 in case $\chi''(G) = \Delta(G)+1$. Let $f : V \cup E \to K$ be a colouring of G with $\chi''(G)$ colours. Let x be a vertex of G. We define a new colouring f' of G by replacing $f(x)$ with a new colour $0 \notin K$. We show that this colouring personalizes the vertices of G.

For, suppose that there are two similar vertices u and v. Denote by Q a shortest path from u to the vertex x. Consider now the walk Q' starting at v and inducing the same colour sequence as Q. Evidently, the walk Q' should also finish in x. The crucial observation is that since the last edges of Q and Q' are of the same colour, they cannot arrive to the same vertex and, since x is the only vertex of colour 0, we get a contradiction.

3.3 Graphs with $\chi''(G) \geq \Delta(G) + 2$

Now, we shall prove Theorem 3.2 in case $\chi''(G) \geq \Delta(G) + 2$. Let $f : V \cup E \to K$ be a proper total colouring of a graph $G = (V, E)$ with $\chi''(G)$ colours, and let $\chi''(G) \geq \Delta(G) + 2$. Assume for the rest of this subsection that there is no proper total colouring of G using $\chi''(G)$ colours which personalizes the vertices of G. For convenience, we will formulate stages of the proof as observations.

Denote by $N(x)$ and $E(x)$ the set of vertices adjacent to x and the set of edges incident to x, respectively, and let $\tilde{N}(x) = f(N(x))$ and $\tilde{E}(x) = f(E(x))$.

Observation 3.3. For each vertex $x \in V$, the set $\{f(x)\} \cup \tilde{N}(x) \cup \tilde{E}(x)$ contains all colours of K.

Proof. Suppose that there is a vertex x and a colour α such that $\alpha \in K \setminus (\{f(x)\} \cup \tilde{N}(x) \cup \tilde{E}(x))$. We shall show that then f could be modified in such a way that the obtained colouring would personalize the vertices of G.

Denote by Y the set of all vertices y with $W_1(y) = W_1(x)$. If Y contains only the vertex x, we are done. For, we can repeat the reasoning from the previous subsection by considering the walks ending with x.

If Y contains more vertices, we replace $f(y)$ by α in each vertex $y \in Y$, $y \neq x$. In this way, x becomes the only vertex of G with the palette $W_1(x)$. Again, we can repeat the reasoning from the previous subsection by considering the walks ending with x. □

Observation 3.4. For each edge $xy \in E$ the set $\{f(x)\} \cup \{f(y)\} \cup \tilde{E}(x) \cup \tilde{E}(y)$ contains all colours of K.

Proof. Let us suppose that there is an edge xy and a colour α such that $\alpha \in K \setminus (\{f(x)\} \cup \{f(y)\} \cup \tilde{E}(x) \cup \tilde{E}(y))$.

Consider now the set F of all edges $x'y'$ such that $f(x'y') = f(xy)$ and $W_1(x) = W_1(x')$ and $W_1(y) = W_1(y')$. Assume first that there exists only one such edge, namely

xy. Then, our colouring personalizes the vertices of G. For, suppose that there are two similar vertices u and v. Denote by Q a shortest path joining u with the edge xy. Consider now the walk Q' starting at v and inducing the same colour sequence as Q. Evidently, the walk Q' should also attain the edge xy.

Since the last edges of Q and Q' are of the same colour, they cannot arrive at the same vertex. So, one of the walks Q and Q' finishes at x and the other one at y. Since the palettes at x and y are distinct, we are done by Proposition 3.1.

If F contains more edges, we replace $f(x'y')$ by α for all edges of F exept for the edge xy. In this way, xy becomes the only edge of G coloured with $f(xy)$ and having the palettes $W_1(x)$ and $W_1(y)$ on its ends. Therefore, we can repeat the reasoning from above. □

A vertex x is *α-free* if $\alpha \notin \{f(x)\} \cup \tilde{E}(x)$.

Observation 3.5. For each vertex x, there is a colour, say α, such that x is *α-free*.

Proof. It suffices to observe that the set $\{f(x)\} \cup \tilde{E}(x)$ contains exactly $d(x)+1$ elements while the number of colours is greater than $\Delta(G)+1$. □

We say that a set of edges incident to a vertex x of G forms a *cyclic structure of size* $p \geq 2$ (with respect to the colouring f) if these edges can be ordered as xy_i, $i = 1, \dots, p$, such that the vertex y_i is $f(xy_{i+1})$-free, for $i = 1, \dots, p$, where the indexes are taken modulo p. Then the vertex x is called *central* while the vertices y_i are *leaves* of the cyclic structure.

The significance of a cyclic structure is shown by the next two observations. The proof of the first one follows immediately from the definition of the cyclic structure.

Observation 3.6. If the edges xy_i, $i = 1, \dots, p$, form a cyclic structure, then we can *rotate* the colours of edges, i.e., replace the colour $f(xy_i)$ on the edge xy_i by the colour $f(xy_{i+1})$, and the obtained colouring of G remains proper.

Observation 3.7. For each vertex x, the set $E(x)$ contains a cyclic structure.

Proof. Let x be a vertex of G and denote $f(x)$ by 0. Since the set $\{x\} \cup N(x)$ has at most $\Delta(G) + 1 < \chi''(G)$ elements, there is a colour, say α, which does not belong to the set $\{f(x)\} \cup \tilde{N}(x)$. Then, by Observation 3.3, $\alpha \in \tilde{E}(x)$. Denote by y_0 the second end of the edge incident to x and coloured by α. By Observation 3.5, there is a colour, say γ_1, such that the vertex y_0 is γ_1-free.

If $\gamma_1 = 0$ we can put the colour 0 on the edge xy_0 and the colour α on the vertex x. In consequence, we are able to reduce the number of vertices having the same palette as x by one, and then eventually get only one such vertex. This would provide a proper total colouring personalizing the vertices of G.

So, we may assume that $\gamma_1 \neq 0$. Then, by Observation 3.4, $\gamma_1 \in \tilde{E}(x)$. Let xy_1 be the edge coloured with γ_1. Again, by Observation 3.5, there is a colour, say γ_2, such that the vertex y_1 is γ_2-free.

If $\gamma_2 = 0$ we can put the colour 0 on the edge xy_1, the colour γ_1 on the edge xy_0 and the colour α on the vertex x (see Figure 2). In consequence, we are able to reduce the number of vertices having the same palette as x to obtain eventally only one such vertex. This would provide a colouring personalizing vertices of G.

If $\gamma_2 = \alpha$, the edges xy_1, xy_2 form a cyclic structure of size two.

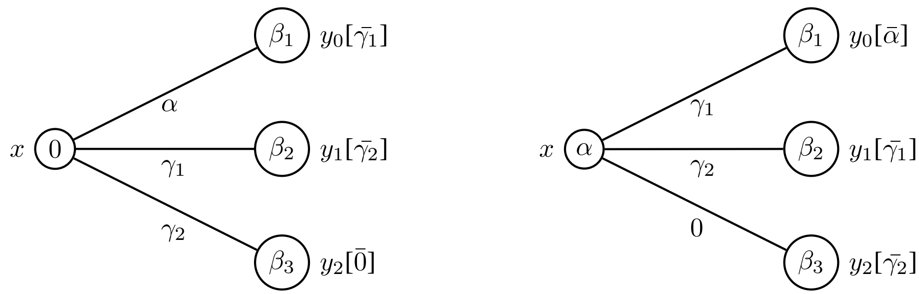

Figure 2: Before and after the change described in the proof of Observation 3.7

If $\gamma_2 \neq 0$ and $\gamma_2 \neq \alpha$, we continue the procedure of choosing at each step, as the missing colour, the first possible colour from the sequence $0, \alpha, \gamma_1, \gamma_2, \ldots$. If such a choice is possible, we can either exchange the colours and get a situation where x has a unique total palette, or we obtain a cyclic structure.

If the procedure finishes without finding 0 as a missing colour and without finding a cyclic structure, then the last vertex y_{d-1}, where $d = d(x)$, is γ_d-free for some $\gamma_d \notin \{0, \alpha, \gamma_1, \ldots, \gamma_{d-1}\}$. It means, in particular, that also the vertex x is γ_d-free, a contradiction with Observation 3.4. □

Let the set Cyc_1 of edges xy_i, $i = 1, \ldots, p$, incident to a vertex x of G, be a cyclic structure of size p (with respect to the colouring f). If all the vertices y_i, $i = 1, \ldots, p$, have the same colour, say β, then the palette at x remains unchanged after the rotation described in Observation 3.6. Therefore, we need a somewhat more complicated structure.

Suppose that a set Cyc_2 is another cyclic structure of size q with a central vertex $\hat{x}$ distinct from x. If Cyc_1 and Cyc_2 have a leave in common then we say that the sets Cyc_1 and Cyc_2 form a *double cyclic structure*.

Observation 3.8. If G has at least one double cyclic structure with respect to the colouring f then this colouring can be modified such that a new colouring personalizes the vertices of G.

Proof. Suppose that two sets of edges $\mathrm{Cyc}_1 = \{xy_i : i = 1, \ldots, p\}$ and $\mathrm{Cyc}_2 = \{\hat{x}z_j : j = 1, \ldots, q\}$ form a double cyclic structure. Without loss of generality we may assume that $y_1 = z_1$. Denote $f(y_1) = f(z_1) = \beta$ and $f(z_1\hat{x}) = \delta_1$.

Let Y be the set of all vertices y with $W_2(x) = W_2(y)$. If Y contains only the vertex x, we are done by repeating the reasoning from the previous subsection with the walks ending at x.

If Y contains more than one vertex, then each vertex y belonging to Y and different from x, is a central vertex of a cyclic structure of size p which is a part of a double cyclic structure with the second part being of size q.

Now, for each vertex $y \in Y \setminus \{x\}$, we rotate the colours of edges of the cyclic structure of size q forming the second part of a double cyclic structure. In this new colouring f' the set $W_2'(y)$ does not contain the sequence $(f(x), \gamma_1, \beta, \delta_1, f(\hat{x}))$ which was and still is present in $W_2'(x)$. In consequence, f' is a colouring such that $W_2'(x) \neq W_2'(y)$ for every vertex y distinct from x. It follows that f' personalizes the vertices of G. □

The next observation finishes the proof of Theorem 3.2.

Observation 3.9. Each graph G has at least one double cyclic structure.

Proof. For each vertex x we choose one cyclic structure $\mathrm{Cyc}(x)$ having x as a central vertex. The existence of such a structure is assured by Observation 3.7.

Consider now an auxiliary digraph Γ defined in the following way. The vertex set $V(\Gamma)$ coincides with the vertex set $V(G)$ and the arcs of Γ are the edges of G belonging to all sets $\mathrm{Cyc}(x)$ oriented from a central vertex of a structure towards the leaves of it.

By definition of a cyclic structure we have $d^+_\Gamma(x) \geq 2$ for each x. This implies, in particular, that there exists at least one vertex, say u, with $d^-_\Gamma(u) \geq 2$. Denote by z and $\hat{z}$ two of its in-neighbours in Γ. Then, the set $\mathrm{Cyc}(z) \cup \mathrm{Cyc}(\hat{z})$ forms a double cyclic structure. □

4 Total distinguishing chromatic number

The following lemma exhibits a relationship between $\tau(G)$ and $\chi''_D(G)$.

Lemma 4.1. *Every connected graph G of order $n \geq 3$ fulfils the inequality*

$$\chi''_D(G) \leq \tau(G).$$

Proof. Let f be a proper total colouring personalizing the vertices of G by colour walks, i.e., $W(x) \neq W(y)$ if $x \neq y$. Suppose φ is a nontrivial automorphism of G preserving f. Then there exists a vertex x such that $x \neq \varphi(x)$. An automorphism φ preserves the colouring, so every sequence $(\alpha_i) \in W(x)$ belongs also to $W(\varphi(x))$. And every sequence (β_i) starting at $\varphi(x)$, starts also at $\varphi^{-1}(\varphi(x)) = x$. Hence, x and $\varphi(x)$ are not distinguished by colour walks in this colouring. □

□

As a consequence of Lemma 4.1 and Theorem 3.2 we obtain a sharp upper bound for the distinguishing chromatic number of connected graphs.

Theorem 4.2. *Every connected graph G fulfils the inequality*

$$\chi''_D(G) \leq \chi''(G) + 1.$$

Moreover, $\chi''_D(G) = \chi''(G)$ if $\chi''(G) \geq \Delta(G) + 2$.

A total proper colouring of G with $\chi''(G)$ colours is called *minimal*. This theorem immediately implies the following interesting result.

Corollary 4.3. *Every connected graph G with $\chi''(G) \geq \Delta(G) + 2$ admits a minimal total colouring that is not preserved by any nontrivial automorphism.*

For graphs with $\chi''(G) = \Delta(G) + 1$, we sometimes need one colour more for $\chi''_D(G)$ than $\chi''(G)$.

For instance, cycles of order $6k$, for all $k \geq 1$, have a unique (up to a permutation of colours) colouring with three colours and this colouring is preserved by some rotations. Thus $\chi''_D(C_{6k}) = \chi''(C_{6k}) + 1$, by Theorem 4.2.

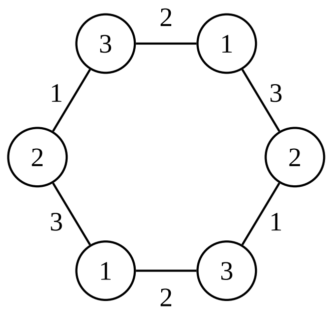

Figure 3: A minimal proper total colouring of C_6 with three colours.

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Accola theorem on hyperelliptic graphs

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Abstract

In this paper, we prove the following theorem: If a graph X is a degree 2 (unramified) covering of a hyperelliptic graph of genus $g \geq 2$, then X is γ-hyperelliptic for some $\gamma \leq \left[\frac{g-1}{2}\right]$. This is a discrete analogue of the corresponding theorem for Riemann surfaces. The Bass-Serre theory of coverings of graphs of groups is employed to get the main result.

Keywords: Riemann surface, graph, hyperelliptic graph, fundamental group, automorphism group, harmonic map, branched covering, graph of groups.

Math. Subj. Class.: 05C10, 20E08, 57M12

1 Introduction

Let M be a compact Riemann surface and let G be its finite group of conformal automorphisms, admitting a partition. That is, G can be expressed as a set-theoretic union of its certain subgroups with trivial pairwise intersections. In [2], R. D. M. Accola proved a formula which relates the genera of M, M/G and M/G_i where subgroups G_i, $i = 1, 2, \ldots, s$, form a partition. This formula is as follows:

$$(s-1)g(M) + |G|g(M/G) = \sum_{i=1}^{s} |G_i|g(M/G_i). \tag{1.1}$$

Demonstrating the applications of the formula, in the same paper Accola proved the following theorem, first proved by H. M. Farkas [7] using theta functions: *If M is a compact*

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Riemann surface of genus three which is a two-fold unramified covering of a genus $g = 2$ hyperelliptic Riemann surface, then M is hyperelliptic. The case $g > 2$ was considered in papers [1], [5]. For example, in the case of $g = 3$ it turns out that M is hyperelliptic or 1-hyperelliptic (M is a two-fold covering of a torus).

In this paper, we find a discrete version of results obtained in [1] and [5]. Finite connected graphs here play the role of Riemann surfaces, and harmonic maps between graphs play the role of holomorphic maps between Riemann surfaces. It turns out that the category of graphs, together with harmonic maps between them, closely mirrors the category of Riemann surfaces, together with the holomorphic maps between them. Namely, we prove that *if a graph X is a degree 2 (unramified) covering of a hyperelliptic graph Y of genus $g \geq 2$, then X is γ-hyperelliptic for some $\gamma \leq \left[\frac{g-1}{2}\right]$.*

Graph X, from the statement above, has the property that its automorphism group contains the Klein four-subgroup. In the proof, we use the fact that the Klein four-group admits a partition, and apply an analogue of (1.1) from [14].

Also we employ the theory of graphs of groups (or the Bass-Serre theory) to uniformize the coverings of a graph just as it works for Riemann surfaces. This approach was proposed by A. Mednykh and I. Mednykh [12].

In his dissertation [8], M. T. Green generalized the Bass-Serre theory and for coverings of graphs of groups obtained results similar to those in the topological theory of coverings. We use some results from this Ph.D. thesis.

2 Preliminaries

2.1 Graphs

In the present paper, a graph is a finite connected multigraph. We allow a graph to have loops. Denote by $V(X)$ the set of vertices of X and by $E(X)$ the set of directed edges of X. Following J.-P. Serre [13], we introduce two maps ∂_0, $\partial_1 : E(X) \to V(X)$ (endpoints) and a fixed point free involution $e \to \bar{e}$ of $E(X)$ (reversal of orientation) such that $\partial_i \bar{e} = \partial_{1-i} e$. We put

$$\mathrm{St}(a) = \mathrm{St}^X(a) = \partial_0^{-1}(a) = \{e \in E(X) \mid \partial_0 e = a\},$$

the *star* of a, and call $\deg(a) = |St(a)|$ the *degree* (or *valency*) of a. A *morphism* of graphs $\varphi : X \to Y$ carries vertices to vertices, edges to edges, and, for $e \in E(X)$, $\varphi(\partial_i e) = \partial_i \varphi(e)$ $(i = 0, 1)$ and $\varphi(\bar{e}) = \overline{\varphi(e)}$. Note that a morphism of graphs carries loops to loops. Working with loops in a graph, one may encounter some problems. On those occasions, one can use the approach with semiedges being developed in [9].

For $a \in X$ we have the local map

$$\varphi_a : \mathrm{St}^X(a) \to \mathrm{St}^Y(\varphi(a)).$$

A map φ is *locally bijective* if φ_a is bijective for all $a \in X$. We call φ a *covering* if φ is surjective and locally bijective. A bijective morphism is called an *isomorphism*, and an isomorphism $\varphi : X \to X$ is called an *automorphism.*

Remark 2.1. Note that the definition of a morphism of graphs given by M. Baker and S. Norine in [3] and our definition differ in the following sense. Let $\varphi : X \to Y$ be morphism of graphs and for some edge $e \in E(X)$ let $\varphi(\partial_0 e) = \varphi(\partial_1 e) = b \in V(Y)$. Then morphism φ, in the sense of [3], sends edge e to vertex b. In our case, morphism φ must send edge e to a loop based at vertex b.

2.2 Harmonic maps and harmonic actions

In this paragraph, we specify the class of morphisms of graphs, called harmonic maps, that share most properties with holomorphic maps between Riemann surfaces. The notion of harmonic maps between graphs was introduced by H. Urakawa [15] for simple graphs and was generalized by M. Baker and S. Norine [3] for multigraphs.

Definition 2.2. A morphism $\varphi : X \to Y$ of graphs is said to be a *harmonic map* or *branched covering* if, for all $x \in V(X)$, $y \in V(Y)$ such that $y = \varphi(x)$, the quantity

$$|e \in E(X) : x = \partial_0 e,\ \varphi(e) = e'|$$

is the same for all edges $e' \in E(Y)$ such that $y = \partial_0 e'$.

One can check directly from the definition that the composition of two harmonic morphisms is again harmonic. Therefore the class of all graphs, together with the harmonic morphisms between them, forms a category. We note also that an arbitrary covering of graphs is a harmonic map.

Let $\varphi : X \to Y$ be harmonic and $x \in V(X)$. We define the *multiplicity* of φ at x by

$$m_\varphi(x) = |e \in E(X) : x = \partial_0 e,\ \varphi(e) = e'|$$

for any edge $e' \in E(X)$ such that $\varphi(x) = \partial_0 e'$. By the definition of a harmonic morphism, $m_\varphi(x)$ is independent of the choice of e'. If $m_\varphi(x) > 1$ for some vertex $x \in V(X)$, such a vertex is called a *ramification point* of φ. The image $\varphi(x)$ of a ramification point is called a *branch point*.

We define the degree of a harmonic map $\varphi : X \to Y$ by the formula

$$\deg(\varphi) := |e \in E(X) : \varphi(e) = e'| \tag{2.1}$$

for any edge $e' \in E(Y)$. From the definition of a harmonic map of graphs and connectivity of the graphs, it follows that the right-hand side of (2.1) does not depend on the choice of e' and therefore $\deg(\varphi)$ is well defined.

Let $G < \mathrm{Aut}(X)$ be a group of automorphisms of a graph X. An edge $e \in E(X)$ is called *invertible* if there is $h \in G$ such that $h(e) = \bar{e}$. Let G act without invertible edges. Define the quotient graph X/G so that its vertices and edges are G-orbits of the vertices and edges of X. Note that if the endpoints of an edge $e \in E(X)$ lie in the same G-orbit then the G-orbit of e is a loop in the quotient graph X/G. Following S. Corry [6], we say that the group G acts *harmonically* on a graph X if for all subgroups $H < G$, the canonical projection $\varphi_H : X \to X/H$ is harmonic. If G acts harmonically and without invertible edges, we say that G acts *purely harmonically* on X.

The *genus* of a graph is defined as the rank of the first homology group of the graph (that is, its cyclomatic number). Let X be a graph of genus g' and let a group $G < \mathrm{Aut}(X)$ act purely harmonically on X. Denote by g the genus of the quotient graph X/G. There is an analogue of the Riemann-Hurwitz relation for graphs introduced in [3]. For the graph morphism under consideration, the relation is proved in [11], and has the following form:

$$g' - 1 = |G|(g - 1) + \sum_{a \in V(X)} (|G^a| - 1), \tag{2.2}$$

where G^a stands for the stabilizer of $a \in V(X)$. Here $|G|$ coincides with the degree of the harmonic map $\varphi : X \to X/G$ and $|G^a|$ coincides with the multiplicity $m_\varphi(a)$ of φ at a.

Remark 2.3. A graph X of genus $g' \geq 2$ is said to be *hyperelliptic*, if there is a degree 2 harmonic map $F : X \to Y$, where graph Y is a tree (that is, a graph of genus 0). Since at every ramification point $x \in V(X)$ the multiplicity $m_F(x) = 2$, by (2.2) the number of ramification points of F is equal to $g' + 1$.

A finite group G is said to admit a *partition* $\{G_1, \ldots, G_s\}$, where $G_i < G$ and $s \geq 2$, if $G = \bigcup_{i=1}^{s} G_i$ and $G_i \cap G_j = \{1\}$, $i, j = 1, 2, \ldots, s$, $i \neq j$. Let $G < \mathrm{Aut}(X)$ act purely harmonically on a graph X and admit a partition $\{G_1, \cdots, G_s\}$. Recall that the Euler characteristic $\chi(X)$ of a graph X is related to the genus $g(X)$ of X via $\chi(X) = 1 - g(X)$. By Corollary 1 in [14], we have

$$(s-1)\, g(X) + |G|\, g(X/G) = \sum_{i=1}^{s} |G_i|\, g(X/G_i). \tag{2.3}$$

2.3 Graphs of groups

The theory of graphs of groups is employed in this paper to uniformize harmonic maps between graphs. Following [4], we give the definition.

Definition 2.4. A *graph of groups* $\mathbb{X} = (X, \mathcal{A})$ consists of

(i) a connected graph X;

(ii) an assignment $\mathcal{A}$ to every vertex $a \in V(X)$ a group $\mathcal{A}_a$, and to every edge $e \in E(X)$ a group $\mathcal{A}_e = \mathcal{A}_{\bar{e}}$;

(iii) monomorphisms $\alpha_e : \mathcal{A}_e \to \mathcal{A}_a$, where $a = \partial_0 e$.

In this paper we restrict ourselves to a class of graphs of groups having trivial groups $\mathcal{A}_e = \{1\}$ for all edges $e \in E(X)$ and finite groups $\mathcal{A}_a$ for all vertices $a \in V(X)$. It will be enough for application to the theory of harmonic maps between graphs.

There are two equivalent definitions of the notion of a fundamental group of a graph of groups: the first is a direct algebraic definition via an explicit group presentation, and the second one using the language of groupoids. The algebraic definition is easier to state. Choose a spanning tree T in X. The fundamental group of X with respect to T, denoted $\pi_1(\mathbb{X}, T)$, is defined as the quotient of the free product

$$\left[\left(\mathop{*}_{a \in V(X)} \mathcal{A}_a \right) * F(E(X)) \right] / R,$$

where $F(E(X))$ denotes the free group with basis $E(X)$ and R is the following set of relations:

(i) $\bar{e} = e^{-1}$ for every e in $E(X)$;

(ii) $e = 1$ for every e in $E(T)$.

There is also a notion of the fundamental group of $\mathbb{X}$ with respect to a base-vertex a in X, denoted $\pi_1(\mathbb{X}, a)$, which is defined using the formalism of groupoids (see [8] and [4] for details). It turns out that for every choice of a base-vertex a and every spanning tree T in X, the groups $\pi_1(\mathbb{X}, T)$ and $\pi_1(\mathbb{X}, a)$ are naturally isomorphic. We note also ([4], section 1.22) that for given $a, b \in X$ the groups $\pi_1(\mathbb{X}, a)$ and $\pi_1(\mathbb{X}, b)$ are conjugate in the fundamental groupoid of $\mathbb{X}$. In what follows we will use notation $\pi_1(\mathbb{X})$, ignoring the way the fundamental group was constructed.

It follows from the above definition that if X is a graph of genus g then $F(E(X))/R = F_g$ is the free group of rank g. Then

$$\pi_1(\mathbb{X}) = \left(\mathop{*}_{a \in V(X)} \mathcal{A}_a \right) * F_g.$$

To every graph of groups $\mathbb{X}$, with a specified choice of a base-vertex $a \in X$, one can associate a *Bass-Serre universal covering tree* $\widetilde{\mathbb{X}} = \widetilde{(\mathbb{X}, a)}$, which is a tree admitting a natural group action of the fundamental group $\pi_1(\mathbb{X}) = \pi_1(\mathbb{X}, a)$ without edge-inversions. Moreover, the quotient graph of groups $\widetilde{\mathbb{X}}/\pi_1(\mathbb{X})$ is naturally isomorphic to $\mathbb{X}$.

2.4 Coverings of graphs of groups and harmonic maps

Let us take graph morphisms in the definition of a covering of graphs of groups, given in [8] or [4], to be the class of all harmonic maps. Taking into consideration the fact that a trivial group is assigned to any edge, the definition of a covering of graphs of groups can be formulated as follows.

Definition 2.5. Let $\mathbb{X} = (X, \mathcal{A})$ and $\mathbb{Y} = (Y, \mathcal{B})$ be graphs of groups with trivial edge groups. A *covering* $\mathbb{F} = (F, \Phi) : \mathbb{X} \to \mathbb{Y}$ of graphs of groups consists of

(i) a harmonic morphism $F : X \to Y$;

(ii) a set Φ of monomorphisms $F_a : \mathcal{A}_a \to \mathcal{B}_{F(a)}$, $a \in V(X)$, such that $m_F(a)|\mathcal{A}_a| = |\mathcal{B}_{F(a)}|$, where $m_F(a)$ is the multiplicity of F at the point a.

This definition was introduced in [12]. To illustrate the notion of a covering in the category of graphs of groups, we provide a basic example.

Example 2.6. Let G be a group of automorphisms of a finite connected graph X. Suppose that G acts on the set $E(X)$ of directed edges of X freely and without edge inversions. Consider the canonical map $F : X \to Y = X/G$. Denote by $\mathrm{St}_G(a)$ the stabilizer of a vertex a in group G. Then F is a harmonic map with $m_F(a) = |\mathrm{St}_G(a)|$, $a \in V(X)$. Denote by $\mathbb{X}$ the graph of groups obtained from X by prescribing a trivial group to each vertex and each edge of X. Graph of groups $\mathbb{Y}$ is defined by prescribing to each vertex $b = F(a)$ of Y a group $\mathcal{B}_{F(a)}$ isomorphic to $\mathrm{St}_G(a)$ and assign a trivial group to each edge of Y. Since G acts transitively on each fibre of F, the group $\mathcal{B}_{F(a)}$ is well defined. Let Φ be the set of trivial monomorphisms $F_a : \mathcal{A}_a \to \mathcal{B}_{F(a)}$, $a \in V(X)$. We have $m_F(a)|\mathcal{A}_a| = |\mathcal{B}_{F(a)}|$. Then

$$\mathbb{F} = (F, \Phi) : \mathbb{X} \to \mathbb{Y} = \mathbb{X}/G$$

is the covering of graphs of groups.

3 Main result

A graph X of genus $g' \geq 2$ is said to be *γ-hyperelliptic* if there is a degree 2 harmonic map $F : X \to Y$ onto a graph Y of genus γ. Each edge of Y has two pre-images under F and there is an order 2 automorphism τ of X, which swaps these pre-images. This automorphism is called *γ-hyperelliptic* involution. Note that γ-hyperelliptic involution acts on X purely harmonically. The case $\gamma = 0$ coincides with the definition of a hyperelliptic graph. The main result is stated in the following theorem.

Theorem 3.1. *Let X be a degree 2 (unramified) covering of a hyperelliptic graph Y of genus $g \geq 2$. Then X is γ-hyperelliptic for some $\gamma \leq \left[\frac{g-1}{2}\right]$.*

In the proof of Theorem 3.1 we use the following algebraic result.

Lemma 3.2. *Let Γ be a free product of $n > 1$ copies of $\mathbb{Z}_2$. If $F < \Gamma$ is a torsion-free subgroup of index 4, then $F \lhd \Gamma$ and Γ/F is isomorphic to the Klein four-group.*

Proof. The given group Γ has the presentation

$$\Gamma = \left\langle x_1, x_2, \ldots, x_n \mid x_1^2, x_2^2, \ldots, x_n^2 \right\rangle. \tag{3.1}$$

Let $F \leq \Gamma$ be any torsion-free subgroup of index 4. The action of Γ on the right cosets $\{F, Fy_1, Fy_2, Fy_3\}$ of F in Γ gives a transitive representation $\theta : \Gamma \to S_4$. If some x_i in the presentation (3.1) of Γ has a fixed point, then for some $y \in \Gamma$ we have $y\, x_i\, y^{-1} \in F$ and F is not torsion-free, because $\left(y\, x_i\, y^{-1}\right)^2 = 1$. Hence x_i has no fixed points, so it is represented in S_4 by a double transposition (that is, by a permutation of cyclic type $(2\,2)$). So long as we deal with the transitive representation, we get an epimorphism $\theta : \Gamma \to V_4$, where V_4 is the Klein four-group.

Let us show that $F \leq \ker\theta$. Take any $w \in F$. Since w fixes the coset F, and there are only double transposition actions and the trivial action, w must fix the remaining cosets. So, $w \in \ker\theta$.

The reverse inclusion $\ker\theta \leq F$ is obvious. Thus, we get $F = \ker\theta \lhd \Gamma$. □

Proof of Theorem 3.1. Let $\phi : X \to Y$ denote the covering from the theorem. The graph Y is hyperelliptic, that is, there is an order two harmonic automorphism $\tau \in \mathrm{Aut}(Y)$, such that the factor graph $T = Y/\langle\tau\rangle$ is a tree. Let $\psi : Y \to T$ be the corresponding harmonic map. Let F stand for the composite harmonic map $\psi \circ \phi$.

Now we are going to find a group G_0 of deck transformations of the harmonic map $F : X \to T$. To do that, we apply the Bass-Serre theory. Turn graphs X and T into graphs of groups as follows. Let $\mathbb{X} = (X, \mathcal{A})$ be a graph of groups based on graph X, and where $\mathcal{A}$ assigns a trivial group $\mathcal{A}_z = \{1\}$ to each vertex and each edge z of X. Let $\mathbb{T} = (T, \mathcal{B})$ be a graph of groups based on tree T, and where $\mathcal{B}$ assigns the group $\mathcal{B}_z = \mathbb{Z}_2$ to each of $g + 1$ branch points z of map ψ, and a trivial group $\mathcal{B}_z = \{1\}$ to every other vertex and edge z of T.

Let us show that the map $F : X \to T$ can be extended to the covering $\mathbb{F} : \mathbb{X} \to \mathbb{T}$ of graph of groups. Since F is harmonic, it remains to check that, for any $a \in V(X)$, the trivial monomorphism $\mathcal{A}_a \to \mathcal{B}_{F(a)}$ satisfies the condition $m_F(a)|\mathcal{A}_a| = |\mathcal{B}_{F(a)}|$ or, since all $\mathcal{A}_a = \{1\}$, the condition

$$m_F(a) = |\mathcal{B}_{F(a)}|. \tag{3.2}$$

The map ϕ is a covering, and so locally bijective. Hence, for any $a \in V(X)$, $m_F(a) = m_\psi(\phi(a))$. If $\phi(a)$ is a ramification point of ψ, then $m_\psi(\phi(a)) = 2$, $\mathcal{B}_{F(a)} = \mathbb{Z}_2$, and so (3.2) is correct. If $\phi(a)$ is not a ramification point of ψ, then $m_\psi(\phi(a)) = 1$, $\mathcal{B}_{F(a)}$ is trivial, hence (3.2) is correct as well.

Let $\mathrm{H} = \pi_1(\mathbb{X})$ and $\Gamma = \pi_1(\mathbb{T})$ be fundamental groups, and $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{T}}$ be universal covering trees of graphs of groups $\mathbb{X}$ and $\mathbb{T}$ respectively. Note that since ϕ has no ramification points, by the Riemann-Hurwitz relation (2.2), X has genus $2g-1$ and so H is a free group on $2g-1$ generators; group Γ is a free product of $g+1$ copies of $\mathbb{Z}_2$.

By the Bass uniformization theorem ([3], Proposition 2.4) there exists a lift of $\mathbb{F}$ to an isomorphism $\widetilde{\mathbb{F}} : \widetilde{\mathbb{X}} \to \widetilde{\mathbb{T}}$ between the covering trees equivariant under the action of H and Γ on $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{T}}$ respectively. Note that $\mathbb{X} \cong \widetilde{\mathbb{X}}/\mathrm{H}$ and $\mathbb{T} \cong \widetilde{\mathbb{T}}/\Gamma$. Identifying $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{T}}$ via $\widetilde{\mathbb{F}}$ we replace the covering $\mathbb{F} : \mathbb{X} \to \mathbb{T}$ by the covering $\mathbb{F}' : \widetilde{\mathbb{X}}/\mathrm{H} \to \widetilde{\mathbb{X}}/\Gamma$ induced by the group inclusion $H < \Gamma$, where H is of index 4 in Γ. By Lemma 3.2, since any free group is a torsion-free group, H is a normal subgroup in Γ. Therefore, by Theorem 8.1 in [8], covering $\mathbb{F}'$ is regular and its covering transformation group is $G_0 = \Gamma/H$. Returning to the category of graphs, we get the underlying harmonic map of graphs $X \to X/G_0$ coinciding with $F : X \to T$ where $X/G_0 \cong T$. Group G_0 is isomorphic to the Klein four-group. So it admits a partition $\{G_1, G_2, G_3\}$ into three subgroups of order two. Note that every subgroup $G_i \leq G_0$ corresponds to a harmonic map $X \to X/G_i$ and one of X/G_i is isomorphic to Y. Let g' and g_i be the genera of X and X/G_i, $i \in \{0, 1, 2, 3\}$, respectively. By (2.3) we have

$$g' + 2g_0 = g_1 + g_2 + g_3.$$

Here $g_0 = 0$, $g' = 2g - 1$ and one of g_i must be g, so we get

$$g - 1 = g_1 + g_2.$$

The possible cases for g_1 and g_2 (up to a symmetry) are

g_1	g_2
0	$g-1$
1	$g-2$
2	$g-3$
$\dots$	$\dots$
$\left[\frac{g-1}{2}\right]$	$\left[\frac{g-1}{2}\right]$ $(+1,$ if g is even$)$.

Taking γ to be the minimum of g_1 and g_2 in each case, we get that X is γ-hyperelliptic for some $\gamma \leq \left[\frac{g-1}{2}\right]$.

Finally, we show that the bound is sharp. That is, for any $g \geq 2$ there exists a graph X of genus $2g-1$, and the smallest genus of graphs Y, such that $X \to Y$ is a degree 2 harmonic morphism, is equal to $\left[\frac{g-1}{2}\right]$. Let g be odd. Consider graph X_1 of genus $2g-1$, depicted on Figure 1 in the case $g = 5$. Its automorphism group contains five involutions. Their actions on X_1 are horizontal and vertical flips, h, v, two diagonal flips, d_1, d_2, and the rotation r on π around the center of the graph. The corresponding factor-graphs have genera $\frac{g-1}{2}$, $\frac{g-1}{2}$, $g-1$, $g-1$ and g respectively.

Now let g be even. Consider graph X_2 of genus $2g - 1$, depicted on Figure 2 in the case $g = 6$. Its automorphism group contains three involutions. They act on X_2 as horizontal and vertical flips, h, v, and the rotation r on π around the center of the graph.

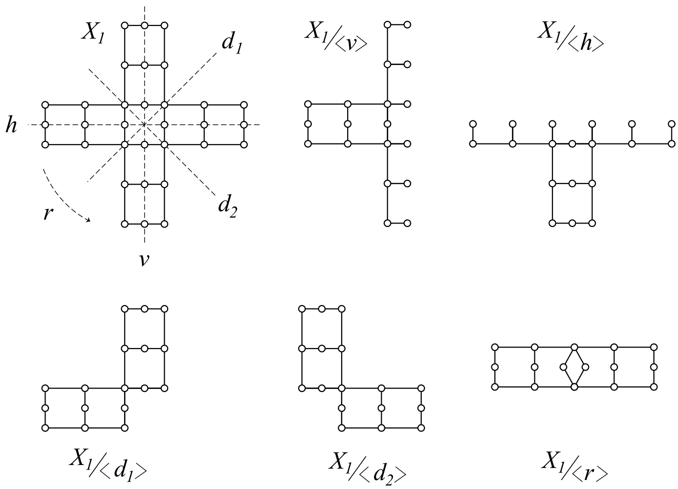

Figure 1: Graph X_1 in the case $g = 5$ and its factor-graphs.

The corresponding factor-graphs have genera $\left[\frac{g-1}{2}\right]$, $\left[\frac{g-1}{2}\right] + 1$ and g respectively. Hence, the bound in the theorem is sharp. □

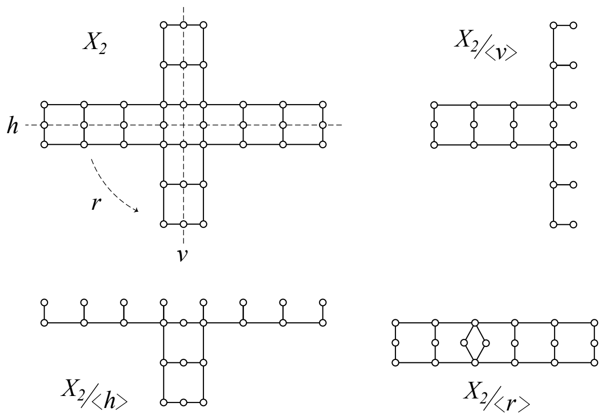

Figure 2: Graph X_2 in the case $g = 6$ and its factor-graphs.

The immediate consequences of the theorem are the assertions below. The first one has been proved by I. Mednykh [10] by exhaustive search.

Corollary 3.3. *Suppose X is a graph of genus 3 which is a degree 2 (unramified) covering of a hyperelliptic graph Y of genus 2. Then X is hyperelliptic.*

Corollary 3.4. *If X is a graph of genus 5 which is a degree 2 (unramified) covering of a hyperelliptic graph of genus 3, then X is hyperelliptic or 1-hyperelliptic.*

Remark 3.5. In both corollaries, the genus of X is not an extra hypothesis, but a necessary consequence of the degree 2 cover due to Riemann-Hurwitz relation (2.2).

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A note on automorphisms of halved Cayley graphs of Coxeter systems

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Abstract

We consider the halved Cayley graphs of Coxeter systems and show that every automorphism of such a graph can be uniquely extended to an automorphism of the corresponding Cayley graph.

Keywords: Coxeter system, Cayley graph.

Math. Subj. Class.: 20F55, 05C12

1 Introduction

Let W be a group generated by a finite set S whose elements are involutions. For distinct $s, s' \in S$ we denote by $m(s, s')$ the order of the element ss'. Then $m(s, s') = m(s', s)$ and the condition $m(s, s') = 2$ is equivalent to the commuting of s and s'. Suppose that (W, S) is a *Coxeter system*, i.e W is the quotient of the free group over S by the normal subgroup generated by all $(ss')^{m(s,s')}$ with $m(s, s') < \infty$.

The *Cayley graph* $\mathrm{C}(W, S)$ is the graph whose vertex set is W and $w, v \in W$ are adjacent vertices of the graph if $v = sw$ for a certain $s \in S$ (since S consists of involutions, the adjacency relation is symmetric). For the dihedral Coxeter system $\mathsf{I}_2(n)$ this graph is the $(2n)$-cycle and we get an infinite path if $n = \infty$. The Cayley graph of A_n is the *permutohedron* [6]. See [1, Figures 3.3] for the Cayley graph of H_3. Also, $\mathrm{C}(W, S)$ can be identified with the graph whose vertices are maximal simplices of the Coxeter complex $\Sigma(W, S)$ and two maximal simplices are adjacent vertices if their intersection consists of $|S| - 1$ elements [5].

In almost all cases, the automorphism group of $\mathrm{C}(W, S)$ is known. For every $w \in W$ the right multiplication $R_w : v \to vw$ is an automorphism of the graph. If the diagram of our Coxeter system does not contain adjacent edges labeled by ∞ then the automorphism

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group of $\mathrm{C}(W,S)$ is the semidirect product of W and the automorphism group of the diagram [1, Corollary 3.2.6].

The *length* $l(w)$ of an element $w \in W$ is the smallest number m such that w has an expression

$$w = s_1 \dots s_m, \quad s_1, \dots, s_m \in S. \tag{1.1}$$

It is clear that $l(w)$ is the distance between 1 and w in the Cayley graph. Since every right multiplication is an automorphism of the graph, the distance $d(w,v)$ between $w, v \in W$ is equal to $l(wv^{-1}) = l(vw^{-1})$. Recall the following remarkable property of Coxeter systems called the *exchange condition*: if (1.1) is a *reduced* expression, i.e. $l(w) = m$, then for every $s \in S$ satisfying $l(sw) \leq m$ there exists $k \in \{1, \dots, m\}$ such that

$$sw = s_1 \dots \hat{s}_k \dots s_m$$

(the symbol $\hat{}$ means that the corresponding term is omitted).

The group W can be presented as the disjoint union of the following subsets

$$W_1 := \{\, w \in W : l(w) \text{ is odd} \,\} \text{ and } W_2 := \{\, w \in W : l(w) \text{ is even} \,\}.$$

Using the exchange condition we establish the following:

- the distance between any two elements of W_i, $i \in \{1, 2\}$ is even,
- the distance between every element of W_1 and every element of W_2 is odd.

Note that W_2 is a subgroup of W. Consider the graph Γ_i, $i \in \{1, 2\}$ whose vertex set is W_i and two elements of W_i are adjacent vertices if the distance between them (in the Cayley graph) is equal to 2. The right multiplication R_w preserves both W_i in the case when $w \in W_2$. If $w \in W_1$ then R_w transfers W_1 to W_2 and conversely. The latter implies that Γ_1 and Γ_2 are isomorphic.

The main result of the note is the following.

Theorem 1.1. *If $|S| \geq 5$ then every isomorphism between Γ_i and Γ_j $i, j \in \{1, 2\}$ can be uniquely extended to an automorphism of the Cayley graph.*

The same fails if $|S| \in \{3, 4\}$ (Remark 3.2), but the statement holds for $|S| = 2$ (the Cayley graph is a cycle or an infinite path) and the case $|S| = 1$ is trivial.

Theorem 1.1 easily follows from the description of maximal 2-cliques of Cayley graph, i.e. maximal cliques of the halved Cayley graph, given in Lemma 2.5.

If S consists of n mutually commuting involutions then $\mathrm{C}(W,S)$ is the n-dimensional hypercube graph H_n and every Γ_i is the half-cube graph $\frac{1}{2}H_n$. So, it is natural to ask which properties of the hypercube and half-cube graphs can be extended to $\mathrm{C}(W,S)$ and Γ_i, respectively?

2 Maximal 2-cliques

Two vertices in a graph are said to be 2-*adjacent* if the distance between them is equal to 2. Recall that a *clique* is a subset of the vertex set, where any two distinct vertices are adjacent. We say that a subset in the vertex set is a 2-*clique* if any two distinct elements of this subset are 2-adjacent vertices.

Consider examples of 2-cliques in $\mathrm{C}(W,S)$.

Example 2.1 (First type)**.** Any two distinct elements of S are 2-adjacent and S is a 2-clique. Since the right multiplication R_w is an automorphism of the Cayley graph, Sw is a 2-clique for every $w \in W$.

Remark 2.2. Suppose that $S = Sw$. Then for any $s_1, s_2 \in S$ there exist $s'_1, s'_2 \in S$ such that $s_1 = s'_1 w$ and $s_2 = s'_2 w$. If $w \neq 1$ then $s_1 \neq s'_1$ and $s_2 \neq s'_2$. We have

$$s'_1 s_1 = w = s'_2 s_2 \text{ and } s'_2 s'_1 s_1 = s_2$$

Since W cannot be generated by a proper subset of S, the latter means that $s_2 = s'_1$. Therefore, $S = \{s_1, s_2\}$ and $s_1 s_2 = s_2 s_1$. So, the equality $Sw = Sw'$ implies that $w = w'$ except the case when our Coxeter system is $\mathsf{I}_2(2)$.

Example 2.3 (Second type)**.** Let s, s', s'' be three distinct mutually commuting elements of S. Then $ss's''$ is 2-adjacent to s, s', s'' and $\{sw, s'w, s''w, ss's''w\}$ is a 2-clique for every $w \in W$.

Example 2.4 (Third type)**.** Suppose that $s, s' \in S$ and $m(s, s') = 3$. Then $ss's = s'ss'$ and we denote this element by $w(s, s')$. It is 2-adjacent to s, s' and for every $w \in W$ the set $\{sw, s'w, w(s, s')w\}$ is a 2-clique.

Lemma 2.5. *Every maximal* 2*-clique of* $\mathrm{C}(W, S)$ *is one of the* 2*-cliques described above.*

Remark 2.6. The n-dimensional hypercube graph contains only maximal 2-cliques of the first and second types if $n \geq 4$ [3]. In the case when $n = 3$, there are precisely two maximal 2-cliques of the second type and 2-cliques of the first type are not maximal.

To prove Lemma 2.5 we use the following properties of Coxeter systems:

(P1) for every $w \in W$ there is a subset $S_w \subset S$ such that every reduced expression of w is formed by all elements of S_w,

(P2) the group W cannot be generated by a proper subset of S.

Lemma 2.7. *If* $u \in W \setminus S$ *is* 2*-adjacent to three distinct* $s, s', s'' \in S$ *then* s, s', s'' *are mutually commuting and* $u = ss's''$.

Proof. Since u is 2-adjacent to s, s', s'' and $u \notin S$, there are three reduced expressions

$$u = s_1 s_2 s, \quad u = s'_1 s'_2 s', \quad u = s''_1 s''_2 s'',$$

where $s_1, s_2, s'_1, s'_2, s''_1, s''_2 \in S$. By (P1), we have $S_u = \{s, s', s''\}$ and

$$\{s_1, s_2\} = \{s', s''\}, \ \{s'_1, s'_2\} = \{s, s''\}, \ \{s''_1, s''_2\} = \{s, s'\}.$$

Thus there are the following possibilities for the first and second expressions:

(1) $u = s''s's = s''ss'$,

(2) $u = s''s's = ss''s'$,

(3) $u = s's''s = s''ss'$,

(4) $u = s's''s = ss''s'$.

Case (1). The involutions s, s' are commuting and the third expression is

$$u = ss's'' = s'ss''. \tag{2.1}$$

Then $s''s's = u = s'ss''$ and $s's''s's = ss''$. We apply the exchange condition to $w = s''s's$ and get the following three possibilities:

- $s's = ss''$,
- $s''s = ss''$,
- $s''s' = ss''$.

The first and third contradict (P2). So, s and s'' are commuting. Similarly, the equality $s''s's = u = ss's''$ shows that $ss''s's = s's''$. Using the above arguments we establish that s' and s'' are commuting.

Case (2). The equality $s''s's = ss''s'$ implies that $s's = s''ss''s'$. As in the previous case, we show that s and s' are commuting. Then the third expression is (2.1) which implies that $ss''s' = u = ss's''$ and s', s'' are commuting. The equality

$$s''ss' = s''s's = u = ss''s'$$

guarantees that s and s'' are commuting.

Case (3). We have $s's''s = s''ss'$ and $s''s's''s = ss'$. As above, this means that s, s' are commuting and the third expression is (2.1). Then $s's''s = u = s'ss''$ and s, s'' are commuting. The equality

$$s's''s = u = s''ss' = s''s's$$

shows that s' and s'' are commuting.

Case (4). Since $s's''s = ss''s'$, we have $s''s = s'ss''s'$ and $ss's''s = s''s'$. By the standard arguments, s'' is commuting with both s and s'. Then

$$s''ss' = ss''s' = u = s's''s = s''s's$$

which implies that s and s' are commuting.

□

Remark 2.8. If s, s', s'' are distinct elements of S then each of the equalities

$$s''s's = ss''s', \quad s's''s = s''ss', \quad s's''s = ss''s'$$

implies that s and s' are commuting; moreover, the third equality guarantees that s, s', s'' are mutually commuting. See the cases (2)–(4) in the proof of Lemma 2.7.

Lemma 2.7 shows that for any three distinct mutually commuting $s, s', s'' \in S$ the 2-clique formed by s, s', s'' and $ss's''$ is maximal. Therefore, every 2-clique of the second type is maximal.

Lemma 2.9. *If $u \in W \setminus S$ is 2-adjacent to distinct $s, s' \in S$ then one of the following possibilities is realized:*

- $m(s, s') = 3$ *and* $u = w(s, s')$,
- s, s' *are commuting and* $u = s''s's$ *for a certain* $s'' \in S$.

Proof. Since u is 2-adjacent to s, s' and $u \notin S$, there are two reduced expressions

$$u = s_1 s_2 s \text{ and } u = s'_1 s'_2 s',$$

where $s_1, s_2, s'_1, s'_2 \in S$. By (P1), we have $\{s, s_1, s_2\} = S_u = \{s', s'_1, s'_2\}$. If $|S_u| = 2$ then $S_u = \{s, s'\}$ and $u = ss's = s'ss'$ which implies that $m(s, s') = 3$, i.e. the first possibility is realized.

If $|S_u| = 3$ then $S_u = \{s, s', s''\}$ and, as in the proof of Lemma 2.7, we have the following possibilities for the above expressions:

(1) $u = s''s's = s''ss'$,

(2) $u = s''s's = ss''s'$,

(3) $u = s's''s = s''ss'$,

(4) $u = s's''s = ss''s'$.

It is clear that s and s' are commuting in the case (1). By Remark 2.8, the same holds for the cases (2) – (4) and s, s', s'' are mutually commuting in the case (4). So, we get the second possibility. □

By Lemma 2.9, for any $s, s' \in S$ satisfying $m(s, s') = 3$ the 2-clique formed by s, s' and $w(s, s')$ is maximal. Thus every 2-clique of the third type is maximal.

Proof of Lemma 2.5. Let C be a maximal 2-clique. For any distinct $u, u' \in C$ there exist $w \in W$ and $s, s' \in S$ such that $u = sw$ and $u' = s'w$. The maximal 2-clique Cw^{-1} contains s and s'. Thus we can suppose that C contains at least two distinct elements of S. Let s and s' be elements of S belonging to C. Suppose that $C \neq S$, i.e. there is $u \in C \setminus S$.

If there is a third element $s'' \in S$ contained in C then, by Lemma 2.7, s, s', s'' are mutually commuting and C is the 2-clique of the second type formed by s, s', s'' and $u = ss's''$. In the case when C contains precisely two elements of S, Lemma 2.9 shows that $m(s, s') = 3$ and $C = \{s, s', w(s, s')\}$ or s, s' are commuting and $u = s''s's$ for a certain $s'' \in S$. The latter means that the maximal 2-clique $Cs's$ contains s, s', s'', i.e. it coincides with S or $\{s, s', s'', ss's''\}$. Then C is a 2-clique of the first type or the second type. □

3 Proof of Theorem 1.1

We consider the case when $i = j = 1$. Let $f : W_1 \to W_1$ be an automorphism of Γ_1. Then f preserves the family of maximal cliques of Γ_1. Every maximal clique of Γ_1 is a maximal 2-clique of $\mathrm{C}(W, S)$ contained in W_1. By Lemma 2.5, there are precisely three types of such subsets. They contain $|S|$ vertices, 4 vertices and 3 vertices, respectively. The condition $|S| \geq 5$ guarantees that f preserves the types of maximal cliques.

If $w \in W_2$ then Sw is a maximal clique of Γ_1 and $f(Sw) = Sw'$ for a certain $w' \in W_2$. We set $f(w) := w'$ and get a bijective transformation of W.

If $w, v \in W$ are adjacent vertices of the Cayley graph then one of these vertices belongs to W_1 and the other is an element of W_2. Suppose that $v \in W_1$ and $w \in W_2$. Then $v \in Sw$

and $f(v) \in f(Sw) = Sf(w)$ which implies that $f(v)$ and $f(w)$ are adjacent vertices of the Cayley graph. The apply the same arguments to f^{-1} and establish that f is an automorphism of $\mathrm{C}(W,S)$.

The uniqueness of such extension follows from the fact that w is the unique vertex of the Cayley graph adjacent to all vertices from Sw (Remark 2.2).

Remark 3.1. A similar idea was exploited in [4, Section 4.8] for an alternative proof of Cooperstein–Kasikova–Shult's characterization of apartments in half-spin Grassmannians [2].

Remark 3.2. If $\mathrm{C}(W,S)$ is H_4 then there are automorphisms of $\Gamma_i = \frac{1}{2}H_4$ which change the types of 2-cliques contained in W_i. Such automorphisms are not extendable to automorphisms of H_4. Similarly, if (W,S) is the direct product of $\mathsf{I}_2(3)$ and the group spanned by an involution then every maximal 2-clique of $\mathrm{C}(W,S)$ is of the first or of the third type and there are automorphisms of Γ_i changing the types of 2-cliques contained in W_i.

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Involutes of polygons of constant width in Minkowski planes

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Abstract

Consider a convex polygon P in the plane, and denote by U a homothetical copy of the vector sum of P and $-P$. Then the polygon U, as unit ball, induces a norm such that, with respect to this norm, P has constant Minkowskian width. We define notions like Minkowskian curvature, evolutes and involutes for polygons of constant U-width, and we prove that many properties of the smooth case, which is already completely studied, are preserved. The iteration of involutes generates a pair of sequences of polygons of constant width with respect to the Minkowski norm and its dual norm, respectively. We prove that these sequences are converging to symmetric polygons with the same center, which can be regarded as a central point of the polygon P.

Keywords: Area evolute, Barbier's theorem, center symmetry set, curvature, curves of constant width, discrete differential geometry, evolutes, Minkowski geometry, normed plane, equidistants, involutes, support function, width function.

Math. Subj. Class.: 52A10, 52A21, 53A15, 53A40

1 Introduction

A *Minkowski* or *normed plane* is a 2-dimensional vector space with a norm. This norm is induced by its *unit ball* U, which is a compact, convex set centered at the origin (or, shortly, *centered*). Thus, we write $(\mathbb{R}^2, U)$ for a Minkowski plane with unit ball U, whose boundary is the *unit circle* of $(\mathbb{R}^2, U)$. The geometry of normed planes and spaces, usually

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called *Minkowski Geometry* (see [21], [14], and [13]), is strongly related to and influenced by the fields of Convexity, Banach Space Theory, Finsler Geometry and, more recently, Discrete and Computational Geometry. The present paper can be considered as one of the possibly first contributions to Discrete Differential Geometry in the spirit of Minkowski Geometry. The study of special types of curves in Minkowski planes is a promising subject (see the survey [15]), and the particular case of curves of constant Minkowskian width has been studied for a long time (see [3], [4], [11], and § 2 of [13]). A curve γ has constant Minkowskian width with respect to the unit ball U or, shortly, *constant U-width*, if $h(\gamma) + h(-\gamma)$ is constant with respect to the norm induced by U, where $h(\gamma)$ denotes the support function of γ. Another concept from the classical theory of planar curves important for our paper is that of *involutes and evolutes*; see, e.g., Chapter 5 of [8] and, respectively, [9]. For natural generalizations of involutes, which also might be extended from the Euclidean case to normed planes, we refer to [18] and [2]. And in [20] it is shown how the concept of evolutes and involutes can help to construct curves of constant width in the Euclidean plane.

In this paper, we consider convex polygons P of constant Minkowskian width in a normed plane, for short calling them *CW-polygons*. If P is a CW polygon, then the unit ball U is necessarily a centered polygon whose sides and diagonals are suitably parallel to corresponding sides and diagonals of P (sometimes with diagonals suitably meaning also sides; see §§ 2.1 below). If, in particular, U is homothetic to $P + (-P)$, then, and only then, P is of constant U-width in the Minkowski plane induced by U.

There are many results concerning *smooth CW curves* in normed planes: Barbier's theorem fixing their circumference only by the diameter of the curve (cf. [16] and [12]); relations between curvature, evolutes, involutes, and equidistants (see [19] and, for applications of Minkowskian evolutes in computer graphics, [1]); mixed areas, and the relation between the area and length of a CW curve cut off along a diameter (see [3], (2.1)). In this paper we prove corresponding results for *CW polygons*. We note that our results are direct discretizations of the corresponding results for the smooth case, where the derivatives and integrals are replaced by differences and sums. It is meant in this sense that the results of this paper can be considered as one of the first contributions to Discrete Differential Geometry in the framework of normed planes.

Among the U-equidistants of a smooth CW curve γ, there is a particular one called *central equidistant*. The central equidistant of γ coincides with its *area evolute*, while the evolute of γ coincides with its *center symmetry set* (see [6] and [7]). We show that for a CW polygon P the same results hold: The central equidistant M coincides with the area evolute, and the evolute E coincides with the central symmetry set (see [5]). Since the equidistants of P are the involutes of E, we shall choose the central equidistant as a representative of them, and we write $M = Inv(E)$.

For a Minkowski plane whose unit ball U is a centered convex (2n)-gon, the *dual unit ball* V is also a centered convex (2n)-gon with diagonals parallel to the sides of U, and the sides parallel to diagonals of U. As in the smooth case (cf. [6]), the involutes of the central equidistant of P form a one-parameter family of polygons having constant V-width. This one-parameter family consists of the V-equidistants of any of its members, and we shall choose the central equidistant N as its representative. Thus we write $N = Inv(M)$. In [6] it is proved that, for smooth curves, the analogous involute N is contained in the region bounded by M and has smaller or equal signed area. In this paper we prove the corresponding fact for polygons, namely, that N is contained in the region bounded by M

and the signed area of N is not larger than the signed area of M.

What happens if we iterate the involutes? Let $N(0) = E$, $M(0) = M$, $N(1) = N$ and define $M(k) = Inv(N(k))$, $N(k+1) = Inv(M(k))$. Then we obtain two sequences $M(k)$ and $N(k)$, the first being of constant U-width and the second of constant V-width. Moreover, we have

$$\overline{N(0)} \supset \overline{M(0)} \supset \overline{N(1)} \supset \overline{M(1)} \supset \ldots,$$

where $\overline{R}$ denotes the closure of the region bounded by R. Denoting by $O = O(P)$ the intersection of all these sets, we shall prove that O is in fact a single point. Another form of describing the convergence of $M(k)$ and $N(k)$ to O is as follows: For fixed c and d, consider the sequences $M(k) + cU$ of polygons of constant U-width, and the sequences $N(k) + dV$ of polygons of constant V-width. Then these sequences are converging to $O + cU$ and $O + dV$, respectively, which are U- and V-balls centered at O. For smooth curves the analogous results were proved in [6].

Our paper is organized as follows: In Section 2 we describe geometrically the unit ball of a Minkowski plane for which a given convex polygon has constant Minkowskian width. In Section 3, we define Minkowskian curvature, evolutes and involutes for CW polygons and prove many properties of them. In Section 4 we consider the involute of the central equidistant, and in Section 5 we prove that the involutes iterates are converging to a single point.

2 Polygonal Minkowskian balls, their duals, and constant Minkowskian width

Since faces and also width functions of convex sets behave additively under (vector or) Minkowski addition, it is clear that a polygon P is of constant Minkowskian width if and only if $P + (-P)$ is a homothetical copy of the unit ball U of the respective normed plane; see, e.g., §§ 2.3 of [13]. If, moreover, the homothety of U and $P + (-P)$ is only possible when P itself is already centrally symmetric, then the only sets of constant U-width are the balls of that norm; cf., e.g., [22]. In the following we will have a closer look at various geometric relations between polygons P of constant U-width and the unit ball U, since we need them later.

Thus, let P be an arbitrary planar convex polygon. By an abuse of notation, we shall denote by the same letter P also the set of vertices of the polygon, the closed polygonal arc formed by the union of its sides, and the convex region bounded by P.

2.1 A centered polygon with parallel sides and diagonals

Assume that $P = \{P_1, ..., P_{2n}\}$ is a planar convex polygon with parallel opposite sides, i.e., the segments P_iP_{i+1} and $P_{i+n}P_{i+n+1}$, $1 \leq i \leq n$, are parallel.

Lemma 2.1. *Fix an origin Z and take U_1 such that $U_1 - Z = \frac{1}{2a}(P_1 - P_{1+n})$, for some $a > 0$. Consider the polygon U whose vertices are*

$$U_i = Z + \frac{1}{2a}(P_i - P_{i+n}), \tag{2.1}$$

$1 \leq i \leq 2n$. Then U is convex, symmetric with respect to Z, $U_{i+1} - U_i \parallel P_{i+1} - P_i$ and $U_i - Z \parallel P_i - P_{i+n}$ for $1 \leq i \leq n$ (see Figure 1). Moreover, U is the unique polygon with these properties.

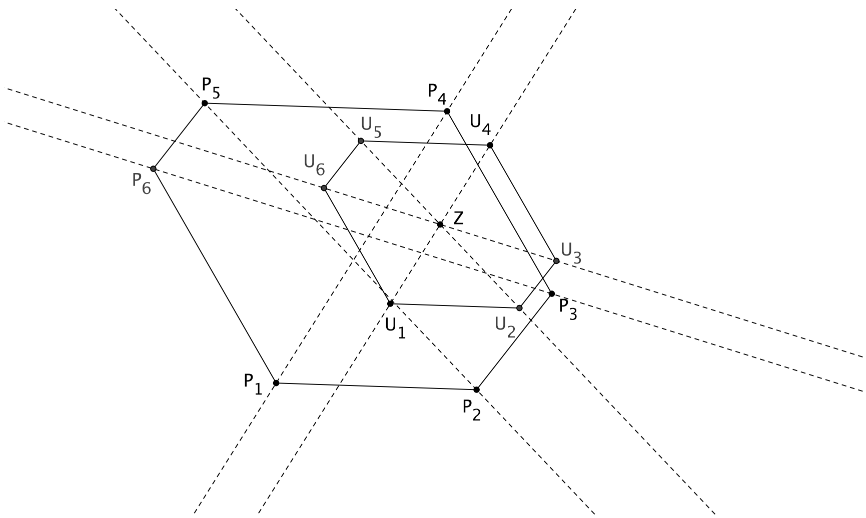

Figure 1: A hexagon P with parallel opposite sides and the corresponding homothet U of $P + (-P)$.

Proof. It is clear that U is symmetric with respect to Z, $U_{i+1} - U_i \parallel P_{i+1} - P_i$ and $U_i - Z \parallel P_i - P_{i+n}$ for $1 \leq i \leq n$. Moreover $U_{i+1} - U_i$ has the same orientation as $P_{i+1} - P_i$, which implies that U is convex.

To prove the uniqueness of U, observe that the point U_2 is obtained as the intersection of the lines parallel to P_1P_2 through U_1 and parallel to P_2P_{2+n} through Z. The points $U_3, ..., U_n$ are obtained inductively in a similar way, while $U_{n+1}, .., U_{2n}$ are reflections of $U_1, ...U_n$ with respect to Z. □

Consider now a convex polygon $\tilde{P} = \{\tilde{P}_1, ..., \tilde{P}_k\}$ that has not necessarily all opposite sides parallel. Suppose that exactly $0 \leq j \leq \frac{k}{2}$ pairs are parallel. Our next lemma shows that the list of vertices of this polygon can be re-written as $P = \{P_1, P_2, .., P_{2n}\}$, $n = k-j$, with "parallel opposite sides" in a broader sense.

Lemma 2.2. *We may re-write the list of vertices of $\tilde{P}$ as $\{P_1, P_2, .., P_{2n}\}$ such that, for each $1 \leq i \leq n$, P_iP_{i+1} is parallel to $P_{i+n}P_{i+n+1}$ or else one of these sides, say $P_{i+n}P_{i+n+1}$, degenerates to a point, in which case the other side P_iP_{i+1} is not degenerated and the line through $P_{i+n} = P_{i+n+1}$ parallel to P_iP_{i+1} is outside P (see Figure 2).*

Proof. The polygon $\tilde{P} = \{\tilde{P}_1, ..., \tilde{P}_k\}$ defines exactly $n = k - j$ directions $\theta_1, ..., \theta_n$, in increasing order, in the plane. We may assume that $\tilde{P}_1\tilde{P}_2$ is in direction θ_1 and define $P_1 = \tilde{P}_1$, $P_2 = \tilde{P}_2$. For the induction step write $P_i = \tilde{P}_l$. If $P_i\tilde{P}_{l+1}$ is in direction θ_i, define $P_{i+1} = \tilde{P}_{l+1}$, otherwise define $P_{i+1} = \tilde{P}_l$. It is now easy to verify that the polygon $P = \{P_1, P_2, .., P_{2n}\}$ satisfies the properties of the lemma. □

The construction of Lemma 2.1 can be applied to the polygon P obtained in Lemma 2.2 (see Figure 2). If, for example, P is a triangle, then $P + (-P)$ is an affinely regular hexagon (see Figure 3). From now on, we shall assume that Z coincides with the origin of $\mathbb{R}^2$ and that $P = \{P_1, ..., P_{2n}\}$, with P_iP_{i+1} parallel to U_iU_{i+1}.

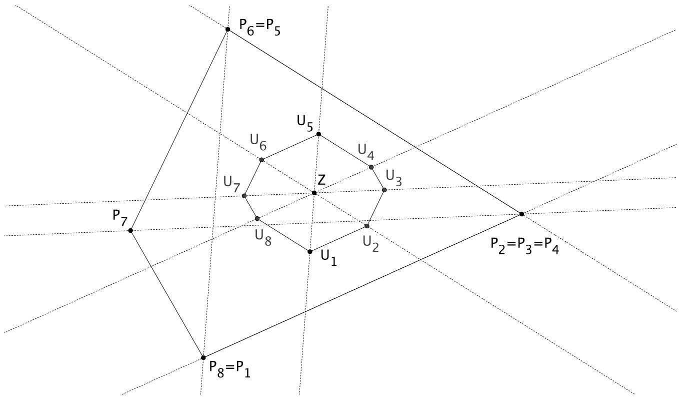

Figure 2: A quadrangle and the corresponding symmetric octagon.

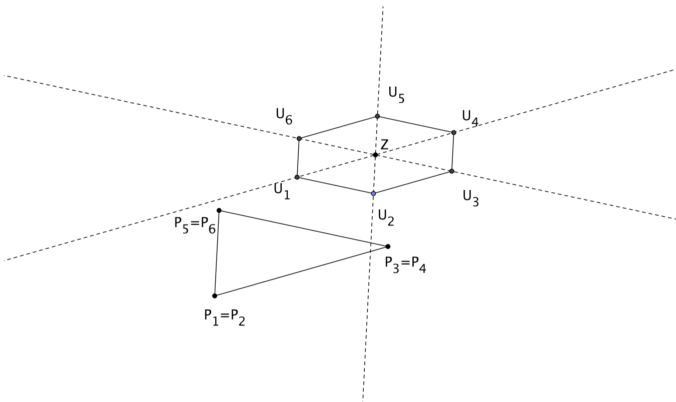

Figure 3: When P is a triangle of constant U-width, then U is an affinely regular hexagon.

2.2 The dual Minkowskian ball

Now we introduce the type of duality which is very useful for our investigations. Let $(\mathbb{R}^2)^*$ denote the space of linear functionals in $\mathbb{R}^2$. The dual norm in $(\mathbb{R}^2)^*$ is defined as

$$||f|| = \sup\{f(u), u \in U\}.$$

We shall identify $(\mathbb{R}^2)^*$ with $\mathbb{R}^2$ by $f(\cdot) = [\cdot, v]$, where $[\cdot,\cdot]$ denotes the determinant of a pair of planar vectors. Under this identification, the dual norm in $\mathbb{R}^2$ is given by

$$||v|| = \sup\{[u,v], u \in U\}.$$

We shall construct below a centered polygon V such that, for v in any side of V, we have $||v|| = 1$. Such a polygon defines a Minkowski norm equivalent to the dual norm of U.

Now assume that the unit ball U is a centered polygon with vertices $\{U_1, ..., U_{2n}\}$, $U_{i+n} = -U_i,\ 1 \le i \le n$. Define the polygon V with vertices

$$V_{i+\frac{1}{2}} = \frac{U_{i+1} - U_i}{[U_i, U_{i+1}]}.$$

Observe that $V_{i+n+\frac{1}{2}} = -V_{i+\frac{1}{2}}$, i.e., V is centered. Now $[V_{i+\frac{1}{2}} - V_{i-\frac{1}{2}}, U_i] = 0$, which implies that $V_{i+\frac{1}{2}} - V_{i-\frac{1}{2}} = -aU_i$. Multiplying both sides by $V_{i+\frac{1}{2}}$ we obtain

$$U_i = -\frac{V_{i+\frac{1}{2}} - V_{i-\frac{1}{2}}}{[V_{i-\frac{1}{2}}, V_{i+\frac{1}{2}}]},$$

for $1 \leq i \leq 2n$.

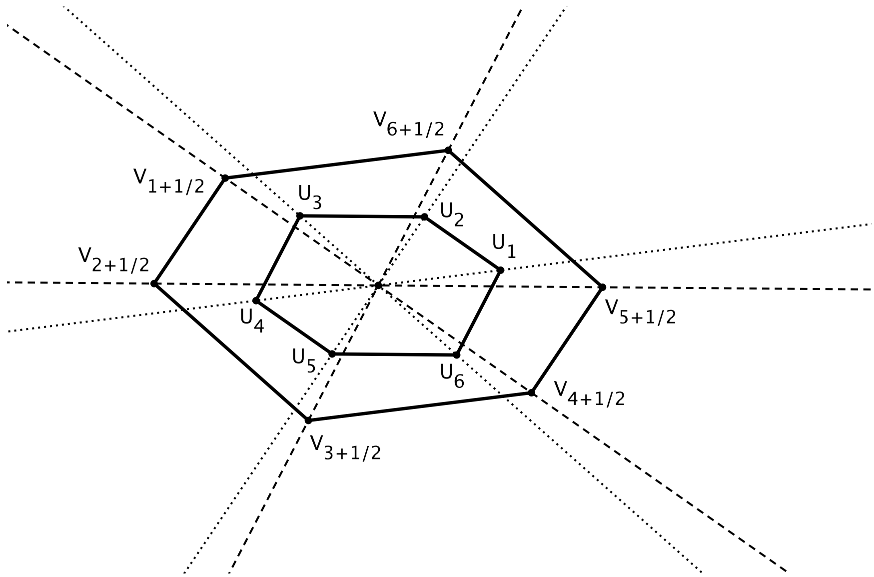

Figure 4: The centered hexagon U and its dual V.

Lemma 2.3. *The polygon V is the dual unit ball.*

Proof. We have that, for $1 \leq i \leq 2n$,

$$[tU_i + (1-t)U_{i+1}, V_{i+\frac{1}{2}}] = 1, \tag{2.2}$$

for any $t \in \mathbb{R}$ and for $j \notin \{i, i+1\}$, $[U_j, V_{i+\frac{1}{2}}] \leq 1$. This implies that the vertex $V_{i+\frac{1}{2}}$ is from the dual unit circle. Moreover,

$$[U_i, tV_{i-\frac{1}{2}} + (1-t)V_{i+\frac{1}{2}}] = 1, \tag{2.3}$$

and for $j \neq i$ we have $[U_j, tV_{i-\frac{1}{2}} + (1-t)V_{i+\frac{1}{2}}] \leq 1$, which implies that also the side $tV_{i-\frac{1}{2}} + (1-t)V_{i+\frac{1}{2}}$ is from the dual unit circle. □

2.3 Polygons of constant Minkowskian width

Consider a Minkowski plane $(\mathbb{R}^2, U)$, and let P be a convex curve. For f in the dual unit ball, the *support function* $h(P)(f)$ of P at f is defined as

$$h(P)(f) = \sup\{f(p), p \in P\}. \tag{2.4}$$

The *width* of P in the direction f is defined as $w(P)(f) = h(P)(f) + h(P)(-f)$. We say that P is of *constant Minkowskian width* if $w(P)(f)$ does not depend on f.

Consider now a Minkowski plane whose unit ball U is a centered polygon, and let P be a polygon with parallel corresponding sides and diagonals.

Lemma 2.4. *In the Minkowski plane $(\mathbb{R}^2, U)$, P has constant U-width.*

Proof. By Lemma 2.1, we have that $P_i - P_{i+n} = a(U_i - U_{i+n})$, for some constant a. Since

$$w(P)(V_{i+\frac{1}{2}}) = h(P)(V_{i+\frac{1}{2}}) + h(P)(-V_{i+\frac{1}{2}}) = [P_i - P_{i+n}, V_{i+\frac{1}{2}}],$$

we obtain

$$w(P)(V_{i+\frac{1}{2}}) = 2a,$$

$1 \leq i \leq 2n$, thus proving the lemma. □

Our next corollary says that in fact U is homothetic to the Minkowski sum $P + (-P)$ (see [21], Th. 4.2.3).

Corollary 2.5. *Let P be a convex planar polygon and let U be as in Lemma 2.1. Then U is homothetic to $P + (-P)$.*

Proof. We have that $2a = h(P) + h(-P) = h(P + (-P)) = h(2aU)$, which implies that $P + (-P)$ is homothetic to U. □

Corollary 2.6. *Consider a centered polygon U and a polygon P whose sides are parallel to the corresponding sides of U. The following statements are equivalent:*

1. *P has constant U-width.*
2. *$P + (-P)$ is homothetic to U.*
3. *The corresponding diagonals of U and P are parallel to each other.*
4. *$P_i - P_{i+n} = 2a(U_i - U_{i+n})$, $1 \leq i \leq n$, for some constant a.*

3 Geometric properties of polygons of constant Minkowskian width

Consider a convex polygon $P = \{P_1, ..., P_{2n}\}$ with parallel opposite sides and let $U = \{U_1, ..., U_{2n}\}$ be the symmetric polygon obtained from P by the construction of Lemma 2.1.

3.1 Central Equidistant, *V*-length, and Barbier's theorem

Central equidistant Any equidistant can be written as $P_i(c) = P_i + cU_i$, $1 \leq i \leq 2n$. If we take $c = -a$, we obtain

$$M_i = P_i + \frac{c}{2a}(P_i - P_{i+n}) = \frac{1}{2}(P_i + P_{i+n}), \ 1 \leq i \leq 2n, \tag{3.1}$$

called the *central equidistant* of P. It is characterized by the condition $M_i = M_{i+n}$ (see Figure 5). If we re-scale the one-parameter family of equidistants as

$$P_i(c) = M_i + cU_i, \ 1 \leq i \leq 2n, \tag{3.2}$$

we get that the 0-equidistant is exactly the central equidistant.

A vertex M_i of the central equidistant is called a *cusp* if M_{i-1} and M_{i+1} are in the same half-plane defined by the diagonal at P_i. The central equidistant coincides with the *area evolute* of polygons defined in [5]. There it is proved that it has an odd number of cusps, at least three (see Figures 5 and 7).

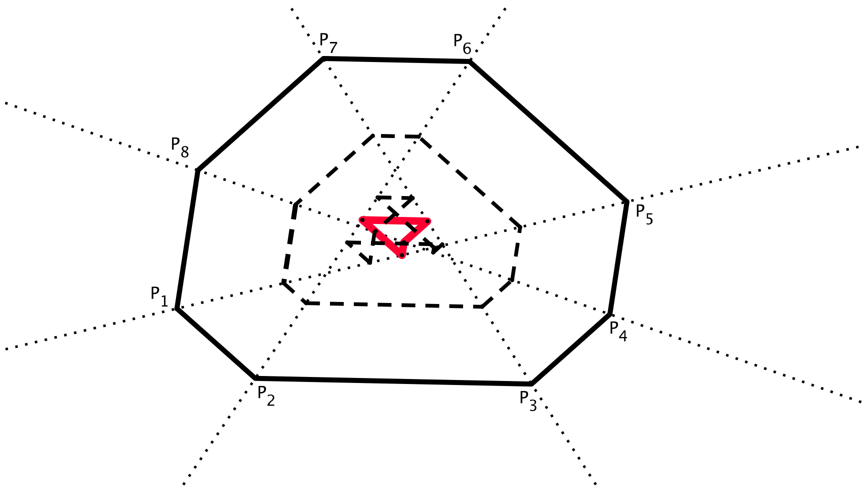

Figure 5: The two traced octagons are ordinary equidistants. The thick quadrangle is the central equidistant.

V-Length Let P be a polygonal arc whose sides are parallel to the corresponding ones of U. More precisely, we shall denote by $\{P_s, ..., P_t\}$ the vertices of P and assume that $P_{i+1} - P_i$ is parallel to $V_{i+\frac{1}{2}}$. We can write

$$P_{i+1} - P_i = \lambda_{i+\frac{1}{2}} V_{i+\frac{1}{2}} \tag{3.3}$$

for some $\lambda_{i+\frac{1}{2}} \geq 0$. Then the V-*length* of the edge $P_i P_{i+1}$ is exactly $\lambda_{i+\frac{1}{2}}$, and we write

$$L_V(P) = \sum_{i=s}^{t-1} \lambda_{i+\frac{1}{2}}. \tag{3.4}$$

Barbier's theorem The classical Theorem of Barbier on curves of constant width in the Euclidean plane says that any such curve of diameter d has circumference $d\pi$. For Minkowski planes, it appears in [16], Th. 6.14(a), and in [12]. We prove here the version of this theorem for polygons.

Define $\alpha_{i+\frac{1}{2}}$, $1 \leq i \leq 2n$, by the equation

$$M_{i+1} - M_i = \alpha_{i+\frac{1}{2}} (U_{i+1} - U_i) = \alpha_{i+\frac{1}{2}} [U_i, U_{i+1}] V_{i+\frac{1}{2}}. \tag{3.5}$$

Proposition 3.1. Let $P(c)$ be defined by equation (3.2). Then the V-length of $P(c)$ is

$$L_V(P) = 2cA(U), \tag{3.6}$$

where $A(U)$ denotes the area of the polygon U.

Proof. The V-length of the polygon $P(c)$ is given by

$$L_V(P(c)) = \sum_{i=1}^{2n} (\alpha_{i+\frac{1}{2}} + c)[U_i, U_{i+1}].$$

Since $\alpha_{i+n+\frac{1}{2}} = -\alpha_{i+\frac{1}{2}}$, we obtain

$$L_V(P(c)) = c \sum_{i=1}^{2n} [U_i, U_{i+1}],$$

which proves the proposition. □

If we admit signed lengths, equation (3.6) holds even for equidistants with cusps. In particular, for $c = 0$ we obtain

$$L_V(M) = 0. \tag{3.7}$$

For smooth closed curves this result was obtained in [19] .

3.2 Curvature and evolutes

Minkowskian normals and evolutes In the smooth case, the Minkowskian normal at a point P is the line $P + sU$, where P and U have parallel tangents (see [19]). The evolute is the envelope of Minkowskian normals. For a polygon P, define the *Minkowskian normal* at a vertex P_i as the line $P_i + sU_i$, $1 \leq i \leq 2n$, and the *evolute* as the polygonal arc whose vertices are the intersections of $P_i + sU_i$ and $P_{i+1} + sU_{i+1}$. These intersections are given by

$$E_{i+\frac{1}{2}} = P_i - \mu_{i+\frac{1}{2}} U_i = P_{i+1} - \mu_{i+\frac{1}{2}} U_{i+1}, \tag{3.8}$$

where $\mu_{i+\frac{1}{2}}$, $1 \leq i \leq 2n$, is defined by

$$P_{i+1} - P_i = \mu_{i+\frac{1}{2}} \left(U_{i+1} - U_i \right). \tag{3.9}$$

Curvature center and radius In [16], three different notions of Minkowskian curvature are defined, where the circular curvature is directly related to evolutes. The circular center E and the corresponding radius of curvature μ are defined by the condition that $E + \mu U$ has a 3-order contact with the curve at a given point (see [19]).

For polygons, we define the *center of curvature* $E_{i+\frac{1}{2}}$ and the *curvature radius* $\mu_{i+\frac{1}{2}}$ of the side P_iP_{i+1} by the condition that the $(i + \frac{1}{2})$-side of $E_{i+\frac{1}{2}} + \mu_{i+\frac{1}{2}} U$ matches exactly P_iP_{i+1} (see Figure 6). Thus we get equations (3.8) and (3.9). From equations (3.3) and (3.9) we obtain that the curvature radius of the side P_iP_{i+1} is also given by

$$\mu_{i+\frac{1}{2}} = \frac{\lambda_{i+\frac{1}{2}}}{[U_i, U_{i+1}]}. \tag{3.10}$$

A vertex $E_{i+\frac{1}{2}}$ is a cusp of the evolute if the vertices $E_{i-\frac{1}{2}}$ and $E_{i+\frac{3}{2}}$ are in the same half-plane defined by the parallel to P_iP_{i+1} through $E_{i+\frac{1}{2}}$. The evolute of a CW polygon coincides with its *center symmetry set* as defined in [5], where it is proved that it coincides with the union of cusps of all equidistants of P. It is also proved in [5] that the number of cusps of the evolute is odd and at least the number of cusps of the central equidistant (see Figure 7).

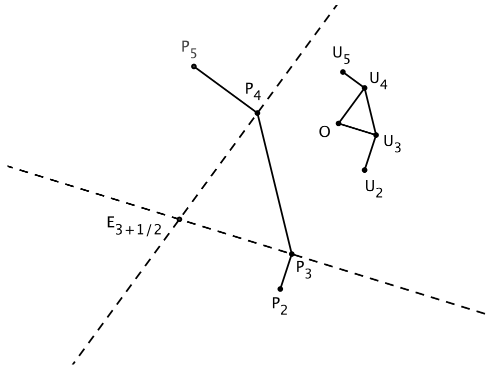

Figure 6: The center of curvature of the side P_3P_4.

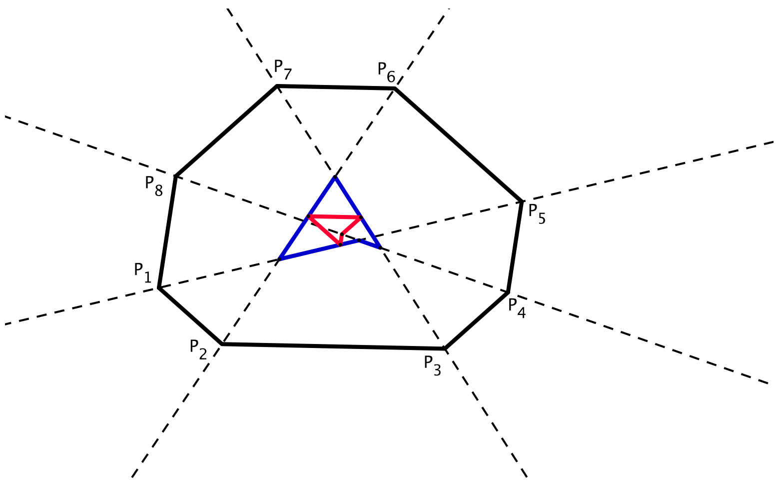

Figure 7: The inner polygonal arc is the central equidistant M of P, and the outer polygonal arc is its evolute E.

Sum of curvature radii Consider equation (3.9) for two opposite sides, and sum up to obtain, for $1 \leq i \leq n$,

$$P_{i+1} - P_{i+n+1} + P_{i+n} - P_i = (\mu_{i+\frac{1}{2}} + \mu_{i+n+\frac{1}{2}})(U_{i+1} - U_i).$$

Since P has constant Minkowskian width,

$$2c(U_{i+1} - U_i) = (\mu_{i+\frac{1}{2}} + \mu_{i+n+\frac{1}{2}})(U_{i+1} - U_i).$$

We conclude that

$$\mu_{i+\frac{1}{2}} + \mu_{i+n+\frac{1}{2}} = 2c. \tag{3.11}$$

The corresponding result for smooth curves is given in [16], Th. 6.14.(c).

Involutes and equidistants Consider the one-parameter family of equidistants given by equation (3.2). The radius of curvature of $P_i(c)P_{i+1}(c)$ is the radius of curvature of

M_iM_{i+1} plus c. Thus, for $1 \le i \le 2n$,

$$E_{i+\frac{1}{2}}(c) = M_i + cU_i - \left(\mu_{i+\frac{1}{2}} + c\right) U_i = E_{i+\frac{1}{2}}. \tag{3.12}$$

We conclude that the evolute of any equidistant of P is equal to the evolute of P. Reciprocally, any polygonal arc whose evolute is equal to $E(P)$ is an equidistant of P. We define an *involute* of E as any polygonal arc whose evolute is E. Thus the involutes of E are the equidistants of P.

3.3 The signed area of the central equidistant

For a simple closed curve P, denote by $A(P)$ the area of the region bounded by P. Given two closed curves P and Q, their mixed area is defined by the equation

$$A(P + tQ) = A(P) + 2tA(P,Q) + t^2 A(Q),$$

(see [17, §§ 5.1]). The Minkowski inequality says that $A(P,Q)^2 \ge A(P)A(Q)$. The next lemma is well-known, see [10, §§ 6.3].

Lemma 3.2. *Take P and Q as convex polygons with k parallel corresponding sides. The mixed area of P and Q is given by*

$$A(P,Q) = \frac{1}{2}\sum_{i=1}^{k}[Q_i, P_{i+1} - P_i] = \frac{1}{2}\sum_{i=1}^{k}[P_{i+1}, Q_{i+1} - Q_i].$$

Assume that P is a closed convex polygon whose sides are parallel to the sides of the centered polygon U, and take $Q = U$ in Lemma 3.2. We obtain

$$A(P,U) = \frac{1}{2}\sum_{i=1}^{2n}[U_i, P_{i+1} - P_i] = \frac{1}{2}\sum_{i=1}^{2n}\lambda_{i+\frac{1}{2}} = \frac{1}{2}L_V(P),$$

where we have used (3.3) and (3.4). Moreover, the Minkowski inequality becomes

$$L_V^2(P) \ge 4A(U)A(P). \tag{3.13}$$

Lemma 3.3. *Let M be the central equidistant of a CW-polygon P. Then the mixed area $A(M,M)$ is non-positive.*

Proof. Let $P(c)$ be defined by equation (3.2). Then

$$A(P(c), P(c)) = A(M,M) + 2cA(M,U) + c^2 A(U,U).$$

Now equation (3.7) says that $A(M,U) = 0$. Moreover, the isoperimetric inequality (3.13) for curves of constant width says that

$$A(P) \le c^2 A(U).$$

We conclude that

$$A(M,M) \le 0.$$

□

Define the *signed area* of M as $SA(M) = -A(M,M)$. In general, the signed area is a sum of positive and negative areas, but when M is a simple curve, it coincides with the area bounded by M.

3.4 Relation between length and area of a half polygon

Define β_i by

$$\beta_i = \frac{1}{2}\sum_{j=i}^{n+i-1} \alpha_{j+\frac{1}{2}}[U_j, U_{j+1}]. \tag{3.14}$$

Observe that $\beta_{i+n} = -\beta_i$, $1 \le i \le n$, and

$$\beta_{i+1} - \beta_i = -\alpha_{i+\frac{1}{2}}[U_i, U_{i+1}]. \tag{3.15}$$

Denote by $A_1(i,c)$ and $A_2(i,c)$ the areas of the polygons with vertices $\{P_i, P_{i+1}, ..., P_{i+n}\}$ and $\{P_{i+n}, P_{i+n+1}, ..., P_i\}$. Observe that these polygons are bounded by P and the diagonal P_iP_{i+n}.

Proposition 3.4. We have that

$$A_1(i,c) - A_2(i,c) = 4c\beta_i,$$

for $1 \le i \le 2n$.

Proof. Lemma 4.1. of [5] says that

$$A_1(i,c) - A_2(i,c) = -2\sum_{j=i}^{i+n-1} [M_{j+1} - M_j, cU_j]$$

$$= -2c\sum_{j=i}^{i+n-1} [\alpha_{j+\frac{1}{2}}[U_j, U_{j+1}]V_{j+\frac{1}{2}}, U_j].$$

Thus

$$A_1(i,c) - A_2(i,c) = 2c\sum_{j=i}^{i+n-1} \alpha_{j+\frac{1}{2}}[U_j, U_{j+1}] = 4c\beta_i.$$

□

Denote by $L_V(i,c)$ the V-length of the polygonal arc whose vertices are $\{P_i(c), P_{i+1}(c), ..., P_{i+n}(c)\}$. Then

$$L_V(i,c) = \sum_{j=i}^{i+n-1} (\alpha_{i+\frac{1}{2}} + c)[U_j, U_{j+1}] = 2cA(U) + 2\beta_i. \tag{3.16}$$

Corollary 3.5. *For $1 \le i \le 2n$, the expression $A_1(i,c) - cL_V(i,c)$ is independent of i.*

Proof. By equation (3.16) and Proposition 3.4, we get

$$2cL_V(i,c) - 2A_1(i,c) = 4c^2A(U) + 4c\beta_i - 2A_1(i,c) = 4c^2A(U) - A(P),$$

which proves the corollary. □

The above corollary presents the "polygonal analogue" of a known theorem holding for strictly convex curves (see [4], eq. (2.1)).

4 The involute of the central equidistant

Recall that $P = \{P_1, ..., P_{2n}\}$ is a convex polygon with parallel opposite sides and $U = \{U_1, ..., U_{2n}\}$ is the Minkowski ball obtained from P by the construction of Lemma 2.1. The polygon $V = \{V_1, ..., V_{2n}\}$ represents the dual Minkowski ball (see Lemma 2.3) and $M = \{M_1, ..., M_n\}$ is the central equidistant of P (see equation (3.1)).

4.1 Basic properties of the involute N of M

Define the polygon N by

$$N_{i+\frac{1}{2}} = M_i + \beta_i V_{i+\frac{1}{2}}, \tag{4.1}$$

$1 \le i \le 2n$. Observe that $N_{i+\frac{1}{2}} = N_{i+n+\frac{1}{2}}$. Due to equations (3.5) and (3.15), we can also write

$$N_{i+\frac{1}{2}} = M_{i+1} + \beta_{i+1} V_{i+\frac{1}{2}}. \tag{4.2}$$

Lemma 4.1. *The polygon N has constant V-width, and the evolute of N is M.*

Proof. Since

$$N_{i+\frac{1}{2}} - N_{i-\frac{1}{2}} = \beta_i \left(V_{i+\frac{1}{2}} - V_{i-\frac{1}{2}} \right), \tag{4.3}$$

$1 \le i \le n$, the sides of N are parallel to the sides of V. Moreover, the diagonals of N are zero, so they are multiples of the diagonals of V. We conclude from Corollary 2.6 that N has constant V-width. Finally, from equation (4.1) we conclude that the evolute of N is M. □

The equidistants of N, which are the involutes of M, are curves of constant V-width (see Figure 8). In [5], these polygons were called the Parallel Diagonal Transforms of P.

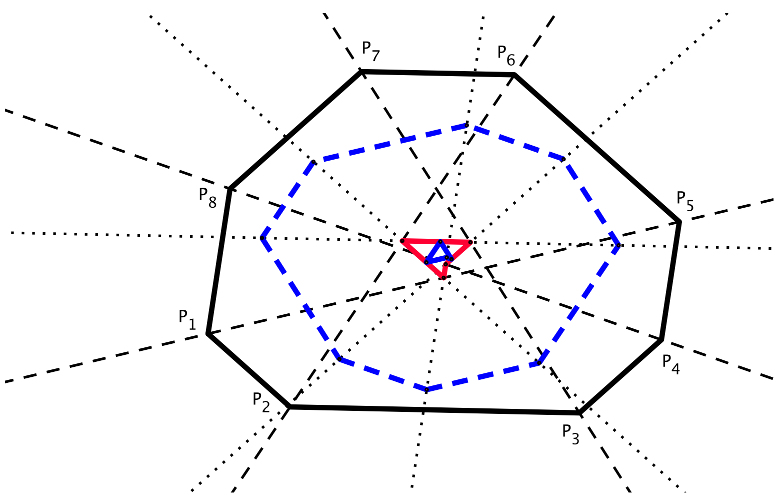

Figure 8: The central equidistant M together with two involutes of M: The inner curve is the central equidistant N, and the traced curve is an ordinary involute.

4.2 The signed area of the involute of the central equidistant

For smooth convex curves of constant Minkowskian width, the signed area of N is not larger than the signed area of M (see [6]). We prove here the corresponding result for polygons.

Proposition 4.2. Denoting by $SA(M)$ and $SA(N)$ the signed areas of M and N, we have

$$SA(M) - SA(N) = \sum_{i=1}^{n} \beta_i^2 \left[V_{i-\frac{1}{2}}, V_{i+\frac{1}{2}}\right].$$

Proof. Observe that

$$[M_i, M_{i+1}] = \left[N_{i+\frac{1}{2}} - \beta_i V_{i+\frac{1}{2}}, \alpha_{i+\frac{1}{2}}(U_{i+1} - U_i)\right] = \alpha_{i+\frac{1}{2}}[N_{i+\frac{1}{2}}, U_{i+1} - U_i] =$$

$$-(\beta_{i+1} - \beta_i)[N_{i+\frac{1}{2}}, V_{i+\frac{1}{2}}], \quad \left[N_{i-\frac{1}{2}}, N_{i+\frac{1}{2}}\right] = \beta_i \left[N_{i+\frac{1}{2}}, V_{i+\frac{1}{2}} - V_{i-\frac{1}{2}}\right],$$

and so

$$-[M_i, M_{i+1}] + \left[N_{i-\frac{1}{2}}, N_{i+\frac{1}{2}}\right] = [N_{i+\frac{1}{2}}, \beta_{i+1} V_{i+\frac{1}{2}} - \beta_i V_{i-\frac{1}{2}}].$$

Thus

$$SA(M) - SA(N) = \sum_{i=1}^{n} -[M_i, M_{i+1}] + \left[N_{i-\frac{1}{2}}, N_{i+\frac{1}{2}}\right] =$$

$$= -\sum_{i=1}^{n} \left[N_{i+\frac{1}{2}} - N_{i-\frac{1}{2}}, \beta_i V_{i-\frac{1}{2}}\right] = \sum_{i=1}^{n} \beta_i^2 \left[V_{i-\frac{1}{2}}, V_{i+\frac{1}{2}}\right],$$

where we have used that the difference

$$[N_{i+\frac{1}{2}}, \beta_{i+1} V_{i+\frac{1}{2}}] - [N_{i-\frac{1}{2}}, \beta_i V_{i-\frac{1}{2}}]$$

is equal to

$$[N_{i+\frac{1}{2}} - N_{i-\frac{1}{2}}, \beta_i V_{i-\frac{1}{2}}] + [N_{i+\frac{1}{2}}, \beta_{i+1} V_{i+\frac{1}{2}} - \beta_i V_{i-\frac{1}{2}}],$$

the discrete version of "integration by parts". □

4.3 The involute is contained in the interior of the central equidistant

We prove now that the region bounded by the central equidistant M contains its involute N. For smooth convex curves, this result was proved in [6].

The exterior of the curve M is defined as the set of points of the plane that can be reached from a point of P by a path that does not cross M. The region $\overline{M}$ bounded by M is the complement of its exterior. It is well known that a point in the exterior of M is the center of exactly one chord of P (see [5]).

Proposition 4.3. The involute N is contained in the region $\overline{M}$ bounded by M.

The proof is based on two lemmas. For a fixed index i, denote by $l(i)$ the line parallel to $P_{i+n} - P_i$ through $N_{i-\frac{1}{2}}$ and $N_{i+\frac{1}{2}}$. Then $l(i)$ divides the interior of P into two regions of areas $B_1 = B_1(i)$ and $B_2 = B_2(i)$, where the second one contains P_i and P_{i+n}.

Lemma 4.4. *We have that* $B_1(i) \geq B_2(i)$, $1 \leq i \leq n$.

Proof. We have that

$$B_1(i) = A_1(i) - (2c\beta_i - \delta_i - \eta_i), \ \ B_2(i) = A_2(i) + (2c\beta_i - \delta_i - \eta_i),$$

where δ_i is the area of the regions outside P and between $l(i)$, P_iP_{i+n} and the support lines of P_iP_{i+1} and $P_{i+n-1}P_{i+n}$, and η_i is the area of the triangle $M_iN_{i+\frac{1}{2}}N_{i-\frac{1}{2}}$ (see Figure 9). Since, by Proposition 3.4, $4c\beta_i = A_1 - A_2$, we conclude that

$$B_1(i) = \frac{A(P)}{2} + \delta_i + \eta_i, \ \ B_2(i) = \frac{A(P)}{2} - \delta_i - \eta_i,$$

which proves the lemma. □

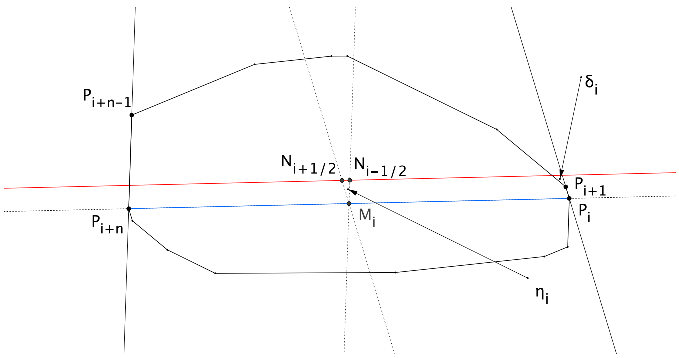

Figure 9: The line through $N_{i+\frac{1}{2}}$ and $N_{i-\frac{1}{2}}$ divides the polygon into two regions of areas B_1 and B_2.

Lemma 4.5. *Choose* C *in the segment* $N_{i-\frac{1}{2}}N_{i+\frac{1}{2}}$. *Then* C *is in the region bounded by* M.

Proof. By an affine transformation of the plane, we may assume that $l(i)$ and M_iC are orthogonal. Consider polar coordinates (r, ϕ) with center C and describe P by $r(\phi)$. Assume that $\phi = 0$ at the line $l(i)$ and that $\phi = -\phi_0$ at P_i. Denote the area of the sector bounded by P and the rays ϕ_1, ϕ_2 by

$$A(\phi_1, \phi_2) = \frac{1}{2} \int_{\phi_1}^{\phi_2} r^2(\phi) d\phi.$$

Consider a line parallel to M_iC and passing through the point Q_0 of P corresponding to $\phi = 0$, and denote by Q_1 and Q_2 its intersection with the rays $\phi = -\phi_0$ and $\phi = \phi_0$, respectively (see Figure 10). By convexity, we have that

$$A(0, \phi_0) \leq A(CQ_0Q_1) = A(CQ_0Q_2) \leq A(-\phi_0, 0).$$

A similar reasoning shows that $A(\pi - \phi_0, \pi) \leq A(\pi, \pi + \phi_0)$. Observe also that, by convexity, $r(\phi_0) \leq r(\phi_0 + \pi)$ and $r(\pi - \phi_0) \leq r(-\phi_0)$.

Now, if $r(\phi + \pi) > r(\phi)$ for any $\phi_0 < \phi < \pi - \phi_0$, we would have $B_1(C) < B_2(C)$, contradicting the previous lemma. We conclude that $r(\phi + \pi) = r(\phi)$ for at least two values of $\phi_0 < \phi < \pi - \phi_0$. Since equality holds also for some $\pi - \phi_0 < \phi < \pi + \phi_0$, there are at least three chords of γ having C as midpoint. Thus C is contained in the region bounded by M. □

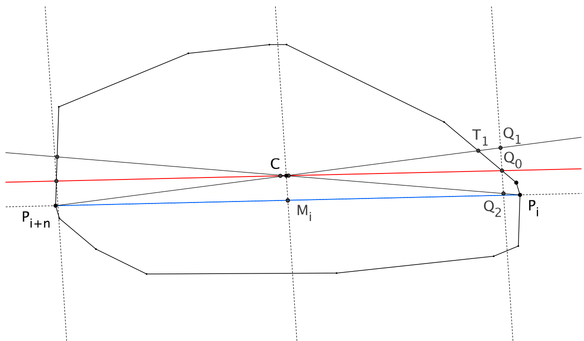

Figure 10: The line parallel to M_iC through Q_0 determines the points Q_1 and Q_2.

We can now complete the proof of Proposition 4.3. In fact, from Lemma 4.5 we have that each side $N_{i-\frac{1}{2}}N_{i+\frac{1}{2}}$ is contained in the region $\overline{M}$ bounded by M. Therefore, no point on the boundary of N can be connected with the boundary of P by a curve that does not intersect M. This implies that the region $\overline{N}$ bounded by N is contained in $\overline{M}$.

5 Iterating involutes

Starting with the central equidistant $M = M(0)$ and its involute $N = N(1)$, we can iterate the involute operation. We obtain two sequences of n-gons $M(k)$ and $N(k)$ defined by $M(k) = \mathcal{I}nv(N(k))$ and $N(k+1) = \mathcal{I}nv(M(k))$. For smooth curves of constant Minkowskian width, it is proved in [6] that these sequences converge to a constant. We prove here the corresponding result for polygons.

From Proposition 4.3, we have

$$\overline{M(0)} \supset \overline{N(1)} \supset \overline{M(1)} \supset ...,$$

and we denote by $O = O(P)$ the intersection of all these sets.

If we represent a polygon by its vertices, we can embed the space $\mathcal{P}_n$ of all n-gons in $(\mathbb{R}^2)^n$. In $\mathcal{P}_n$ we consider the topology induced by $\mathbb{R}^{2n}$.

Theorem 5.1. *The set $O = O(P)$ consists of a unique point, and the polygons $M(k)$ and $N(k)$ are converging to O in $\mathcal{P}_n$.*

We shall call $O = O(P)$ the *central point* of P. A natural question that arises is the following.

Question Is there a direct method to obtain the central point O from the polygon P?

For fixed c and d construct the sequences of convex polygons $P(k,c)$ and $Q(k,d)$ whose vertices are

$$P_i(k) = M_i(k) + cU_i(k), \quad Q_{i+\frac{1}{2}}(k) = N_{i+\frac{1}{2}}(k) + dV_{i+\frac{1}{2}}(k),$$

respectively. The polygons $P(k,c)$ are of constant U-width, while the polygons $Q_{i+\frac{1}{2}}(k,d)$ are of constant V-width. We can re-state Theorem 5.1 as follows:

Theorem 5.2. *The sequences of polygons $P(k,c)$ and $Q(k,d)$ are converging in $\mathcal{P}_{2n}$ to $O + c\partial\mathcal{U}$ and $O + d\partial\mathcal{V}$, respectively.*

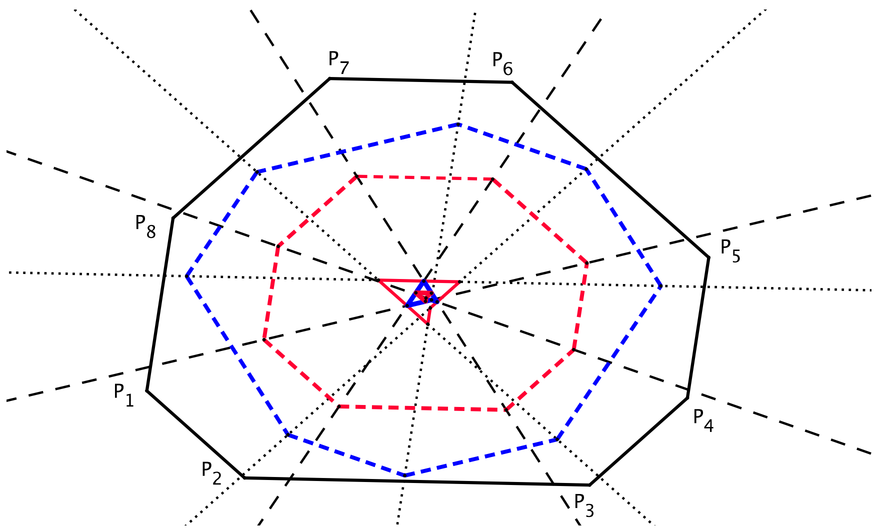

Figure 11: The inner curves are $M = M(0)$, $N = N(1)$ and $M(1)$. One traced curve is an ordinary V-equidistant of N, and the other one is an ordinary U-equidistant of $M(1)$.

We shall prove now Theorem 5.1.

Proof. Denote the signed areas of $M(k)$ and $N(k)$ by $SA(M(k))$ and $SA(N(k))$, respectively. By Section 3.3, $SA(M(k)) \geq 0$, $SA(N(k)) \geq 0$, and Proposition 4.2 implies that

$$SA(M(k)) - SA(N(k+1)) = \sum_{i=1}^{n} \beta_i^2(k)[U_i, U_{i+1}],$$

$$SA(N(k)) - SA(M(k)) = \sum_{i=1}^{n} \alpha_{i+\frac{1}{2}}^2(k)[V_{i-\frac{1}{2}}, V_{i+\frac{1}{2}}],$$

where $\alpha_{i+\frac{1}{2}}(k)$ and $\beta_i(k)$ are defined by

$$M_{i+1}(k) - M_i(k) = \alpha_{i+\frac{1}{2}}(k)(U_{i+1} - U_i),$$
$$N_{i+\frac{1}{2}}(k) - N_{i-\frac{1}{2}}(k) = \beta_i(k)(V_{i+\frac{1}{2}} - V_{i-\frac{1}{2}}).$$

We conclude that

$$\sum_{k=1}^{\infty}\sum_{i=1}^{n}\beta_i^2(k)[U_i,U_{i+1}]+\sum_{k=0}^{\infty}\sum_{i=1}^{n}\alpha_{i+\frac{1}{2}}^2(k)[V_{i-\frac{1}{2}},V_{i+\frac{1}{2}}]\leq SA(M(0)). \tag{5.1}$$

From the above equation, we obtain that the sequences $\alpha_{i+\frac{1}{2}}(k)$ and $\beta_i(k)$ are converging to 0 in $\mathbb{R}^n$. So the diameters of $M(k)$ and $N(k)$ are converging to zero, and thus O is in fact a set consisting of a unique point. □

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Two-arc-transitive two-valent digraphs of certain orders

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Abstract

The topic of this paper is digraphs of in-valence and out-valence 2 that admit a 2-arc-transitive group of automorphisms. We classify such digraphs that satisfy certain additional conditions on their order. In particular, a classification of those with order kp or kp^2 where $k \leq 14$ and p is a prime can be deduced from the results of this paper.

Keywords: Graph, digraph, arc-transitive, order.

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1 Introduction

This paper is about finite connected arc-transitive digraphs of in- and out-valence 2 the order of which has a specific prime factorisation. We refer the reader to Section 2.1 for exact definitions of notions such as *digraph*, *arc-transitive*, *valence* etc. To simplify exposition, we tacitly assume throughout the paper (even where not stated explicitly) that all digraphs are finite and connected.

Studying arc-transitive graphs and digraphs of orders with a specific prime factorisation has a long history and has become increasingly popular in the last decade or two. For example, arc-transitive graphs and digraphs of order p or $2p$, where p is a prime, were classified in [3] and [4], respectively; later, using the classification of finite simple groups, all arc-transitive graphs and digraphs of order a product of two distinct primes were characterised

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in [36], and independently in [26], and those that are 2-arc-transitive were determined in [23].

Once the prime factorisation of the order becomes more complex, results of this type become considerably more complicated (see [38] for an illustration of the difficulties that can arise when the order is a product of three distinct primes). However, when one fixes the valence (and perhaps imposes some further restrictions), further analysis becomes possible (see for example [5, 8, 10]).

Since every connected digraph of valence 1 is isomorphic to a directed cycle, valence 2 is the smallest interesting valence in the context of arc-transitive digraphs. In the literature, arc-transitive 2-valent digraphs often arise in disguise as undirected 4-valent graphs admitting a group of automorphisms acting transitively on the edges, vertices, but not on the arcs of the graph; such group actions are usually called $\frac{1}{2}$*-arc-transitive*. Namely, if Γ is a G-arc-transitive 2-valent digraph, then its underlying (undirected) graph Γ' admits a $\frac{1}{2}$-arc-transitive action of the group G; and conversely, if the automorphism group of an undirected 4-valent graph Γ' contains a subgroup G acting $\frac{1}{2}$-arc-transitively on Γ', then there exists an orientation of the edges of Γ' that gives rise to a G-arc-transitive 2-valent digraph whose underlying graph is Γ' (in fact, there are precisely two such orientations giving rise to a pair of opposite digraphs). In this sense, the study of G-arc-transitive 2-valent digraphs is equivalent to the study of $(G, \frac{1}{2})$-arc-transitive graphs of valence 4. There is a substantial literature about the latter class of graphs (see for example [6, 18, 19, 20, 21, 25, 39, 40]).

If Γ is an arc-transitive 2-valent digraph, then, for some positive integer s, the automorphism group $\mathrm{Aut}(\Gamma)$ acts regularly on the set of all s-arcs of the digraph. If $s = 1$, then the automorphism group acts regularly on the arc-set, and if the order of the digraph has a simple prime factorisation, one is usually able to classify all possible automorphism groups and use this information to determine all digraphs upon which such groups can act. An instructive example of how this can be done (in the case of undirected 4-valent graphs) can be found in [10]. Here, we will avoid this case and restrict ourselves to the case $s \geq 2$; that is, we will assume that our digraphs are all 2-arc-transitive.

The two main results of the paper are Theorems 1.1 and 1.2, stated below and proved in Section 3. The digraphs $\overrightarrow{\mathrm{PX}}(t, s)$ appearing in the statements are defined in Section 2.5.

Theorem 1.1. *Let p and q be distinct odd primes, and let a, b, c be integers satisfying $a \in \{0, 1, 2, 3\}$, $b, c \in \{0, 1, 2\}$, and $(b, c) \neq (2, 2)$. If Γ is a connected $(G, 2)$-arc-transitive 2-valent digraph of order $2^a q^b p^c$ and G is non-solvable, then the order of Γ is at most 1224 and Γ is isomorphic to one of the sixty-seven digraphs in Table 1.*

Remark. Exact descriptions of the sixty-seven exceptional digraphs of Theorem 1.1 are available in [29] (for the digraphs of order up to 1000) and [1] (for digraphs of larger order). The digraphs are given there in a form readable by `Magma` [2].

Theorem 1.2. *Let Γ be a connected $(G, 2)$-arc-transitive 2-valent digraph and suppose that G is solvable. Let n be the order of Γ, and suppose that one of the following holds:*

(i) *n is odd and cube-free;*

(ii) *$n = 2^a m$, where $a \in \{1, 2, 3\}$ and m is an odd, square-free integer;*

(iii) *$n = 2^a q^b p^2$, where $a \in \{1, 2, 3\}$, $b \in \{0, 1\}$ and p, q are distinct odd primes.*

Then one of the following conclusions holds:

Order	Name	$\lvert \mathrm{Aut}_v \rvert$	$\lvert S \rvert$	$\mathrm{soc}(\mathrm{Aut})$
$2 \cdot 3 \cdot 5$	ATD[30;6]	4	1	$\mathrm{Alt}(5)$
$2 \cdot 3 \cdot 7$	ATD[42;3]	8	1	$\mathrm{PSL}(2,7)$
$2^2 \cdot 3 \cdot 5$	ATD[60;16]	4	1	$\mathrm{Alt}(5) \times C_2$
$2^2 \cdot 3 \cdot 7$	ATD[84;20]	8	1	$\mathrm{PSL}(2,7) \times C_2$
$2^2 \cdot 3 \cdot 7$	ATD[84;23]	4	1	$\mathrm{PSL}(2,7)$
$2^2 \cdot 3 \cdot 7$	ATD[84;24]	4	1	$\mathrm{PSL}(2,7)$
$2 \cdot 3^2 \cdot 5$	ATD[90;12]	4	1	$\mathrm{Alt}(5) \times C_3$
$2 \cdot 3^2 \cdot 5$	ATD[90;13]	16	3	$\mathrm{Alt}(6)$
$2^3 \cdot 3 \cdot 5$	ATD[120;11]	4	1	$\mathrm{Alt}(5) \times C_2$
$2^3 \cdot 3 \cdot 5$	ATD[120;54]	4	1	$\mathrm{Alt}(5) \times C_2$
$2^3 \cdot 3 \cdot 5$	ATD[120;56]	4	1	$\mathrm{Alt}(5) \times C_2$
$2 \cdot 3^2 \cdot 7$	ATD[126;15]	8	1	$\mathrm{PSL}(2,7) \times C_3$
$2 \cdot 3 \cdot 5^2$	ATD[150;16]	4	1	$\mathrm{Alt}(5) \times C_5$
$2^3 \cdot 3 \cdot 7$	ATD[168;53]	8	1	$\mathrm{PSL}(2,7) \times C_2$
$2^3 \cdot 3 \cdot 7$	ATD[168;64]	4	1	$\mathrm{PSL}(2,7) \times C_2$
$2^3 \cdot 3 \cdot 7$	ATD[168;65]	4	1	$\mathrm{PSL}(2,7) \times C_2$
$2^3 \cdot 3 \cdot 7$	ATD[168;81]	4	1	$\mathrm{PSL}(2,7) \times C_2$
$2^3 \cdot 3 \cdot 7$	ATD[168;82]	4	1	$\mathrm{PSL}(2,7) \times C_2$
$2^2 \cdot 3^2 \cdot 5$	ATD[180;42]	4	1	$\mathrm{Alt}(6)$
$2^2 \cdot 3^2 \cdot 5$	ATD[180;45]	4	1	$\mathrm{Alt}(5) \times C_2 \times C_3$
$2^2 \cdot 3^2 \cdot 5$	ATD[180;57]	8	3	$\mathrm{Alt}(6)$
$2^2 \cdot 3^2 \cdot 5$	ATD[180;58]	16	3	$\mathrm{Alt}(6) \times C_2$
$2^2 \cdot 3^2 \cdot 7$	ATD[252;59]	8	1	$\mathrm{PSL}(2,7) \times C_2 \times C_3$
$2^2 \cdot 3^2 \cdot 7$	ATD[252;69]	4	1	$\mathrm{PSL}(2,7) \times C_3$
$2^2 \cdot 3^2 \cdot 7$	ATD[252;70]	4	1	$\mathrm{PSL}(2,7) \times C_3$
$2 \cdot 3 \cdot 7^2$	ATD[294;19]	8	1	$\mathrm{PSL}(2,7) \times C_7$
$2^2 \cdot 3 \cdot 5^2$	ATD[300;66]	4	1	$\mathrm{Alt}(5) \times C_2 \times C_5$
$2 \cdot 3^2 \cdot 17$	ATD[306;11]	8	1	$\mathrm{PSL}(2,17)$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;146]	4	1	$\mathrm{Alt}(6) \times C_2$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;148]	4	1	$\mathrm{Alt}(6)$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;150]	8	3	$\mathrm{Alt}(6) \times C_2$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;153]	8	3	$\mathrm{Alt}(6) \times C_2$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;154]	4	1	$\mathrm{Alt}(6)$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;158]	4	1	$\mathrm{Alt}(5) \times C_2^2$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;163]	4	1	$\mathrm{Alt}(5) \times C_2 \times C_3$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;172]	4	1	$\mathrm{Alt}(5) \times C_2 \times C_3$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;174]	4	1	$\mathrm{Alt}(5) \times C_2 \times C_3$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;201]	8	3	$\mathrm{Alt}(6) \times C_2$
$2^3 \cdot 3^2 \cdot 5$	ATD[360;202]	16	3	$\mathrm{Alt}(6) \times C_2$
$2^3 \cdot 3^2 \cdot 7$	ATD[504;162]	8	1	$\mathrm{PSL}(2,7) \times C_2 \times C_3$
$2^3 \cdot 3^2 \cdot 7$	ATD[504;180]	4	1	$\mathrm{PSL}(2,7) \times C_2 \times C_3$
$2^3 \cdot 3^2 \cdot 7$	ATD[504;182]	4	1	$\mathrm{PSL}(2,7) \times C_2 \times C_3$
$2^3 \cdot 3^2 \cdot 7$	ATD[504;232]	4	1	$\mathrm{PSL}(2,7) \times C_2 \times C_3$
$2^3 \cdot 3^2 \cdot 7$	ATD[504;233]	4	1	$\mathrm{PSL}(2,7) \times C_2 \times C_3$
$2^2 \cdot 3 \cdot 7^2$	ATD[588;87]	8	1	$\mathrm{PSL}(2,7) \times C_2 \times C_7$
$2^2 \cdot 3 \cdot 7^2$	ATD[588;90]	4	1	$\mathrm{PSL}(2,7) \times C_7$
$2^2 \cdot 3 \cdot 7^2$	ATD[588;91]	4	1	$\mathrm{PSL}(2,7) \times C_7$
$2^3 \cdot 3 \cdot 5^2$	ATD[600;199]	4	1	$\mathrm{Alt}(5) \times C_2 \times C_5$
$2^3 \cdot 3 \cdot 5^2$	ATD[600;201]	4	1	$\mathrm{Alt}(5) \times C_2 \times C_5$
$2^3 \cdot 3 \cdot 5^2$	ATD[600;204]	4	1	$\mathrm{Alt}(5) \times C_2 \times C_5$
$2^2 \cdot 3^2 \cdot 17$	ATD[612;48]	4	1	$\mathrm{PSL}(2,17)$
$2^2 \cdot 3^2 \cdot 17$	ATD[612;49]	8	1	$\mathrm{PSL}(2,17)$
$2^3 \cdot 3 \cdot 7^2$	X_1	4	1	$\mathrm{PSL}(2,7) \times C_2 \times C_7$
$2^3 \cdot 3 \cdot 7^2$	X_2	4	1	$\mathrm{PSL}(2,7) \times C_2 \times C_7$
$2^3 \cdot 3 \cdot 7^2$	X_3	4	1	$\mathrm{PSL}(2,7) \times C_2 \times C_7$
$2^3 \cdot 3 \cdot 7^2$	X_4	4	1	$\mathrm{PSL}(2,7) \times C_2 \times C_7$
$2^3 \cdot 3 \cdot 7^2$	X_5	8	1	$\mathrm{PSL}(2,7) \times C_2 \times C_7$
$2^3 \cdot 3^2 \cdot 17$	X_6	4	1	$\mathrm{PSL}(2,17) \times C_2$
$2^3 \cdot 3^2 \cdot 17$	X_7	8	1	$\mathrm{PSL}(2,17) \times C_2$
$2^3 \cdot 3^2 \cdot 17$	X_8	4	1	$\mathrm{PSL}(2,17) \times C_2$
$2^3 \cdot 3^2 \cdot 17$	X_9	4	1	$\mathrm{PSL}(2,17) \times C_2$
$2^2 \cdot 3^2 \cdot 17$	X_{10}	4	1	$\mathrm{PSL}(2,17)$
$2^2 \cdot 3^2 \cdot 17$	X_{11}	4	1	$\mathrm{PSL}(2,17)$
$2^2 \cdot 3^2 \cdot 17$	X_{12}	4	1	$\mathrm{PSL}(2,17)$
$2^2 \cdot 3^2 \cdot 17$	X_{13}	4	1	$\mathrm{PSL}(2,17)$
$2^2 \cdot 3^2 \cdot 17$	X_{14}	4	1	$\mathrm{PSL}(2,17)$
$2^2 \cdot 3^2 \cdot 17$	X_{15}	4	1	$\mathrm{PSL}(2,17)$

Table 1: Exceptional digraphs for Theorem 1.1. The column "Name" refers to the digraph names as given in [28] (up to order 1000) or [1] (for orders greater than 1000). The number of non-solvable 2-arc-transitive subgroups of $\mathrm{Aut}(\Gamma)$ (up to conjugacy) is given in the column called $|S|$.

(a) $\Gamma \cong \overrightarrow{\mathrm{PX}}(t,s)$ *for some* $t \geq 1$ *and* $s \geq 0$*;*

(b) condition (iii) holds, G *has a normal Sylow* p*-subgroup* P*, which is elementary abelian of order* p^2*, and* $\Gamma/P \cong \overrightarrow{\mathrm{PX}}(t,s)$ *for some* $t \geq 1$ *and* $s \geq 0$.

Remark. Let us spend a few words on the seemingly unfinished case (b) of Theorem 1.2. The digraphs appearing in this case arise from regular covering projections onto the digraphs $\overrightarrow{\mathrm{PX}}(t,s)$ of order $2^a q^b$ where the groups of covering transformations are elementary abelian of order p^2, along which a 2-arc-transitive group of automorphisms of $\overrightarrow{\mathrm{PX}}(t,s)$ lifts. The theory of lifting groups along elementary abelian covering projections was developed in [14] and illustrated in several papers (see for example [15, 31]). If desired, one could use this theory to determine all the resulting covering digraphs for fixed (a,q,b). In particular, we could easily obtain a complete classification in the case of order kp or kp^2 for every $k \leq 14$ and prime p.

Recently, numerous papers have been written in which authors classified arc-transitive graphs and digraphs of fixed valence and orders with a simple prime factorisation (usually kp or kp^2 for a fixed small k and variable prime p). Unlike in many of the above mentioned papers, we have tried to prove our results in as general a form as our approach allowed. Slight improvements are certainly possible (for example, using the classification of finite simple groups whose order is divisible by four primes only [13], one could extend Theorem 1.1 to orders divisible by a third odd prime). However, it seems that major improvements would require new ideas.

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2 Preliminaries

2.1 On graphs and digraphs

Even though we are mainly interested in simple digraphs, it will be convenient in the proofs to allow digraphs to be non-simple. We therefore define a *digraph* Γ as a quadruple $(V, A, \mathrm{head}, \mathrm{tail})$ where V and A are finite non-empty sets and head and tail are functions mapping from A to V; we call the sets V and A the *vertex-set* and the *arc-set* of Γ and denote them by $\mathrm{V}(\Gamma)$ and $\mathrm{A}(\Gamma)$, respectively. We then think of an arc to *point* from its *tail* to its *head*. The cardinality of $\mathrm{V}(\Gamma)$ is called the *order* of Γ.

Similarly, a *graph* Γ is determined by a vertex-set $\mathrm{V}(\Gamma)$, edge-set $\mathrm{E}(\Gamma)$ and a function $\mathrm{end}\colon \mathrm{E}(\Gamma) \to \{X \subseteq \mathrm{V}(\Gamma) : |X| \in \{1,2\}\}$, assigning a pair of *endvertices* to each edge of Γ. An edge e of a graph Γ is a *loop* provided that $|\mathrm{end}(e)| = 1$, and two edges y and x are *parallel* if $\mathrm{end}(x) = \mathrm{end}(y)$. A graph Γ without loops and parallel edges is *simple* and is uniquely determined by $\mathrm{V}(\Gamma)$ and the set $\{\mathrm{end}(e) : e \in \mathrm{E}(\Gamma)\}$.

If Γ is a digraph, then the *underlying graph* of Γ is the graph with vertex-set $\mathrm{V}(\Gamma)$, edge-set $\mathrm{A}(\Gamma)$ and the end-function defined by $\mathrm{end}(x) = \{\mathrm{tail}(x), \mathrm{head}(x)\}$. A digraph is *simple* provided that its underlying graph is simple.

A sequence $(x_1, \ldots, x_s)$ of arcs of a digraph Γ is called an s-arc of Γ provided that $\mathrm{head}(x_i) = \mathrm{tail}(x_{i+1})$ for every $i \in \{1, \ldots, s-1\}$. The set of all s-arcs of Γ is denoted by $\mathrm{A}_s(\Gamma)$.

An automorphism of a digraph Γ is a permutation of $\mathrm{V}(\Gamma) \cup \mathrm{A}(\Gamma)$ that preserves $\mathrm{V}(\Gamma)$ set-wise and commutes with the functions head and tail. If G is a subgroup of the automorphism group $\mathrm{Aut}(\Gamma)$, then Γ is said to be G-arc-transitive (or (G, s)-arc-transitive) provided that G acts transitively on $\mathrm{A}(\Gamma)$ (or $\mathrm{A}_s(\Gamma)$, respectively). When $G = \mathrm{Aut}(\Gamma)$, the symbol G can be omitted from this notation.

If v is the tail and u the head of some arc x, then we say that u is an *out-neighbour* of v and v an *in-neighbour* of u. For a vertex $v \in \mathrm{V}(\Gamma)$, we let $\Gamma^+(v) = \{x \in \mathrm{A}(\Gamma) : \mathrm{tail}(x) = v\}$ and $\Gamma^-(v) = \{x \in \mathrm{A}(\Gamma) : \mathrm{head}(x) = v\}$, and call the sizes of these two sets the *out-valence* and the *in-valence* of v in Γ, respectively. (Note that when the digraph is not simple the out-valence does not necessarily equal the number of out-neighbours of v, and similarly for the in-valence). If for some integer k, the in-valence (out-valence) of every vertex equals k, then we say that the digraph has in-valence (out-valence, respectively) k. A digraph is called *k-valent* if it is of out-valence and in-valence k.

Observe that every arc-transitive digraph without vertices of out-valence 0 (in particular, every connected arc-transitive digraph) is vertex-transitive.

2.2 Non-simple arc-transitive 2-valent digraphs

In this section, we characterise arc-transitive 2-valent digraphs that are not simple. To formulate the characterisation (Lemma 2.1), we first need to introduce the digraphs $\vec{\mathrm{C}}_n^{(2)}$ and $\overleftrightarrow{\mathrm{C}}_n$ for $n \geq 1$. Both digraphs arise from an undirected cycle with each edge doubled, and their vertex-sets and arc-sets can be taken to be $\mathbb{Z}_n$ and $\mathbb{Z}_n \times \mathbb{Z}_2$, respectively. In $\vec{\mathrm{C}}_n^{(2)}$ the functions head and tail are defined with $\mathrm{tail}(i, \epsilon) = i$ and $\mathrm{head}(i, \epsilon) = i + 1$ for every arc $(i, \epsilon) \in \mathbb{Z}_n \times \mathbb{Z}_2$. Similarly, in $\overleftrightarrow{\mathrm{C}}_n$, the functions head and tail are defined with $\mathrm{tail}(i, 0) = i$, $\mathrm{head}(i, 0) = i+1$, $\mathrm{tail}(i, 1) = i+1$, and $\mathrm{head}(i, 1) = i$. Note that $\vec{\mathrm{C}}_1^{(2)}$ and $\overleftrightarrow{\mathrm{C}}_1$ are both isomorphic to a digraph with a single vertex and two directed loops attached to it, while $\vec{\mathrm{C}}_2^{(2)}$ and $\overleftrightarrow{\mathrm{C}}_2$ consist of two vertices and four arcs between them, two pointing in each of the two possible directions. The proof of the following lemma is straightforward and is left to the reader.

Lemma 2.1. *If Γ is a connected non-simple arc-transitive 2-valent digraph of order n, then $\Gamma \cong \vec{\mathrm{C}}_n^{(2)}$ or $\Gamma \cong \overleftrightarrow{\mathrm{C}}_n$, and if in addition Γ is 2-arc-transitive, then $\Gamma \cong \vec{\mathrm{C}}_n^{(2)}$ for some $n \geq 2$.*

The following result will be needed in the proof of Theorem 1.2.

Lemma 2.2. *Let G be a subgroup of $\mathrm{Aut}(\vec{\mathrm{C}}_n^{(2)})$ acting transitively on the s-arcs but not on the $(s+1)$-arcs of $\vec{\mathrm{C}}_n^{(2)}$ and let v be a vertex of $\vec{\mathrm{C}}_n^{(2)}$. Then G_v is an elementary abelian 2-group of order 2^s and is normal in G. If G_v has order 4 and contains a non-trivial central element of G, then n is even.*

Proof. Observe that every automorphism of $\vec{\mathrm{C}}_n^{(2)}$ that fixes v fixes every vertex of $\vec{\mathrm{C}}_n^{(2)}$, implying that G_v is the kernel of the action of G on the vertex-set of $\vec{\mathrm{C}}_n^{(2)}$, and is therefore normal in G. Furthermore, G_v preserves set-wise each pair of arcs with the same tail (and thus the same head). In particular, G_v is an elementary abelian 2-group. Since G is transitive on the s-arcs but not on the $(s+1)$-arcs, it is an easy exercise to show that G_v acts regularly on the s-arcs starting at v, and since there are 2^s of them, it follows that $|G_v| = 2^s$.

Suppose now that n is odd, that $|G_v| = 4$, and that τ is a non-trivial central element of G contained in G_v. Without loss of generality, we may assume that τ acts non-trivially on the pair of arcs pointing out of v. Furthermore, since the index of G_v in G is n, it follows that G_v is the unique Sylow 2-subgroup of G, and thus $G = G_v \rtimes H$, where H is a group of order n. Moreover, since G_v is the kernel of the action of G on the vertices of $\overrightarrow{\mathrm{C}}_n^{(2)}$, it follows that H acts regularly on the vertices of $\overrightarrow{\mathrm{C}}_n^{(2)}$; in particular, $H = \langle g \rangle$ where g is an automorphism of order n that maps every vertex to its unique out-neighbour.

Since $\tau = \tau^g$, the element τ acts non-trivially on every pair of arcs sharing the same tail. In particular, τ is the unique non-trivial central element of G contained in G_v. Since $G = G_v H$ and since G_v is abelian, this shows that H centralises no element of $G_v \setminus \{1, \tau\}$. However, this is impossible since H has odd order and $|G_v \setminus \{1, \tau\}| = 2$. This contradiction completes the proof of the lemma. □

2.3 Alter-relations, alter-exponent, radius and perimeter

In this section, we present a very useful tool for studying digraphs, based on the orientation of arcs in the walks of a digraph. The concepts presented in this section were first introduced in [24] (for a generalisation to infinite digraphs, see [16]). All the facts stated below were proved in [24] for simple digraphs and extend without any change to digraphs with loops and multiple arcs.

A *walk* from a vertex v_0 to a vertex v_s of *length* s in a digraph Γ is a sequence $(v_0, x_1, v_1, \ldots, v_{s-1}, x_s, v_s)$ of arcs $x_i \in \mathrm{A}(\Gamma)$ and vertices $v_j \in \mathrm{V}(\Gamma)$ such that for any $i \in \{1, \ldots, s\}$ the pair $\big(\mathrm{tail}(x_i), \mathrm{head}(x_i)\big)$ equals either (v_{i-1}, v_i) or (v_i, v_{i-1}). In the former case, we say that x_i is *positively oriented*, while in the latter case we say that x_i is *negatively oriented* in the walk. A walk is *directed* if all of its arcs are positively oriented and is *alternating* if the orientation of the arcs in the walk alternates. A digraph Γ is *(strongly) connected* provided that for any two vertices $u, v \in \Gamma$ there exists a (directed) walk from u to v. A vertex-transitive digraph is strongly connected if and only if it is connected (see, for example, [27, Lemma 2]). A walk is closed provided that it begins and ends in the same vertex.

Let $W = (v_0, x_1, v_1, x_2, \ldots, x_n, v_n)$ be a walk in a digraph Γ. The *sum* $s(W)$ is the difference between the number of positively oriented arcs in W and the number of negatively oriented arcs in W. The k-th partial sum $s_k(W)$ is defined as the sum of the initial walk $(v_0, x_1, v_1, \ldots, v_k)$ of length k. The set $\{s_k(W), 0 \le k \le n\}$ is the *tolerance* of W and vertices u and v are *alter-equivalent* with tolerance $\mathcal{J}$ (written $uA_{\mathcal{J}}v$) if there exists a walk from u to v with sum 0 and tolerance contained in $\mathcal{J}$. It transpires that $A_{\mathcal{J}}$ is an equivalence relation (called an *alter-relation*) for every interval $\mathcal{J}$ containing 0 and that it is invariant under every automorphism of Γ. We will denote the equivalence class containing a vertex v with $A_{\mathcal{J}}(v)$ and use the shorthand $A_i(v)$ to mean $A_{[0,i]}(v)$ (when $i \ge 0$) or $A_{[i,0]}(v)$ (when $i < 0$). Note that since Γ is a finite digraph, there exists a non-negative integer e such that $A_e = A_{e+1}$ and (by induction) $A_e = A_\infty$. The smallest such integer e is called the *alter-exponent* of Γ and denoted $\exp(\Gamma)$. It can be shown that $\exp(\Gamma)$ also equals the smallest non-negative integer i for which $A_{-i} = A_{-i-1}$ as well as the smallest i such that $A_{[-i,i]} = A_{[-i-1,i+1]}$. When we consider alter-relations in several different digraphs, we shall use the symbol $A^{\Gamma}_{\mathcal{J}}$ (instead of $A_{\mathcal{J}}$) to denote the one in the digraph Γ.

The number of equivalence classes of the alter-relation A_∞ is called the *perimeter* of

Γ and denoted $\mathrm{perim}(\Gamma)$. If the in-valence and the out-valence of each vertex is positive, then the equivalence classes B_i of A_∞ can be indexed by $\mathbb{Z}_p$ (where $p = \mathrm{perim}(\Gamma)$) in such a way that every arc of Γ having its tail in B_i, has its head in B_{i+1}.

We will be particularly interested in the sets $A_1(v)$ and $A_{-1}(v)$. Note that these sets consists of precisely those vertices that can be reached from v by alternating walks of even length starting with a positively (negatively, respectively) oriented arc. The intersection $A_1(v) \cap A_{-1}(v)$ will be denoted $\mathrm{Att}(v)$ and called the *attachment set* (at vertex v).

Suppose henceforth that Γ is a G-arc-transitive digraph. Then the sets $A_{\mathcal{J}}(v)$ (as well as $\mathrm{Att}(v)$) are all blocks for the action of G on $\mathrm{V}(\Gamma)$ and their size depends only on $\mathcal{J}$ (but not on v). One can thus define the *radius* of Γ (denoted $\mathrm{rad}(\Gamma)$) to be the cardinality of $|A_1(v)|$ for any $v \in \Gamma$, and the *attachment number* of Γ (denoted $\mathrm{att}(\Gamma)$) to be the cardinality of $\mathrm{Att}(v)$ for any $v \in \mathrm{V}(\Gamma)$. Since $\mathrm{Att}(v) \subseteq A_1(v) \subseteq A_2(v) \subseteq \ldots$, we see that $\mathrm{att}(\Gamma)$ divides $\mathrm{rad}(\Gamma)$, and that $|A_i(v)|$ divides $|A_{i+1}(v)|$ for every $i \geq 1$.

Suppose now that Γ is a 2-valent arc-transitive digraph. Then the sub-digraph of Γ induced by a closed alternating walk of sum 0 that traverses every arc of Γ at most once is called an *alternating cycle*. The *length* of an alternating cycle is defined to be the length of the closed alternating walk that induces it. (Alternating cycles were introduced in [19] in the context of simple $(G, \frac{1}{2})$-arc-transitive 4-valent graphs.)

Note that an alternating cycle is uniquely determined by any of its arcs, implying that the set of alternating cycles induces a decomposition of the arc-set of Γ. Furthermore, this decomposition is preserved by every automorphism of Γ, implying that all alternating cycles in Γ have the same length.

In addition to the assumption that Γ is a 2-valent arc-transitive digraph, assume for the rest of the section that Γ is not isomorphic to any $\overleftrightarrow{\mathrm{C}}_n$ with n odd. Then an alternating cycle is indeed a cycle (in the sense that the walk that generates it traverses every vertex of the digraph at most once), and Γ contains at least two alternating cycles.

Furthermore, observe that $A_1(v)$ consists of every second vertex of an alternating cycle starting with a positively oriented arc with its tail in v, and similarly, $A_{-1}(v)$ consist of every second vertex of an alternating cycle starting with a negatively oriented arc with its head in v. In particular, $|A_1(v)| = |A_{-1}(v)|$ and the length of each alternating cycle is twice the radius of Γ. Note also that there are precisely two alternating cycles meeting in a given vertex v and the set of vertices that are contained in both of these alternating cycles is precisely $\mathrm{Att}(v)$. Two alternating cycles therefore meet in either 0 or $\mathrm{att}(\Gamma)$ vertices.

Suppose now that $\mathrm{att}(\Gamma) \geq 3$ and let $g \in \mathrm{Aut}(\Gamma)$ fix an arc x of Γ. Then g fixes point-wise the alternating cycle C containing x. Since $\mathrm{att}(\Gamma) \geq 3$, g fixes also at least three vertices of each alternating cycle intersecting C, and therefore fixes each of these cycles point-wise. But then by connectivity, g fixes each alternating cycles of Γ point-wise. In particular, g is trivial. This proves the following easy, but very useful result.

Lemma 2.3. *If* Γ *is a connected* 2*-valent* 2*-arc-transitive digraph, then* $\mathrm{att}(\Gamma) \leq 2$.

We finish this section with another useful result.

Lemma 2.4. *If* Γ *is a connected* 2*-valent* 2*-arc-transitive digraph and* $\exp(\Gamma) = 1$*, then* $\Gamma \cong \overrightarrow{\mathrm{PX}}(m, 1)$ *for some integer* m.

Proof. If Γ is not simple, then by Lemma 2.1, $\Gamma \cong \overrightarrow{\mathrm{C}}_n^{(2)}$ for some $n \geq 2$, implying that $\exp(\Gamma) = 0$; a contradiction. Hence Γ is simple, and we can apply [30, Theorem 7.1] to conclude that $\mathrm{rad}(\Gamma) = 2$. Since $\exp(\Gamma) = 1$, it is then easy to see that $\mathrm{att}(\Gamma) = 2$, and also that $\Gamma \cong \overrightarrow{\mathrm{PX}}(m, 1)$ for some m (see, for example, [19, Proposition 3.1]). □

2.4 Covers and quotients

The second tool that we will use extensively is the concept of (di)graph coverings. This tool is usually defined in the setting of undirected graphs, but extends naturally to digraphs. In this section, we present a few basic facts and results and refer the reader to [14, 17] for more details.

Let Γ and Λ be two digraphs. A *morphism* from Γ onto Λ is a function $f : \mathrm{V}(\Gamma) \cup \mathrm{A}(\Lambda) \to \mathrm{V}(\Lambda) \cup \mathrm{A}(\Lambda)$ mapping $\mathrm{V}(\Gamma)$ to $\mathrm{V}(\Lambda)$ and $\mathrm{A}(\Gamma)$ to $\mathrm{A}(\Lambda)$ such that $f(\mathrm{tail}(x)) = \mathrm{tail}(f(x))$ and $f(\mathrm{head}(x)) = \mathrm{head}(f(x))$ for every $x \in \mathrm{A}(\Gamma)$. A morphism is an *epimorphism* or *isomorphism* if it is *surjective* or *bijective*, respectively. (Note that an automorphism of a digraph is precisely an isomorphism from the digraph onto itself.)

An epimorphism $\wp\colon \Gamma \to \Lambda$ is a *covering projection* provided that for every $v \in \mathrm{V}(\Gamma)$ the restrictions $\wp_v^+\colon \Gamma^+(v) \to \Lambda^+(\wp(v))$ and $\wp_v^-\colon \Gamma^-(v) \to \Lambda^-(\wp(v))$ of $\wp$ to the out- and in-neighbourhoods of v are bijective. For simplicity, we shall also require both Γ and Λ to be connected. The preimage $\wp^{-1}(x)$ of a vertex or an arc x of Λ is called a *fibre* of the covering projection $\wp$ and the group of all automorphisms of Γ that preserve each fibre set-wise is called the *group of covering transformations*. If the latter is transitive on each fibre, then the covering projection is *regular*.

Normal quotients of simple graphs were introduced in [33, 34] and have now become a standard tool in studying symmetric graphs. Here we adapt this concept slightly to fit into the setting of digraphs admitting loops and multiple arcs. This adaptation will prove most useful in the proofs of our main results.

Let Γ be a digraph and let $N \le \mathrm{Aut}(\Gamma)$. Let $A_N = \{x^N : x \in \mathrm{A}(\Gamma)\}$ and $V_N = \{v^N : v \in \mathrm{V}(\Gamma)\}$ denote the sets of N-orbits on the arcs and vertices of Γ, respectively. Further, let $\mathrm{tail}_N\colon A_N \to V_N$ and $\mathrm{head}_N\colon A_N \to V_N$ be defined by $\mathrm{tail}_N(x^N) = \mathrm{tail}(x)^N$ and $\mathrm{head}_N(x^N) = \mathrm{head}(x)^N$. This defines the *quotient digraph* ${}^{\Gamma}/_{\mathrm{N}} = (V_N, A_N, \mathrm{head}_N, \mathrm{tail}_N)$, together with the obvious epimorphism $\wp_N\colon \Gamma \to {}^{\Gamma}/_{\mathrm{N}}$ satisfying $\wp_N(x) = x^N$ for every $x \in \mathrm{V}(\Gamma) \cup \mathrm{A}(\Gamma)$, called the *normal quotient projection relative to* N. If $N \trianglelefteq G \le \mathrm{Aut}(\Gamma)$, then there is an obvious, but not necessarily faithful action of the quotient group ${}^{G}/_{\mathrm{N}}$ on the digraph ${}^{\Gamma}/_{\mathrm{N}}$. Note also that if G acts transitively on vertices, arcs or s-arcs of Γ, then so does ${}^{G}/_{\mathrm{N}}$ on ${}^{\Gamma}/_{\mathrm{N}}$. If the quotient projection $\wp_N$ is a covering projection, then the situation is particularly nice; for example:

Lemma 2.5. *Let Γ be a digraph, let $G \le \mathrm{Aut}(\Gamma)$ and let N be a normal subgroup of G. If the quotient projection $\wp\colon \Gamma \to {}^{\Gamma}/_{\mathrm{N}}$ is a covering projection, then the action of ${}^{G}/_{\mathrm{N}}$ on $\mathrm{V}({}^{\Gamma}/_{\mathrm{N}}) \cup \mathrm{A}({}^{\Gamma}/_{\mathrm{N}})$ is faithful, and moreover, the stabilisers G_v and $({}^{G}/_{\mathrm{N}})_{v^N}$ are isomorphic for every $v \in \mathrm{V}(\Gamma)$.*

We say in this case that the group ${}^{G}/_{\mathrm{N}}$ *lifts along* $\wp$. More precisely, a group $H \le \mathrm{Aut}({}^{\Gamma}/_{\mathrm{N}})$ *lifts along* $\wp$ if there exists some $G \le \mathrm{Aut}(\Gamma)$, containing N as a normal subgroup, such that ${}^{G}/_{\mathrm{N}} = H$.

We now state two very useful sufficient and necessary conditions for a normal quotient projection to be a regular covering projection. (We shall call a group N of automorphisms of Γ *semiregular* provided that the stabiliser N_v is trivial for every $v \in \mathrm{V}(\Gamma)$.)

Lemma 2.6. *Let* Γ *be a connected digraph, let* $N \le \mathrm{Aut}(\Gamma)$ *and let* $\wp\colon \Gamma \to {}^{\Gamma}/_{N}$ *be the corresponding quotient projection. Then the following statements are equivalent:*

(a) N *is semiregular;*

(b) *the in-valence as well as the out-valence of* v *and* $\wp(v)$ *coincide for every* $v \in V(\Gamma)$*;*

(c) $\wp$ *is a regular covering projection.*

The rest of the section is devoted to the interplay between the concepts of alter-relations and covering projections.

Lemma 2.7. *Let* $\wp\colon \Gamma \to \Lambda$ *be a covering projection, let* v *be a vertex of* Γ *and let* $\mathcal{J}$ *be an interval of integers containing* 0*. Then* $\wp(A^{\Gamma}_{\mathcal{J}}(v)) = A^{\Lambda}_{\mathcal{J}}(\wp(v))$.

Proof. Suppose that $\tilde{u} \in \wp(A^{\Gamma}_{\mathcal{J}}(v))$. Then there exists $u \in \mathrm{V}(\Gamma)$ such that $\wp(u) = \tilde{u}$ and a walk $(v, x_1, v_1, \ldots, x_n, u)$ in Γ of sum 0 and tolerance within $\mathcal{J}$. But then the projected walk $(\wp(v), \wp(x_1), \wp(v_1), \ldots, \wp(x_n), \tilde{u})$ is also a walk of sum 0 and tolerance within $\mathcal{J}$, implying that $\tilde{u} \in A^{\Lambda}_{\mathcal{J}}(\wp(v))$.

Conversely, suppose that $\tilde{u} \in A^{\Lambda}_{\mathcal{J}}(\wp(v))$. Then there exists a walk $(\wp(v), \tilde{x}_1, \tilde{v}_1, \ldots, \tilde{x}_n, \tilde{u})$ of sum 0 and tolerance within $\mathcal{J}$. Since $\wp$ is a local bijection, one can then construct a lift $(v, x_1, v_1, \ldots, x_n, u)$ such that $\wp(x_i) = \tilde{x}_i$, $\wp(v_i) = \tilde{v}_i$, and $\wp(u) = \tilde{u}$. Note that this lift will also have sum 0 and tolerance within $\mathcal{J}$, implying that $u \in A^{\Gamma}_{\mathcal{J}}(v)$, and therefore $\tilde{u} \in \wp(A^{\Gamma}_{\mathcal{J}}(v))$. □

Lemma 2.8. *Let* Γ *be a* G*-vertex-transitive digraph, let* N *be a semiregular normal subgroup of* G*, let* $\Lambda = {}^{\Gamma}/_{N}$ *and let* $\wp\colon \Gamma \to \Lambda$ *be the corresponding covering projection. Further, let* v *be a vertex of* Γ*, and let* $\mathcal{J}$ *be an interval of integers containing* 0*. Then* $|A^{\Gamma}_{\mathcal{J}}(v)|$ *divides* $|N||A^{\Lambda}_{\mathcal{J}}(\wp(v))|$.

Proof. In view of Lemma 2.7, we see that $A^{\Gamma}_{\mathcal{J}}(v) \subseteq \wp^{-1}(\wp(A^{\Gamma}_{\mathcal{J}}(v))) = \wp^{-1}(A^{\Lambda}_{\mathcal{J}}(\wp(v)))$. Since $A^{\Lambda}_{\mathcal{J}}(\wp(v))$ is a block for the action of ${}^{G}/_{N}$ on Λ, it follows easily that $\wp^{-1}(A^{\Lambda}_{\mathcal{J}}(\wp(v)))$ is a block for the action of G on Γ. Since $A^{\Gamma}_{\mathcal{J}}(v)$ is also a block for G, it follows that $|A^{\Gamma}_{\mathcal{J}}(v)|$ divides $|\wp^{-1}(A^{\Lambda}_{\mathcal{J}}(\wp(v)))|$. However, since the $\wp$-preimage of a vertex in Λ is an N-orbit on Γ, it follows that the latter equals $|N||A^{\Lambda}_{\mathcal{J}}(\wp(v))|$. □

Lemma 2.9. *Let* Γ *be a connected,* $(G, 2)$*-arc-transitive* 2*-valent digraph and let* N *be a normal subgroup of* G*. If* N *has odd prime order, then* $\mathrm{rad}({}^{\Gamma}/_{N}) = \mathrm{rad}(\Gamma)$.

Proof. Let q be the order of N, let $\Lambda = {}^{\Gamma}/_{N}$ and let $\wp\colon \Gamma \to \Lambda$ be the corresponding quotient projection. Suppose that the conclusion of the lemma is false, that is, $\mathrm{rad}({}^{\Gamma}/_{N}) \neq \mathrm{rad}(\Gamma)$.

Since G_v is a 2-group (see Lemma 3.1) and N is of odd order, N acts semiregularly on $\mathrm{V}(\Gamma)$. By Lemma 2.6, the quotient projection $\wp$ is then a regular covering projection. Choose a vertex v of Γ and $\epsilon \in \{-1, 1\}$, and consider the set $T = A^{\Gamma}_{\epsilon}(v)$. Recall that $|T| = \mathrm{rad}(\Gamma)$. By Lemma 2.7, $\wp(T) = A^{\Lambda}_{\epsilon}(\wp(v))$. Since the size of the latter is $\mathrm{rad}(\Lambda)$, it follows by our initial assumption that $|\wp(T)| \neq |T|$, implying that T contains at least two elements of the orbit v^N. Since both T and v^N are blocks for the action of G on $\mathrm{V}(\Gamma)$, so is their intersection. However, v^N is of prime size, implying that $v^N = v^N \cap T$, and thus $v^N \subseteq T$. Since this is true for any choice of ϵ, it follows that $v^N \subseteq A^{\Gamma}_{1}(v) \cap A^{\Gamma}_{-1}(v) = \mathrm{Att}(v)$. But then by Lemma 2.3 it follows that Γ is not 2-arc-transitive, a contradiction. □

2.5 Partial line graphs and digraphs of Praeger and Xu

In this section, we give a brief overview of the very useful concept of *partial line graph construction*, which was invented in [21] to analyse G-arc-transitive 2-valent digraphs of radius 2, and was further developed in [30].

For a digraph Γ and a positive integer s, the *s-th partial line graph* $\mathrm{Pl}_s(\Gamma)$ of Γ is the digraph with vertex-set being the set of s-arcs $\mathrm{A}_s(\Gamma)$, the arc-set being $\mathrm{A}_{s+1}(\Gamma)$, and the functions tail and head defined by the rules $\mathrm{tail}(x_1,\ldots,x_{s+1}) = (x_1,\ldots,x_s)$ and $\mathrm{head}(x_1,\ldots,x_{s+1}) = (x_2,\ldots,x_{s+1})$ for every $(s+1)$-arc $(x_1,\ldots,x_{s+1})$ of Γ. Moreover, we let $\mathrm{Pl}_0(\Gamma) = \Gamma$ and write Pl instead of Pl_1. Note that if Γ is a 2-valent digraph, then so is $\mathrm{Pl}_s(\Gamma)$ for every $s \geq 0$. The following formula (which appeared as [30, Lemma 3.2(i)] in the context of simple digraphs), provides an alternative, recursive definition of the Pl_s operator:

$$\mathrm{Pl}_s(\Gamma) \cong \mathrm{Pl}(\mathrm{Pl}_{s-1}(\Gamma)) \quad \text{for } s \geq 1. \tag{2.1}$$

The lemma below follows from [30, Lemma 3.1(iv)] and [30, Lemma 3.2(ii)] in the context of simple digraphs. The proof remains unchanged in the case of non-simple digraphs.

Lemma 2.10. *If* Γ *is a vertex-transitive digraph, then* $\exp(\mathrm{Pl}(\Gamma)) = \exp(\Gamma) + 1$.

The following result appeared as [30, Lemma 5.1] in the context of simple digraphs, and extends to general digraphs via Lemma 2.1.

Lemma 2.11. *If* Γ *is a* 2*-valent* $(G,2)$*-arc-transitive digraph such that* $\mathrm{rad}(\Gamma) = 2$*, then* $\Gamma \cong \mathrm{Pl}(\Lambda)$*, where* Λ *is a* 2*-valent* $(G,3)$*-arc-transitive digraph of order half that of* Γ.

The Pl operator can be used to define a very important class of digraphs, first studied by Praeger and Xu [37] in the context of simple graphs, and by Praeger [35] in the context of simple digraphs. For integers n and s, $n \geq 1$, $s \geq 0$, let

$$\overrightarrow{\mathrm{PX}}(n,s) = \begin{cases} \overrightarrow{\mathrm{C}}_n^{(2)} & \text{if } s = 0 \\ \mathrm{Pl}(\overrightarrow{\mathrm{PX}}(n,s-1)) & \text{if } s \geq 1 \end{cases} \tag{2.2}$$

We shall call a graph isomorphic to some $\overrightarrow{\mathrm{PX}}(n,s)$ simply a $\overrightarrow{\mathrm{PX}}$*-digraph*. Note that, in view of (2.1), we have

$$\overrightarrow{\mathrm{PX}}(n,s) \cong \mathrm{Pl}_s(\overrightarrow{\mathrm{C}}_n^{(2)}). \tag{2.3}$$

The automorphism group of $\overrightarrow{\mathrm{C}}_n^{(2)}$ acts naturally as a group of automorphisms on each $\overrightarrow{\mathrm{PX}}(n,s)$ for $s \geq 1$. The following surprising characterisation of $\overrightarrow{\mathrm{PX}}$-digraphs was proved in [35, Theorem 2.9] in the context of simple digraphs. In view of Lemma 2.1, the result extends to non-simple digraphs.

Lemma 2.12. *Let* Γ *be a connected* 2*-valent* G*-arc-transitive digraph and let* $v \in \mathrm{V}(\Gamma)$*. If* G *contains an abelian normal subgroup* N *that is not semiregular, then* Γ *is a* $\overrightarrow{\mathrm{PX}}$*-digraph.*

The following lemma is an analogue of a similar result for the undirected graphs (see [9, Lemma 3.1]). Our proof is just a slight modification of the proof given there.

Lemma 2.13. *Let Γ be a connected 2-valent, G-arc-transitive digraph and let N be a minimal normal subgroup of G. Suppose that N is a 2-group and that $\Gamma/\mathrm{N} \cong \overrightarrow{\mathrm{C}}_n^{(2)}$ for some $n \geq 1$. Then Γ is a $\overrightarrow{\mathrm{PX}}$-digraph.*

Proof. Since N is a minimal normal subgroup of G and a 2-group, it is elementary abelian. Let K be the kernel of the action of G on the set of N-orbits on $\mathrm{V}(\Gamma)$, and observe that G/K acts faithfully on $\mathrm{V}(\Gamma/\mathrm{N})$. Let C be the centraliser of N in K. Then $N \leq C \leq K$. Since N and K are normal in G, so is C. Since N and K have the same orbits on $\mathrm{V}(\Gamma)$, so does C, implying that $K = NK_v$ and $C = NC_v$ for any vertex v.

Since the quotient Γ/N is 2-valent, Lemma 2.6 implies that the quotient projection $\Gamma \to \Gamma/\mathrm{N}$ is a covering projection, and also that N is semiregular (for otherwise the valence of the quotient Γ/N would be less than that of Γ). Therefore, $N \cap C_v \leq N_v = \mathbf{1}$, and since C_v centralises N, we see that $C = N \times C_v$. Since the quotient projection $\Gamma \to \Gamma/\mathrm{N}$ is a covering projection, Lemma 2.5 implies that G_v embeds into a vertex-stabiliser in $\mathrm{Aut}(\overrightarrow{\mathrm{C}}_n^{(2)})$. In particular, G_v (and thus C_v) is an elementary abelian 2-group, implying that C is an abelian normal subgroup of G.

Let us now show that $C_v \neq \mathbf{1}$. By way of contradiction, assume that $C_v = \mathbf{1}$, and thus that $C = NC_v = N$. Now recall that $K = NK_v$ and $N \cap K_v = \mathbf{1}$. Since both N and K_v are 2-groups, so is K. In particular, the centre $Z(K)$ is non-trivial. On the other hand, since $Z(K) \leq C$ and since $C = N$, we see that $Z(K) \leq N$. Since N is a minimal normal subgroup of G, this implies that $N = Z(K)$. But then $K = NK_v = N \times K_v$, and thus K is an elementary abelian 2-group. In particular, N, being the centre of K, equals K. Now recall that G/K acts faithfully on $\mathrm{V}(\Gamma/\mathrm{N})$. On the other hand, G/K equals G/N, which is clearly unfaithful on $\mathrm{V}(\Gamma/\mathrm{N})$. This contradiction shows that $C_v \neq \mathbf{1}$, and by Lemma 2.12, Γ is a $\overrightarrow{\mathrm{PX}}$-digraph, as claimed. □

Lemma 2.14. *Let n and s be integers, $n \geq 1$, $s \geq 0$, let $\Lambda = \overrightarrow{\mathrm{PX}}(n, s)$ and let v be a vertex of Λ. Then $\exp(\Lambda) = s$, $|A_\infty^\Lambda(v)| = 2^s$ and $\mathrm{perim}(\Lambda) = n$. Suppose G is a group acting transitively on the arcs of Λ and let $K = \langle G_u : u \in \mathrm{V}(\Lambda)\rangle$, that is, the group generated by all the vertex-stabilisers in G. Then K is the kernel of the action of G on the partition $\{A_\infty^\Lambda(u) : u \in \mathrm{V}(\Lambda)\}$ and $v^K = A_\infty^\Lambda(v)$; in particular, K is normal in G. Furthermore, the group K is elementary abelian of order $2^s|G_v|$, the quotient digraph Λ/K is isomorphic to a directed cycle of length n, and G/K is a cyclic group of order n.*

Proof. Observe first that $\exp(\overrightarrow{\mathrm{PX}}(n, 0)) = \exp(\overrightarrow{\mathrm{C}}_n^{(2)}) = 0$. On the other hand, by formula (2.2), $\Lambda = \mathrm{Pl}(\overrightarrow{\mathrm{PX}}(n, s-1))$, and thus by induction and Lemma 2.10, $\exp(\Lambda) = s$, as claimed.

By formula (2.3), a vertex of Λ is an s-arc of $\overrightarrow{\mathrm{C}}_n^{(2)}$. Now recall that $\mathrm{V}(\overrightarrow{\mathrm{C}}_n^{(2)}) = \mathbb{Z}_n$ and that there is an arc pointing from i to j if and only if $j - i = 1$. It is now clear that if v is an s-arc of $\overrightarrow{\mathrm{C}}_n^{(2)}$ starting in a vertex i of $\overrightarrow{\mathrm{C}}_n^{(2)}$, and W is a walk in Λ of sum k starting in v, then the end-point of W will be an s-arc of $\overrightarrow{\mathrm{C}}_n^{(2)}$ starting in $i + k$; in particular, every member of $A_\infty^\Lambda(v)$ is one of the 2^s s-arcs of $\overrightarrow{\mathrm{C}}_n^{(2)}$ starting in i. On the other hand, if w and u are arbitrary s-arcs of $\overrightarrow{\mathrm{C}}_n^{(2)}$ starting in i and $i + s$, respectively, then there clearly exists a directed walk in Λ of length s from v to u. By combining two such walks from v to u and from an arbitrary w to u, one gets a walk from v to w of sum 0. This shows that $A_\infty^\Lambda(v)$ is precisely the set of all s-arcs of $\overrightarrow{\mathrm{C}}_n^{(2)}$ starting in i. In particular, $|A_\infty^\Lambda(v)| = 2^s$, as claimed. Since $|\operatorname{V}(\Lambda)| = 2^s n$ and since $\operatorname{perim}(\Lambda) = |\operatorname{V}(\Lambda)|/|A_\infty^\Lambda(v)|$, it follows that $\operatorname{perim}(\Lambda) = n$.

The equality $v^K = A_\infty^\Lambda(v)$ follows directly from [30, Lemma 4.1] and [30, Corollary 4.2]. In particular, K fixes every class $A_\infty^\Lambda(u)$, $u \in \Lambda$, set-wise, implying that K is contained in the kernel (call it M) of the action of G on the partition $\{A_\infty^\Lambda(u) : u \in \operatorname{V}(\Lambda)\}$. Moreover, $v^K = v^M$, and since $K_v = G_v = M_v$, it follows that $K = M$. In particular, $|K| = |v^K|\,|K_v| = 2^s|G_v|$, as claimed.

The fact that Λ/K is isomorphic to the directed cycle of length $\operatorname{perim}(\Lambda)$ and that G/K is a cyclic group of order $\operatorname{perim}(\Lambda)$ is now a direct consequence of either [24, Propositions 3.2 and 3.5] or [35, Proposition 2.1].

Finally, to see that K is elementary abelian, recall that a vertex of Λ is an s-arc in $\overrightarrow{\mathrm{C}}_n^{(2)}$, and thus the stabiliser of a vertex in $\operatorname{Aut}(\Lambda)$ equals the stabiliser of an s-arc in $\operatorname{Aut}(\overrightarrow{\mathrm{C}}_n^{(2)})$. However, each stabiliser of an s-arc in $\operatorname{Aut}(\overrightarrow{\mathrm{C}}_n^{(2)})$ is contained in the kernel of the action of $\operatorname{Aut}(\overrightarrow{\mathrm{C}}_n^{(2)})$ on $\operatorname{V}(\overrightarrow{\mathrm{C}}_n^{(2)})$, which is elementary abelian of order 2^n. Since K is generated by the vertex-stabilisers G_u, $u \in \operatorname{V}(\Lambda)$, and thus by the stabilisers of the s-arcs of $\overrightarrow{\mathrm{C}}_n^{(2)}$ in G, it follows that K is also elementary abelian. □

3 Proofs of the main results

3.1 Auxilliary results

We start this section by a folklore fact about the vertex-stabilisers in arc-transitive 2-valent digraphs (see for example [22, Theorem 1.1] or [30, Theorem 1.2]).

Lemma 3.1. *If Γ be a connected 2-valent (G, s)-arc-transitive but not $(G, s+1)$-arc-transitive digraph and $v \in \operatorname{V}(\Gamma)$, then G_v is a group of order 2^s, generated by s involutions, and acts regularly upon the set of all s-arcs starting in v.*

The following is a well-known fact about the general linear groups $\operatorname{GL}(2, q)$.

Lemma 3.2. *If q is a power of an odd prime, and H an elementary abelian 2-subgroup of $\operatorname{GL}(2, q)$, then $|H| \le 4$, and if $|H| = 4$, then H contains the central involution of $\operatorname{GL}(2, q)$, namely the minus identity matrix.*

Proof. Recall that the group $\operatorname{SL}(2, q)$ contains a unique involution, namely the minus identity matrix. This implies that the intersection $H \cap \operatorname{SL}(2, q)$ is of order at most 2. On the other hand, the quotient $\operatorname{GL}(2, q)/\operatorname{SL}(2, q)$ is cyclic (of order $q - 1$), implying that $H\operatorname{SL}(2, q)/\operatorname{SL}(2, q)$ is cyclic; but this cyclic group is isomorphic to $H/(\operatorname{SL}(2, q) \cap H)$, and so of order at most 2. Hence the order of H is at most 4; and if it is of order 4, then $\operatorname{SL}(2, q) \cap H$ is non-trivial and thus contains the minus identity matrix. □

The following situation will occur several times in the proofs of the main results of the paper. To avoid repetition, we formulate it as a lemma.

Lemma 3.3. *Let Z be a group containing normal subgroups X and Y, such that X is abelian and contained in Y. Let C be the centraliser of X in Y. If the order of X is coprime to its index in C, then $C = X \times T$ for some normal subgroup T of Z. Moreover, T is isomorphic to a normal subgroup of Y/X.*

Proof. Since both X and Y are normal in Z, so is C. Moreover, since X is abelian, it is contained in C. Since, by assumption, the order of X is coprime to its index in C, the Schur-Zassenhaus theorem implies that X has a complement, say T. However, since X is centralised by C, it follows that $C = X \times T$. Now observe that T consists of all the elements of C of order coprime to $|X|$, implying that T is characteristic in C. Since C is normal in Z, so is T. Furthermore, since $X \le C \trianglelefteq Y$, the quotient Y/X contains a normal subgroup isomorphic to C/X, which is isomorphic to T. □

Lemma 3.4. *Let m be an odd positive integer and G an arc-transitive group of automorphisms of the digraph $\overrightarrow{\mathrm{PX}}(m,s)$. Then G contains a normal cyclic subgroup, the order of which divides m and is at least $m/(2^s|G_v|-1)$.*

Proof. Let $\Lambda = \overrightarrow{\mathrm{PX}}(m,s)$. By Lemma 3.1, the vertex-stabiliser G_v has order 2^r for some positive integer r. Let $K = \langle G_v : v \in \mathrm{V}(\Lambda)\rangle$, and recall that by Lemma 2.14, K is an elementary abelian normal subgroup of G of order 2^{s+r} and G/K is a cyclic group of order m. In particular, K is a normal Sylow 2-subgroup of G.

Let C be the centraliser of K in G. By applying Lemma 3.3 with $Z = Y = G$ and $X = K$, we can conclude that $C = K \times T$ for some normal subgroup T of G, isomorphic to a subgroup of the quotient G/K. Since G/K is cyclic of order m, T is cyclic of order dividing m.

Further, the quotient G/C is isomorphic to the quotient of G/K by C/K, and since $C/K \cong T$ and G/K is cyclic of order m, the quotient G/C is a cyclic group of order $m/|T|$. However, G/C embeds into $\mathrm{Aut}(K)$, which is isomorphic to $\mathrm{GL}(r+s,2)$. It is well known that every cyclic subgroup of $\mathrm{GL}(r+s,2)$ is of order at most $2^{r+s}-1$ (see for example [11, Corollary 2.7]), implying that $m/(2^{s+r}-1) \le |T|$. □

3.2 Proof of Theorem 1.1

As in Theorem 1.1, let Γ be a connected $(G,2)$-arc-transitive 2-valent digraph of order $2^a q^b p^c$, where p and q are distinct odd primes, $a \in \{0,1,2,3\}$, $b,c \in \{0,1,2\}$, $(b,c) \neq (2,2)$, and G is non-solvable. We need to show that Γ is isomorphic to one of the digraphs in Table 1.

All such digraphs of order up to 1000 can be found by inspecting the census [28] of arc-transitive digraphs of valence 2. It transpires that there are precisely fifty-two of them, and they are all listed in Table 1 as digraphs labelled ATD. We may thus assume throughout the proof that $|\mathrm{V}(\Gamma)| > 1000$.

Suppose that G acts transitively on the s-arcs but not on the $(s+1)$-arcs of Γ. Then $|G_v| = 2^s$ (see Lemma 3.1) and therefore $|G| = 2^{a+s}q^b p^c$. Now consider a composition series $\mathbf{1} = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_k = G$ of G, and the corresponding set of composition factors $F_i = G_i/G_{i-1}$ for $i \in \{1,\ldots,k\}$. Recall that F_i are simple groups. Since G is non-solvable, there exists $j \in \{1,\ldots,k\}$ such that F_j is non-abelian. Let $T = F_j$ and note that $|T|$ divides $|G|$, which equals $2^{a+s}q^b p^c$.

It is known that there are precisely eight non-abelian simple groups whose orders are divisible by at most three distinct primes (see, for example, [12]); these are $\mathrm{Alt}(5)$, $\mathrm{PSL}(2,7)$, $\mathrm{Alt}(6)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,17)$, $\mathrm{PSL}(3,3)$, $\mathrm{PSU}(3,3)$, and $\mathrm{PSU}(4,2)$. Out of these, only the first five are such that the odd primes appear with multiplicity at most 2; these five groups, together with their orders and the orders of their automorphism groups are listed in Table 2.

T	$\lvert T\rvert$	$\lvert \mathrm{Aut}(T)\rvert$
$\mathrm{Alt}(5)$	$2^2 \cdot 3 \cdot 5$	$2^2 \cdot 3 \cdot 5 = 120$
$\mathrm{PSL}(2,7)$	$2^3 \cdot 3 \cdot 7$	$2^4 \cdot 3 \cdot 7 = 336$
$\mathrm{Alt}(6)$	$2^3 \cdot 3^2 \cdot 5$	$2^5 \cdot 3^2 \cdot 5 = 1440$
$\mathrm{PSL}(2,8)$	$2^3 \cdot 3^2 \cdot 7$	$2^3 \cdot 3^3 \cdot 7 = 1512$
$\mathrm{PSL}(2,17)$	$2^4 \cdot 3^2 \cdot 17$	$2^5 \cdot 3^2 \cdot 17 = 4896$

Table 2: Simple groups of orders divisible by three primes only, with odd part cube-free

Observe that the order of each of these groups is divisible by 3 and that the other odd prime divisor is $5, 7$, or 17. We may thus assume without loss of generality that $q = 3$ and $p \in \{5, 7, 17\}$.

If $p = 5$, then $|\mathrm{V}(\Gamma)| \leq 8 \cdot 3 \cdot 5^2 = 600$, contradicting our initial assumption. This rules out the groups $\mathrm{Alt}(5)$ and $\mathrm{Alt}(6)$ as possibilities for T.

If $p = 7$, then the order $2^a 3^b 7^c$ of Γ is larger than 1000 only when $a = 3$, $b = 1$ and $c = 2$. Since 9 divides the order of $\mathrm{PSL}(2,8)$, this implies that $T \not\cong \mathrm{PSL}(2,8)$, and therefore $T \cong \mathrm{PSL}(2,7)$ and $|V(\Gamma)| = 8 \cdot 3 \cdot 7^2 = 1176$.

Finally, if $p = 17$, then $T \cong \mathrm{PSL}(2,17)$, and since 3^2 divides the order of $\mathrm{PSL}(2,17)$, it follows that the order of Γ is $8 \cdot 3^2 \cdot 17 = 1224$.

We shall now distinguish two cases, depending on whether G contains a non-trivial abelian normal subgroup or not.

Case I. Suppose that G contains a non-trivial abelian normal subgroup. Then G contains a minimal normal subgroup N that is abelian. Since G is non-solvable, Γ is not isomorphic to a $\overrightarrow{\mathrm{PX}}$-digraph. In view of Lemma 2.12, N is then semiregular, and thus $\wp\colon \Gamma \to {}^{\Gamma}/_{\mathrm{N}}$ is a regular covering projection.

If N is a 2-group, then, since the 2-part of $|\mathrm{V}(\Gamma)|$ is 8, we see that $|N| \in \{2, 4, 8\}$. The possible orders of ${}^{\Gamma}/_{\mathrm{N}}$ are then 147 and 153 (when $|N| = 8$, and $T \cong \mathrm{PSL}(2,7)$ and $\mathrm{PSL}(2,17)$, respectively), 294 and 306 (when $|N| = 4$), and 588 and 612 (when $|N| = 2$).

Now suppose that N is of odd order. Since N is solvable, T is a composition factor of ${}^{G}/_{\mathrm{N}}$ and thus $|T|$ divides $|G|/|N| = 2^{a+s}q^b p^c/|N|$. Since $|N|$ is odd and $b + c \leq 3$, it follows that the odd part of $|T|$ is of the form $q^{b'}p^{c'}$ where $b' + c' \leq 2$; in particular, $T \not\cong \mathrm{PSL}(2,17)$, and therefore $T \cong \mathrm{PSL}(2,7)$, $|N| = 3$ or $|N| = 7$, and $|\mathrm{V}({}^{\Gamma}/_{\mathrm{N}})| = 2^a \cdot 3 \cdot 7 \leq 168$. In fact, since we have already established that $|\mathrm{V}(\Gamma)| = 8 \cdot 3 \cdot 7^2$ when $T \cong \mathrm{PSL}(2,7)$, it follows that $|N| = 7$ and $|\mathrm{V}({}^{\Gamma}/_{\mathrm{N}})| = 168$.

We have thus shown that in Case I, we have $|\mathrm{V}({}^{\Gamma}/_{\mathrm{N}})| \in \{147, 153, 168, 294, 306, 588, 612\}$ and therefore the quotient digraph ${}^{\Gamma}/_{\mathrm{N}}$ appears in the census [28]. By searching the census for 2-arc-transitive digraphs of these orders with a non-solvable automorphism group, one sees that the triple $(T, |N|, {}^{\Gamma}/_{\mathrm{N}})$ is as one given in Table 3 (here the data in the last column corresponds to the names of digraphs as given in [28]).

Using the methods described in, say, [14, 32], for each of the digraphs ${}^{\Gamma}/_{\mathrm{N}}$ from Table 3, all the corresponding N-regular covers were computed for which a 2-arc-transitive subgroup of $\mathrm{Aut}({}^{\Gamma}/_{\mathrm{N}})$ lifts, and the resulting nine covering digraphs were included in Table 1 under the names $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_9$.

Case II. Suppose now that G contains no non-trivial abelian normal subgroups. Let us now consider the group generated by all minimal normal subgroups of G, called the *socle* of G and denoted $\mathrm{soc}(G)$. Since G contains no non-trivial abelian normal subgroups, it

T	$\|N\|$	Γ/N
$\mathrm{PSL}(2,7)$	2	ATD[588;87], ATD[588;90], ATD[588;91]
$\mathrm{PSL}(2,7)$	4	ATD[294;19]
$\mathrm{PSL}(2,7)$	8	order 147; none
$\mathrm{PSL}(2,7)$	7	ATD[168;53], ATD[168;64], ATD[168;65], ATD[168;81], ATD[168;82]
$\mathrm{PSL}(2,17)$	2	ATD[612;48], ATD[612;49]
$\mathrm{PSL}(2,17)$	4	ATD[306;11]
$\mathrm{PSL}(2,17)$	8	order 153; none

Table 3: Possible quotients of Γ by a minimal abelian normal subgroup

follows that $\mathrm{soc}(G)$ is a direct product of non-abelian simple groups (see, for example, [7, Theorem 4.3A]). Since the order of every non-abelian simple group is divisible by at least three distinct primes, and since not both b and c are 2, $\mathrm{soc}(G)$ is a simple normal subgroup of G and is therefore isomorphic to the non-abelian composition factor T of G.

Moreover, G acts faithfully by conjugation on $\mathrm{soc}(G)$ and thus embeds into its automorphism group. Since $\mathrm{soc}(G)$ is isomorphic to either $\mathrm{PSL}(2,7)$ or $\mathrm{PSL}(2,17)$, we see that G is isomorphic to one of $\mathrm{PSL}(2,7)$, $\mathrm{PGL}(2,7)$, $\mathrm{PSL}(2,17)$ or $\mathrm{PGL}(2,17)$. On the other hand, recall that $|G| = 2^{a+s}q^bp^a$ and that $a = 3$ and $s \geq 2$, implying that $|G|$ is divisible by 2^5. This rules out all but the last possibility, that is $G \cong \mathrm{PGL}(2,17)$. Since, in this case, $|\mathrm{V}(\Gamma)| = 2^3 \cdot 3^2 \cdot 17$ and $|G| = 2^5 \cdot 3^2 \cdot 17$, it follows that $|G_v| = 4$. By Lemma 3.1, G_v is elementary abelian. In particular, Γ is a coset digraph of G with respect to an elementary abelian subgroup of order 4 and a non-self-paired suborbit of length 2. A direct inspection of the appropriate subgroups of $\mathrm{PGL}(2,17)$ and their coset digraphs reveals that there are six pairwise non-isomorphic digraphs arising in this way. They are listed in Table 1 as digraphs $\mathrm{X}_{10}, \mathrm{X}_{11}, \ldots, \mathrm{X}_{15}$. This concludes the proof of Theorem 1.1.

3.3 Proof of Theorem 1.2

We shall say that a positive integer n satisfies condition (i), (ii) or (iii), respectively, if the following holds:

(i) n is odd and cube-free;

(ii) $n = 2^a m$, where $a \in \{1,2,3\}$ and m is an odd, square-free integer;

(iii) $n = 2^a q^b p^2$, where $a \in \{1,2,3\}$, $b \in \{0,1\}$ and p, q are distinct odd primes.

As in the statement of Theorem 1.2, we assume that Γ is a connected 2-valent $(G,2)$-arc-transitive digraph with G solvable, and that one of the conditions (i), (ii) or (iii) holds for $n = |\mathrm{V}(\Gamma)|$. We need to show that either:

(a) Γ is a $\overrightarrow{\mathrm{PX}}$-digraph; or that

(b) n satisfies the condition (iii) and G contains a normal Sylow p-subgroup P, which is elementary abelian of order p^2 and such that Γ/P is a $\overrightarrow{\mathrm{PX}}$-digraph.

Suppose that the theorem is false and let Γ be a minimal counter-example (in terms of n). In particular, Γ is not a $\overrightarrow{\mathrm{PX}}$-digraph. By Lemma 2.1, Γ is then simple. Since G acts transitively on the vertex-set of Γ and since the vertex-stabiliser G_v is of order 2^s for some $s \geq 2$ (see Lemma 3.1), it follows that $|G| = |G_v|n = 2^s n$.

We shall now prove a few facts about Γ and G, finally resulting in a contradiction.

Fact 0: *If N is a semiregular normal subgroup of G, then Γ/N is a $\overrightarrow{\mathrm{PX}}$-digraph, or $n/|N|$ (and thus also n) satisfies the condition (iii) and the Sylow p-subgroup of G/N is elementary abelian of order p^2 and normal in G/N.*

Proof: Since N is semiregular, by Lemma 2.6, $\Gamma \to \Gamma/\mathrm{N}$ is a covering projection, and by Lemma 2.5, Γ/N is a connected 2-valent $(G/\mathrm{N}, 2)$-arc-transitive digraph. Moreover, since every divisor of an integer satisfying one of the conditions (i), (ii), or (iii) also satisfies one of these conditions, the minimality of the counterexample Γ implies that either Γ/N is a $\overrightarrow{\mathrm{PX}}$-digraph or that $n/|N|$ satisfies the condition (iii) and the Sylow p-subgroup of G/N is indeed as claimed.

Fact 1: *n does not satisfy the condition (i); in particular, n is even.*

Proof: Assume the contrary (that is, n is odd and cube-free). Since n is odd, the vertex-stabiliser in G is a Sylow 2-subgroup of G, and every 2-subgroup of G is contained in some vertex-stabiliser in G.

Let N be a minimal normal subgroup of G. Since G is solvable, N is elementary abelian. If N is a 2-group, then $N \leq G_v$ for some vertex v, and thus the action of G on the vertices of Γ is not faithful, implying that Γ is not simple, a contradiction.

Hence N is an elementary abelian group of odd order, and thus acts semiregularly on the vertices of Γ. By Fact 0, Γ/N is a $\overrightarrow{\mathrm{PX}}$-digraph, and since its order is odd, it must be isomorphic to $\overrightarrow{\mathrm{PX}}(n', 0)$ where $n' = n/|N|$. Further, by Lemma 2.9 (note that $\mathrm{rad}(\Gamma/\mathrm{N}) = 1 \neq \mathrm{rad}(\Gamma)$), we see that N is not of prime order. Since the order of Γ is cube-free, it follows that N is elementary abelian of order p^2 for some odd prime p.

Let us now consider the group G/N acting on Γ/N. Since $\Gamma/N \cong \overrightarrow{\mathrm{PX}}(n', 0)$, by Lemma 2.2, the stabiliser $(G/N)_{v^N}$ of a vertex v^N of Γ/N is elementary abelian and normal in G/N. Note also that $(G/N)_{v^N} = G_vN/N$, implying that G_vN is normal in G.

Let C be the centraliser of N in G_vN. If we apply Lemma 3.3 with $X = N$, $Y = G_vN$ and $Z = G$, we see that $C = N \times T$ for some normal subgroup T of G, isomorphic to a subgroup of $Y/N \cong G_v$. In particular, T is a 2-group. Since the order of Γ is odd and T is a 2-group, T fixes a vertex of Γ, and being normal in G, it acts trivially on the vertex-set of Γ. Since Γ is a simple digraph, it follows that $T = 1$, and thus $C = N$.

Since G_vN is normal in G and contains G_v, it contains G_u for every vertex $u \in \mathrm{V}(\Gamma)$. In particular, G_vN contains every involution of G. Together with the fact that N is self-centralising in G_vN this implies that no involution of G centralises N.

Now consider the centraliser D of N in G. We have just shown that D has odd order, implying that D is semiregular, and thus, $\Gamma/D \cong \overrightarrow{\mathrm{PX}}(n'', 0)$ for some odd integer n''. Moreover, since G/D acts 2-arc-transitively on Γ/D, the Sylow 2-subgroup S of G/D is the vertex-stabiliser of every vertex of Γ/D, and is thus normal in Γ/D, elementary abelian, and of order at least 4. On the other hand G/D embeds into $\mathrm{Aut}(N) \cong \mathrm{GL}(2, p)$. By Lemma 3.2, it follows that S is of order 4 and contains an involution that is central in G/D. However, by Lemma 2.2, this implies that n'' is even. This contradiction concludes the proof of Fact 1.

Fact 2: *The group G does not contain a normal elementary abelian subgroup of order p^2 for any odd prime p.*

Proof: Assume the contrary and note that in view of Fact 1, n then satisfies the condition (iii); that is, $n = 2^a q^b p^2$ for some $a \leq 3$ and $b \leq 1$. Moreover, G contains a normal elementary abelian subgroup P of order p^2. Since p is odd, P is semiregular, and by Fact 0, either $^{\Gamma}/_{P}$ is a $\overrightarrow{\mathrm{PX}}$-digraph, or $n/|P| = 2^a q^b$ satisfies the condition (iii). The latter is clearly false, while the former implies that the conclusion (b) of Theorem 1.2 holds for Γ, a contradiction.

Fact 3: $\mathrm{rad}(\Gamma) \geq 3$*; that is, the alternating cycles of* Γ *are of length at least* 6.

Proof: Assume the contrary; that is, $\mathrm{rad}(\Gamma) < 3$. Since Γ is simple, we have $\mathrm{rad}(\Gamma) \neq 1$. Hence $\mathrm{rad}(\Gamma) = 2$, and by Lemma 2.11, it follows that $\Gamma \cong \mathrm{Pl}(\Lambda)$ for some connected 2-valent $(G, 3)$-arc-transitive digraph Λ of order $\frac{1}{2}n$. If Λ is a $\overrightarrow{\mathrm{PX}}$-digraph, then by formula (2.2), so is Γ, a contradiction. By the minimality of the counterexample Γ, this implies that conclusion (b) holds for the pair (Λ, G) in place of (Γ, G), and in particular, that G contains a normal elementary abelian subgroup of order p^2 for some odd prime p. However, the latter contradicts Fact 2.

Fact 4: *The group G contains no normal subgroup of odd prime order.*

Proof: Suppose the contrary and let N be a normal subgroup of G of odd prime order q. Since G_v is a 2-group, N is semiregular. By Fact 3 and Lemma 2.9, $^{\Gamma}/_{N}$ is not a $\overrightarrow{\mathrm{PX}}$-digraph. But then Fact 0 implies that $n = 2^a q p^2$ and the Sylow p-subgroup $\tilde{P}$ of $^{G}/_{N}$ is normal in $^{G}/_{N}$ and isomorphic to $\mathbb{Z}_p^2$.

Let Q be the preimage of $\tilde{P}$ with respect to the quotient projection $G \to {}^{G}/_{N}$. Then Q is a normal subgroup of G of order qp^2, containing the normal subgroup N of order q. Let C be the centraliser of N in Q. Since N is abelian and since N has order coprime to its index in Q, we may apply Lemma 3.3 with $Z = G$, $Y = Q$ and $Z = N$, to conclude that $C = N \times P$ for some normal subgroup P of G, isomorphic to a normal subgroup of $^{Q}/_{N}$. Since the latter is isomorphic to $\tilde{P}$, we see that P is either trivial, cyclic of order p, or isomorphic to $\mathbb{Z}_p^2$.

If P is trivial, then $C = N$, and $^{Q}/_{N}$ embeds into $\mathrm{Aut}(N)$, implying that $^{Q}/_{N}$ is cyclic. However, $^{Q}/_{N}$ is isomorphic to $\tilde{P}$, which is isomorphic to $\mathbb{Z}_p^2$, a contradiction. Further, by Fact 2, the order of P is not p^2. This leaves us with the possibility that $|P| = p$.

Now consider the quotient $^{\Gamma}/_{P}$. Since the order of $^{\Gamma}/_{P}$ is $2^a qp$, Fact 0 implies that $^{\Gamma}/_{P}$ is a $\overrightarrow{\mathrm{PX}}$-digraph. But then, by Lemma 2.9, $\mathrm{rad}(\Gamma) = \mathrm{rad}(^{\Gamma}/_{P})$, which is at most 2, since $^{\Gamma}/_{P}$ is a $\overrightarrow{\mathrm{PX}}$-digraph, contradicting Fact 3.

Fact 5: *If N is a minimal normal subgroup of G, then N is semiregular and of order* 2 *or* 4. *If* $|N| = 2$, *then* $^{\Gamma}/_{N} \cong \overrightarrow{\mathrm{PX}}(m, 2)$, *and if* $|N| = 4$, *then* $^{\Gamma}/_{N} \cong \overrightarrow{\mathrm{PX}}(m, 1)$ *for some odd integer m. Moreover,* $\exp(\Gamma) = 2$.

Proof: Let m be the odd part of n. By Fact 1, $n = 2^a m$ where $a \geq 1$ and m is cube-free. Let N be a minimal normal subgroup of G. Since G is solvable, N is elementary abelian, and since $|G| = 2^{a+s} m$, Facts 2 and 4 imply that N is a 2-group. If N is not semiregular, then by Lemma 2.12, Γ is a $\overrightarrow{\mathrm{PX}}$-digraph, contradicting our assumptions. Hence N is semiregular, and thus $|N|$ divides n, and therefore $|N| = 2^t$ for some integer t satisfying $1 \leq t \leq a$.

By Fact 0, either $^{\Gamma}/_{N}$ is a $\overrightarrow{\mathrm{PX}}$-digraph, or $n/|N|$ satisfies the condition (iii) and the group $^{G}/_{N}$ contains a normal elementary abelian subgroup $\tilde{P}$ of order p^2.

Suppose first that the latter case occurs. Then $n/|N| = 2^{a-t} q^b p^2$ where $a - t \geq 1$. Since $a \leq 3$, this implies that $t \in \{1, 2\}$. As in the proof of Fact 4, let Q be the preimage of $\tilde{P}$ with respect to the quotient projection $G \to {}^{G}\!/\mathrm{N}$. Then Q is a normal subgroup of G of order $2^t p^2$, containing the normal subgroup N of order 2^t. Now consider the centraliser C of N in Q, apply Lemma 3.3 with $Z = G$, $Y = Q$ and $X = N$, and conclude that $C = N \times P$ for some (possibly trivial) p-group P which is normal in G. If P is trivial, then ${}^{Q}\!/\mathrm{N} \cong \tilde{P} \cong \mathbb{Z}_p^2$ embeds into $\mathrm{Aut}(N) \cong \mathrm{GL}(t, 2)$. Since $t \le 2$, this is clearly not the case. Hence P is non-trivial, contradicting either Fact 2 or Fact 4.

This contradiction shows that the former case occurs, that is ${}^{\Gamma}\!/\mathrm{N} \cong \overrightarrow{\mathrm{PX}}(2^{a-t-r}m, r)$ for some integer r such that $0 \le r \le a - t$. Let $\Lambda = {}^{\Gamma}\!/\mathrm{N}$ and let $\wp\colon \Gamma \to \Lambda$ be the corresponding quotient projection. Since $a \le 3$ and $t \ge 1$, we see that $r \le 2$.

If $r = 0$, then Lemma 2.13 implies that Γ is a $\overrightarrow{\mathrm{PX}}$-digraph, a contradiction.

If $r = 1$, either $a = 2$ and $t = 1$, or $a = 3$ and $t \in \{1, 2\}$. Let $v \in \mathrm{V}(\Gamma)$ and let $v' = \wp(v)$. Observe that $\exp(\Lambda) = 1$ (see Lemma 2.14) and $|A_i^{\Lambda}(v')| = 2$ for every $i \ge 1$. By Lemma 2.8, it follows that $|A_i^{\Gamma}(v)|$ divides $2^t|A_i^{\Lambda}(v')| = 2^{t+1} \le 8$ for every $i \ge 1$. Since $|A_1^{\Gamma}(v)| = \mathrm{rad}(\Gamma) \ge 3$, it follows that $|A_1^{\Gamma}(v)| \in \{4, 8\}$. If $|A_1^{\Gamma}(v)| = |A_2^{\Gamma}(v)|$, then $\exp(\Gamma) = 1$, and by Lemma 2.4, Γ is a $\overrightarrow{\mathrm{PX}}$-digraph, a contradiction. Hence $|A_1^{\Gamma}(v)| < |A_2^{\Gamma}(v)|$, implying that $|A_1^{\Gamma}(v)| = 4$ and $|A_i^{\Gamma}(v)| = 8$ for every $i \ge 2$ (hence $\exp(\Gamma) = 2$). Moreover, since $8 = |A_2^{\Gamma}(v)|$ divides 2^{t+1}, we see that $t = 2$ and $a = 3$, implying that $\Lambda = \overrightarrow{\mathrm{PX}}(m, 1)$, as claimed.

Similarly, if $r = 2$, then $a = 3$, $t = 1$ and $\Lambda = \overrightarrow{\mathrm{PX}}(m, 2)$. Hence $\exp(\Lambda) = 2$, $|A_1^{\Lambda}(v')| = 2$ and $|A_i^{\Lambda}(v')| = 4$ for every $i \ge 2$. Moreover, as above, $|A_1^{\Gamma}(v)| \ge 3$ and $|A_1^{\Gamma}(v)| < |A_2^{\Gamma}(v)|$. In view of Lemma 2.8, it thus follows that $|A_1^{\Gamma}(v)| = 2|A_1^{\Lambda}(v')| = 4$, and $|A_2^{\Gamma}(v)| = |A_{\infty}^{\Gamma}(v)| = 8$. In particular, $\exp(\Gamma) = 2$, as claimed. This concludes the proof of Fact 5.

Fact 6: *The order* n *of* Γ *is at most* 744.

Proof: Let N be a minimal normal subgroup of Γ and recall Fact 5. Since $\exp(\Gamma) = 2$, [30, Theorem 7.1] implies that $|G_v| \le 2^4$. By Lemma 2.5, also the stabiliser $({}^{G}\!/\mathrm{N})_{v^N}$ has order at most 2^4. By Lemma 3.4, this implies that ${}^{G}\!/\mathrm{N}$ contains a normal cyclic group $\tilde{Y}$ whose order is ℓ for some odd integer ℓ satisfying

$$\ell \ge m/(2^{\alpha+4} - 1), \qquad (*)$$

where α is either 1 or 2, depending on whether $|N| = 4$ or $|N| = 2$, respectively. Let $Y \le G$ be the preimage of $\tilde{Y}$ with respect to the quotient projection $G \to {}^{G}\!/\mathrm{N}$, let C be the centraliser of N in Y, and apply Lemma 3.3 to deduce that $C = N \times T$ for some cyclic group $T \trianglelefteq G$ of order dividing ℓ. Since T is cyclic, every subgroup of T is characteristic in T and thus normal in G. If T is non-trivial, this implies that G contains a normal subgroup of odd prime order, contradicting Fact 4. Hence $T = 1$, and $C = N$.

If $|N| = 2$, then $\alpha = 2$, N is central in Y, and $Y = C = N$. However, $\ell = |Y|/|N|$, and thus $\ell = 1$. In view of $(*)$, we see that $m \le 2^{2+4} - 1 \le 63$, and therefore $n = |N||\,\mathrm{V}({}^{\Gamma}\!/\mathrm{N})| = 2|\,\mathrm{V}(\overrightarrow{\mathrm{PX}}(m, 2))| = 8m \le 504$.

If $|N| = 4$, then $\alpha = 1$, and by $(*)$, we see that $m \le 31\ell$, and thus $n = 4|\mathrm{V}(\overrightarrow{\mathrm{PX}}(m, 1))| = 8m \le 248\ell$. On the other hand, since $C = N$, the cyclic group Y/N of order ℓ embeds into $\mathrm{Aut}(N) \cong \mathrm{GL}(2, 2) \cong \mathrm{Sym}(3)$, and thus $\ell \le 3$. But then $n \le 3 \cdot 248 = 744$. This concludes the proof of Fact 6.

Since a census of all simple arc-transitive digraphs of valence 2 is available in [28], we can easily see that no counter-example to the theorem of order at most 1000 exists. This, however, contradicts Fact 6, and thus proves the theorem.

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Testing whether the lifted group splits

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Abstract

Let a group of automorphisms lift along a regular covering projection of connected graphs given combinatorially by means of voltages. The data that determine the lifted group and its action are then conveniently encoded in terms of voltages as well. Along these lines, an algorithm for testing whether the lifted group is a split extension of the group of covering transformations has recently been proposed in the case when the group of covering transformations is solvable. It consists of decomposing the covering into a series of coverings with elementary abelian groups of covering transformations, and inductively solving the problem at every elementary abelian step. Although the explicit construction of the lifted group is not needed, it still involves time and space consuming constructions of certain subgroups in the lifted group at every step except at the final one.

In this paper, an improved version that completely avoids such constructions is presented. From voltage distribution we first compute the weak action and the factor set that determine the lifted group, and we then carry out the test by extracting the necessary information only from the corresponding weak actions and factor sets at every step. An experimental comparison is made against the previous version.

Keywords: Algorithm, graph, group extension, lifting automorphisms, regular covering projection, voltages.

Math. Subj. Class.: 05C50, 05E18, 20B40, 20B25, 20K35, 57M10

1 Introduction

Group extensions arising from lifting groups of automorphisms along regular graph coverings play a significant role in analyzing symmetry properties of graphs; see, for example, [5, 6, 9, 10, 13, 16, 19]. One therefore frequently needs to answer questions regarding structural properties of such extensions.

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Specifically, let a group G of automorphisms of a graph X lift along a regular covering projection $\wp\colon \tilde{X} \to X$ to a group $\tilde{G}$ of automorphisms of the covering graph $\tilde{X}$. Then the lifted group $\tilde{G}$ is an extension of the group of covering transformations $\mathrm{CT}(\wp)$ by G. Often, all of the data about the lifted group and its action are conveniently encoded on X by means of voltages that determine $\wp$. In such a situation we can always reconstruct $\tilde{G}$ as a permutation group acting on $\tilde{X}$, and then apply the known algorithms for permutation groups in order to investigate its structure. However, taking into account complexity issues, this reconstruction is expensive whenever $\mathrm{CT}(\wp)$ is large. Instead, we wish to reduce the investigation of structural properties of $\tilde{G}$ to the study of voltage distribution on X. A natural question of interest is then the following: for a group G that lifts along $\wp$ given by means of voltages, is the lifted group $\tilde{G}$ a split extension of $\mathrm{CT}(\wp)$ by G?

There are efficient algorithms in computational group theory for testing whether a given group extension splits (see, for example, [3] and [8, Chapters 7 and 8]), and these functions have also been implemented in MAGMA [1]. Unfortunately, the algorithms as well as the implementations address the case when extensions are input as permutation groups.

In [15], an algorithm for testing whether the lifted group $\tilde{G}$ splits is described in the case when $\mathrm{CT}(\wp)$ is (elementary) abelian. It is based on extracting all the necessary information about $\tilde{G}$ from voltage distribution, rather then explicitly constructing $\tilde{G}$ as a permutation group.

This idea is taken further in [17] to deal with the case of a solvable $\mathrm{CT}(\wp)$. The algorithm consists of decomposing $\wp$ into a series of regular covering projections with elementary abelian groups of covering transformations, and inductively applying the algorithm from [15] at every elementary abelian step. Although the explicit construction of $\tilde{G}$ is not needed, the algorithm still involves time and space consuming constructions of certain subgroups isomorphic to G in the lifted group at (possibly) every step except at the finale one.

In this paper, we improve the algorithm from [17] by avoiding such constructions entirely. The approach is based on the fact that a group extension can be recaptured by have it written as a crossed product extension in terms of the corresponding weak action and a factor set. As a first step we compute the weak action and the factor set corresponding to $\tilde{G}$ from voltage distribution. At each step, we then carry out our test by extracting all the necessary information only from the corresponding weak actions and the factor sets.

The paper is organized as follows. In Section 2 we review some preliminary concepts about regular graph coverings and lifting automorphisms as well as group extensions. In Section 3 we discuss the problem of testing whether an extension splits in terms of weak actions and factor sets. In Section 4 we then propose an improved algorithm for testing whether the lifted group splits. Finally, we evaluate the performance of our algorithm in comparison with the previous version [17] in Section 5. Experimental results confirm the effectiveness of the improvements made.

2 Preliminaries

We begin with a review of some basic concepts in order to fix the notation and terminology.

2.1 Regular graph covers and lifts of automorphism

Throughout the paper, graphs are finite, simple and undirected. For a graph X we denote by $V(X)$, $A(X)$ its vertex and arc set, respectively. The full automorphism group of X is denoted by $\mathrm{Aut}(X)$. For a detailed treatment of graph coverings and lifting automorphism we refer the reader to [7, 12, 14].

A surjective graph homomorphism $\wp\colon \tilde{X} \to X$ is called a *regular covering projection* if there exists a semiregular subgroup $S_\wp$ of $\mathrm{Aut}(\tilde{X})$ such that its vertex orbits coincide with the *vertex fibres* $\wp^{-1}(v)$, $v \in V(X)$. In this setting we call X a *base graph*, and $\tilde{X}$ a *covering graph* (or a *cover*). Regular covering projections $\wp\colon \tilde{X} \to X$ and $\wp'\colon \tilde{X}' \to X$ are *equivalent* if there exists a graph isomorphism $\tilde{g}\colon \tilde{X} \to \tilde{X}'$ such that $\wp = \tilde{g}\wp'$.

An automorphism $g \in \mathrm{Aut}(X)$ *lifts along* $\wp\colon \tilde{X} \to X$ if there exists an automorphism $\tilde{g} \in \mathrm{Aut}(\tilde{X})$, called a *lift* of g, such that $\tilde{g}\wp = \wp g$. A group $G \le \mathrm{Aut}(X)$ lifts if each $g \in G$ lifts. The collection of all lifts of all elements in G forms a subgroup $\tilde{G} \le \mathrm{Aut}(\tilde{X})$, called the *lift* of G or the *lifted group*. In particular, the lift of the trivial group, denoted by $\mathrm{CT}(\wp)$, is known as the *group of covering transformations*. If $\mathrm{CT}(\wp)$ is an elementary abelian or a solvable group, the regular covering projection $\wp$ is called *elementary abelian* or *solvable*, respectively. Observe that $\mathrm{CT}(\wp)$ is a normal subgroup of $\tilde{G}$ and that $\tilde{G}/\mathrm{CT}(\wp) \cong G$, so $\tilde{G}$ is an extension of $\mathrm{CT}(\wp)$ by G.

Regular covering projections can be grasped combinatorially as follows. Let N be a (finite) group. Define a *voltage function* $\zeta\colon A(X) \to N$ such that $\zeta(v_2, v_1) = (\zeta(v_1, v_2))^{-1}$ for each $(v_1, v_2) \in A(X)$; that is, a function assigning mutually inverse elements in N to mutually inverse arcs in X. We call N the *voltage group*, while the values of ζ are called *voltages*. Further, construct the *derived graph* $X \times_\zeta N$ with vertex set $V(X) \times N$ and adjacency relation $(v_1, n) \sim (v_2, n\zeta(v_1, v_2))$ whenever $v_1 \sim v_2$. The projection

$$\wp_\zeta\colon X \times_\zeta N \to X, \;\; (v, n) \mapsto v,$$

is then the *derived regular covering projection*, where the required semiregular subgroup $S_{\wp_\zeta}$ of $\mathrm{Aut}(X \times_\zeta N)$ arises from the action of N on the second coordinate by left multiplication on itself. Conversely, with any regular covering projection $\wp\colon \tilde{X} \to X$ there is an *associated* voltage function ζ on X such that the derived covering projection $\wp_\zeta$ is equivalent to $\wp$. Since both graphs $\tilde{X}$ and X are connected, the voltage function ζ associated with the projection $\wp$ is valued in $N \cong \mathrm{CT}(\wp)$ (viewed as an abstract group).

The fact that an automorphism lifts along a projection $\wp$ if and only if it lifts along along any covering projection equivalent to $\wp$ allows us to study lifts of automorphisms combinatorially in terms of voltage functions. Let $\zeta\colon A(X) \to N$ be a voltage function associated with a regular covering projection $\wp\colon \tilde{X} \to X$ of connected graphs. We note that ζ can be naturally extended to walks: if $W = v_1 v_2 \cdots v_{n-1} v_n$ is a walk in X, then $\zeta W = \zeta(v_1, v_2) \cdots \zeta(v_{n-1}, v_n)$. By the *basic lifting lemma*, see [12, 14], $g \in \mathrm{Aut}(X)$ lifts along $\wp$ if and only if there exists an automorphism $g^{\#_v}$ of N such that

$$g^{\#_v}(\zeta W) = \zeta g(W)$$

for all closed walks W in X rooted at a fixed vertex v. Of course, if g lifts, $g^{\#_v}$ is uniquely determined by a map $\zeta W^* \mapsto \zeta g(W^*)$, where W^* ranges over all fundamental closed walks in X rooted at v.

2.2 Group extensions

A group E is called a (group) *extension* of a group N by a group G if there is a short exact sequence

$$1 \to N \xrightarrow{i} E \xrightarrow{q} G \to 1.$$

It is called a *split extension* if there is a homomorphism $j\colon G \to E$ with $qj = \mathrm{id}$. In particular, the group E having a normal subgroup N is an extension of N by E/N, and it is a split extension if there is a *transversal* of N in E – a system of representatives in E of cosets of N in E – that forms a group. Such a group is called a *complement* of N in E. Group extensions E and E' of N by G are *equivalent* if there exists an isomorphism $\alpha\colon E \to E'$ such that the diagram

$$\begin{array}{ccccc} N & \longrightarrow & E & \longrightarrow & G \\ {\scriptstyle \mathrm{id}}\big\downarrow & & {\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ N & \longrightarrow & E' & \longrightarrow & G \end{array}$$

is commutative. Of course, if E and E' are equivalent extensions, then E is split if and only if E' is split.

Suppose that the group E has a normal subgroup N. All of the data that determine the group operation in E can be, up to equivalence of extensions, given in terms of N and $G = E/N$. The approach is known and goes back to Schreier [11]. For each $g \in G$ fix a coset representative $\bar{g}$ in E such that $\bar{g}N = g$. Since N is normal, the element $\bar{g}$ gives rise to an automorphism $g^{\#}$ of N defined by $g^{\#}(n) = \bar{g}\, n\, \bar{g}^{\,-1}$. Clearly, this definition depends on the choice of $\bar{g}$, and hence the function

$$\#\colon G \to \mathrm{Aut}(N), \quad g \mapsto g^{\#},$$

called *a weak action*, is not a group homomorphism in general. Further, the fact that the elements $\{\bar{g} \mid g \in G\}$ form a transversal of N in E implies that for any $g_1, g_2 \in G$ we have $\overline{g_1}\,\overline{g_2} = \mathcal{F}(g_1, g_2)\overline{g_1 g_2}$ for some unique $\mathcal{F}(g_1, g_2) \in N$. The function

$$\mathcal{F}\colon G \times G \to N, \quad (g_1, g_2) \mapsto \overline{g_1}\,\overline{g_2}\,\overline{g_1 g_2}^{\,-1},$$

for this choice of coset representatives is called a *factor set*. It is natural to choose $\bar{1} = 1$. Then $\mathcal{F}(1, 1) = 1$, and such a factor set is called *normalized*. This will be our standard assumption without loss of generality. The weak action $\#$ and the factor set $\mathcal{F}$ defined above determine a group operation on the set $N \times G$; namely, $N \times G$ becomes a group, denoted by $N \operatorname{ext}_{\#,\mathcal{F}} G$, under the multiplication

$$(n_1, g_1) * (n_2, g_2) = (n_1\, g_1^{\#}(n_2)\, \mathcal{F}(g_1, g_2), g_1 g_2). \tag{2.1}$$

In fact, $N \operatorname{ext}_{\#,\mathcal{F}} G$ is an extension of N by G, called the *crossed product extension*, and is equivalent to E. More precisely, there exists an isomorphism

$$N \operatorname{ext}_{\#,\mathcal{F}} G \to E, \quad (n, g) \mapsto n\bar{g}, \tag{2.2}$$

mapping $N \times 1$ onto N and $1 \times G$ onto the transversal $\{\bar{g} \mid g \in G\}$.

3 Testing whether an extension splits

Let N be a normal subgroup of a finite group E, and let $G = E/N$. We first briefly describe a general strategy for testing whether E is a split extension of N by G. In principal we follow [3] and [8, Chapters 7 and 8], however, for reasons that will become apparent in Section 4, we extract the necessary information from the crossed product extension $N \operatorname{ext}_{\#,\mathcal{F}} G$ that reconstructs E.

Let $G = \langle S \mid R \rangle$ be a finite presentation of G, where $S = \{g_1, \ldots, g_n\}$ is a set of generators and $R = \{r_1(g_1, \ldots, g_n), \ldots, r_m(g_1, \ldots, g_n)\}$ is a set of relators – that is, a set of words in generators representing the identity element in G. We note that neither $\#$ is determined uniquely by its values $g_i^{\#}$ for $g_i \in S$, nor $\mathcal{F}$ is determined uniquely by its values $\mathcal{F}(g_i, g_j)$ for $g_i, g_j \in S$. But this is not a problem; as we shall see in (3.2) and (3.3) below, it is enough to only know the images $g_i^{\#}$ of the generators $g_i \in S$ under $\#$, along with some particular images under $\mathcal{F}$.

A general transversal of $N \times 1$ in $N \operatorname{ext}_{\#,\mathcal{F}} G$ has the form $\{(\delta(g), g) \mid g \in G\}$ for a function $\delta\colon G \to N$. The same function also determines a transversal of N in E, namely $\{\delta(g)\bar{g} \mid g \in G\}$, where $\{\bar{g} \mid g \in G\}$ is a transversal of N in E giving rise to the isomorphism $N \operatorname{ext}_{\#,\mathcal{F}} G \to E, (n, g) \mapsto n\bar{g}$, see (2.2).

As it is known, E splits if and only if there exist coset representatives in E of the generators of G satisfying the defining relators of G. More precisely, if and only if, for each g_i in S, there exists an element $\overline{g_i}$ in E such that $\overline{g_i}N = g_i$ and that, for each relator r_j in R, the word $r_j(\overline{g_1}, \ldots, \overline{g_n})$ obtained from r_j by replacing each g_i by $\overline{g_i}$ whenever it appears is a relator of E. In the context of a crossed product extension, $N \operatorname{ext}_{\#,\mathcal{F}} G$ splits if and only if there exists a function $\delta\colon S \to N$ such that, for all $r_j \in R$,

$$r_j((\delta(g_1), g_1), \ldots, (\delta(g_n), g_n)) = (1, 1) \tag{3.1}$$

in $N \operatorname{ext}_{\#,\mathcal{F}} G$. Then the function δ defined on the generators extends to $\delta\colon G \to N$, and a complement is generated by the set $\{(\delta(g_1), g_1), \ldots, (\delta(g_n), g_n)\}$.

Let us now rewrite (3.1) explicitly in terms of the weak action and the factor set. Suppose $r_j = g_{j_1} \cdots g_{j_t} \in R$. Taking into account the multiplication rule (2.1) in $N \operatorname{ext}_{\#,\mathcal{F}} G$, denoted by $*$, and considering $(\delta(g), g)$ as $(\delta(g), 1) * (1, g)$, the condition (3.1) becomes

$$(\delta(g_{j_1}) \prod_{k=2}^{t} g_{j_1}^{\#} \cdots g_{j_{k-1}}^{\#}(\delta(g_{j_k})), 1) * r_j((1, g_1), \ldots, (1, g_n)) = (1, 1). \tag{3.2}$$

In this expression we can explicitly compute $r_j((1, g_1), \ldots, (1, g_n))$ as

$$(\prod_{k=2}^{t-1} g_{j_1}^{\#} \cdots g_{j_{t-k}}^{\#}(\mathcal{F}(g_{j_{t-k+1}}, g_{j_{t-k+2}} \cdots g_{j_t})) \cdot \mathcal{F}(g_{j_1}, g_{j_2} \cdots g_{j_t}), 1). \tag{3.3}$$

Think of values $\delta(g_i)$ as being variables for the moment. Then each relation (3.2) gives rise to an equation in N. It is important to stress out that for the construction of such an equation we only need to know the values $\mathcal{F}(g_{j_k}, g_{j_{k+1}} \cdots g_{j_t})$ and the automorphisms $g_{j_k}^{\#}$ for $k = 1, \ldots, t-1$. Considering all relators $r_j \in R$ thus yields a system of equations, whose solutions correspond to complements. However, solving such a system is rather hopeless in general.

3.1 Elementary abelian case

Let us therefore assume that N is an elementary abelian p-group of rank d. In this case, N can be identify with d-dimensional vector space $\mathbb{Z}_p^d$, the function $\#$ is a homomorphism that defines an action of G on N, and the automorphisms $g^{\#}$ of N are invertible $d \times d$ matrices. We search for a complement by considering each $\delta(g_i)$ in N as a vector with variable entries $x_{i,1}, \ldots, x_{i,d}$. Then each relation gives rise to d linear equations in the variables $x_{i,1}, \ldots, x_{i,d}$. Putting all together we obtain a non-homogeneous system of md equations, whose set of all solutions is in bijective correspondence with all the complements.

3.2 Solvable case

The case when N is solvable can be dealt with by choosing a characteristic series

$$N = N_0 \rhd N_1 \rhd \cdots \rhd N_r = 1$$

such that each factor N_{j-1}/N_j is elementary abelian. The problem reduces into the same problem on N_{j-1}/N_j and N_j inductively down the series. The following theorem is a first step towards this reduction when the extension E is reconstructed as a crossed product extension $N \operatorname{ext}_{\#,\mathcal{F}} G$.

Theorem 3.1. *Let M, N be normal subgroups of a finite group E with $M < N$, and let $G = E/N$.*

(i) If $N \operatorname{ext}_{\#,\mathcal{F}} G$ reconstructs E, then $N/M \operatorname{ext}_{\#_{N/M},\mathcal{F}_{N/M}} G$ reconstructs E/M with

$$\begin{aligned} g^{\#_{N/M}}(nM) &= g^{\#}(n)M \\ \mathcal{F}_{N/M}(g_1, g_2) &= \mathcal{F}(g_1, g_2)M. \end{aligned}$$

(ii) In particular, suppose that E/M splits, and let L/M be a complement of N/M in E/M determined by a function $\delta\colon G \to N/M$. Let T be a transversal of M in N and, for each $\delta(g)$, let $\overline{\delta(g)}$ be the representative in T such that $\overline{\delta(g)}M = \delta(g)$. Then $M \operatorname{ext}_{\#_\delta,\mathcal{F}_\delta} G$ reconstructs L with

$$\begin{aligned} g^{\#_\delta}(m) &= \overline{\delta(g)}\, g^{\#}(m)\, \overline{\delta(g)}^{\,-1} \\ \mathcal{F}_\delta(g_1, g_2) &= \overline{\delta(g_1)}\, g_1^{\#}(\overline{\delta(g_2)})\, \mathcal{F}(g_1, g_2)\, \overline{\delta(g_1 g_2)}^{\,-1}. \end{aligned}$$

Proof. Let $M, N \lhd E$ with $M < N$, and suppose that E is reconstructed in a form of a crossed product extension $N \operatorname{ext}_{\#,\mathcal{F}} G$ by taking a transversal $\{\bar{g} \mid g \in G\}$. Then $(E/M)/(N/M) \cong E/N = G$ and $\{\bar{g}M \mid g \in G\}$ is a transversal of N/M in E/M. For each $g \in G$ we have the automorphism $g^{\#_{N/M}}$ of N/M defined by

$$g^{\#_{N/M}}(nM) = \bar{g}MnM\bar{g}^{\,-1}M = \bar{g}\, n\, \bar{g}^{\,-1}M = g^{\#}(n)M,$$

and hence the weak action $\#_{N/M}\colon G \to \operatorname{Aut}(N/M)$ is given by $\#_{N/M}\colon g \mapsto g^{\#_{N/M}}$. Furthermore,

$$\overline{g_1}M\, \overline{g_2}M\, \overline{g_1 g_1}^{\,-1}M = \overline{g_1}\, \overline{g_2}\, \overline{g_1 g_2}^{\,-1}M = \mathcal{F}(g_1, g_2)M$$

shows that the factor set $\mathcal{F}_{N/M}\colon G \times G \to N/M$ is given by

$$\mathcal{F}_{N/M}\colon (g_1, g_2) \mapsto \mathcal{F}(g_1, g_2)M.$$

This proves (i).

As for (ii), let L/M be a complement of N/M in E/M determined by $\delta\colon G \to N/M$; that is, L/M has the form $\{\delta(g)\bar{g}M, \mid g \in G\}$. Fix a transversal T of M in N. For each $\delta(g)$ in N/M choose the representative $\overline{\delta(g)}$ in T such that $\overline{\delta(g)}M = \delta(g)$. Then $\{\overline{\delta(g)}\,\bar{g} \mid g \in G\}$ is a transversal of M in L. For $g \in G$ the corresponding automorphism $g^{\#_\delta}$ of M is defined by

$$g^{\#_\delta}(m) = \overline{\delta(g)}\,\bar{g}\, m\, \bar{g}^{\,-1}\, \overline{\delta(g)}^{\,-1} = \overline{\delta(g)}\, g^{\#}(m)\, \overline{\delta(g)}^{\,-1}.$$

Hence the weak action $\#_\delta\colon G \to \mathrm{Aut}(M)$ is given by $\#_\delta\colon g \mapsto g^{\#_\delta}$. It remains to compute the corresponding factor set. We have

$$\begin{aligned}
\overline{\delta(g_1)}\,\overline{g_1}\,\overline{\delta(g_2)}\,\overline{g_2}\,(\overline{\delta(g_1g_2)}\,\overline{g_1g_2})^{-1} &= \overline{\delta(g_1)}\,\overline{g_1}\,\overline{\delta(g_2)}\,\overline{g_2}\,\overline{g_1g_2}^{-1}\,\overline{\delta(g_1g_2)}^{-1}\\
&= \overline{\delta(g_1)}\,\overline{g_1}\,\overline{\delta(g_2)}\,\overline{g_1}^{-1}\,\overline{g_1}\,\overline{g_2}\,\overline{g_1g_2}^{-1}\,\overline{\delta(g_1g_2)}^{-1}\\
&= \overline{\delta(g_1)}\,g_1^{\#}(\overline{\delta(g_2)})\,\mathcal{F}(g_1,g_2)\,\overline{\delta(g_1g_2)}^{-1},
\end{aligned}$$

and so $\mathcal{F}_\delta\colon G \times G \to M$ is given by

$$\mathcal{F}_\delta\colon (g_1,g_2) \mapsto \overline{\delta(g_1)}\,g_1^{\#}(\overline{\delta(g_2)})\,\mathcal{F}(g_1,g_2)\,\overline{\delta(g_1g_2)}^{-1}.$$

This completes the proof. □

To start the reduction we first need to test whether E/N_1 is a split extension of N/N_1 by G. By Theorem 3.1(i) we reconstruct E/N_1 in a form of a crossed product extension $N/N_1\,\mathrm{ext}_{\#_{N/N_1},\mathcal{F}_{N/N_1}}\,G$, and test whether it is a split extension of N/N_1 by G. Since N/N_1 is elementary abelian, this is done by solving a non-homogeneous system of linear equations described in Subsection 3.1. If the system has no solution, then E does not split. Otherwise, each solution δ uniquely determines a complement L/N_1 of N/N_1 in E/N_1. We further need to test each L (corresponding to each δ) for being a split extension of N_1 by G. Using Theorem 3.1(ii) we reconstruct each such L in a form of a crossed product extension $N_1\,\mathrm{ext}_{\#_\delta,\mathcal{F}_\delta}\,G$, and continue down the series.

Suppose inductively that, for some $j < r$, we have complements $\overline{L}/N_j$ of N/N_j in E/N_j, and that each $\overline{L}$ is reconstructed as a crossed product $N_j\,\mathrm{ext}_{\overline{\#},\overline{\mathcal{F}}}\,G$. In order to test whether each such $\overline{L}/N_{j+1}$ is a split extension of N_j/N_{j+1} by G we reconstruct $\overline{L}/N_{j+1}$ in a form

$$N_j/N_{j+1}\,\mathrm{ext}_{\overline{\#}_{N_j/N_{j+1}},\overline{\mathcal{F}}_{N_j/N_{j+1}}}\,G,$$

and test whether the latter is a split extension of N_j/N_{j+1} by G. Again, N_j/N_{j+1} is elementary abelian, so we need to solve an appropriate linear system. If none of $\overline{L}/N_{j+1}$ are split extensions, then neither is E. Otherwise, for each $\overline{L}/N_{j+1}$ that splits, solutions δ^* uniquely determine complements L^*/N_{j+1} of N_j/N_{j+1} in $\overline{L}/N_{j+1}$. Clearly, each L^*/N_{j+1} is also a complement of N/N_{j+1} in E/N_{j+1}. Finally, we reconstruct each L^* in a form $N_{j+1}\,\mathrm{ext}_{\overline{\#}_{\delta^*},\overline{\mathcal{F}}_{\delta^*}}\,G$, and proceed to the next step.

Observe that at each step it is enough to consider complements only up to conjugacy. Reduction up to conjugacy can be described by an action on the set of solutions δ^* that determine complements, see [3] and [8, Chapter 8] for more details.

4 An improved algorithm for testing whether the lifted group splits

The general method described in Section 3 will be now applied in the context of lifting automorphisms along regular covering projections.

Let $\zeta\colon A(X) \to N$ be a voltage function associated with a solvable regular covering projection $\wp\colon \tilde{X} \to X$ of connected graphs, and let $G \le \mathrm{Aut}(X)$ lift to $\tilde{G}$. We derive an algorithm for testing whether the lifted group $\tilde{G}$ is a split extension of $\mathrm{CT}(\wp)$ by G. In contrast with [17] we avoid the combinatorial reconstruction not only of the covering graph $\tilde{X}$ and the lifted group $\tilde{G}$, but also of the all intermediate elementary abelian regular covering projections $\wp_j \colon X_j \to X_{j-1}$ in the decomposition

$$\tilde{X} = X_n \stackrel{\wp_n}{\to} X_{n-1} \to \cdots \to X_1 \stackrel{\wp_1}{\to} X_0 = X$$

of $\wp$ arising from a characteristic series $N = N_0 \rhd N_1 \rhd \cdots \rhd N_r = 1$ with elementary abelian factors N_{j-1}/N_j. Consequently, we neither reconstruct the graphs X_j nor the intermediate complements acting on X_j.

Instead, we first reconstruct $\tilde{G}$ in a form of a crossed product extension $N \operatorname{ext}_{\#,\mathcal{F}} G$ derived from the voltage function $\zeta \colon A(X) \to N$. Recall from Preliminaries that, since G lifts, for each $g \in G$, there exists an automorphism $g^{\#_v}$ of N uniquely determined by a map $\zeta W^* \mapsto \zeta g(W^*)$, where W^* ranges over all fundamental closed walks in X rooted at v. As it is proved in [15], choosing a base vertex v, the function $\# \colon G \to \operatorname{Aut}(N)$, given by

$$\# \colon g \mapsto g^{\#_v},$$

is in fact the weak action, while the factor set $\mathcal{F} \colon G \times G \to N$ is given by

$$\mathcal{F} \colon (g_1, g_2) \mapsto g_1^{\#_v}(\zeta Q)(\zeta g_1(Q))^{-1}, \quad \text{for a walk } Q \text{ from } g_2(v) \text{ to } v.$$

In view of the approach in Section 3, if G has a presentation $\langle S \mid R \rangle$ we actually only need to know the automorphisms $g_i^{\#_v}$ for all $g_i \in S$ and, for each $r_j = g_{j_1} \cdots g_{j_t} \in R$, the values $\mathcal{F}(g_{j_k}, g_{j_{k+1}} \cdots g_{j_t})$ for $k = 1, \ldots, t-1$. As each $g_i^{\#_v}$ is uniquely determined by $\zeta W^* \mapsto \zeta g_i(W^*)$, we only store the voltages ζW^* of the fundamental closed walks W^* at v together with the voltages $\zeta g_i(W^*)$ of the mapped walks. All these data can be efficiently computed, for instance, by using breadth first search on X that starts at root v. Finally, with these data in hand we simply follow the approach described in Subsection 3.2.

5 Performance

In order to verify the effectiveness of the proposed algorithm we compare its performance with the previous version (called ISA, see [17]). The new version, called ISAI from now on, has been implemented in MAGMA. The source code of both versions is available online [18].

A test has been performed on a subset of the database described in [17]. In particular, we have selected solvable regular covering projections for the complete graph K_5, the Petersen graph $GP(5, 2)$, the Ljubljana graph $\mathcal{L}$ [4], and the graph F258A [2] along which the full automorphism group lifts. Elementary abelian coverings have been eliminated since ISAI actually coincides with ISA on such coverings. Both algorithms were run on an 2.93 GHz Quad-Core Intel® Xeon® processor X7350 at the Faculty of Mathematics and Physics, University of Ljubljana.

Results are gathered in Tables 1-4. The first column shows the order of the covering graph, while the second one describes the type of the voltage group: solvable, but not abelian; or, abelian, but not elementary abelian. Further, the notation used in the third column for identifying the voltage group is the library number in the database of small groups in MAGMA. Execution times given in seconds (CPU time) are displayed in the fourth and the fifth column (for ISA and ISAI, respectively). The last column indicates whether the corresponding lift of the full automorphism group splits. As can be seen from results, ISAI is clear winner of the comparison.

Table 1: Performance comparison for the complete graph K_5

Order of covering graph	Type of voltage group	Library number of voltage group	$t_{\text{ISA}}(s)$	$t_{\text{ISAI}}(s)$	Split?
30	Solvable	$\langle 6, 1\rangle$	0.010	0.010	true
120	Solvable	$\langle 24, 12\rangle$	0.050	0.040	true
240	Solvable	$\langle 48, 28\rangle$	0.520	0.090	false
480	Solvable	$\langle 96, 230\rangle$	0.350	0.040	true
640	Solvable	$\langle 128, 2326\rangle$	1.530	0.050	true
960	Solvable	$\langle 192, 1542\rangle$	1.530	0.060	true
1250	Abelian	$\langle 250, 15\rangle$	0.020	0.050	false
1280	Solvable	$\langle 256, 55642\rangle$	1.670	0.070	true

Table 2: Performance comparison for the Petersen graph

Order of covering graph	Type of voltage group	Library number of voltage group	$t_{\text{ISA}}(s)$	$t_{\text{ISAI}}(s)$	Split?
80	Solvable	$\langle 8, 4\rangle$	0.020	0.060	false
360	Solvable	$\langle 36, 10\rangle$	0.020	0.020	true
720	Solvable	$\langle 72, 24\rangle$	0.020	0.020	false
1080	Solvable	$\langle 108, 17\rangle$	0.610	0.040	true
1280	Solvable	$\langle 128, 2321\rangle$	1.770	0.020	false
1620	Solvable	$\langle 162, 54\rangle$	0.020	0.020	true
2160	Solvable	$\langle 216, 33\rangle$	0.030	0.030	false
2500	Abelian	$\langle 250, 15\rangle$	0.030	0.030	false
2560	Solvable	$\langle 256, 55628\rangle$	1.810	0.030	false

Table 3: Performance comparison for the Ljubljana graph $\mathcal{L}$

Order of covering graph	Type of voltage group	Library number of voltage group	$t_{\text{ISA}}(s)$	$t_{\text{ISAI}}(s)$	Split?
896	Solvable	$\langle 8, 4\rangle$	0.650	0.030	true
1344	Solvable	$\langle 12, 3\rangle$	0.560	0.040	true
1792	Abelian	$\langle 16, 2\rangle$	0.630	0.030	true
2352	Solvable	$\langle 21, 1\rangle$	0.600	0.030	true
2688	Solvable	$\langle 24, 11\rangle$	3.090	0.040	true

Table 4: Performance comparison for the graph F258A

Order of covering graph	Type of voltage group	Library number of voltage group	$t_{\text{ISA}}(s)$	$t_{\text{ISAI}}(s)$	Split?
2064	Solvable	$\langle 8, 4\rangle$	2.660	0.120	true
3096	Abelian	$\langle 12, 5\rangle$	2.720	0.150	false
4128	Abelian	$\langle 16, 2\rangle$	2.670	0.130	true

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On factorisations of complete graphs into circulant graphs and the Oberwolfach problem

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Abstract

Various results on factorisations of complete graphs into circulant graphs and on 2-factorisations of these circulant graphs are proved. As a consequence, a number of new results on the Oberwolfach Problem are obtained. For example, a complete solution to the Oberwolfach Problem is given for every 2-regular graph of order $2p$ where $p \equiv 5 \pmod 8$ is prime.

Keywords: Oberwolfach problem, graph factorisations, graph decompositions, 2-factorisations.

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1 Introduction

The Oberwolfach problem was posed by Ringel in the 1960s and is first mentioned in [16]. It concerns graph factorisations. A *factor* of a graph is a spanning subgraph and a *factorisation* is a decomposition into edge-disjoint factors. A factor that is regular of degree k is called a k*-factor*. If each factor of a factorisation is a k-factor, then the factorisation is called a k*-factorisation*, and if each factor is isomorphic to a given graph F, then we say it is a *factorisation into* F.

Let F be an arbitrary 2-regular graph and let n be the order of F. If n is odd, then the *Oberwolfach Problem* $\mathrm{OP}(F)$ asks for a 2-factorisation of K_n into F, and if n is even, then $\mathrm{OP}(F)$ asks for a 2-factorisation of $K_n - I$ into F, where $K_n - I$ denotes the graph obtained from K_n by removing the edges of a 1-factor.

The Oberwolfach Problem has been solved completely when F consists of isomorphic components [1, 3, 18], when F has exactly two components [29], when F is bipartite [5, 17] and in numerous special cases. See [7] for a survey of results up to 2006. It is known that there is no solution to $\mathrm{OP}(F)$ for $F \in \{C_3 \cup C_3, C_4 \cup C_5, C_3 \cup C_3 \cup C_5, C_3 \cup C_3 \cup C_3 \cup C_3\}$, but a solution exists for every other 2-regular graph of order at most 40 [13].

In [8], it was shown that the Oberwolfach Problem has a solution for every 2-regular graph of order $2p$ where p is any of the infinitely many primes congruent to $5 \pmod{24}$, and for every 2-regular graph whose order is in an infinite family of primes congruent to $1 \pmod{16}$. In this paper we extend these results as follows. We show that $\mathrm{OP}(F)$ has a solution for every 2-regular graph of order $2p$ where p is any prime congruent to $5 \pmod 8$ (see Theorem 4.2), and we obtain solutions to $\mathrm{OP}(F)$ for broad classes of 2-regular graphs in many other cases (see Theorems 4.3 and 4.4). We also obtain results on the generalisation of the Oberwolfach Problem to factorisations of complete multigraphs into isomorphic 2-factors (see Theorem 5.4). Our results are obtained by constructing various factorisations of complete graphs into circulant graphs in Section 2, and then showing in Section 3 that these circulant graphs can themselves be factored into isomorphic 2-regular graphs in a wide variety of cases.

2 Factorising complete graphs into circulant graphs

Let $G = (G, \cdot)$ be a finite group with identity e and let S be a subset of G such that $e \notin S$ and $s \in S$ implies $s^{-1} \in S$. The *Cayley graph on G with connection set S*, denoted $\mathrm{Cay}(G\,; S)$, has the elements of G as its vertices and g is adjacent to $g \cdot s$ for each $s \in S$ and each $g \in G$. A Cayley graph on a cyclic group is called a *circulant graph*. We use the following standard notation. The ring of integers modulo n is denoted by $\mathbb{Z}_n$, the multiplicative group of units modulo n is denoted by $\mathbb{Z}_n^*$ and, when b divides $|\mathbb{Z}_n^*|$, the subgroup $\{x^b : x \in \mathbb{Z}_n^*\}$ of index b in $\mathbb{Z}_n^*$ is denoted by $(\mathbb{Z}_n^*)^b$.

In this section we consider factorisations of K_n for n odd (in Section 2.1) and of $K_n - I$ for n even (in Section 2.2) into circulant graphs. A 2-regular graph is a circulant if and only if its components are all isomorphic. Thus, for each 2-regular circulant graph F, there exists a factorisation of K_n (if F has odd order) or of $K_n - I$ (if F has even order) into F; except that there is no such factorisation when $F \in \{C_3 \cup C_3, C_3 \cup C_3 \cup C_3 \cup C_3\}$. Considerably less is known for factorisations into circulant graphs of degree greater than 2. Some factorisations into $\mathrm{Cay}(\mathbb{Z}_n\,; \pm\{1, 2\})$ and $\mathrm{Cay}(\mathbb{Z}_n\,; \pm\{1, 2, 3, 4\})$ are given in [4] and [8] respectively, and some further results, including results on self-complementary and almost self-complementary circulant graphs, appear in [2, 14, 15, 26].

2.1 Factorising complete graphs of odd order

In this subsection we will construct factorisations of complete graphs of odd order into isomorphic circulant graphs by finding certain partitions of cyclic groups. Problems concerning such partitions have been well studied, for example see [28], and existing results overlap with some of the results in this subsection. In particular, Theorem 2.3 below is a consequence of Lemma 3.1 of [24].

Lemma 2.1. *Let s be an integer, let $p \equiv 1 \pmod{2s}$ be prime, and let $S = \pm\{d_1, d_2, \dots, d_s\} \subseteq \mathbb{Z}_p^*$. Further, suppose a and b are integers such that $2abs = p-1$, let $G = (\mathbb{Z}_p^*)^b$, and let $H = (\mathbb{Z}_p^*)^{bs}$. If $d_1, d_2, \dots, d_s$ represent the s distinct cosets of G/H, then there exists a $2s$-factorisation of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,; S)$.*

Proof. For each $x \in \mathbb{Z}_p$ let $xS = \{xy : y \in S\}$. Since p is prime, $\mathrm{Cay}(\mathbb{Z}_p\,; xS) \cong \mathrm{Cay}(\mathbb{Z}_p\,; S)$ for any $x \in \mathbb{Z}_p \setminus \{0\}$. If there is a partition of $\mathbb{Z}_p^*$ into sets $x_1S, x_2S, \dots, x_{ab}S$ where $x_i \in \mathbb{Z}_p \setminus \{0\}$ for $i = 1, 2, \dots, ab$, then $\{\mathrm{Cay}(\mathbb{Z}_p\,; x_iS) : i = 1, 2, \dots, ab\}$ is the required $2s$-factorisation of K_p. We now present such a partition.

Let ω be a generator of $\mathbb{Z}_p^*$. Thus, $H = \omega^0, \omega^{bs}, \omega^{2bs}, \dots, \omega^{(2a-1)bs}$, and $\omega^{abs} = -1 \in H$. Let $A = \omega^0, \omega^{bs}, \omega^{2bs}, \dots, \omega^{(a-1)bs}$, so that $H = A \cup -A$ (A is a set of representatives for the cosets in H of the order 2 subgroup of H). Since $d_1, d_2, \dots, d_s$ represent distinct cosets of G/H, it is easy to see that $\{xS : x \in A\}$ is a partition of G. Thus, if B is a set of representatives for the cosets of $\mathbb{Z}_p^*/G$, then $\{xyS : x \in A, y \in B\}$ is the required partition of $\mathbb{Z}_p^*$. □

Note that upon putting $s = 1$ in Lemma 2.1 we obtain the Hamilton decomposition

$$\{\mathrm{Cay}(\mathbb{Z}_p\,; \{\pm 1\}), \mathrm{Cay}(\mathbb{Z}_p\,; \{\pm 2\}), \dots, \mathrm{Cay}(\mathbb{Z}_p\,; \{\pm \tfrac{p-1}{2}\})\}$$

of K_p. We will be mostly interested in applications of Lemma 2.1 where the connection set S is $\pm\{1,2\}$, $\pm\{1,2,3\}$, $\pm\{1,3,4\}$ or $\pm\{1,2,3,4\}$. The factorisations given by Lemma 2.1 have the property that each factor is invariant under the action of $\mathbb{Z}_p$. It is worth mentioning that for $S \in \{\pm\{1,2\}, \pm\{1,2,3\}, \pm\{1,3,4\}, \pm\{1,2,3,4\}\}$, the construction given in Lemma 2.1 yields every $2s$-factorisation of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,; S)$ with this property. This follows from the results in [9] and [22], together with Turner's result [30] that for p prime $\mathrm{Cay}(\mathbb{Z}_p\,; S) \cong \mathrm{Cay}(\mathbb{Z}_p\,; S')$ if and only if there exists an $\alpha \in \mathbb{Z}_p^*$ such that $S' = \alpha S$.

Theorem 2.2. *If $p \equiv 1 \pmod 4$ is prime and 4 divides the order of k in $\mathbb{Z}_p^*$, then there is a factorisation of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1, k\})$.*

Proof. Apply Lemma 2.1 with $S = \pm\{1, k\}$ taking G to be the subgroup of $\mathbb{Z}_p^*$ generated by k, and H to be the index 2 subgroup of G. □

Theorem 2.3. *If $p \equiv 1 \pmod 6$ is prime such that $2, 3 \notin (\mathbb{Z}_p^*)^3$ and $6 \in (\mathbb{Z}_p^*)^3$, then there is a factorisation of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1,2,3\})$.*

Proof. It follows from $2, 3 \notin (\mathbb{Z}_p^*)^3$ and $6 \in (\mathbb{Z}_p^*)^3$ that 1, 2 and 3 represent the three cosets of $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^3$. Thus, we obtain the required factorisation by applying Lemma 2.1 with $b = 1$. □

Theorem 2.4. *If $p \equiv 1 \pmod 6$ is prime such that $2, 3, 6 \notin (\mathbb{Z}_p^*)^3$, then there is a factorisation of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1,3,4\})$.*

Proof. It follows from $2, 3, 6 \notin (\mathbb{Z}_p^*)^3$ that 1, 3 and 4 represent the three cosets of $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^3$. Thus, we obtain the required factorisation by applying Lemma 2.1 with $b = 1$. □

The primes less than 1000 to which Theorem 2.3 applies are

$$7, 37, 139, 163, 181, 241, 313, 337, 349, 379, 409, 421, 541, 571, 607, 631, 751, 859, 877, 937,$$

and the primes less than 1000 to which Theorem 2.4 applies are

$$13, 19, 79, 97, 199, 211, 331, 373, 463, 487, 673, 709, 769, 823, 829, 883, 907.$$

In the next theorem we show that there are infinitely many primes to which Theorem 2.3 applies, and also infinitely many primes to which Theorem 2.4 applies.

Theorem 2.5. *There are infinitely many values of* p *such that* p *is prime,* $p \equiv 1 \pmod 6$, $2, 3 \notin (\mathbb{Z}_p^*)^3$ *and* $6 \in (\mathbb{Z}_p^*)^3$*, and there are infinitely many values of* p *such that* p *is prime,* $p \equiv 1 \pmod 6$ *and* $2, 3, 6 \notin (\mathbb{Z}_p^*)^3$.

Proof. Assume $p \equiv 1 \pmod 6$. Let $\mathbb{F}_p$ be the field with p elements. We use standard definitions and results from algebraic number theory, as found in [20]. The result essentially follows from the Chebotarev Density Theorem.

Let ω be a primitive cube root of unity, $\lambda = \sqrt[3]{2}$ be a cube root of 2 and $\rho = \sqrt[3]{3}$ a cube root of 3. Consider the following tower of fields:

$$M = \mathbb{Q}(\omega, \lambda, \rho) \supseteq L = \mathbb{Q}(\omega, \lambda) \supseteq K = \mathbb{Q}(\omega) \supseteq \mathbb{Q}.$$

Let $\mathbb{O}_K$, $\mathbb{O}_L$ denote the rings of integers of K and L respectively. We may ignore the finitely many ramified primes. Thus let p be a prime number, sufficiently large that it is unramified in M, let $\mathfrak{p}$ be a prime in K extending p and $\mathfrak{P}$ a prime in L extending $\mathfrak{p}$. Let $\mathbb{K} = \mathbb{O}_K/\mathfrak{p}$ and $\mathbb{L} = \mathbb{O}_L/\mathfrak{P}$ be the residue fields. We view $\mathbb{K}$ as embedded in $\mathbb{L}$ via the map $x + \mathfrak{p} \mapsto x + \mathfrak{P}$. As $p \equiv 1 \pmod 6$, p splits in K and $\mathbb{K} = \mathbb{O}_K/\mathfrak{p} \simeq \mathbb{F}_p$.

Since M and L are splitting fields, M/K and L/K are Galois extensions. The Galois group of M/K is $\mathsf{Gal}(M/K) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$ generated by the maps $\alpha\colon \lambda \mapsto \lambda\omega$ and $\beta\colon \rho \mapsto \rho\omega$. The *Frobenius map* of $\mathbb{L}/\mathbb{K}$ is the map $x \mapsto x^{|\mathbb{L}|}$. The *Frobenius element* $\sigma_\mathfrak{p}^L$ is the element of $\mathsf{Gal}(L/K)$ inducing the Frobenius map on $\mathbb{L}/\mathbb{K}$. (A priori $\sigma_\mathfrak{p}^L$ could also depend on the choice of $\mathfrak{P}$ extending $\mathfrak{p}$, but this is not the case since Gal(L/K) is abelian; see [20, III.2.1].) Define $\sigma_\mathfrak{p}^M \in \mathsf{Gal}(M/K)$ analogously. Then $\sigma_\mathfrak{p}^L$ is the restriction of $\sigma_\mathfrak{p}^M$ to L by [20, III.2.3].

By definition of $\mathbb{L}$, for all sufficiently large $p \equiv 1 \pmod 6$, $2 \in (\mathbb{Z}_p^*)^3$ if and only if $\mathbb{L} = \mathbb{K}$. But $\mathbb{L} = \mathbb{K}$ if and only if $\sigma_\mathfrak{p}^L$ is the identity map, and it follows that $2 \in (\mathbb{Z}_p^*)^3$ if and only if $\sigma_\mathfrak{p}^M \in \langle\beta\rangle$. Similarly, $3 \in (\mathbb{Z}_p^*)^3$ if and only if $\sigma_\mathfrak{p}^M \in \langle\alpha\rangle$ and $6 \in (\mathbb{Z}_p^*)^3$ if and only if $\sigma_\mathfrak{p}^M \in \langle\alpha\beta\rangle$. In summary:

$$\begin{aligned} 2, 3 \notin (\mathbb{Z}_p^*)^3,\ 6 \in (\mathbb{Z}_p^*)^3 \quad &\Longleftrightarrow \quad \sigma_\mathfrak{p}^M \in \{\alpha\beta, \alpha^2\beta^2\}. \\ 2, 3, 6 \notin (\mathbb{Z}_p^*)^3 \quad &\Longleftrightarrow \quad \sigma_\mathfrak{p}^M \in \{\alpha^2\beta, \alpha\beta^2\}. \end{aligned}$$

The Chebotarev Density Theorem [20, V.10.4] implies that for each $\theta \in \mathsf{Gal}(M/K)$, the set of primes $\mathfrak{p}$ of K (unramified in M) for which $\sigma_\mathfrak{p}^M = \theta$ is infinite. Thus each of the two conditions for $\sigma_\mathfrak{p}^M$ displayed above holds infinitely often. □

It is possible to describe the primes in Theorem 2.5 more explicitly. Given $p \equiv 1 \pmod 6$, factoring the ideal $p\mathbb{O}_K$ and taking norms, one shows there exist unique $c, d \in$

$\mathbb{Z}$ with $d > 0$, $\gcd(c,d) = 1$, $c \equiv 2 \pmod 3$ and $4p = (2c-3d)^2 + 27d^2$. Let $t(p) = (c \pmod 6),\ d \pmod 6))$. There are 9 possible values for $t(p)$: $(2,1)$, $(2,3)$, $(2,5)$, $(5,0)$, $(5,1)$, $(5,2)$, $(5,3)$, $(5,4)$ and $(5,5)$. The Chebotarev density theorem implies that each of the 9 possible $t(p)$ values occurs "equally often" (that is, for a subset of the primes $p \equiv 1 \pmod 6$ of relative density $1/9$). Using cubic reciprocity [19, Ch. 9] one calculates that $2, 3 \notin (\mathbb{Z}_p^*)^3$ and $6 \in (\mathbb{Z}_p^*)^3$ if and only if $t(p) = (2,1)$ or $(5,5)$, while $2,3,6 \notin (\mathbb{Z}_p^*)^3$ if and only if $t(p) = (2,5)$ or $(5,1)$. Each case occurs for $2/9$ of the primes that are $1 \pmod 6$.

The above applications of Lemma 2.1 have all been with $b = 1$. We note however that the conditions of Lemma 2.1 are never satisfied when $S = \pm\{1,2,3,4\}$ and $b = 1$. This is because 2 is a quadratic residue when $p \equiv 1 \pmod 8$, which means that both 1 and 4 are in H. The factorisations of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,;\pm\{1,2,3,4\})$ in [8] were obtained by applying Lemma 2.1 with $b = 2$ so that G and H have index 2 and 8, respectively, in $\mathbb{Z}_p^*$. Another example where Lemma 2.1 can be applied with $b \neq 1$ is when $p = 919$, $S = \pm\{1,2,3\}$, $a = 51$ and $b = 3$. This yields a factorisation of K_{919} into $\mathrm{Cay}(\mathbb{Z}_{919}\,;\pm\{1,2,3\})$. Such a factorisation cannot be obtained by applying Lemma 2.1 with $b = 1$ because 1, 2 and 3 are all cubes in $\mathbb{Z}_{919}^*$.

The following lemma can be used to obtain factorisations of K_p, for certain values of p, in which some of the factors are isomorphic to $\mathrm{Cay}(\mathbb{Z}_p\,;\pm\{1,2,3\})$ and the others are isomorphic to $\mathrm{Cay}(\mathbb{Z}_p\,;\pm\{1,2,3,4\})$.

Lemma 2.6. *Let p be prime, let H be the subgroup of $\mathbb{Z}_p^*$ generated by $\{-1,6\}$, and let d be the order of $2H$ in $\mathbb{Z}_p^*/H$. If there exist nonnegative integers α and β such that $d = 3\alpha + 4\beta$, then there is a factorisation of K_p into $\frac{\alpha(p-1)}{2d}$ copies of $\mathrm{Cay}(\mathbb{Z}_p\,;\pm\{1,2,3\})$ and $\frac{\beta(p-1)}{2d}$ copies of $\mathrm{Cay}(\mathbb{Z}_p\,;\pm\{1,2,3,4\})$.*

Proof. It is sufficient to partition $\mathbb{Z}_p^*$ into $\frac{\alpha(p-1)}{2d}$ 6-tuples of the form $\pm\{x,2x,3x\}$ and $\frac{\beta(p-1)}{2d}$ 8-tuples of the form $\pm\{x,2x,3x,4x\}$. Since $d = 3\alpha + 4\beta$, there is a partition

$$\begin{array}{l}\{\{2^{r_i-1}H, 2^{r_i}H, 2^{r_i+1}H\} : i = 1,\dots,\alpha\} \cup \\ \{\{2^{r_i-1}H, 2^{r_i}H, 2^{r_i+1}H, 2^{r_i+2}H\} : i = \alpha+1,\dots,\alpha+\beta\}\end{array}$$

of $\{H, 2H, \dots, 2^{d-1}H\}$. But $6 \in H$ implies $2^{r_i-1}H = 3 \cdot 2^{r_i}H$ for $i = 1,2,\dots,\alpha+\beta$. Thus, we can rewrite our partition of $\{H, 2H, \dots, 2^{d-1}H\}$ as

$$\{\{H_i, 2H_i, 3H_i\} : i = 1,\dots,\alpha\} \cup \{\{H_i, 2H_i, 3H_i, 4H_i\} : i = \alpha+1,\dots,\alpha+\beta\},$$

where $H_i = 2^{r_i}H$ for $i = 1,\dots,\alpha+\beta$.

Since $-1 \in H$, for $i = 1,\dots,\alpha$, $H_i \cup 2H_i \cup 3H_i$ can be partitioned into $\frac{|H|}{2}$ 6-tuples of the form $\pm\{x,2x,3x\}$, and for $i = \alpha+1,\dots,\alpha+\beta$, $H_i \cup 2H_i \cup 3H_i \cup 4H_i$ can be partitioned into $\frac{|H|}{2}$ 8-tuples of the form $\pm\{x,2x,3x,4x\}$. If $\mathcal{R}$ is the set of all $\alpha\frac{|H|}{2}$ of these 6-tuples and $\mathcal{S}$ is the set of all $\beta\frac{|H|}{2}$ of these 8-tuples, then $\mathcal{R} \cup \mathcal{S}$ is a partition of the subgroup $G = H \cup 2H \cup \cdots \cup 2^{d-1}H$ of $\mathbb{Z}_p^*$. Thus, if $g_1, g_2, \dots, g_t$ ($t = \frac{p-1}{d|H|}$) represent the cosets of $\mathbb{Z}_p^*/G$, then

$$\{g_iR : R \in \mathcal{R}, i = 1,\dots,t\} \cup \{g_iS : S \in \mathcal{S}, i = 1,\dots,t\}$$

is a partition of $\mathbb{Z}_p^*$ into $t\alpha\frac{|H|}{2} = \frac{\alpha(p-1)}{2d}$ 6-tuples of the form $\pm\{x, 2x, 3x\}$ and $t\beta\frac{|H|}{2} = \frac{\beta(p-1)}{2d}$ 8-tuples of the form $\pm\{x, 2x, 3x, 4x\}$. This is the required partition of $\mathbb{Z}_p^*$. □

Notice that any 6-factorisation of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1, 2, 3\})$ given by Lemma 2.1 can also be obtained via Lemma 2.6. For if $1, 2, 3$ represent the three distinct cosets of G/H (where $G = (\mathbb{Z}_p^*)^b$ and $H = (\mathbb{Z}_p^*)^{3b}$, and $p - 1 = 6ab$), then it follows that $\{-1, 6\} \subseteq H$ and $2H$ has order 3 in G/H. This means that if H' is the subgroup of $\mathbb{Z}_p^*$ generated by $\{-1, 6\}$, then $H' \leq H$ and 3 divides the order d of $2H'$ in $\mathbb{Z}_p^*/H'$. Thus, we can obtain our 6-factorisation of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1, 2, 3\})$ by applying Lemma 2.6 with $\alpha = \frac{d}{3}$ and $\beta = 0$. Similarly, any 8-factorisation of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1, 2, 3, 4\})$ given by Lemma 2.1 can be obtained by applying Lemma 2.6 with $\alpha = 0$ and $\beta = \frac{d}{4}$.

However, Lemma 2.6 gives us additional factorisations such as the following. When $p = 101$ we have $H = \pm\{1, 6, 14, 17, 36\}$, and $2H$ has order $d = 10$ in $\mathbb{Z}_p^*/H$. Taking $\alpha = 2$ and $\beta = 1$, we obtain a factorisation of K_{101} into 10 copies of $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1, 2, 3\})$ and 5 copies of $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1, 2, 3, 4\})$. Of course, 101 is neither $1 \pmod 6$ nor $1 \pmod 8$, so there is neither a 6-factorisation nor an 8-factorisation of K_{101}.

2.2 Factorising complete graphs of even order

In this section we construct factorisations of $K_{2p} - I$ where the factors are all isomorphic to $\mathrm{Cay}(\mathbb{Z}_{2p}\,; \pm\{1, 2\})$ or all isomorphic to $\mathrm{Cay}(\mathbb{Z}_{2p}\,; \pm\{1, 2, 3, 4\})$. We do this by considering $K_{2p} - I$ as a Cayley graph on a dihedral group and partitioning its connection set to generate the factors. The dihedral group D_{2p} of order $2p$ has elements $r_0, r_1, r_2, \ldots, r_{p-1}, s_0, s_1, s_2, \ldots, s_{p-1}$ and satisfies

$$r_i \cdot r_j = r_{i+j}, \quad r_i \cdot s_j = s_{i+j}, \quad s_i \cdot r_j = s_{i-j}, \quad s_i \cdot s_j = r_{i-j}$$

where arithmetic of subscripts is carried out modulo p.

Lemma 2.7. *If $p \geq 3$ is prime, then*

$$\mathrm{Cay}(D_{2p}\,; \{r_{\pm i}, s_j, s_{i+j}\}) \cong \mathrm{Cay}(\mathbb{Z}_{2p}\,; \pm\{1, 2\})$$

for all $i \in \mathbb{Z}_p \setminus \{0\}$ and all $j \in \mathbb{Z}_p$.

Proof. An isomorphism is given by

$$\begin{array}{ccccccccccccc} r_0 & r_i & r_{2i} & r_{3i} & \cdots & r_{(p-1)i} & s_j & s_{i+j} & s_{2i+j} & s_{3i+j} & \cdots & s_{(p-1)i+j} \\ \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ 0 & 2 & 4 & 6 & \cdots & 2p-2 & 2p-1 & 1 & 3 & 5 & \cdots & 2p-3 \end{array}$$

□

Lemma 2.8. *If $p \geq 5$ is prime, then*

$$\mathrm{Cay}(D_{2p}\,; \{r_{\pm i}, r_{\pm 2i}, s_j, s_{i+j}, s_{2i+j}, s_{3i+j}\}) \cong \mathrm{Cay}(\mathbb{Z}_{2p}\,; \pm\{1, 2, 3, 4\})$$

for all $i \in \mathbb{Z}_p \setminus \{0\}$ and all $j \in \mathbb{Z}_p$.

Proof. An isomorphism is given by

$$\begin{array}{cccccccccccc} r_0 & r_i & r_{2i} & r_{3i} & \dots & r_{(p-1)i} & s_j & s_{i+j} & s_{2i+j} & s_{3i+j} & \dots & s_{(p-1)i+j} \\ \downarrow & \downarrow & \downarrow & \downarrow & \dots & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \dots & \downarrow \\ 0 & 2 & 4 & 6 & \dots & 2p-2 & 2p-3 & 2p-1 & 1 & 3 & \dots & 2p-5 \end{array}$$

□

Theorem 2.9. *For each odd prime p, there is a factorisation of $K_{2p} - I$ into* $\mathrm{Cay}(\mathbb{Z}_{2p}\,;\pm\{1,2\})$.

Proof. The required factorisation is $\mathcal{F} = \{X_i : i \in \mathbb{Z}_p \setminus \{0\}\}$ where

$$X_i = \mathrm{Cay}(D_{2p}\,;\{r_{\pm 2i}, s_i, s_{-i}\})$$

for $i \in \mathbb{Z}_p \setminus \{0\}$. Note that $X_i = X_{-i}$ so $|\mathcal{F}| = \frac{p-1}{2}$ as required. Lemma 2.7 guarantees that $X_i \cong \mathrm{Cay}(\mathbb{Z}_{2p}\,;\pm\{1,2\})$ for each $i \in \mathbb{Z}_p \setminus \{0\}$. Also, r_0 is the identity of D_{2p} and each element of $D_{2p} \setminus \{r_0, s_0\}$ occurs in exactly one X_i. Thus, $\mathcal{F}$ is a factorisation of $\mathrm{Cay}(D_{2p}\,; D_{2p} \setminus \{r_0, s_0\}) \cong K_{2p} - I$ where the 1-factor I is $\mathrm{Cay}(D_{2p}\,;\{s_0\})$. □

Following work of Davenport [10, Theorem 5] and Weil, a special case of a result due to Moroz [23] yields the following. If $p \equiv 1 \pmod 4$ is prime and $p > 8 \times 10^6$, then there exists an integer x such that x, $x+1$, $x+2$, $x+3$ represent all four distinct cosets of $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^4$. A computer search using PARI/GP [25] verifies in a few minutes that such an x also exists for all $p < 8 \times 10^6$ with $p \equiv 1 \pmod 4$, with the exceptions $p = 13$ and $p = 17$. Thus, we have the following result.

Lemma 2.10. *If $p \equiv 1 \pmod 4$ is prime with $p \notin \{13, 17\}$, then there exists an $x \in \mathbb{Z}_p^*$ such that x, $x+1$, $x+2$ and $x+3$ represent all four distinct cosets of $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^4$.*

Theorem 2.11. *If $p \equiv 5 \pmod 8$ is prime, then there is a factorisation of $K_{2p} - I$ into* $\mathrm{Cay}(\mathbb{Z}_{2p}\,;\pm\{1,2,3,4\})$*; except that there is no factorisation of $K_{26} - I$ into* $\mathrm{Cay}(\mathbb{Z}_{2p}\,;\pm\{1,2,3,4\})$.

Proof. We first observe that there is no factorisation of $K_{26} - I$ into graph $\mathrm{Cay}(\mathbb{Z}_{2p}\,;\pm\{1,2,3,4\})$. If such a factorisation exists, then we can assume without loss of generality that the vertex set is $\mathbb{Z}_{26}$ and that $\mathrm{Cay}(\mathbb{Z}_{26}\,;\pm\{1,2,3,4\})$ is a factor. But no edge of $\mathrm{Cay}(\mathbb{Z}_{26}\,;\pm\{7\})$ (for example) occurs in a complete subgraph of order 5 in $\mathrm{Cay}(\mathbb{Z}_{26}\,;\pm\{5,6,7,8,9,10,11,12,13\})$. Since $\mathrm{Cay}(\mathbb{Z}_{26}\,;\pm\{1,2,3,4\})$ contains a complete subgraph of order 5, it follows that there is no factorisation of $K_{26} - I$ into graph $\mathrm{Cay}(\mathbb{Z}_{2p}\,;\pm\{1,2,3,4\})$.

Let $p \equiv 5 \pmod 8$ be prime with $p \neq 13$. By Lemma 2.10, there exists an $x \in \mathbb{Z}_p^*$ such that x, $x+1$, $x+2$ and $x+3$ represent all four distinct cosets of $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^4$. By Lemma 2.8,

$$\mathrm{Cay}(D_{2p}\,;\{r_{\pm 1}, r_{\pm 2}, s_x, s_{x+1}, s_{x+2}, s_{x+3}\}) \cong \mathrm{Cay}(\mathbb{Z}_{2p}\,;\pm\{1,2,3,4\}).$$

Now let $H = (\mathbb{Z}_p^*)^4$ act on the subscripts of the connection set $\{r_{\pm 1}, r_{\pm 2}, s_x, s_{x+1}, s_{x+2}, s_{x+3}\}$ and consider the collection $S_1, S_2, \dots, S_{\frac{p-1}{4}}$ of subsets of D_{2p} thus formed.

We show that $\{\mathrm{Cay}(D_{2p}\,; S_i) : i = 1, 2, \ldots, \frac{p-1}{4}\}$ is a factorisation of $K_{2p} - I$ into $\mathrm{Cay}(\mathbb{Z}_{2p}\,; \pm\{1, 2, 3, 4\})$. If $h \in H$, then

$$\mathrm{Cay}(D_{2p}\,; \{r_{\pm h}, r_{\pm 2h}, s_{hx}, s_{h(x+1)}, s_{h(x+2)}, s_{h(x+3)}\}) \cong \mathrm{Cay}(\mathbb{Z}_{2p}\,; \pm\{1, 2, 3, 4\})$$

by Lemma 2.8 (indeed this is true for any $h \in \mathbb{Z}_p^*$) so it remains only to verify that we have a decomposition of $K_{2p} - I$. To do this we observe that $S_1, S_2, \ldots, S_{\frac{p-1}{4}}$ partitions $D_{2p} \setminus \{r_0, s_0\}$ (r_0 is the identity in D_{2p} and $\mathrm{Cay}(D_{2p}\,; \{s_0\})$ is a 1-factor in K_{2p}). We have $Hx \cup H(x+1) \cup H(x+2) \cup H(x+3) = \mathbb{Z}_p \setminus \{0\}$. Also, since $p \equiv 5 \pmod 8$ we have $-1 \in (\mathbb{Z}_p^*)^2$, $-1 \notin (\mathbb{Z}_p^*)^4$ and $2 \notin (\mathbb{Z}_p^*)^2$ (by the law of quadratic reciprocity). Thus, $\{\pm h : h \in H\} \cup \{\pm 2h : h \in H\} = \mathbb{Z}_p \setminus \{0\}$. So $S_1, S_2, \ldots, S_{\frac{p-1}{4}}$ does indeed partition $D_{2p} \setminus \{r_0, s_0\}$ and we have the required decomposition. □

3 2-factorisations of circulant graphs

In this section we present various results on 2-factorisations of circulant graphs, beginning with a couple of known results. Lemma 3.1 was proved independently in [4] and [27], and is a special case of a result in [6]. Lemma 3.2 was proved in [8].

Lemma 3.1. ([4, 27]) *If $n \geq 5$ and F is any 2-regular graph of order n, then there is a 2-factorisation of $\mathrm{Cay}(\mathbb{Z}_n\,; \pm\{1, 2\})$ into a copy of F and a Hamilton cycle.*

Lemma 3.2. ([8]) *If $n \geq 9$ and F is a 2-regular graph of order n, then there is a 2-factorisation of $\mathrm{Cay}(\mathbb{Z}_n\,; \pm\{1, 2, 3, 4\})$ into F with the definite exceptions of $F = C_4 \cup C_5$ and $F = C_3 \cup C_3 \cup C_3 \cup C_3 \cup C_3$, and the following possible exceptions.*

(1) $F = C_3 \cup C_3 \cup \cdots \cup C_3$ *when* $n \equiv 3, 6 \pmod 9$, $n \geq 21$.

(2) $F = C_4 \cup C_4 \cup \cdots \cup C_4$ *when* $n \equiv 4 \pmod 8$, $n \geq 20$.

(3) $F = C_3 \cup C_3 \cup \cdots \cup C_3 \cup C_4$ *when* $n \equiv 1 \pmod 3$, $n \geq 19$.

(4) $F = C_3 \cup C_4 \cup C_4 \cup \cdots \cup C_4$ *when* $n \equiv 7 \pmod 8$, $n \geq 23$.

We now obtain results on 2-factorisations of $\mathrm{Cay}(\mathbb{Z}_n\,; \pm\{1, 2, 3\})$, but first we need some definitions and notation. For each $m \geq 1$, the graph with vertex set $\{0, 1, \ldots, m+2\}$ and edge set $\{\{i, i+1\}, \{i+1, i+3\}, \{i, i+3\} : i = 0, 1, \ldots, m-1\}$ is denoted by $J_m^{1,2,3}$. If F is a 2-regular graph of order m, and there exists a decomposition $\{H_1, H_2, H_3\}$ of $J_m^{1,2,3}$ into F such that

(1) $V(H_1) = \{0, 1, \ldots, m+2\} \setminus \{m, m+1, m+2\}$,

(2) $V(H_2) = \{0, 1, \ldots, m+2\} \setminus \{0, 2, m+1\}$, and

(3) $V(H_3) = \{0, 1, \ldots, m+2\} \setminus \{0, 1, m+2\}$,

then we shall write $J_m^{1,2,3} \mapsto F$. Notice that for $i = 1, 2, 3$, the subgraph H_i of $J_m^{1,2,3}$ contains exactly one vertex from each of $\{0, m\}$, $\{1, m+1\}$ and $\{2, m+2\}$.

Lemma 3.3. *If $n \geq 7$ and F is a 2-regular graph of order n such that there exists a decomposition $J_n^{1,2,3} \mapsto F$, then there exists a 2-factorisation of $\mathrm{Cay}(\mathbb{Z}_n\,; \pm\{1, 2, 3\})$ into F.*

Proof. For each $i \in \{0,1,2\}$, identify vertex i of $J_n^{1,2,3}$ with vertex $n+i$. The resulting graph is $\mathrm{Cay}(\mathbb{Z}_n\,;\pm\{1,2,3\})$ and the 2-regular graphs in the decomposition $J_n^{1,2,3} \mapsto F$ become the required 2-factors. □

Lemma 3.4. *If F and F' are vertex-disjoint 2-regular graphs and there exist decompositions $J_{|V(F)|}^{1,2,3} \mapsto F$ and $J_{|V(F')|}^{1,2,3} \mapsto F'$, then there exists a decomposition $J_{|V(F)|+|V(F')|}^{1,2,3} \mapsto F \cup F'$.*

Proof. Let r and s be the respective orders of F and F', let $\{H_1, H_2, H_3\}$ be a decomposition $J_r^{1,2,3} \mapsto F$ and let $\{H_1', H_2', H_3'\}$ be a decomposition $J_s^{1,2,3} \mapsto F'$. Apply the translation $x \mapsto x+r$ to the decomposition $\{H_1', H_2', H_3'\}$ to obtain a decomposition $\{H_1'', H_2'', H_3''\}$ of a copy of $J_s^{1,2,3}$ having vertex set $r, r+1, \ldots, r+s+2$ (H_i'' being the translation of H_i' for $i \in \{1,2,3\}$). It is clear that $\mathcal{D} = \{H_1 \cup H_1'', H_2 \cup H_2'', H_3 \cup H_3''\}$ is a decomposition $J_{r+s}^{1,2,3} \mapsto F \cup F'$. Properties (1)-(3) in the definition of $J_r^{1,2,3} \mapsto F$ ensure that H_i and H_i'' are vertex-disjoint for $i \in \{1,2,3\}$, and that

(1) $V(H_1 \cup H_1'') = \{0,1,\ldots,r+s+2\} \setminus \{r+s, r+s+1, r+s+2\}$,

(2) $V(H_2 \cup H_2'') = \{0,1,\ldots,r+s+2\} \setminus \{0,2,r+s+1\}$, and

(3) $V(H_3 \cup H_3'') = \{0,1,\ldots,r+s+2\} \setminus \{0,1,r+s+2\}$.

□

Lemma 3.5. *For each $m \geq 4$, $J_m^{1,2,3} \mapsto C_m$.*

Proof. For $m \in \{4,5,6\}$, H_1, H_2, H_3 are as defined in the following table.

m	H_1	H_2	H_3
4	$(0,1,2,3)$	$(1,3,6,4)$	$(2,4,3,5)$
5	$(0,1,2,4,3)$	$(1,3,5,7,4)$	$(2,3,6,4,5)$
6	$(0,1,2,5,4,3)$	$(1,3,5,8,6,4)$	$(2,4,7,5,6,3)$

For $m \geq 7$ and odd

- H_1 contains the edges $\{0,1\}$, $\{1,2\}$, $\{0,3\}$, $\{m-2, m-1\}$ and $\{i, i+2\}$ for $i \in \{2,3,\ldots,m-3\}$,
- H_2 contains the edges $\{1,3\}$, $\{m-2, m\}$, $\{m, m+2\}$, $\{m-1, m+2\}$, $\{i, i+1\}$ for $i \in \{4,6,\ldots,m-3\}$ and $\{i, i+3\}$ for $i \in \{1,3,\ldots,m-4\}$, and
- H_3 contains the edges $\{2,3\}$, $\{m-2, m+1\}$, $\{m-1, m\}$, $\{m-1, m+1\}$, $\{i, i+1\}$ for $i \in \{3,5,\ldots,m-4\}$ and $\{i, i+3\}$ for $i \in \{2,4,\ldots,m-3\}$.

For $m \geq 8$ and even

- H_1 contains the edges $\{0,1\}$, $\{1,2\}$, $\{3,4\}$, $\{0,3\}$, $\{2,5\}$, $\{m-2, m-1\}$ and $\{i, i+2\}$ for $i \in \{4,5,\ldots,m-3\}$,
- H_2 contains the edges $\{1,3\}$, $\{1,4\}$, $\{3,5\}$, $\{m-2, m\}$, $\{m, m+2\}$, $\{m-1, m+2\}$, $\{i, i+1\}$ for $i \in \{5,7,\ldots,m-3\}$ and $\{i, i+3\}$ for $i \in \{4,6,\ldots,m-4\}$, and
- H_3 contains the edges $\{2,4\}$, $\{m-2, m+1\}$, $\{m-1, m\}$, $\{m-1, m+1\}$, $\{i, i+1\}$ for $i \in \{2,4,\ldots,m-4\}$ and $\{i, i+3\}$ for $i \in \{3,5,\ldots,m-3\}$.

□

Lemma 3.6. *For $m = 8$ and for each $m \geq 10$, $J_m^{1,2,3} \mapsto C_3 \cup C_{m-3}$.*

Proof. For $m \in \{8, 10, 11\}$, H_1, H_2, H_3 are as defined in the following table.

m	
8	$H_1 = (4,6,7) \cup (0,1,2,5,3)$ $H_2 = (7,8,10) \cup (1,3,6,5,4)$ $H_3 = (2,3,4) \cup (5,7,9,6,8)$
10	$H_1 = (7,8,9) \cup (0,1,2,4,5,6,3)$ $H_2 = (1,3,4) \cup (5,7,6,9,12,10,8)$ $H_3 = (2,3,5) \cup (4,6,8,11,9,10,7)$
11	$H_1 = (8,9,10) \cup (0,1,2,4,5,7,6,3)$ $H_2 = (1,3,4) \cup (5,6,9,11,13,10,7,8)$ $H_3 = (2,3,5) \cup (4,6,8,11,10,12,9,7)$

For $m \geq 12$ and even

- H_1 consists of the 3-cycle $(m-3, m-2, m-1)$ and the $(m-3)$-cycle with edges $\{0,1\}$, $\{0,3\}$, $\{1,2\}$, $\{2,4\}$, $\{m-5, m-4\}$, $\{i, i+1\}$ for $i \in \{4, 6, \ldots, m-6\}$ and $\{i, i+3\}$ for $i \in \{3, 5, \ldots, m-7\}$,
- H_2 consists of the 3-cycle $(1,3,4)$ and the $(m-3)$-cycle with edges $\{5,7\}$, $\{m-5, m-2\}$, $\{m-4, m-3\}$, $\{m-2, m\}$, $\{m, m+2\}$, $\{m-1, m+2\}$, $\{i, i+1\}$ for $i \in \{5, 7, \ldots, m-7\}$ and $\{i, i+3\}$ for $i \in \{6, 8, \ldots, m-4\}$, and
- H_3 consists of the 3-cycle $(2,3,5)$ and the $(m-3)$-cycle with edges $\{4,6\}$, $\{4,7\}$, $\{m-2, m+1\}$, $\{m-3, m\}$, $\{m-1, m\}$, $\{m-1, m+1\}$ and $\{i, i+2\}$ for $i \in \{6, 7, \ldots, m-4\}$.

For $m \geq 13$ and odd

- H_1 consists of the 3-cycle $(m-3, m-2, m-1)$ and the $(m-3)$-cycle with edges $\{0,1\}$, $\{0,3\}$, $\{1,2\}$, $\{2,4\}$, $\{3,6\}$, $\{4,5\}$, $\{5,7\}$, $\{m-5, m-4\}$, $\{i, i+1\}$ for $i \in \{7, 9, \ldots, m-6\}$ and $\{i, i+3\}$ for $i \in \{6, 8, \ldots, m-7\}$,
- H_2 consists of the 3-cycle $(1,3,4)$ and the $(m-3)$-cycle with edges $\{5,6\}$, $\{m-5, m-2\}$, $\{m-4, m-3\}$, $\{m-2, m\}$, $\{m, m+2\}$, $\{m-1, m+2\}$, $\{i, i+1\}$ for $i \in \{6, 8, \ldots, m-7\}$ and $\{i, i+3\}$ for $i \in \{5, 7, \ldots, m-4\}$, and
- H_3 consists of the 3-cycle $(2,3,5)$ and the $(m-3)$-cycle with edges $\{4,6\}$, $\{4,7\}$, $\{m-2, m+1\}$, $\{m-3, m\}$, $\{m-1, m\}$, $\{m-1, m+1\}$ and $\{i, i+2\}$ for $i \in \{6, 7, \ldots, m-4\}$.

□

Lemma 3.7. *Let $n \geq 7$ and let F be a 2-regular graph of order n. If $\nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F)$ where $\nu_m(F)$ denotes the number of m-cycles in F, then there exists a 2-factorisation of* $\mathrm{Cay}(\mathbb{Z}_n; \pm\{1,2,3\})$ *into F.*

Proof. If $n \geq 7$ and F is a 2-regular graph of order n such that $\nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F)$, then F can be written as a vertex-disjoint union of 2-regular graphs $G_1, G_2, \ldots, G_t$ where each G_i is isomorphic to either

- C_m with $m \geq 4$, or

- $C_3 \cup C_{m-3}$ with $m = 8$ or $m \geq 10$.

By Lemmas 3.5 and 3.6 we have a decomposition $J^{1,2,3}_{|V(G_i)|} \mapsto G_i$ for $i = 1, 2, \ldots, t$. Applying Lemma 3.4 we obtain a decomposition $J^{1,2,3}_n \mapsto F$, and from this we obtain the required 2-factorisation of $\mathrm{Cay}(\mathbb{Z}_n; \pm\{1,2,3\})$ into F by applying Lemma 3.3. □

We can obtain an analogue of Lemma 3.7 for $\mathrm{Cay}(\mathbb{Z}_n; \pm\{1,3,4\})$ by using similar methods, but we will require F to have girth at least 6. The graph with vertex set $\{0, 1, \ldots, m+3\}$ and edge set $\{\{i, i+1\}, \{i+1, i+4\}, \{i, i+4\} : i = 0, 1, \ldots, m-1\}$ is denoted by $J^{1,3,4}_m$. We write $J^{1,3,4}_m \mapsto F$ when there exists a decomposition $\{H_1, H_2, H_3\}$ of $J^{1,3,4}_m$ into a 2-regular graph F such that

(1) $V(H_1) = \{0, 1, \ldots, m+3\} \setminus \{m, m+1, m+2, m+3\}$,

(2) $V(H_2) = \{0, 1, \ldots, m+3\} \setminus \{0, 3, m+1, m+2\}$, and

(3) $V(H_3) = \{0, 1, \ldots, m+3\} \setminus \{0, 1, 2, m+3\}$.

Notice that for $i = 1, 2, 3$, the subgraph H_i of $J^{1,3,4}_m$ contains exactly one vertex from each of $\{0, m\}$, $\{1, m+1\}$, $\{2, m+2\}$ and $\{3, m+3\}$. It is clear that the proofs of Lemmas 3.3 and 3.4 can be easily modified to give the following two results.

Lemma 3.8. *If $n \geq 9$ and F is a 2-regular graph of order n such that there exists a decomposition $J^{1,3,4}_n \mapsto F$, then there exists a 2-factorisation of* $\mathrm{Cay}(\mathbb{Z}_n; \pm\{1,3,4\})$ *into F.*

Lemma 3.9. *If F and F' are vertex-disjoint 2-regular graphs and there exist decompositions $J^{1,3,4}_{|V(F)|} \mapsto F$ and $J^{1,3,4}_{|V(F')|} \mapsto F'$, then there exists a decomposition $J^{1,3,4}_{|V(F)|+|V(F')|} \mapsto F \cup F'$.*

Lemmas 3.8 and 3.9 allow us to obtain 2-factorisations of $\mathrm{Cay}(\mathbb{Z}_n; \pm\{1,3,4\})$ via the same method we used in the case of $\mathrm{Cay}(\mathbb{Z}_n; \pm\{1,2,3\})$, providing we can find appropriate decompositions of $J^{1,3,4}_m$. We now do this.

Lemma 3.10. *For $m = 6$, $m = 7$ and each $m \geq 9$, $J^{1,3,4}_m \mapsto C_m$.*

Proof. For $m \in \{6, 7, 9, 10\}$, H_1, H_2, H_3 are as defined in the following table.

m	H_1	H_2	H_3
6	$(0,1,5,2,3,4)$	$(1,2,6,9,5,4)$	$(3,6,5,8,4,7)$
7	$(0,1,2,3,6,5,4)$	$(1,4,7,10,6,2,5)$	$(3,4,8,5,9,6,7)$
9	$(0,1,2,3,7,6,5,8,4)$	$(1,4,7,8,12,9,6,2,5)$	$(3,4,5,9,8,11,7,10,6)$
10	$(0,1,2,3,6,9,5,8,7,4)$	$(1,4,8,9,13,10,7,6,2,5)$	$(3,4,5,6,10,9,12,8,11,7)$

For $m \geq 11$ and odd

- H_1 contains the edges $\{0,1\}$, $\{0,4\}$, $\{1,2\}$, $\{2,3\}$, $\{3,7\}$, $\{5,6\}$, $\{m-3, m-2\}$, $\{m-5, m-1\}$, $\{m-4, m-1\}$ and $\{i, i+4\}$ for $i \in \{4, 5, \ldots, m-6\}$,
- H_2 contains the edges $\{1,4\}$, $\{1,5\}$, $\{2,5\}$, $\{2,6\}$, $\{4,7\}$, $\{m, m+3\}$, $\{m-1, m+3\}$, $\{m-2, m-1\}$, $\{m-3, m\}$, $\{i, i+1\}$ for $i \in \{7, 9, \ldots, m-4\}$ and $\{i, i+3\}$ for $i \in \{6, 8, \ldots, m-5\}$, and

- H_3 contains the edges $\{3,4\}$, $\{3,6\}$, $\{4,5\}$, $\{m-1,m\}$, $\{m-2,m+1\}$, $\{m-1,m+2\}$, $\{m-4,m\}$, $\{m-3,m+1\}$, $\{m-2,m+2\}$, $\{i,i+1\}$ for $i \in \{6,8,\dots,m-5\}$ and $\{i,i+3\}$ for $i \in \{5,7,\dots,m-6\}$.

For $m \geq 12$ and even

- H_1 contains the edges $\{0,1\}$, $\{0,4\}$, $\{1,2\}$, $\{2,3\}$, $\{3,6\}$, $\{4,7\}$, $\{5,6\}$, $\{5,9\}$, $\{m-5,m-2\}$, $\{m-4,m-3\}$, $\{m-4,m-1\}$, $\{m-2,m-1\}$, $\{i,i+1\}$ for $i \in \{7,9,\dots,m-7\}$ and $\{i,i+3\}$ for $i \in \{8,10,\dots,m-6\}$,
- H_2 contains the edges $\{1,4\}$, $\{1,5\}$, $\{2,5\}$, $\{2,6\}$, $\{4,8\}$, $\{m-6,m-2\}$, $\{m-5,m-4\}$, $\{m-5,m-1\}$, $\{m-3,m-2\}$, $\{m-3,m\}$, $\{m-1,m+3\}$, $\{m,m+3\}$, $\{i,i+1\}$ for $i \in \{6,8,\dots,m-8\}$ and $\{i,i+3\}$ for $i \in \{7,9,\dots,m-7\}$, and
- H_3 contains the edges $\{3,4\}$, $\{3,7\}$, $\{4,5\}$, $\{5,8\}$, $\{6,9\}$, $\{m-6,m-5\}$, $\{m-4,m\}$, $\{m-3,m+1\}$, $\{m-2,m+1\}$, $\{m-2,m+2\}$, $\{m-1,m\}$, $\{m-1,m+2\}$ and $\{i,i+4\}$ for $i \in \{6,7,\dots,m-7\}$.

□

Lemma 3.11. *For each $m \geq 14$, $J_m^{1,3,4} \mapsto C_8 \cup C_{m-8}$.*

Proof. For $m \in \{14,15,16,17\}$, H_1, H_2, H_3 are as defined in the following table.

m	
14	$H_1 = (0,1,2,3,7,8,5,4) \cup (6,9,13,12,11,10)$
	$H_2 = (8,11,14,17,13,10,9,12) \cup (1,4,7,6,2,5)$
	$H_3 = (7,10,14,13,16,12,15,11) \cup (3,4,8,9,5,6)$
15	$H_1 = (0,1,2,3,6,5,8,4) \cup (7,10,14,13,9,12,11)$
	$H_2 = (1,4,7,8,9,6,2,5) \cup (10,11,14,18,15,12,13)$
	$H_3 = (8,11,15,14,17,13,16,12) \cup (3,4,5,9,10,6,7)$
16	$H_1 = (0,1,5,6,2,3,7,4) \cup (8,9,10,11,15,14,13,12)$
	$H_2 = (1,2,5,9,6,7,8,4) \cup (10,13,16,19,15,12,11,14)$
	$H_3 = (3,4,5,8,11,7,10,6) \cup (9,12,16,15,18,14,17,13)$
17	$H_1 = (0,1,2,3,7,6,5,4) \cup (8,9,13,16,12,15,14,10,11)$
	$H_2 = (1,4,8,12,9,6,2,5) \cup (7,10,13,14,17,20,16,15,11)$
	$H_3 = (3,4,7,8,5,9,10,6) \cup (11,12,13,17,16,19,15,18,14)$

For $m \geq 18$ and even

- H_1 consists of the 8-cycle $(0,1,5,6,2,3,7,4)$ and the $(m-8)$-cycle with edges $\{8,9\}$, $\{9,10\}$, $\{10,11\}$, $\{8,12\}$, $\{m-5,m-1\}$, $\{m-4,m-3\}$, $\{m-3,m-2\}$, $\{m-2,m-1\}$ $\{i,i+1\}$ for $i \in \{12,14,\dots,m-6\}$ and $\{i,i+3\}$ for $i \in \{11,13,\dots,m-7\}$,
- H_2 consists of the 8-cycle $(1,2,5,9,6,7,8,4)$ and the $(m-8)$-cycle with edges $\{10,13\}$, $\{11,12\}$, $\{m-6,m-2\}$, $\{m-5,m-2\}$, $\{m-4,m-1\}$, $\{m-3,m\}$, $\{m-1,m+3\}$, $\{m,m+3\}$ and $\{i,i+4\}$ for $i \in \{10,11,\dots,m-7\}$, and
- H_3 consists of the 8-cycle $(3,4,5,8,11,7,10,6)$ and the $(m-8)$-cycle with edges $\{9,12\}$, $\{9,13\}$, $\{m-4,m\}$, $\{m-3,m+1\}$, $\{m-2,m+1\}$, $\{m-2,m+2\}$, $\{m-1,m\}$, $\{m-1,m+2\}$, $\{i,i+1\}$ for $i \in \{13,15,\dots,m-5\}$ and $\{i,i+3\}$ for $i \in \{12,14,\dots,m-6\}$.

For $m \geq 19$ and odd

- H_1 consists of the 8-cycle $(0, 1, 2, 3, 7, 6, 5, 4)$ and the $(m-8)$-cycle with edges $\{8, 9\}$, $\{8, 11\}$, $\{9, 13\}$, $\{10, 11\}$, $\{10, 14\}$, $\{12, 15\}$, $\{12, 16\}$, $\{m-4, m-1\}$, $\{m-3, m-2\}$ and $\{i, i+4\}$ for $i \in \{13, 14, \ldots, m-5\}$,
- H_2 consists of the 8-cycle $(1, 4, 8, 12, 9, 6, 2, 5)$ and the $(m-8)$-cycle with edges $\{7, 10\}$, $\{7, 11\}$, $\{10, 13\}$, $\{11, 15\}$, $\{m-4, m-3\}$, $\{m-3, m\}$, $\{m-2, m-1\}$, $\{m-1, m+3\}$, $\{m, m+3\}$, $\{i, i+1\}$ for $i \in \{13, 15, \ldots, m-6\}$ and $\{i, i+3\}$ for $i \in \{14, 16, \ldots, m-5\}$, and
- H_3 consists of the 8-cycle $(3, 4, 7, 8, 5, 9, 10, 6)$ and the $(m-8)$-cycle with edges $\{11, 12\}$, $\{11, 14\}$, $\{12, 13\}$, $\{m-4, m\}$, $\{m-3, m+1\}$, $\{m-2, m+1\}$, $\{m-2, m+2\}$, $\{m-1, m\}$, $\{m-1, m+2\}$, $\{i, i+1\}$ for $i \in \{14, 16, \ldots, m-5\}$ and $\{i, i+3\}$ for $i \in \{13, 15, \ldots, m-6\}$.

□

Lemma 3.12. $J_{24}^{1,3,4} \mapsto C_8 \cup C_8 \cup C_8$.

Proof. Take

$$\begin{aligned} H_1 &= (0,1,2,3,6,5,8,4) \cup (7,10,9,12,13,14,15,11) \cup (16,17,18,19,23,22,21,20), \\ H_2 &= (1,4,7,8,9,6,2,5) \cup (10,11,12,15,16,13,17,14) \cup (18,21,24,27,23,20,19,22), \text{ and} \\ H_3 &= (3,4,5,9,13,10,6,7) \cup (8,11,14,18,15,19,16,12) \cup (17,20,24,23,26,22,25,21). \end{aligned}$$

□

The following result is an analogue of Lemma 3.7 for 2-factorisations of $\mathrm{Cay}(\mathbb{Z}_n\,; \pm\{1,3,4\})$.

Lemma 3.13. *If $n \geq 9$ and F is a 2-regular graph of order n with girth at least 6, then there exists a 2-factorisation of $\mathrm{Cay}(\mathbb{Z}_n\,; \pm\{1,3,4\})$ into F.*

Proof. If $n \geq 9$ and F is a 2-regular graph of order n with girth at least 6, then F can be written as a vertex-disjoint union of 2-regular graphs $G_1, G_2, \ldots, G_t$ where each G_i is isomorphic to either

- C_m with $m = 6, 7$ or $m \geq 9$,
- $C_8 \cup C_{m-8}$ with $m \geq 14$, or
- $C_8 \cup C_8 \cup C_8$.

By Lemmas 3.10, 3.11 and 3.12 we have a decomposition $J_{|V(G_i)|}^{1,3,4} \mapsto G_i$ for $i = 1, 2, \ldots, t$. Applying Lemma 3.9 we obtain a decomposition $J_n^{1,3,4} \mapsto F$, and from this we obtain the required 2-factorisation of $\mathrm{Cay}(\mathbb{Z}_n;\, \pm\{1,3,4\})$ into F by applying Lemma 3.8. □

4 2-factorisations and the Oberwolfach Problem

In this section we use results from the preceding sections to obtain results on the Oberwolfach Problem (and an additional result on 2-factorisations of $K_n - I$ into a number of specified 2-factors and Hamilton cycles). We will also use the following corollary of Lemma 3.2 which was proved in [8].

Lemma 4.1. ([8]) *If there exists a factorisation of K_n or of $K_n - I$ into $\mathrm{Cay}(\mathbb{Z}_n\,; \pm\{1,2,3,4\})$, then $\mathrm{OP}(F)$ has a solution for each 2-regular graph F of order n, with the exception that there is no solution to $\mathrm{OP}(C_4 \cup C_5)$.*

Theorem 4.2. *If $p \equiv 5 \pmod 8$ is prime, then $\mathrm{OP}(F)$ has a solution for every 2-regular graph F of order $2p$.*

Proof. The case $p = 13$ is covered in [13]. For $p \neq 13$, Theorem 2.11 gives us a factorisation of $K_{2p} - I$ into $\mathrm{Cay}(\mathbb{Z}_{2p}\,; \pm\{1,2,3,4\})$ and the result then follows by Lemma 4.1. □

Theorem 4.3. *Let $\mathcal{P}$ be the set of primes given by $p \in \mathcal{P}$ if and only if $p \geq 7$ and neither 4 nor 32 is in the subgroup of $\mathbb{Z}_p^*$ generated by $\{-1, 6\}$. Then $\mathcal{P}$ is infinite and if $p \in \mathcal{P}$, then $\mathrm{OP}(F)$ has a solution for every 2-regular graph F of order p satisfying $\nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F)$ where $\nu_m(F)$ denotes the number of m-cycles in F.*

Proof. Let p be prime such that $p \equiv 1 \pmod 6$, $2, 3 \notin (\mathbb{Z}_p^*)^3$ and $6 \in (\mathbb{Z}_p^*)^3$. Theorem 2.5 says that there are infinitely many such p. We shall show that $p \in \mathcal{P}$, which shows that $\mathcal{P}$ is also infinite. We have $-1 \in (\mathbb{Z}_p^*)^3$, and this together with the fact that $6 \in (\mathbb{Z}_p^*)^3$ implies that the subgroup of $\mathbb{Z}_p^*$ generated by $\{-1, 6\}$ is a subgroup of $(\mathbb{Z}_p^*)^3$. Since it follows from $2 \notin (\mathbb{Z}_p^*)^3$ that $4, 32 \notin (\mathbb{Z}_p^*)^3$, neither 4 nor 32 is in the subgroup of $\mathbb{Z}_p^*$ generated by $\{-1, 6\}$. That is, $p \in \mathcal{P}$.

Now let p be an arbitrary element of $\mathcal{P}$ and let G be the subgroup of $\mathbb{Z}_p^*$ generated by $\{-1, 6\}$. The condition that neither 4 nor 32 is in G implies that the order d of $2G$ in $\mathbb{Z}_p^*/G$ is neither 1, 2 nor 5, and so there exist non-negative integers α and β such that $d = 3\alpha + 4\beta$. Thus, by Lemma 2.6 there is a factorisation of K_p in which each factor is either $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1,2,3\})$ or $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1,2,3,4\})$.

Let F be a 2-regular graph of order p satisfying $\nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F)$. Lemma 3.7 gives us a 2-factorisation of $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1,2,3\})$ into F, and Lemma 3.2 gives us a 2-factorisation of $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1,2,3,4\})$ (the facts that p is prime and that $\nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F)$ imply that F is not amongst the possible exceptions listed in Lemma 3.2). The result follows. □

Theorem 4.4. *Let $\mathcal{P}$ be the set of primes such that $p \in \mathcal{P}$ if and only if $p \equiv 1 \pmod 6$ and $2, 3, 6 \notin (\mathbb{Z}_p^*)^3$. Then $\mathcal{P}$ is infinite and if $p \in \mathcal{P}$, then $\mathrm{OP}(F)$ has a solution for every 2-regular graph F of order p with girth at least 6.*

Proof. By Theorem 2.5, $\mathcal{P}$ is infinite. If $p \in \mathcal{P}$, then Theorem 2.4 gives us a factorisation of K_p into $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1,3,4\})$, and the result then follows by applying Lemma 3.13 to each factor ($7 \notin \mathcal{P}$ so Lemma 3.13 can indeed be applied). □

For each odd prime p, the following theorem states there is a 2-factorisation of $K_{2p} - I$ into $\frac{p-1}{2}$ prescribed 2-factors and $\frac{p-1}{2}$ Hamilton cycles.

Theorem 4.5. *If p is an odd prime and $G_1, G_2, \ldots, G_{\frac{p-1}{2}}$ are 2-regular graphs of order $2p$, then there is a 2-factorisation $\{F_1, F_2, \ldots, F_{p-1}\}$ of $K_{2p} - I$ such that $F_i \cong G_i$ for $i = 1, 2, \ldots, \frac{p-1}{2}$ and F_i is a Hamilton cycle for $i = \frac{p+1}{2}, \frac{p+3}{2}, \ldots, p-1$.*

Proof. By Theorem 2.9 there is a factorisation of $K_{2p} - I$ into $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1,2\})$. By Lemma 3.1, each copy of $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1,2\})$ can be factored into any specified 2-regular graph of order $2p$ and a Hamilton cycle. The result follows. □

5 Isomorphic 2-factorisations of complete multigraphs

The complete multigraph of order n and multiplicity s is denoted by sK_n. It has s distinct edges joining each pair of distinct vertices.

Lemma 5.1. *If p is an odd prime and $S = \pm\{d_1, d_2, \ldots, d_s\} \subseteq \mathbb{Z}_p^*$, then there exists a $2s$-factorisation of sK_p into $\mathrm{Cay}(\mathbb{Z}_p\,; S)$.*

Proof. The required factorisation is given by $\{\mathrm{Cay}(\mathbb{Z}_p\,; \omega^i S) : i = 0, 1, \ldots, \frac{p-3}{2}\}$ where ω is primitive in $\mathbb{Z}_p$ and $\omega^i S = \{\omega^i s : s \in S\}$. □

Theorem 5.2. *If p is an odd prime and F is any 2-regular graph of order p satisfying $\nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F)$, where $\nu_m(F)$ denotes the number of m-cycles in F, then there exists a 2-factorisation of $3K_p$ into F.*

Proof. The cases $p = 3$ and $p = 5$ are trivial so assume $p \geq 7$. By Lemma 5.1 there exists a 6-factorisation of $3K_p$ into $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1, 2, 3\})$, and by Lemma 3.7 each such 6-factor has a 2-factorisation into F. □

Theorem 5.3. *If p is an odd prime and F is any 2-regular graph of order p, then there exists a 2-factorisation of $4K_p$ into F.*

Proof. The cases $p = 3$ and $p = 5$ are trivial. Since solutions to $\mathrm{OP}(C_7)$ and $\mathrm{OP}(C_3 \cup C_4)$ exist, the case $p = 7$ can be dealt with by taking four copies of these 2-factorisations of K_7. So we may assume $p \geq 11$. By Lemma 5.1 there exists an 8-factorisation of $4K_p$ into $\mathrm{Cay}(\mathbb{Z}_p\,; \pm\{1, 2, 3, 4\})$, and by Lemma 3.2 each such 8-factor has a 2-factorisation into F; except in the case where F is one of the listed exceptions or possible exceptions in Lemma 3.2. These are easily dealt with as follows. Since p is prime the only relevant exceptions are $F = C_3 \cup C_3 \cup \cdots \cup C_3 \cup C_4$ where the number of copies of C_3 is at least 5, and $F = C_3 \cup C_4 \cup C_4 \cup \cdots \cup C_4$ where the number of copies of C_4 is odd and at least 5. However, it is known that for each such F, there is a 2-factorisation of K_p into F; the former case is covered in [11], and the latter case is covered in [21]. Thus, by taking four copies of these 2-factorisations of K_p, we obtain the required 2-factorisations of $4K_p$. □

Theorem 5.4. *Let p be an odd prime and let F be a 2-regular graph of order p. If $\lambda \equiv 0 \pmod 4$, then there exists a 2-factorisation of λK_p into F. Moreover, if F satisfies $\nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F)$, where $\nu_m(F)$ denotes the number of m-cycles in F, then the result also holds for $\lambda = 3$ and for all $\lambda \geq 6$.*

Proof. For the given values of λ, it is trivial to factorise λK_p such that each factor is either $3K_p$ or $4K_p$, and with each factor being $4K_p$ when $\lambda \equiv 0 \pmod 4$. Thus, the result follows by Theorems 5.2 and 5.3. □

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The $2A$-Majorana representations of the Harada-Norton group

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Abstract

We show that all $2A$-Majorana representations of the Harada-Norton group F_5 have the same shape. If $\mathcal{R}$ is such a representation, we determine, using the theory of association schemes, the dimension and the irreducible constituents of the linear span U of the Majorana axes. Finally, we prove that, if $\mathcal{R}$ is based on the (unique) embedding of F_5 in the Monster, U is closed under the algebra product.

Keywords: Majorana representations, association schemes, Monster algebra, Harada-Norton group.

Math. Subj. Class.: 20D08, 20C34, 05E30, 17B69.

1 Introduction

Let $(W, \cdot)$ be a real commutative algebra endowed with a scalar product $(\,,)_W$ and denote with $Aut(W)$ the group of algebra automorphisms of W that preserve the scalar product. We shall assume that, for every $u, v, w \in W$,

(M1) $(\,,)_W$ is *associative* , that is $(u \cdot v, w) = (u, v \cdot w)$,

(M2) the *Norton Inequality*, $(u \cdot u, v \cdot v) \geq (u \cdot v, u \cdot v)$, holds.

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Recall that a *Majorana axis* of W (see [10, Definition 8.6.1] or, equivalently, [9, p. 2423]) is a vector $a \in W$ such that

(M3) a has length 1,

(M4) the adjoint endomorphism $ad(a)$, induced by multiplication by a on the $\mathbb{R}$-vector space W, is semisimple with spectrum contained in $\{1, 0, 2^{-2}, 2^{-5}\}$,

(M5) a spans linearly the eigenspace relative to the eigenvalue 1 of $ad(a)$,

(M6) the linear transformation $a^\tau \colon W \to W$, that inverts the eigenvectors of $ad(a)$ relative to 2^{-5} and centralises the other eigenvectors, preserves the algebra product,

(M7) the linear transformation $a^\sigma \colon C_W(a^\tau) \to C_W(a^\tau)$, that inverts the eigenvectors of $ad(a)$ relative to 2^{-2} and centralises the other eigenvectors contained in $C_W(a^\tau)$, preserves the restriction to $C_W(a^\tau)$ of the algebra product.

Denote with $\mathcal{A}$ the set of Majorana axes of W. If $a \in \mathcal{A}$, the map a^τ is called a *Majorana involution* corresponding to a. Note that, by (M1) and (M4), W decomposes into an orthogonal sum of $ad(a)$-eigenspaces, hence (M6) actually implies that every Majorana involution is an element of $Aut(W)$. Let

$$\tau \colon \mathcal{A} \to Aut(W)$$

be the map $a \mapsto a^\tau$. Note that $\mathcal{A}$ is invariant under $Aut(W)$ and, for $a \in \mathcal{A}$ and $\delta \in Aut(W)$, we have

$$(a^\delta)^\tau = \delta^{-1} a^\tau \delta,$$

so that the set $\mathcal{A}^\tau$ of Majorana involutions is invariant under conjugation by elements of $Aut(W)$.

The fundamental examples of Majorana involutions are given by the $2A_M$-involutions (i.e. those centralised by the double cover of the Baby Monster) of the Monster group M acting on the 196884-dimensional Conway-Norton-Griess algebra W_M. A key result, in this context, is the Norton-Sakuma Theorem, that classifies and describes the *Norton-Sakuma algebras*, i.e. the algebras that are generated by a pair of Majorana axes [19] (see also [9, Section 2.6]). By S. Sakuma's classification, every Norton-Sakuma algebra is isomorphic to a subalgebra of W_M generated by a pair of Majorana axes a_0, a_1 corresponding via τ to $2A_M$-involutions in M. In [17] S. Norton proved that the latter algebras (hence all Norton-Sakuma algebras) fall into nine isomorphism types, labelled $1A$, $2A$, $2B$, $3A$, $3C$, $4A$, $4B$, $5A$, and $6A$, accordingly to the conjugacy class in the Monster of the element $a_0^\tau a_1^\tau$. Further, Norton produced, for each type, a basis (the *Norton basis*), the relative structure constants and the Gram matrix. Table 1 (which is an extract from Table 3 in [9]) summarises the results from the Norton-Sakuma Theorem we need for this paper: more precisely, for each pair of distinct Majorana axes a_0, a_1, we give the Norton basis of the algebra generated by a_0 and a_1, and the relevant (for this paper) scalar products (with the same scaling as in [9]):

Here, for $\rho := a_0^\tau a_1^\tau$ in each Norton-Sakuma algebra,

- $a_{-1} := a_1^{\rho^{-1}}$, $a_{-2} := a_0^{\rho^{-1}}$, $a_2 := a_0^\rho$, $a_3 := a_1^\rho$, in particular they are Majorana axes.

Table 1: Norton bases and relevant scalar products for the Norton-Sakuma algebras.

Type	Norton basis	Scalar Products
$2A$	a_0, a_1, a_ρ	$(a_0, a_1)_W = \frac{1}{2^3}$
$2B$	a_0, a_1	$(a_0, a_1)_W = 0$
$3A$	a_0, a_1, a_{-1}, u_ρ	$(a_0, a_1)_W = \frac{13}{2^8}$
$3C$	a_0, a_1, a_{-1}	$(a_0, a_1)_W = \frac{1}{2^6}$
$4A$	$a_0, a_1, a_{-1}, a_2, v_\rho$	$(a_0, a_1)_W = \frac{1}{2^5}$
$4B$	$a_0, a_1, a_{-1}, a_2, a_{\rho^2}$	$(a_0, a_1)_W = \frac{1}{2^6}$
$5A$	$a_0, a_1, a_{-1}, a_2, a_{-2}, w_\rho$	$(a_0, a_1)_W = \frac{3}{2^7}$
$6A$	$a_0, a_1, a_{-1}, a_2, a_{-2}, a_3, a_{\rho^3}, u_{\rho^2}$	$(a_0, a_1)_W = \frac{5}{2^8}$

- The vectors u_ρ, v_ρ, $\pm w_\rho$, resp. u_{ρ^2}, appearing in the algebras of type $3A$, $4A$, $5A$, resp. $6A$, are called $3A$-, $4A$-, $5A$-, resp. $3A$-, *axes* and, in each Norton-Sakuma algebra, they are defined as follows,

$$\begin{aligned} u_\rho &:= \tfrac{2^6}{3^3 5}(2a_0 + 2a_1 + a_{-1}) - \tfrac{2^{11}}{3^3 5} a_0 \cdot a_1, \\ v_\rho &:= a_0 + a_1 + \tfrac{1}{3}(a_{-1} + a_{-1}) - \tfrac{2^6}{3} a_0 \cdot a_1, \\ w_\rho &:= -\tfrac{1}{2^7}(3a_0 + 3a_1 - a_{-1} - a_{-1} - a_{-2}) + a_0 \cdot a_1, \\ u_{\rho^2} &:= \tfrac{2^6}{3^3 5}(2a_0 + 2a_{-1} + a_{-2}) - \tfrac{2^{11}}{3^3 5} a_0 \cdot a_{-1}. \end{aligned}$$

  The indexing with powers of ρ is justified by the fact that, in the action of M on W_M, for $3 \leq N \leq 5$, the NA-axes are essentially determined (up to the sign in the $5A$-case) by the cyclic groups $\langle \rho \rangle$ in M of order N (see [9, p. 2450]). It is not clear if that property follows from Axioms (M1)-(M7), therefore axiom (M8)(b) was added in [3] in the definition of Majorana representations.

- The vectors a_ρ, a_{ρ^2}, resp. a_{ρ^3} appearing in the algebras of type $2A$-, $4B$-, resp. $6A$ are further Majorana axes. As above, the indexing is suggested by the action of M on W_M since, in that case, whenever a_0 and a_1 generate a subalgebra of type $2A$, the product $\rho = a_0^\tau a_1^\tau$ is the Majorana involution corresponding to a_ρ. As in the previous paragraph, that property will be axiomatised in (M8)(a). Finally, by the Norton-Sakuma Theorem (see [9, Lemma 2.20 (iv) and (v)]), a_0 and a_2 (resp. a_0 and a_3) generate a subalgebra of type $2A$ in the algebra of type $4B$ (resp. $6A$) and, for $i \in \{2, 3\}$, by the definition of a_i, the product $a_0^\tau a_i^\tau$ is equal to ρ^i.

The Norton-Sakuma Theorem inspired the definition of Majorana representations, introduced by A. A. Ivanov in [10] in order to provide an axiomtic framework for studying the actions of $2A_M$-generated subgroups of M on W_M.

Let G be a finite group, $\mathcal{T}$ a G-invariant set of involutions generating G,

$$\phi\colon G \to Aut(W)$$

a faithful representation of G on W, and

$$\psi\colon \mathcal{T} \to \mathcal{A}$$

be an injective map such that for every $g \in G$ and $t \in \mathcal{T}$,

$$(t^{\psi})^{\tau} := t^{\phi} \tag{1.1}$$

and

$$(t^{\psi})^{g^{\phi}} = (g^{-1}tg)^{\psi}. \tag{1.2}$$

The quintet

$$\mathcal{R} := (G, \mathcal{T}, W, \phi, \psi)$$

is called a *Majorana representation* (or, to put evidence on the set $\mathcal{T}$, a $\mathcal{T}$-*Majorana representation*) of G, if $\mathcal{R}$ satisfies the following condition (see [3, Axiom **M8**]):

(M8) (a) For t_1 and t_2 in $\mathcal{T}$, the Norton-Sakuma algebra generated by t_1^{ψ} and t_2^{ψ} has type $2A$ if and only if $t_1t_2 \in \mathcal{T}$.

(b) Suppose that t_1, t_2, t_3, and t_4 are elements of $\mathcal{T}$ such that $t_1t_2 = t_3t_4$ and the subalgebras generated by t_1^{ψ}, t_2^{ψ} and t_3^{ψ}, t_4^{ψ} have both type $3A$, $4A$, or $5A$. Then $u_{(t_1t_2)^{\phi}} = u_{(t_3t_4)^{\phi}}$, $v_{(t_1t_2)^{\phi}} = v_{(t_3t_4)^{\phi}}$, or $w_{(t_1t_2)^{\phi}} = w_{(t_3t_4)^{\phi}}$, respectively.

Axiom (M8)(a) and Norton-Sakuma Theorem (see [9, Lemma 2.20]) imply that,

> if t_1^{ψ} and t_2^{ψ} generate a Norton-Sakuma subalgebra of W of type $2A$, $4B$, or $6A$, then t_1t_2, $(t_1t_2)^2$, or $(t_1t_3)^3$ belongs to $\mathcal{T}$, and $(t_1t_2)^{\psi}$, $((t_1t_2)^2)^{\psi}$, or $((t_1t_3)^3)^{\psi}$ coincides with $a_{(t_1t_2)^{\phi}}$, $a_{((t_1t_2)^2)^{\phi}}$, or $a_{((t_1t_2)^3)^{\phi}}$, respectively.

An immediate consequence of that definition is that, given a Majorana representation

$$\mathcal{R} := (G, \mathcal{T}, W, \phi, \psi)$$

of a group G and a nonempty subset $\mathcal{T}_0$ of $\mathcal{T}$, such that $\mathcal{T}_0$ is $\langle \mathcal{T}_0 \rangle$-invariant, the quintet

$$\mathcal{R}_{\langle \mathcal{T}_0 \rangle} := (\langle \mathcal{T}_0 \rangle, \mathcal{T}_0, W, \phi|_{\langle \mathcal{T}_0 \rangle}, \psi|_{\mathcal{T}_0}) \tag{1.3}$$

is a $\mathcal{T}_0$-Majorana representation of $\langle \mathcal{T}_0 \rangle$. Further, if we replace W with the subalgebra $W_{\mathcal{T}_0}$ generated by the set of Majorana axes $\mathcal{T}_0^{\psi}$ in the quintet (1.3), we still have a Majorana representation of $\langle \mathcal{T}_0 \rangle$ provided $\langle \mathcal{T}_0 \rangle$ acts nontrivially on $W_{\mathcal{T}_0}$ (which is the case, e.g., when $\langle \mathcal{T}_0 \rangle$ has trivial centre). In particular, if ϵ is an embedding of a group H in M and H^{ϵ} is generated by a subset $\mathcal{T}$ of $2A_M$, then H inherits a $(\mathcal{T} \cap H^{\epsilon})^{\epsilon^{-1}}$-Majorana representation $\mathcal{R}_{\epsilon}$ obtained by composing ϵ with the restriction of $\mathcal{R}_M$ to H^{ϵ}. In that case, the Majorana representation $\mathcal{R}_{\epsilon}$ of H is said to be *based on the embedding* ϵ. In this paper, whenever a Majorana representation of a group G is based on an embedding ϵ in the Monster, we shall always identify G with G^{ϵ}.

For a pair (a, b) of elements in W, denote the subalgebra they generate with $\langle\langle a, b \rangle\rangle$. Let $\mathcal{R}$ be as above, the *shape* of $\mathcal{R}$ is a function $sh_{\mathcal{R}}$ from the set of the nondiagonal orbitals of G on $\mathcal{T}$ to the set of types of the Norton-Sakuma algebras so that

1. $sh_{\mathcal{R}}((t,s)^G) = NX$ if and only if ts has order N and the algebra $\langle\langle t^\psi, s^\psi\rangle\rangle$ is a Norton-Sakuma algebra of type NX.

2. $sh_{\mathcal{R}}$ must respect the embeddings of the algebras:

$$2A \hookrightarrow 4B, 2A \hookrightarrow 6A, 2B \hookrightarrow 4A, 3A \hookrightarrow 6A$$

in the sense that, for $t, r_1, r_2 \in \mathcal{T}$, if $\langle\langle t^\psi\rangle\rangle < \langle\langle t^\psi, r_1^\psi\rangle\rangle < \langle\langle t^\psi, r_2^\psi\rangle\rangle$, then

$$(sh_{\mathcal{R}}((t,r_1)^G),\ sh_{\mathcal{R}}((t,r_2)^G)) \in \{(2A,4B),(2A,6A),(2B,4A),(3A,6A)\}.$$

Remark: Clearly, if $\mathcal{T}_0$ is a $\langle\mathcal{T}_0\rangle$-invariant nonempty subset of $\mathcal{T}$, the shape of $\mathcal{R}_{\langle\mathcal{T}_0\rangle}$ is the restriction of $sh_{\mathcal{R}}$ to $\mathcal{T}_0 \times \mathcal{T}_0$.

Majorana representations of several groups have already been investigated (see [9, 11, 12, 13, 14, 5, 3, 6]).

In this paper we study the $2A$-Majorana representations of the Harada-Norton group F_5, where $2A$ is the set of the involutions of F_5 whose centraliser is $(2HS)\cdot 2$, the double cover of the Higman-Sims group extended by its outer automorphism group of order 2. We shall show that every $2A$-Majorana representation of F_5 has the same shape as the Majorana representations of F_5 based on its embedding into M as the subgroup generated by the set of involutions in $2A_M$ that centralise an element of type $5A$ (here $2A = 2A_M \cap F_5$, see [4]). By [18, Theorem 21], that one is the unique embedding of F_5 into M (up to conjugation in M), hence, since F_5 is transitive on $2A$, there is (up to conjugation in M) only one Majorana representation of F_5 based on an embedding in M. We prove the following result.

Theorem 1.1. *Let W be as above and $\mathcal{R} := (F_5, 2A, W, \phi, \psi)$ be a $2A$-Majorana representation of F_5 on W. Then*

(i) *$\mathcal{R}$ has the shape given in Table 3;*

(ii) *The $\mathbb{R}$-linear span $\langle 2A^\psi\rangle$ of $2A^\psi$ has dimension* 18 316*;*

(iii) *$\langle 2A^\psi\rangle$ is the direct sum of three irreducible $\mathbb{R}[F_5]$-submodules of dimensions* 1, 8910 *and* 9405*, respectively;*

(iv) *if $\mathcal{R}$ is based on the embedding of F_5 in M, then $W_{2A} = \langle 2A^\psi\rangle$.*

Unless explicitly stated, for the remainder of this paper we shall stick to the notations introduced in this section. We shall also set $\mathcal{T} := 2A$.

2 The First Eigenmatrix

By [4, p. 166], we have $|\mathcal{T}| = 1539000$, and it seems hard, at present, to perform a direct computation of the dimension of the linear span of $\mathcal{T}^\psi$. We therefore apply the theory of association schemes as in [14] and [6] to reduce ourselves to a more manageable situation. The first step is to compute the first eigenmatrix of the association scheme relative to the permutation action of F_5 on $\mathcal{T}$ (see [1, pp. 59-60]). For that purpose, we need to recover some information about the action F_5 induces by conjugation on $\mathcal{T}$.

Let $n := |\mathcal{T}|$ and let $t_1, \ldots, t_n$ be the distinct elements of $\mathcal{T}$, so that

$$\mathcal{B} := (t_1, \ldots, t_n)$$

is an ordered basis for the complex permutation module V of F_5 on $\mathcal{T}$. With respect to $\mathcal{B}$, we identify $End_{\mathbb{C}}(V)$ with the set of $n \times n$ matrices with complex entries. Let $T_0, \ldots, T_8$ be the orbitals of F_5 on $\mathcal{T}$ and, for every $k \in \{0, \ldots, 8\}$, let A_k be the *adjacency matrix* associated to the orbital T_k, that is

$$(A_k)_{ij} = \begin{cases} 1 & \text{if the pair } (t_i, t_j) \text{ is in } T_k \\ 0 & \text{otherwise.} \end{cases}$$

By [1, Theorem 1.3], the 9-tuple $(A_0, \ldots, A_8)$ is a basis for the *centralizer algebra*

$$\mathcal{C} := End_{\mathbb{C}[F_5]}(V).$$

For $i, j, k \in \{0, \ldots, 8\}$, let p_{ij}^k be the number of elements z in $\mathcal{T}$ such that for a fixed pair (x, y) in T_k we have $(x, z) \in T_i$ and $(z, y) \in T_j$. By definition, the p_{ij}^k's are all non negative integers and, by [1, §2.2], they are the structure constants of $\mathcal{C}$ relative to the basis $(A_0, \ldots, A_8)$, that is

$$A_i A_j = \sum_{k=0}^{8} p_{ij}^k A_k. \tag{2.1}$$

The matrix B_i of size 9 whose j, k entry is p_{ij}^k is called *ith intersection matrix*. Clearly, B_i^t is the matrix associated to the endomorphism induced by A_i on $\mathcal{C}$ via left multiplication with respect to the basis $(A_0, \ldots, A_8)$, in particular B_i has the same eigenvalues as A_i. By [8, Lemma 2.18.1*(ii)*] we may choose the indexes of the orbitals $T_0, \ldots, T_8$ in such a way that T_0 is the diagonal orbital (hence B_0 is the identity matrix), T_1 is the non-diagonal orbital of smallest size, and the first intersection matrix B_1 is as follows:

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1408 & 53 & 32 & 18 & 4 & 2 & 0 & 0 & 0 \\ 0 & 50 & 0 & 2 & 12 & 0 & 2 & 0 & 0 \\ 0 & 450 & 32 & 100 & 32 & 50 & 32 & 0 & 0 \\ 0 & 350 & 672 & 112 & 160 & 100 & 92 & 160 & 0 \\ 0 & 504 & 0 & 504 & 288 & 356 & 312 & 320 & 0 \\ 0 & 0 & 672 & 672 & 552 & 650 & 720 & 640 & 1280 \\ 0 & 0 & 0 & 0 & 360 & 250 & 240 & 288 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 128 \end{pmatrix}.$$

By [1, Theorem 3.1], we have that V decomposes into the direct sum

$$V = V_0 \oplus \ldots \oplus V_8 \tag{2.2}$$

of nine irreducible $\mathbb{C}[F_5]$-submodules. Since F_5 is transitive on $\mathcal{T}$, the subspace linearly spanned by the sum of all elements of $\mathcal{T}$ is the unique trivial submodule of V. As usual, we shall denote it by V_0. Since the action of F_5 on $\mathcal{T}$ is multiplicity free (see [8, Lemma 2.18.1.(*ii*)]), the V_j's are minimal common eigenspaces for the adjacency matrices A_i. It follows that there is a complex invertible matrix D that simultaneously diagonalises the matrices A_i's. By the definition of the adjacency matrices, we have that, for each i, the sums (say k_i) of the entries in each row of the matrices A_i are constant, whence V_0 is a k_i-eigenspace for A_i, for each i.

For $i, j \in \{0, \ldots, 8\}$, let p_{ij} be the eigenvalue of A_j on V_i. The 9×9 matrix $P := (p_{ij})$ is called the *first eigenmatrix* of the association scheme $(\mathcal{T}, \{T_0, \ldots, T_8\})$.

Lemma 2.1. *With the above notations,*

$$P = \begin{pmatrix} 1 & 1408 & 2200 & 35200 & 123200 & 354816 & 739200 & 277200 & 5775 \\ 1 & 128 & 200 & 0 & 1600 & -2304 & 0 & 0 & 375 \\ 1 & 28 & -50 & -50 & -100 & 396 & -750 & 450 & 75 \\ 1 & 16 & 4 & -56 & -136 & -288 & 504 & 0 & -45 \\ 1 & -32 & 40 & -80 & 80 & 576 & -240 & -360 & 15 \\ 1 & -47 & -50 & 250 & 350 & -504 & 0 & 0 & 0 \\ 1 & -112 & 300 & 1000 & -2200 & -864 & -1800 & 3600 & 75 \\ 1 & 208 & -50 & 2200 & -2800 & 2016 & 4200 & -6300 & 525 \\ 1 & 208 & 100 & 1000 & 1400 & 2016 & -4200 & 0 & -525 \end{pmatrix}.$$

Proof. Note that, since A_0 is the identity matrix, $p_{i0} = 1$ for all i's. Straightforward computation shows that the eigenvalues of B_1 are 1408, 128, 28, 16, -32, -47, -112, 208, and 208, giving the first two columns of P. Set

$$(\lambda_0, \ldots, \lambda_8) = (1408, 128, 28, 16, -32, -47, -112, 208, 208).$$

For each $h \in \{0, \ldots, 8\}$, let $\mathcal{S}_h$ be the linear system

$$(B_1 - \lambda_h Id)\,{}^t(1, \lambda_h, x_2, \ldots, x_8) = 0 \tag{2.3}$$

in the indeterminates $x_2, \ldots, x_8$. Taking $i = 1$ in Equation (2.1) and multipling each term by D on the right and by D^{-1} on the left, we get

$$(D^{-1}A_1D)(D^{-1}A_jD) = \sum_{h=0}^{8} p_{1j}^h (D^{-1}A_hD). \tag{2.4}$$

Since the matrices $D^{-1}A_hD$ are diagonal with eigenvalues p_{kh} on the common eigenspaces V_k, for each $k \in \{0, \ldots, 8\}$, from Equation (2.4) we obtain that the relation

$$\lambda_k p_{kj} = \sum_{h=0}^{8} p_{1j}^h p_{kh} \tag{2.5}$$

holds for every $k \in \{0, \ldots, 8\}$. Note that the second member is the jth entry of the vector $B_1{}^t(1, \lambda_k, p_{k2}, \ldots, p_{k8})$, therefore Equation (2.5) implies that the 9-tuple

$$(1, \lambda_k, p_{k2}, \ldots, p_{k8})$$

is an eigenvector for B_1 relative to the eigenvalue λ_k, for every $k \in \{0, \ldots, 8\}$. Since, for $k \neq 7, 8$, the eigenvalue λ_k has multiplicity 1, it follows that the first seven rows of the matrix P can be obtained computing the unique solution $(p_{k2}, \ldots, p_{k8})$ of the system $\mathcal{S}_k$ for each $k \in \{1, \ldots, 6\}$.

We are now left with the last two rows of the matrix P, corresponding to the eigenvalue 208 of B_1. The set of solutions of the system $\mathcal{S}_7$,

$$(B_1 - 208 Id)\,{}^t(1, 208, x_2, \ldots, x_8) = 0,$$

is

$$\{(25 - \frac{x}{7}, 1600 + \frac{8x}{7}, -700 - 4x, 2016, 8x, -3150 - 6x, x) \mid \text{ where } x \in \mathbb{R}\}.$$

Therefore, for suitable $x, y \in \mathbb{R}$, we can write the last two rows of the matrix P as follows

$$1, 208, 25 - \frac{x}{7}, 1600 + \frac{8x}{7}, -700 - 4x, 2016, 8x, -3150 - 6x, x$$

$$1, 208, 25 - \frac{y}{7}, 1600 + \frac{8y}{7}, -700 - 4y, 2016, 8y, -3150 - 6y, y.$$

Set $m_i = \dim_{\mathbb{R}}(V_i)$. Then $m_0 = 1$ and, for $1 \leq i \leq 6$, m_i can be computed from the rows of P using the following formula (see [1, Theorem 4.1]):

$$m_i = \frac{n}{\sum_{j=0}^{8} k_j^{-1} p_{ij}^2}$$

from which we get $m_1 = 16929$, $m_2 = 267520$, $m_3 = 653125$, $m_4 = 365750$, $m_5 = 214016$, $m_6 = 8910$, whence

$$m_7 + m_8 = n - \sum_{i=0}^{6} m_i = 12749.$$

Comparing that value with the decomposition of the permutation module of F_5 on $\mathcal{T}$ into irreducible submodules given in [8, Lemma 2.18.1.(*ii*)], we obtain that, modulo interchanging the indices 7 and 8,

$$m_7 = 3344 \text{ and } m_8 = 9405.$$

By the Column Orthogonality Relation of the first eigenmatrix,

$$\sum_{k=0}^{8} m_k p_{ki} p_{kj} = n k_i \delta_{ij}$$

(see [1, Theorem 3.5]), applied with $(i, j) = (0, 8)$ and $(i, j) = (8, 8)$, we get the quadratic system

$$\begin{cases} 3344x + 9405y = -3182025 \\ 3344x^2 + 9405y^2 = 3513943125 \end{cases}$$

whose solutions are

$$(x, y) = (525, -525) \text{ or } (x, y) = (1575/61, 62475/61).$$

By [2, Theorem 3.5(b)], the matrices A_i's are symmetric, since, by [4], the Frobenius-Schur indices of the irreducible constituents of the permutation character of F_5 on $\mathcal{T}$ is $+1$ (and the action is multiplicity free). Thus, recalling that the p_{ij}^k's are all non negative integers, in order to determine which of the two solutions is the right one, we may use the formula

$$p_{ij}^h = \frac{1}{nk_h} tr(A_i A_j A_h) \tag{2.6}$$

(see [1, Theorem 3.6(ii)]). Since the trace is invariant by matrix conjugation, $tr(A_i A_j A_h)$ can be obtained by multiplying, entry-wise, the ith, jth, and hth columns of the matrix P and adding the entries of the resulting column. In that way, we get that the entries p_{2j}^k are integers only in the case when $(x, y) = (525, -525)$. □

3 The shape

We continue with the notations of the last section. The next lemma recalls some known facts about conjugacy classes in M and F_5 (see [16, 15]). For the remainder of this paper let H be the centraliser in M of an A_5-subgroup of type $(2A, 3A, 5A)$. By [15], we have that $H \cong A_{12}$ and we may w.l.o.g. assume that F_5 centralizes a $5A$-element in that A_5-subgroup, in particular $H \leq F_5$.

Lemma 3.1. *Denoting the conjugacy classes of M and F_5 as in [4], the correspondences between the conjugacy classes of the elements of order less or equal to 6 in M, F_5 and H are as in Table 2.*

Table 2: Correspondences between the conjugacy classes of the elements of order at most 6 in M, F_5, and H.

Conj. class in M	$2A$	$2B$	$3A$	$4A$	$4B$	$5A$	$6A$
Conj. class in F_5	$2A$	$2B$	$3A$	$4A$	$4B$	$5A$	$6A$
Cycle type in H	$2^2, 2^6$	2^4	$3, 3^2, 3^4$	4^2, $4^2 \cdot 2^2$	$4 \cdot 2$, $4 \cdot 2^2$	$5, 5^2$	$3 \cdot 2^2$, $6 \cdot 2^3$, $6^2, 3^2 \cdot 2^2$

Let $(t_1, \ldots, t_n)$ be as in the previous section. For $i, j \in \{1, \ldots n\}$, set

$$\gamma_{ij} := (t_i^\psi, t_j^\psi)_W.$$

Lemma 3.2. *If (t_i, t_j) and (t_h, t_k) belong to the same orbital of F_5 on $\mathcal{T}$, then $\gamma_{ij} = \gamma_{hk}$.*

Proof. That follows immediately from Equation (1.2) and the definition of γ_{ij}. □

Thus, we can set, for $k \in \{0, \ldots, 8\}$ and $(t, s) \in T_k$,

$$\gamma_k := (t^\psi, s^\psi)_W. \tag{3.1}$$

Lemma 3.3. *For every $x \in \{2^2, 3, 4 \cdot 2, 2^4, 5\}$ there are pairs of involutions of type 2^2 in A_{12} such that their product has cycle type x. Every element of cycle type $4^2 \cdot 2^2$ in A_{12} is the product of two elements of cycle type 2^6.*

Proof. That is an elementary computation (note that two elements of cycle type 2^6 whose product has cycle type $4^2 \cdot 2^2$ are explicitely given in the proof of Lemma 3.4). □

Lemma 3.4. *With the above notations, for every $k \in \{0, \ldots, 8\}$ and $(t, s) \in T_k$, the scalar products γ_k's are given in Table 3.*

Proof. The first two columns of Table 3 follow from Lemma 2.1. The correspondence that associates to each orbital T_k of F_5 on $\mathcal{T}$ the F_5-conjugacy class x_k of the products ts, where $(t, s) \in T_k$, has been determined by Segev in [20], giving the third column.

Table 3: Valencies, shapes, and scalar products related to the orbitals of F_5 on the set of its $2A$-involutions.

k	$\|t^{C_{F_5}(s)}\|$	$(st)^{F_5}$	$sh_{\mathcal{R}}(T_k)$	γ_k
0	1	1	–	1
1	1408	$5A$	$5A$	$3/2^7$
2	2200	$2A$	$2A$	$1/2^3$
3	35200	$3A$	$3A$	$13/2^8$
4	123200	$4B$	$4B$	$1/2^6$
5	354816	$5E$	$5A$	$3/2^7$
6	739200	$6A$	$6A$	$5/2^8$
7	277200	$4A$	$4A$	$1/2^5$
8	5775	$2B$	$2B$	0

Assume $sh_{\mathcal{R}}(T_k) = NX$, where $N \in \{1, \ldots, 6\}$ and $X \in \{A, B, C\}$. By the definition of shape, for $(t, s) \in T_k$, we have that $|st| = N$. In particular, for k equal to 1, 5 and 6, we have that $sh_{\mathcal{R}}(T_k)$ is equal to $5A$, $5A$, and $6A$, respectively.

Let $k \in \{2, 3, 4, 8\}$. By the second and third rows of Table 2 and Lemma 3.3 there are involutions s and t of cycle type 2^2 in $\mathcal{T} \cap H$ such that $st \in x_k$, whence, by the first and third columns of Table 3,

$$(s, t) \in T_k \cap (H \times H).$$

By the remark in the introduction, we have that

$$sh_{\mathcal{R}}(T_k) = sh_{\mathcal{R}_H}((s, t)^H),$$

whence Lemma 8 and Table 10 in [6] give the entry in the fourth column corresponding to k.

Assume now $k = 7$. Choose the elements

$$s = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12) \text{ and } t = (1, 3)(2, 4)(5, 7)(6, 9)(8, 11)(10, 12)$$

in H. Then st has cycle type $4^2{\cdot}2^2$. By Table 2, s and t are contained in $\mathcal{T}$ and $(st)^{F_5} = 4A$, hence, by the third column of Table 3, $(s, t) \in T_7$ and, by the Norton-Sakuma Theorem, $sh_{\mathcal{R}}(T_7) \in \{4A, 4B\}$. By Equation (1.2),

$$(t^{\psi})^{(ts)^{\phi}} = (t^{ts})^{\psi} = (t^s)^{\psi},$$

so we have that t^{ψ} and $(t^s)^{\psi}$ are contained in the subalgebra generated by t^{ψ} and s^{ψ}, which is $\langle s, t\rangle$-invariant. Since tt^s has cycle type 2^4, by Table 2 it belongs to the class $2B$ of F_5, whence, by the third column of Table 3, $(t, t^s) \in T_8$ and the subalgebra generated by $t^{\psi}, (t^s)^{\psi}$ is of type $2B$, by the previous paragraph. By the second condition of the definition of the shape, $sh_{\mathcal{R}}(T_7) = 4A$.

Finally, the last column follows from Table 1. □

4 Closure

Lemma 4.1. *Suppose that $\mathcal{R}$ is based on the embedding of F_5 in M. Then*

$$\langle \mathcal{T}^{\psi} \rangle = W_{\mathcal{T}}.$$

Proof. Let H be the subgroup of F_5 isomorphic to A_{12} defined as in the previous section. Let t, s be distinct elements of $\mathcal{T}$, set $\rho = (ts)^{\phi}$ and let N be the order of ρ. Let U be the Norton-Sakuma algebra generated by t^{ψ} and s^{ψ}, and let NX be its type. By Table 1, if NX is contained in $\{2A, 2B, 4B\}$, then U is linearly spanned by elements in $\mathcal{T}^{\psi}$, otherwise, by Lemma 3.4, $NX \in \{3A, 4A, 5A, 6A\}$ and U has a basis all of whose elements but the NX-axis are Majorana axes. Therefore, with the notations of Table 1, we may assume that $NX \in \{3A, 4A, 5A, 6A\}$ and show that, in all those cases, the NX-axes u_{ρ}, v_{ρ}, w_{ρ}, u_{ρ^2} are contained in $\langle \mathcal{T}^{\psi} \rangle$.

If ts has order $3, 4$, or 5, then, by Lemma 3.1, there is $g \in F_5$, depending on ts, such that ts is an element of cycle type respectively 3, $4^2 \cdot 2^2$, and 5 in H^g. By Lemma 3.3, there are elements t' and s' of cycle type 2^2 or 2^6 in H^g such that $ts = t's'$. By Lemma 3.1, $(t')^{\psi}$ and $(s')^{\psi}$ generate a Norton-Sakuma algebra of the same type as U, thus, by Axiom (M8)(b), we have that $u_{\rho} = u_{(t's')^{\phi}}$, $v_{\rho} = v_{(t's')^{\phi}}$, and $w_{\rho} = w_{(t's')^{\phi}}$, respectively.

Assume $NX = 3A$. By [3, Corollary 3.2], $u_{(t's')^{\phi}}$ is a linear combination of elements of $(\mathcal{T} \cap H^g)^{\psi}$ and we are done.

Similarly, assume $NX = 4A$ (resp. $NX = 5A$). By [3], second formula in the abstract, or Section 6 (resp. Lemma 5.1), we have that $v_{(t's')^{\phi}}$ (resp. $w_{(t's')^{\phi}}$) is a linear combination of elements in $(\mathcal{T} \cap H^g)^{\psi}$ and $3A$-axes, and we are done by the previous case.

Finally assume $NX = 6A$. Then, by the remarks after Table 1, u_{ρ^2} is a $3A$-axis and again we are done by the $3A$ case. □

Note that in the previous proof we require that $\mathcal{R}$ is based on the embedding of F_5 in M only to deal with the case $4A$, all the other cases following from results of [3] that depend only on the shape of that representation of A_{12}.

5 Proof of Theorem 1.1

The first claim of Theorem 1.1 follows from Lemma 3.4 and the last is the content of Lemma 4.1. To prove the second and the third claims, let

$$\Gamma = (\gamma_{ij})$$

be the Gram matrix of $(\,,\,)_W$ associated to the n-tuple $(t_1^{\psi}, \dots, t_n^{\psi})$. By an elementary result on Euclidean spaces, we have that

$$\mathrm{rank}(\Gamma) = \dim_{\mathbb{R}}(\langle \mathrm{t}^{\psi} \mid \mathrm{t} \in \mathcal{T} \rangle). \tag{5.1}$$

Since $T_0, \dots, T_8$ is a partition of $\mathcal{T} \times \mathcal{T}$ and, by Equation (3.1), $\gamma_k = \gamma_{ij}$, for $(t_i, t_j) \in T_k$, we have that

$$\Gamma = \sum_{k=0}^{8} \gamma_k A_k. \tag{5.2}$$

Let D be as in Section 2. From Equation (5.2) we get:

$$\overline{\Gamma} := D^{-1}\Gamma D = \sum_{k=0}^{8} \gamma_k D^{-1} A_k D, \tag{5.3}$$

where all the matrices $\overline{\Gamma}$, and $\overline{A}_k := D^{-1}A_k D$ for $k \in \{0, \dots, 8\}$, are diagonal. Now, clearly, the rank of Γ is equal to the rank of $\overline{\Gamma}$, hence (being $\overline{\Gamma}$ diagonal) to the number of nonzero entries of $\overline{\Gamma}$. By Lemma 3.4 (Table 3), Equation (5.3) becomes

$$\overline{\Gamma} = \overline{A}_0 + \frac{3}{2^7}\overline{A}_1 + \frac{1}{8}\overline{A}_2 + \frac{13}{2^8}\overline{A}_3 + \frac{1}{2^6}\overline{A}_4 + \frac{3}{2^7}\overline{A}_5 + \frac{5}{2^8}\overline{A}_6 + \frac{1}{2^5}\overline{A}_7 + 0\overline{A}_8,$$

which, by Lemma 2.1, gives the eigenvalues

$$70875/2, 0, 0, 0, 0, 0, 875/8, 0, 225/4$$

of $\overline{\Gamma}$ on the subspaces $V_0, \dots, V_8$, respectively. Hence

$$\dim_{\mathbb{R}}(\langle \mathcal{T}^{\psi} \rangle) = m_0 + m_6 + m_8 = 1 + 9405 + 8910 = 18\,316.$$

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On colour-preserving automorphisms of Cayley graphs

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Abstract

We study the automorphisms of a Cayley graph that preserve its natural edge-colouring. More precisely, we are interested in groups G, such that every such automorphism of every connected Cayley graph on G has a very simple form: the composition of a left-translation and a group automorphism. We find classes of groups that have the property, and we determine the orders of all groups that do not have the property. We also have analogous results for automorphisms that permute the colours, rather than preserving them.

Keywords: Cayley graph, automorphism, colour-preserving, colour-permuting.

Math. Subj. Class.: 05C25

1 Introduction

Definitions 1.1. Let S be a subset of a group G, such that $S = S^{-1}$. (All groups and all graphs in this paper are finite.)

- The *Cayley graph* of G, with respect to S, is the graph $\mathrm{Cay}(G; S)$ whose vertices are the elements of G, and with an edge $x — xs$, for each $x \in G$ and $s \in S$.

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- $\mathrm{Cay}(G;S)$ has a natural edge-colouring. Namely, each edge of the form $x \,—\, xs$ is coloured with the set $\{s, s^{-1}\}$. (In order to make the colouring well-defined, it is necessary to include s^{-1}, because $x \,—\, xs$ is the same as the edge $xs \,—\, x$, which is of the form $y \,—\, ys^{-1}$, with $y = xs$.)

Note that $\mathrm{Cay}(G;S)$ is connected if and only if S generates G. Also note that a permutation φ of G is a colour-preserving automorphism of $\mathrm{Cay}(G;S)$ if and only if we have $\varphi(xs) \in \big\{\varphi(x)\, s^{\pm 1}\big\}$, for each $x \in G$ and $s \in S$.

For any $g \in G$, the left translation $x \mapsto gx$ is a colour-preserving automorphism of $\mathrm{Cay}(G;S)$. In addition, if α is an automorphism of the group G, such that $\alpha(s) \in \{s^{\pm 1}\}$ for all $s \in S$, then α is also a colour-preserving automorphism of $\mathrm{Cay}(G;S)$. We will see that, in many cases, every colour-preserving automorphism of $\mathrm{Cay}(G;S)$ is obtained by composing examples of these two obvious types.

Definition 1.2. Let G be a group.

1. A function $\varphi\colon G \to G$ is said to be *affine* if it is the composition of an automorphism of G with left translation by an element of G. This means $\varphi(x) = \alpha(gx)$, for some $\alpha \in \mathrm{Aut}\, G$ and $g \in G$.
2. A Cayley graph $\mathrm{Cay}(G;S)$ is *CCA* if all of its colour-preserving automorphisms are affine functions on G. (CCA is an abbreviation for the *Cayley Colour Automorphism* property.)
3. We say that G is *CCA* if every connected Cayley graph on G is CCA.

Here are some of our main results:

Theorem 1.3.

1. *There is a non-CCA group of order n if and only if $n \geq 8$ and n is divisible by either 4, 21, or a number of the form $p^q \cdot q$, where p and q are prime* (*see Corollary 6.13 and Remark 6.14*)*.*
2. *An abelian group is not CCA if and only if it has a direct factor that is isomorphic to either $\mathbb{Z}_4 \times \mathbb{Z}_2$ or a group of the form $\mathbb{Z}_{2^k} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, with $k \geq 2$* (*see Proposition 4.1*)*.*
3. *Every dihedral group is CCA* (*see Corollary 5.4*)*.*
4. *No generalized dicyclic group or semidihedral group is CCA, except that $\mathbb{Z}_4$ is dicyclic, but is CCA* (*see Corollary 2.8*)*.*
5. *Every non-CCA group of odd order has a section that is isomorphic to either the nonabelian group of order 21 or a certain generalization of a wreath product* (*called a semi-wreathed product*) (*see Theorem 6.8*)*.*
6. *If $G \times H$ is CCA, then G and H are both CCA* (*see Proposition 3.1*)*. The converse is not always true* (*for example, $\mathbb{Z}_4 \times \mathbb{Z}_2$ is not CCA*)*, but it does hold if $\gcd\big(|G|, |H|\big) = 1$* (*see Proposition 3.2*)*.*

We also consider automorphisms of $\mathrm{Cay}(G;S)$ that permute the colours, rather than preserving them:

Definitions 1.4.

- An automorphism α of a Cayley graph $\mathrm{Cay}(G; S)$ is *colour-permuting* if it respects the colour classes; that is, if two edges have the same colour, then their images under α must also have the same colour. This means there is a permutation π of S, such that $\alpha(gs) \in \{\alpha(g)\, \pi(s)^{\pm 1}\}$ for all $g \in G$ and $s \in S$ (and $\pi(s^{-1}) = \pi(s)^{-1}$).
- We say that a group G is *strongly CCA* if every colour-permuting automorphism of every connected Cayley graph on G is affine.

Note that any strongly CCA group is CCA, since colour-preserving automorphisms are colour-permuting (with π being the identity map on S). The converse is not true. For example, any dihedral group is CCA (as was mentioned above), but it is not strongly CCA if its order is of the form $8k + 4$ (see Proposition 5.6). However, the converse does hold for at least two natural families of groups:

Theorem 1.5. *A CCA group is strongly CCA if either:*

1. *it is abelian* (*see Proposition 4.1*)*, or*
2. *it has odd order* (*see Proposition 6.4*)*.*

Remarks 1.6.

1. It follows from Theorems 1.3(2) and 1.5(1) that every cyclic group is strongly CCA. This is also a consequence of the main theorem of [9].
2. Groups that are not strongly CCA seem to be far more likely to be of even order than of odd order. For example, of the 28 groups of order less than 32 that are not strongly CCA, only one has odd order (see Section 7). In fact, there are only three groups of odd order less than 100 that are not strongly CCA: the non-abelian group G_{21} of order 21, the group $G_{21} \times \mathbb{Z}_3$ of order 63, and the wreath product $\mathbb{Z}_3 \wr \mathbb{Z}_3$, which has order 81 (see Corollary 6.15).
3. If the subgroup consisting of all left-translations is normal in the automorphism group of the Cayley graph $\mathrm{Cay}(G; S)$, then $\mathrm{Cay}(G; S)$ is said to be *normal* [12]. It is not difficult to see that every normal Cayley graph is strongly CCA (cf. Remark 6.2), and that every automorphism of a normal Cayley graph is colour-permuting.
4. The notion of (strongly) CCA generalizes in a natural way to the setting of Cayley digraphs $\overrightarrow{\mathrm{Cay}}(G; S)$, by putting the colour s on each directed edge of the form $x \to xs$. (There is no need to include s^{-1} in the colour.) However, it is very easy to see that if $\overrightarrow{\mathrm{Cay}}(G; S)$ is connected, then every colour-preserving automorphism of $\overrightarrow{\mathrm{Cay}}(G; S)$ is left-translation by some element of G [11, Thm. 4-8, p. 25], and that every colour-permuting automorphism is affine [3, Lem. 2.1]. Therefore, both notions are completely trivial in the directed setting. However, there has been some interest in determining when every automorphism of $\overrightarrow{\mathrm{Cay}}(G; S)$ is colour-permuting [1, 2] (in which case, the Cayley digraph is normal, in the sense of (3)).

2 Examples of non-CCA groups

Remark 2.1. Since automorphisms are the only affine functions that fix the identity element e (and left-translations are colour-preserving automorphisms of any Cayley graph), it is easy to see that if $\mathrm{Cay}(G; S)$ is CCA, then every colour-preserving automorphism that fixes the identity is an automorphism of the group G. More precisely:

A Cayley graph $\mathrm{Cay}(G; S)$ is CCA if and only if, for every colour-preserving automorphism φ of $\mathrm{Cay}(G; S)$, such that $\varphi(e) = e$, we have $\varphi \in \mathrm{Aut}\, G$.

The same is true with "strongly CCA" in the place of "CCA," if "colour-preserving" is replaced with "colour-permuting." This is reminiscent of the CI (Cayley Isomorphism) property [7], and this similarity motivated our choice of terminology.

We thank Gabriel Verret for pointing out that the quaternion group Q_8 is not CCA. In fact, two different groups of order 8 are not CCA:

Example 2.2 (G. Verret)**.** $\mathbb{Z}_4 \times \mathbb{Z}_2$ and Q_8 are not CCA.

Proof. (Q_8) Let $\Gamma = \mathrm{Cay}(Q_8; \{\pm i, \pm j\})$. This is the complete bipartite graph $K_{4,4}$. (See Figure 1 with the labels that are inside the vertices.) Let φ be the graph automorphism that interchanges the vertices k and $-k$ while fixing every other vertex. This is clearly not an automorphism of G since i and j are fixed by φ and generate G, but $\varphi \neq 1$. It is, however, a colour-preserving automorphism of Γ.

($\mathbb{Z}_4 \times \mathbb{Z}_2$) Let $\Gamma = \mathrm{Cay}\big(\mathbb{Z}_4 \times \mathbb{Z}_2; \{\pm(1,0), \pm(1,1)\}\big)$. This is again the complete bipartite graph $K_{4,4}$. (See Figure 1 with the labels that are outside the vertices.) Let φ be the graph automorphism that interchanges the vertices $(0,1)$ and $(2,1)$ while fixing all of the other vertices. This is clearly not an automorphism of G since $(1,0)$ and $(1,1)$ are fixed by φ and generate G, but $\varphi \neq 1$. It is, however, a colour-preserving automorphism of Γ. □

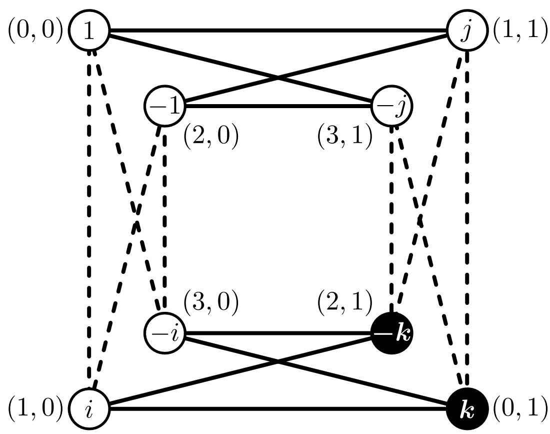

Figure 1: Interchanging the two black vertices while fixing all of the white vertices is a colour-preserving graph automorphism that fixes the identity vertex but is not a group automorphism.

Both of the groups in Example 2.2 are generalized dicyclic (cf. Definition 2.6):

- Q_8 is the generalized dicyclic group over $\mathbb{Z}_4$, and
- $\mathbb{Z}_4 \times \mathbb{Z}_2$ is the generalized dicyclic group over $\mathbb{Z}_2 \times \mathbb{Z}_2$.

More generally, we will see in Corollary 2.8(4) below that no generalized dicyclic group is CCA (unless the cyclic group $\mathbb{Z}_4$ is considered to be dicyclic).

We will see in Theorem 6.8 that the following example is the smallest group of odd order that is not CCA.

Example 2.3. The nonabelian group of order 21 is not CCA.

Proof. Let $G = \langle\, a, x \mid a^3 = e,\ a^{-1}xa = x^2 \,\rangle$. (Since $x = e^{-1}xe = a^{-3}xa^3 = x^8$, the relations imply $x^7 = e$, so G has order 21.) By letting $b = ax$, we see that G also has the presentation

$$G = \langle\, a, b \mid a^3 = e,\ (ab^{-1})^2 = b^{-1}a \,\rangle.$$

As illustrated in Figure 2, every element of G can be written uniquely in the form

$$a^i b^j a^k, \text{ where } i, j, k \in \{0, \pm 1\} \text{ and } j = 0 \Rightarrow k = 0.$$

Define

$$\varphi(a^i b^j a^k) = \begin{cases} b^j a^{-k} & \text{if } i = 0, \\ ab^{-j}a^k & \text{if } i = 1, \\ a^{-1}b^{-j}a^{-k} & \text{if } i = -1. \end{cases}$$

Then φ is a colour-preserving automorphism of $\mathrm{Cay}\big(G; \{a^{\pm 1}, b^{\pm 1}\}\big)$ (see Figure 2). However, φ is not affine, since it fixes e, but is not an automorphism of G (because $\varphi(ab) = ab^{-1} \neq ab = \varphi(a)\,\varphi(b)$). □

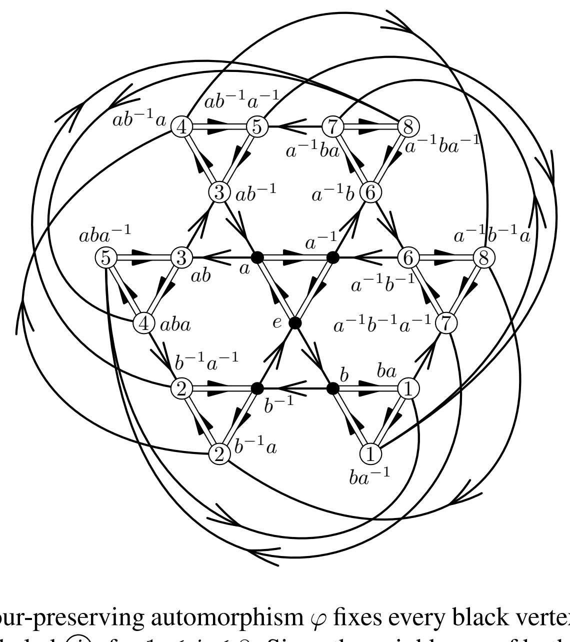

Figure 2: The colour-preserving automorphism φ fixes every black vertex, but interchanges the two vertices labeled ⓘ, for $1 \leq i \leq 8$. Since the neighbours of both copies of ⓘ have the same labels (for example, the vertices labeled ⑦ are connected by a black edge to ① and ⑤, and by a white edge to ⑥ and ⑧), we see that φ is indeed a colour-preserving automorphism of the graph (if the orientations of the edges are ignored).

See Proposition 3.3 for a generalization of the following example.

Example 2.4. The wreath product $\mathbb{Z}_m \wr \mathbb{Z}_n$ is not CCA whenever $m \geq 3$ and $n \geq 2$.

Proof. This group is a semidirect product

$$(\mathbb{Z}_m \times \mathbb{Z}_m \times \cdots \times \mathbb{Z}_m) \rtimes \mathbb{Z}_n.$$

For the generators $a = \big((1,0,0,\ldots,0),0\big)$ and $b = \big((0,0,\ldots,0),1\big)$, the map

$$\big((x_1,x_2,x_3,\ldots,x_n),y\big) \mapsto \big((-x_1,x_2,x_3,\ldots,x_n),y\big)$$

(negate a single factor of the abelian normal subgroup) is a colour-preserving automorphism of $\mathrm{Cay}\big(\mathbb{Z}_m \wr \mathbb{Z}_n; \{a^{\pm 1}, b^{\pm 1}\}\big)$ that fixes the identity element but is not a group automorphism. □

The following construction provides many additional examples of non-CCA groups by generalizing the idea of Example 2.2.

Proposition 2.5. *Suppose there is a generating set S of G, an element τ of G, and a subset T of S, such that:*

- *τ is an element of order 2,*
- *each element of S is either centralized or inverted by τ,*
- *$t^2 = \tau$ for all $t \in T$,*
- *the subgroup $\langle (S \smallsetminus T) \cup \{\tau\} \rangle$ is not all of G, and*
- *either $\big|G : \langle (S \smallsetminus T) \cup \{\tau\} \rangle\big| > 2$ or τ is not in the centre of G.*

Then G is not CCA.

Proof. For convenience, let $H = \langle (S \smallsetminus T) \cup \{\tau\} \rangle$. Since $\langle S \rangle = G$, but, by assumption, $H \neq G$, there exists some $x \in T \smallsetminus H$. Define

$$\varphi(g) = \begin{cases} g\tau & \text{if } g \in xH, \\ g & \text{otherwise.} \end{cases}$$

It is obvious that φ fixes e, since $e \notin xH$.

We claim that φ is is not an automorphism of G. If $|G : H| > 2$, this follows from the fact that a nonidentity automorphism cannot fix more than half of the elements of G. Thus, we may assume $|G : H| = 2$. Then, by assumption, there is some element h of G that does not commute with τ. Since τ commutes with every element of T (because $\tau = t^2$), we see that we may assume $h \in H$. If φ is an automorphism, then, since it is the identity on the normal subgroup H of G, but $x^{-1} = xx^{-2} = x\tau \in xH$, we have:

$$x^{-1}hx = \varphi(x^{-1}hx) = \varphi(x^{-1}) \cdot \varphi(h) \cdot \varphi(x) = x^{-1}\tau \cdot h \cdot x\tau \neq x^{-1}hx\tau^2 = x^{-1}hx.$$

This is a contradiction.

Since each element of S is either centralized or inverted by τ, we know that right-multiplication by τ is a colour-preserving automorphism of $\mathrm{Cay}(G; S)$. Restricting to xH, this tells us that φ preserves colours (and existence) of all edges of $\mathrm{Cay}(G;S)$ that have both endvertices in xH.

Now consider an edge from g to h, where $g \in xH$ and $h \notin xH$. There is some element $t \in T$ such that $gt = h$, and there is an edge of the same colour from $\varphi(g) = g\tau$ to $g\tau t^{-1}$. Since $t^2 = \tau$ and $\tau^2 = e$, we have $t^{-1} = \tau t$. Hence, the edge is from $\varphi(g)$ to

$$g\tau t^{-1} = gt^2 t^{-1} = gt = h = \varphi(h).$$

Thus φ preserves the existence and colour of every edge from a vertex in xH to a vertex outside of xH. Since the only vertices moved by φ are in xH, this shows that φ is a colour-preserving automorphism of $\mathrm{Cay}(G;S)$. □

Here are a few particular examples to which Proposition 2.5 can be applied.

Definition 2.6. Let A be an abelian group of even order. Choose an involution y of A. The corresponding *generalized dicyclic group* is

$$\mathrm{Dic}(y, A) = \langle\, x, A \mid x^2 = y,\ x^{-1}ax = a^{-1},\ \forall a \in A \,\rangle.$$

Definition 2.7. For $n \geq 1$, let

$$\mathrm{SemiD}_{16n} = \langle\, a, x \mid a^{8n} = x^2 = e,\ xa = a^{4n-1}x \,\rangle.$$

This is a *semidihedral* (or *quasidihedral*) group. The term is usually used only when n is a power of 2, but the construction is valid more generally.

Corollary 2.8. *The following groups are not CCA:*

1. $\mathbb{Z}_4 \times \mathbb{Z}_2$,
2. $\mathbb{Z}_{2^k} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, *for any* $k \geq 2$,
3. Q_8,
4. *every generalized dicyclic group except* $\mathbb{Z}_4$ (*this generalizes* (3)), *and*
5. *every semidihedral group.*

Proof. (1) Apply Proposition 2.5 with $\tau = (2,0)$ and $S = T = \{(1,0),(1,1)\}$.

(2) Apply Proposition 2.5 with $\tau = (2^{k-1},0,0)$, $T = \{(2^{k-2},1,0),(2^{k-2},0,1)\}$, and $S = \{(1,0,0)\} \cup T$.

(3) Since $i^2 = j^2 = -1$, we may apply Proposition 2.5 with $\tau = -1$ and $S = T = \{i, j\}$.

(4) For $G = \mathrm{Dic}(y,A) = \langle x, y, A\rangle$, apply Proposition 2.5 with $\tau = y$ and $S = T = xA$. (We have $\big|G : \langle (S \smallsetminus T) \cup \{\tau\}\rangle\big| = |G : \langle\tau\rangle| = |G|/2 > 2$, since $G \not\cong \mathbb{Z}_4$.)

(5) For $G = \mathrm{SemiD}_{16n} = \langle a, x\rangle$, apply Proposition 2.5 with $\tau = a^{4n}$, $T = \{(ax)^{\pm 1}\}$, and $S = \{x\} \cup T$. (Note that $\big|G : \langle (S \smallsetminus T) \cup \{\tau\}\rangle\big| = |G : \langle x, \tau\rangle| = |G|/4 \geq 4$.) □

3 Direct products and semidirect products

Proposition 3.1. *If G_1 is not strongly CCA, and G_2 is any group, then $G_1 \times G_2$ is not strongly CCA. Furthermore, the same is true with "CCA" in the place of "strongly CCA."*

Proof. Since G_1 is not strongly CCA, some connected Cayley graph $\mathrm{Cay}(G_1; S_1)$ on G_1 has a colour-permuting automorphism φ_1 that is not affine. Let π be a permutation of S_1, such that $\varphi_1(g_1 s) \in \{\varphi_1(g_1)\,\pi(s)^{\pm 1}\}$ for all $g_1 \in G_1$. (If G_1 is not CCA, then we may assume π is the identity permutation.) Now, fix any connected Cayley graph $\mathrm{Cay}(G_2; S_2)$ on G_2, and let

$$S = \big(S_1 \times \{e\}\big) \cup \big(\{e\} \times S_2\big),$$

so $\mathrm{Cay}(G_1 \times G_2; S)$ is connected. (It is isomorphic to the Cartesian product $\mathrm{Cay}(G_1; S_1) \square \mathrm{Cay}(G_2; S_2)$.)

Define a permutation φ of $G_1 \times G_2$ by $\varphi(g_1, g_2) = \big(\varphi(g_1), g_2\big)$. For all $(g_1, g_2) \in G_1 \times G_2$ and $s_i \in S_i$, we have

- $\varphi\big((g_1, g_2) \cdot (s_1, e)\big) = \big(\varphi_1(g_1 s_1), g_2\big) \in \big\{\varphi(g_1, g_2) \cdot \big(\pi(s_1), e\big)^{\pm 1}\big\}$, and
- $\varphi\big((g_1, g_2) \cdot (e, s_2)\big) = \big(\varphi_1(g_1), g_2 s_2\big) = \varphi(g_1, g_2) \cdot (e, s_2)$.

Therefore, φ is a colour-permuting automorphism of $\mathrm{Cay}(G_1 \times G_2; S)$ (and it is colour-preserving if π is the identity permutation of S_1).

However, φ is not affine (since its restriction to G_1 is the permutation φ_1, which is not affine). So G is not strongly CCA (and is not CCA if π is the identity permutation of S_1). □

Proposition 3.1 tells us that if $G_1 \times G_2$ is CCA, then G_1 and G_2 must both be CCA. The converse is not true. (For example, $\mathbb{Z}_4$ and $\mathbb{Z}_2$ are both CCA, but Example 2.2 tells us that the direct product $\mathbb{Z}_4 \times \mathbb{Z}_2$ is not CCA.) However, the converse is indeed true when the groups are of relatively prime order:

Proposition 3.2. *Assume* $\gcd\big(|G_1|, |G_2|\big) = 1$. *Then* $G_1 \times G_2$ *is CCA* (*or strongly CCA*) *if and only if* G_1 *and* G_2 *are both CCA* (*or strongly CCA, respectively*).

Proof. (⇒) Proposition 3.1.

(⇐) Let

- $G = G_1 \times G_2$,
- S be a generating set of G,
- φ be a colour-permuting automorphism of $\mathrm{Cay}(G; S)$ that fixes the identity element (see Remark 2.1),
- $\pi_i \colon G_1 \times G_2 \to G_i$ be the natural projection, and
- k be a multiple of $|G_2|$ that is $\equiv 1 \pmod{|G_1|}$, so $g^k = \pi_1(g)$ for all $g \in G$.

Consider some $s \in S$, and let $t = \varphi(s)$, so $\varphi(xs^i) = \varphi(x)\, t^{\pm i}$ for all $x \in G$ and $i \in \mathbb{Z}$. Then, for all $g \in G$, we have

$$\varphi\big(g\,\pi_1(s)\big) = \varphi(gs^k) = \varphi(g)\, t^{\pm k} = \varphi(g) \cdot \pi_1(t)^{\pm 1}. \qquad (*)$$

Since $\pi_1(S)$ generates G_1, this implies there is a well-defined permutation φ_2 of G_2, such that

$$\varphi(G_1 \times \{g_2\}) = G_1 \times \{\varphi_2(g_2)\}.$$

By repeating the argument with the roles of G_1 and G_2 interchanged, we conclude that there is a permutation φ_1 of G_1, such that

$$\varphi(g_1, g_2) = \big(\varphi_1(g_1), \varphi_2(g_2)\big).$$

Now, $(*)$ implies that φ_1 is a colour-permuting automorphism of $\mathrm{Cay}\big(G_1; \pi_1(S)\big)$. Similarly, φ_2 is a colour-permuting automorphism of $\mathrm{Cay}\big(G_2; \pi_2(S)\big)$. Since each G_i is CCA, we conclude that φ_i is an automorphism of G_i. So φ is an automorphism of $G_1 \times G_2$. □

The idea used in Example 2.4 yields the following result that generalizes the CCA part of Proposition 3.1.

Proposition 3.3. *Suppose $G = H \rtimes K$ is a semidirect product, and $\mathrm{Cay}(H; S_0)$ is a connected Cayley graph of H, such that:*

- *S_0 is invariant under conjugation by every element of K, and*
- *there is a colour-preserving automorphism φ_0 of $\mathrm{Cay}(H; S_0)$, such that either*
  - *φ_0 is not affine, or*
  - *$\varphi_0(e) = e$, and there exist $s \in S_0$ and $k \in K$, such that $\varphi_0(k^{-1}sk) \neq k^{-1}\,\varphi_0(s)\,k$.*

Then G is not CCA.

Proof. Define $\varphi \colon G \to G$ by $\varphi(hk) = \varphi_0(h)\,k$. We claim that φ is a colour-preserving automorphism of $\mathrm{Cay}(G; S_0 \cup K)$ that is not affine (so G is not CCA, as desired).

For $k_1 \in K$, we have

$$\varphi(hk\,k_1) = \varphi_0(h)\,kk_1 = \varphi(hk)\,k_1,$$

so φ preserves the colour of K-edges. Now consider some $s \in S_0$ and let ${}^k s = ksk^{-1} \in S_0$. Then, since φ_0 is colour preserving, we have

$$\varphi(hk\,s) = \varphi(h\,{}^k s\,k) = \varphi_0(h\,{}^k s)\,k = \big(\varphi_0(h)\,({}^k s)^{\pm 1}\big)\,k = \varphi_0(h)\,ks^{\pm 1} = \varphi(hk)\,s^{\pm 1},$$

so φ also preserves the colour of S_0-edges. Hence, φ is colour-preserving.

Now, suppose φ is affine. Then the restriction φ_0 of φ to H is also affine, so, by assumption, we must have $\varphi(e) = e$, so φ is an automorphism of G. Hence, for all $s \in S_0$ and $k \in K$, we have

$$\varphi_0(k^{-1}sk) = \varphi(k^{-1}sk) = \varphi(k)^{-1}\,\varphi(s)\,\varphi(k) = k^{-1}\,\varphi(s)\,k = k^{-1}\,\varphi_0(s)\,k.$$

This contradicts the hypotheses of the proposition. □

Remark 3.4. Proposition 3.3 can be generalized slightly: assume $G = HK$ and $H \lhd G$ (but do not assume $H \cap K = \{e\}$, which would make G a semidirect product). Then the above proof applies if we make the additional assumption that $\varphi_0(hk) = \varphi_0(h)\,k$ for all $h \in H$ and $k \in H \cap K$.

4 Abelian groups

The following result shows that all non-CCA abelian groups can be constructed from examples that we have already seen in Corollary 2.8 (and that CCA and strongly CCA are equivalent for abelian groups).

Proposition 4.1. *For an abelian group G, the following are equivalent:*

1. *G has a direct factor that is isomorphic to either $\mathbb{Z}_4 \times \mathbb{Z}_2$ or a group of the form $\mathbb{Z}_{2^k} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, with $k \geq 3$.*
2. *G is not CCA.*
3. *G is not strongly CCA.*

Proof. ($1 \Rightarrow 2$) This is immediate from Corollary 2.8 and Proposition 3.1.

($2 \Rightarrow 3$) Obvious.

($3 \Rightarrow 1$) Let φ be a colour-permuting automorphism of any connected Cayley graph $\mathrm{Cay}(G; S)$ on G, such that $\varphi(0) = 0$. From Proposition 3.2 (and the fact that any abelian group is the direct sum of its Sylow subgroups), we may assume G is a p-group for some prime p. Then

$$G \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}} \times \cdots \times \mathbb{Z}_{p^{k_m}}, \quad \text{with } k_1 \geq k_2 \geq \cdots \geq k_m \geq 1.$$

Since S is a generating set, it is easy to see that there is some $s_1 \in S$, such that $|s_1| = p^{k_1}$. Also, it is a basic fact about finite abelian groups that every cyclic subgroup of maximal order is a direct summand [4, Lem. 1.3.3, p. 10]. Therefore, by induction on i, we see that there exist $s_1, \ldots, s_m \in S$, such that if we let $G_i = \langle s_1, \ldots, s_i \rangle$, then

$$G_i \cong G_{i-1} \times \mathbb{Z}_{p^{k_i}} \quad \text{and} \quad G \cong G_i \times \mathbb{Z}_{p^{k_{i+1}}} \times \cdots \times \mathbb{Z}_{p^{k_m}}, \quad \text{for each } i.$$

It is important to note that each element of G_i can be written uniquely in the form

$$g + rs_i, \text{ with } g \in G_{i-1} \text{ and } -p^{k_i}/2 < r \leq p^{k_i}/2 \text{ (and } r \in \mathbb{Z}). \qquad (\dagger)$$

For convenience, also let

$$t_i = \varphi(s_i) \quad \text{and} \quad H_i = \langle t_1, \ldots, t_i \rangle.$$

We will show, by induction on i, that if G does not have any direct summands of the form specified in the statement of the proposition, then H_i is a direct factor of G, and the restriction of φ to G_i is an isomorphism onto H_i. (Note that this implies $G/G_i \cong G/H_i$, by the uniqueness of the decomposition of G as a direct sum of cyclic groups.) Taking $i = m$ yields the desired conclusion that φ is an automorphism of G.

The base case $i = 0$ is trivial. For the induction step, write $G = G_{i-1} \times \overline{G}$, so

$$\overline{G} \cong G/G_{i-1} \cong \mathbb{Z}_{p^{k_i}} \times \mathbb{Z}_{p^{k_{i+1}}} \times \cdots \times \mathbb{Z}_{p^{k_m}},$$

and let $^{-}\colon G \to \overline{G}$ be the natural projection. Then $\langle \overline{s_i} \rangle = \overline{G_i} \cong \mathbb{Z}_{p^{k_i}}$ is a direct summand of $\overline{G}$. Since φ is colour-permuting (and $H_{i-1} = \varphi(G_{i-1})$ is a subgroup), it is easy to see that the order of t_i in G/H_{i-1} is equal to p^{k_i} (the same as the the order of s_i in G/G_{i-1}), and that $\varphi(p^{k_i} s_i) = p^{k_i} t_i$. This implies that if we define

$$\alpha\colon G_i \to H_i \text{ by } \alpha(g + rs_i) = \varphi(g) + rt_i \quad \text{for } g \in G_{i-1} \text{ and } r \in \mathbb{Z},$$

then α is a well-defined isomorphism. So we need only show that the restriction of φ to G_i is equal to α (unless G has a direct summand of the desired form).

Suppose $\varphi|_{G_i} \neq \alpha$. (This will lead either to a contradiction or to a summand of the desired form.) Since φ is colour-permuting and, by definition, α agrees with φ on G_{i-1}, this implies there is some $g \in G_{i-1}$, such that $\varphi(g + s_i) \neq \alpha(g + s_i)$. However, since φ is colour-permuting, we know

$$\varphi(g + s_i) = \varphi(g) \pm \varphi(s_i) = \alpha(g) \pm t_i.$$

Since $\alpha(g + s_i) = \alpha(g) + t_i$, the preceding two sentences imply

$$\varphi(g + s_i) = \alpha(g) - t_i \in H_{i-1} - t_i.$$

Furthermore, since φ is colour-permuting (and $\varphi(s_j) = t_j$), we know that it maps edges of colour $\{s_1^{\pm 1}\}, \ldots, \{s_{i-1}^{\pm 1}\}$ to edges of colour $\{t_1^{\pm 1}\}, \ldots, \{t_{i-1}^{\pm 1}\}$, so

$$\varphi(x + h) \in \varphi(x) + H_{i-1} \text{ for all } x \in G \text{ and } h \in H_{i-1}.$$

Taking $x = s_i$ and $h = g$ yields

$$\varphi(g + s_i) \in H_{i-1} + \varphi(s_i) = H_{i-1} + t_i.$$

This contradicts the uniqueness of r in the analogue of (†) for H_i, unless $1 = p^{k_i}/2$. Hence, we must have $p^{k_i} = 2$ (so $\mathbb{Z}_2$ is a direct summand of G), which means $p = 2$ and $k_i = 1$.

We have

$$\begin{aligned}
\varphi(g) + 2t_i &= \alpha(g + 2s_i) && \text{(definition of } \alpha) \\
&= \varphi(g + 2s_i) && (g + 2s_i = g + p^{k_i} s_i \in G_{i-1}) \\
&= \varphi(g) - 2t_i && (\varphi(g + s_i) = \alpha(g) - t_i = \varphi(g) - t_i),
\end{aligned}$$

so $4t_i = 0$. Also note that, since

$$\varphi(g) + t_i = \alpha(g + s_i) \neq \varphi(g + s_i) = \varphi(g) - t_i,$$

we must have $2t_i \neq 0$. So $|t_i| = 4$.

Since $\langle s_1, \ldots, s_{i-1} \rangle = G_{i-1}$, there must exist $g' \in G_{i-1}$, and $j < i$, such that

$$\varphi(g' + s_i) = \alpha(g') + t_i, \text{ but } \varphi(g' + s_j + s_i) = \alpha(g' + s_j) - t_i = \alpha(g') + t_j - t_i.$$

Since φ is colour-permuting, we also have

$$\varphi(g' + s_j + s_i) = \varphi(g' + s_i) \pm t_j = \alpha(g') + t_i \pm t_j.$$

Hence, $t_j - t_i = t_i \pm t_j$, so $t_j \mp t_j = 2t_i$. Since $2t_i \neq 0$, we conclude that $2t_j = 2t_i$; hence, $|t_j| = 4$.

Since $2^{k_j} = |H_j : H_{j-1}|$ is a divisor of $|t_j|$, and $|t_j| = 4$, there are two possibilities for k_j:

- If $k_j = 2$, then $\mathbb{Z}_4 \times \mathbb{Z}_2 \cong \mathbb{Z}_{2^{k_j}} \times \mathbb{Z}_{2^{k_i}}$ is a direct summand of G, as desired.
- If $k_j = 1$, then, since $|t_j| = 4$, there must be some $\ell < j$, such that $k_\ell \geq 2$. This implies that $\mathbb{Z}_{2^{k_\ell}} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_{2^{k_\ell}} \times \mathbb{Z}_{2^{k_j}} \times \mathbb{Z}_{2^{k_i}}$ is a direct summand of G, as desired. □

Corollary 4.2. *For $n \in \mathbb{Z}^+$, there is a non-CCA abelian group of order n if and only if n is divisible by* 8.

5 Generalized dihedral groups

Definition 5.1. The *generalized dihedral group* over an abelian group A is the group

$$\langle \sigma, A \mid \sigma^2 = e,\ \sigma a \sigma = a^{-1}\ \forall a \in A \rangle.$$

Lemma 5.2. *Suppose D is the generalized dihedral group over an abelian group A, and φ is a colour-permuting automorphism of a connected Cayley graph* $\mathrm{Cay}(D; S)$*, such that $\varphi(e) = e$. If A is strongly CCA, and $\varphi(S \cap A) = S \cap A$, then φ is an automorphism of D.*

Proof. Label the elements of S as $S = \{a_1, a_2, \ldots, a_k, \sigma_1, \sigma_2, \ldots, \sigma_t\}$, where $a_i \in A$ for $1 \le i \le k$, and $\sigma_i \notin A$ for $1 \le i \le t$ (so each σ_i is an involution that inverts the elements of A). By assumption, $\{a_1, a_2, \ldots, a_k\}$ and $\{\sigma_1, \sigma_2, \ldots, \sigma_t\}$ are invariant under φ. Thus, for each i, we have

- $\varphi(a_i) = a_i'$ for some $a_i' \in \{a_1, a_2, \ldots, a_k\}$, and
- $\varphi(\sigma_i) = \sigma_i'$ for some $\sigma_i' \in \{\sigma_1, \sigma_2, \ldots, \sigma_t\}$.

Notice that since $\sigma_1, \ldots, \sigma_t$ are involutions, each σ_i is its own inverse. Therefore, whenever σ is a word in $\sigma_1, \ldots, \sigma_t$ and $g \in D$, the fact that φ is a colour-permuting automorphism means that $\varphi(g\sigma) = \varphi(g)\sigma'$, where σ' is formed from σ by replacing each instance of σ_i in σ by σ_i'. Therefore, if we let Σ be the subgroup generated by $\{\sigma_1, \ldots, \sigma_t\}$, then φ is a colour-preserving automorphism of the Cayley graph $\mathrm{Cay}(D; S \cup \Sigma)$. Hence, there is no harm in assuming that $S = S \cup \Sigma$, so $\Sigma \subseteq S$.

Since $\langle S \cap A \rangle$ is normal in D (in fact, every subgroup of A is normal, because every element of D either centralizes or inverts it), we have $D = \langle S \cap A \rangle \Sigma$. Therefore $A = \langle S \cap A \rangle (\Sigma \cap A) = \langle S \cap A \rangle$, so $\mathrm{Cay}(A; S \cap A)$ is connected. Since φ is colour-preserving, and $\varphi(S \cap A) = S \cap A$, this implies that $\varphi(A) = A$. So φ is a colour-permuting automorphism of the connected Cayley graph $\mathrm{Cay}(A; S \cap A)$. Since, by assumption, A is strongly CCA, this implies that $\varphi|_A$ is an automorphism of A. So $\varphi(ab^\epsilon) = \varphi(a)\,\varphi(b)^\epsilon$ for all $a, b \in A$ and $\epsilon \in \mathbb{Z}$.

Now we are ready to show that φ is an automorphism of D. Let $g, h \in D$. Then we may write $g = a\sigma$ and $h = b\widetilde{\sigma}$, where $a, b \in A$ and $\sigma, \widetilde{\sigma} \in \{e, \sigma_1\}$. For convenience, let $\epsilon \in \{\pm 1\}$, such that $\sigma c \sigma = c^\epsilon$ for all $c \in A$. Note that, since $\sigma_1' \in \{\sigma_1, \ldots, \sigma_t\}$, we know that σ_1 and σ_1' both invert A, so we also have $\sigma' c \sigma' = c^\epsilon$. Then

$$\varphi(gh) = \varphi(a\sigma \cdot b\widetilde{\sigma}) = \varphi(ab^\epsilon \cdot \sigma\widetilde{\sigma}) = \varphi(a)\,\varphi(b)^\epsilon \cdot \sigma'\widetilde{\sigma}' = \varphi(a)\sigma' \cdot \varphi(b)\widetilde{\sigma}' = \varphi(g) \cdot \varphi(h).$$

Since $g, h \in D$ are arbitrary, this proves that φ is an automorphism of D. □

Proposition 5.3. *The generalized dihedral group D over an abelian group A is CCA if and only if A is CCA.*

Proof. ($\Leftarrow$) Note that if φ is any colour-preserving automorphism of a connected Cayley graph $\mathrm{Cay}(D; S)$, then $\varphi(S \cap A) = S \cap A$, since A is closed under inverses. Furthermore, A is strongly CCA, since it is assumed to be CCA and every CCA abelian group is strongly CCA (see Proposition 4.1). Therefore, Lemma 5.2 implies that φ is a group automorphism. So D is CCA.

($\Rightarrow$) Write $D = A \rtimes \langle \sigma \rangle$. Since A is not CCA, there is a colour-preserving automorphism φ_0 of some connected Cayley graph $\mathrm{Cay}(A; S)$, such that φ_0 is not affine. Since

σ inverts every element of S, it is easy to see that $\mathrm{Cay}\big(D; S \cup \{\sigma\}\big)$ is isomorphic to the Cartesian product $\mathrm{Cay}(A; S) \,\square\, P_2$. So the proof of Proposition 3.1 provides a colour-preserving automorphism φ of $\mathrm{Cay}\big(D; S \cup \{\sigma\}\big)$ whose restriction to A is φ_0, which is not an affine map. Therefore, φ is not affine. □

The following result is the special case where A is cyclic (since Proposition 4.1 implies that every cyclic group is CCA).

Corollary 5.4. *Every dihedral group is CCA.*

Lemma 5.5. *If T is a generating set of a group H, and σ is a nontrivial automorphism of H, such that $\sigma(t) \in \{t^{\pm 1}\}$ for every $t \in T$, then the group $G = (H \rtimes \langle \sigma \rangle) \times \mathbb{Z}_2$ is not strongly CCA.*

Proof. Let $G' = H \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and define $\varphi \colon G \to G'$ by $\varphi(h, \sigma^x, y) = (h, x, y)$ for $h \in H$ and $x, y \in \mathbb{Z}_2$. Since $\sigma(t) \in \{t^{\pm 1}\}$ for every t, it is easy to verify that φ is a colour-respecting isomorphism

$$\begin{aligned} &\text{from } \mathrm{Cay}\big(G; (H, e, 0) \cup \{(e, \sigma, 0), (e, 0, 1)\}\big) \\ &\text{to } \mathrm{Cay}\big(G; (H, 0, 0) \cup \{(e, 1, 0), (e, 0, 1)\}\big). \end{aligned}$$

Permuting the two $\mathbb{Z}_2$ factors of G' provides an automorphism of G' that preserves the generating set, and therefore corresponds to a colour-permuting automorphism of the two Cayley graphs. However, it is not an automorphism of G, since it takes the central element $(e, e, 1)$ to $(e, \sigma, 0)$, which is not central (since the automorphism σ is nontrivial). □

Proposition 5.6. *The generalized dihedral group over an abelian group A is strongly CCA if and only if either A does not have $\mathbb{Z}_2$ as a direct factor, or A is an elementary abelian 2-group (in which case, the generalized dihedral group is also an elementary abelian 2-group).*

Proof. ($\Rightarrow$) Suppose $A = A' \times \mathbb{Z}_2$, and A' is not elementary abelian. Then the generalized dihedral group $A \rtimes \langle \sigma \rangle$ over A is isomorphic to $(A' \rtimes \langle \sigma \rangle) \times \mathbb{Z}_2$, so Lemma 5.5 tells us that it is not strongly CCA.

($\Leftarrow$) Let $D = A \rtimes \langle \sigma \rangle$ be the generalized dihedral group over A, and let φ be a colour-permuting automorphism of a connected Cayley graph $\mathrm{Cay}(D; S)$, such that $\varphi(e) = e$. We may assume A does not have $\mathbb{Z}_2$ as a direct factor (otherwise, the desired conclusion follows from the fact that every elementary abelian 2-group is strongly CCA (see Proposition 4.1)). From Proposition 4.1, we see that A is strongly CCA. Hence, the desired conclusion will follow from Lemma 5.2 if we show that $\varphi(S \cap A) = S \cap A$.

Let $a \in S \cap A$. Since φ is colour-permuting, we have $|\varphi(s)| = |s|$ for all $s \in S$. Also, we know that $|g| = 2$ for all $g \in D \setminus A$. Therefore, it is obvious that $\varphi(a) \in S \cap A$ if $|a| \neq 2$.

So we may assume $|a| = 2$. Since A does not have $\mathbb{Z}_2$ as a direct factor, this implies that a is a square in A: that is, we have $a = x^2$, for some $x \in A$. Also, since $\mathrm{Cay}(D; S)$ is connected, we may write $x = s_1 s_2 \cdots s_n$ for some $s_1, \ldots, s_n \in S$. So $a = (s_1 s_2 \cdots s_n)^2$ can be written as a word in which every element of S occurs an even number of times. Since φ is colour-permuting, this implies that $\varphi(a)$ can be written as a word in which, for each $s \in S$, the total number of occurrences of either s or s^{-1} is even. Since s and s^{-1} both either centralize A or invert it, this implies that $\varphi(a)$ centralizes A. Since every element of $D \setminus A$ inverts A, we conclude that $\varphi(a) \in A$, as desired. □

6 Groups of odd order

The following notation will be assumed throughout this section.

Notation 6.1. For a fixed Cayley graph $\mathrm{Cay}(G; S)$:

- $\mathcal{A}^0$ is the group of all colour-preserving automorphisms of $\mathrm{Cay}(G; S)$.
- $\widehat{G}$ is the subgroup of $\mathcal{A}^0$ consisting of all left translations by elements of G. (Although we do not need this terminology, it is often called the *left regular representation* of G.)
- H_e is the stabilizer of the identity element e in $\mathrm{Cay}(G; S)$, for any subgroup H of $\mathcal{A}^0$.

Remark 6.2. It is well known (and very easy to prove) that a permutation of G is affine if and only if it normalizes $\widehat{G}$ (see, for example [10, Lem. 2]).

Lemma 6.3. *$\mathcal{A}^0_e$ is a 2-group.*

Proof. Let $\varphi \in \mathcal{A}^0_e$, so φ is a colour-preserving automorphism of $\mathrm{Cay}(G; S)$ that fixes e. If C is any monochromatic cycle through e, then either φ is the identity on C or φ reverses the orientation of C. Therefore, φ^2 acts trivially on the union of all monochromatic cycles that contain e. This implies that φ^2 acts trivially on all vertices at distance ≤ 1 from e.

Repeating the argument shows that φ^{2^k} acts trivially on all vertices at distance $\leq k-1$ from e. For k larger than the diameter of $\mathrm{Cay}(G; S)$, this implies that φ^{2^k} is trivial. So the order of φ is a power of 2. □

Proposition 6.4. *Let $\mathrm{Cay}(G; S)$ be a connected Cayley graph on a group G of odd order. If every colour-preserving automorphism of $\mathrm{Cay}(G; S)$ is affine, then every colour-permuting automorphism is affine.*

Proof. Let $\mathcal{A}^\bullet$ be the group of all colour-permuting automorphisms of $\mathrm{Cay}(G; S)$. Since $\mathcal{A}^\bullet$ acts on the set of colours, and $\mathcal{A}^0$ is the kernel of this action (and the kernel of a homomorphism is always normal), it is obvious that $\mathcal{A}^0 \triangleleft \mathcal{A}^\bullet$. Also, since G is CCA, we have $\widehat{G} \triangleleft \mathcal{A}^0$ (cf. Remark 6.2). Furthermore, $|G|$ is odd, $|\mathcal{A}^0_e|$ is a power of 2, and $\mathcal{A}^0 = \widehat{G} \cdot \mathcal{A}^0_e$. Therefore, $\widehat{G}$ is the (unique) largest normal subgroup of odd order in $\mathcal{A}^0$. The uniqueness implies that $\widehat{G}$ is *characteristic* in $\mathcal{A}^0$. (That is, it is fixed by all automorphisms of $\mathcal{A}^0$.) So $\widehat{G}$ is a characteristic subgroup of the normal subgroup $\mathcal{A}^0$ of $\mathcal{A}^\bullet$. Although a normal subgroup of a normal subgroup need not be normal, it is well known (and easy to prove) that any characteristic subgroup of a normal subgroup is normal [4, Thm. 2.1.2(ii), p. 16]. Therefore $\widehat{G} \triangleleft \mathcal{A}^\bullet$. This implies that G is strongly CCA (see Remark 6.2). □

Wreath products $\mathbb{Z}_m \wr \mathbb{Z}_n$ provide examples of non-CCA groups of odd order (see Example 2.4). We will see in Theorem 6.8 that the following slightly more general construction is essential for understanding many of the other non-CCA groups of odd order.

Example 6.5. Let α be an automorphism of a group A, and let $n \in \mathbb{Z}^+$. Then we can define an automorphism $\widetilde{\alpha}$ of A^n by

$$\widetilde{\alpha}(w_1, \ldots, w_n) = \big(\alpha(w_n), w_1, w_2, \ldots, w_{n-1}\big).$$

It is easy to see that the order of $\widetilde{\alpha}$ is n times the order of α, so we may form the corresponding semidirect product $A^n \rtimes \mathbb{Z}_{n|\alpha|}$. Let us call this the *semi-wreathed product* of A by $\mathbb{Z}_n$, with respect to the automorphism α, and denote it $A \wr_\alpha \mathbb{Z}_n$. (If α is the trivial automorphism, then this is the usual wreath product $A \wr \mathbb{Z}_n$.)

Negating the first coordinate, as in Example 2.4, shows that if $n > 1$ and A is abelian, but not an elementary abelian 2-group, then $A \wr_\alpha \mathbb{Z}_n$ is not CCA.

Remark 6.6. Because it may be of interest to find minimal examples, we point out that any semi-wreathed product of odd order satisfying the conditions in the final paragraph of Example 6.5 must contain a subgroup that is isomorphic to a semi-wreathed product $A \wr_\alpha \mathbb{Z}_q$, where A is an elementary abelian p-group, p and q are primes (not necessarily distinct), α is an automorphism of q-power order, and no nontrivial, proper subgroup of A is invariant under α.

Definition 6.7 ([4, p. 5])**.** Let G be a group. For any subgroups H and K of G, such that $K \triangleleft H$, the quotient H/K is said to be a *section* of G.

Theorem 6.8. *Any non-CCA group of odd order has a section that is isomorphic to either:*

1. *a semi-wreathed product $A \wr_\alpha \mathbb{Z}_n$ (see Example 6.5), where A is a nontrivial, elementary abelian group (of odd order) and $n > 1$, or*
2. *the (unique) nonabelian group of order 21.*

Proof. Assume $\mathrm{Cay}(G; S)$ is a connected Cayley graph on a group G of odd order that does not have a section as described in either (1) or (2). We will show, by induction on the order, that if $\mathcal{A}$ is any subgroup of $\mathcal{A}^0$ that contains $\widehat{G}$, then $\widehat{G}$ is a normal subgroup of $\mathcal{A}$. (Then taking $\mathcal{A} = \mathcal{A}^0$ implies that G is CCA (see Remark 6.2).)

It is important to note that this conclusion implies $\widehat{G}$ is a *characteristic* subgroup of $\mathcal{A}$ (because Lemma 6.3 implies that $\widehat{G}$ is the unique largest normal subgroup of odd order). For convenience, we write $\widehat{G} \blacktriangleleft \mathcal{A}$ when $\widehat{G}$ is characteristic.

Let $\mathcal{N}$ be a minimal normal subgroup of $\mathcal{A}$. Then $\mathcal{N}$ is either elementary abelian or the direct product of (isomorphic) nonabelian simple groups [4, Thm. 2.1.5, p. 17], and we consider the two possibilities as separate cases.

Case 1. *Assume $\mathcal{N}$ is elementary abelian.* Since the Sylow 2-subgroup $\mathcal{A}_e$, being the stabilizer of a vertex, does not contain any normal subgroups of $\mathcal{A}$, we know that $\mathcal{N}$ is not contained in a Sylow 2-subgroup. Hence, $\mathcal{N}$ is not a 2-group, so it must be a p-group for some odd prime p. Therefore, since $\widehat{G}$ is a maximal subgroup of odd order, we have $\mathcal{N} \subseteq \widehat{G}$, so

$$\mathcal{N} = \widehat{N}, \text{ for some (elementary abelian) normal subgroup } N \text{ of } G.$$

Let $\mathcal{N}^+$ be the largest normal subgroup of $\mathcal{A}$ that is contained in $\mathcal{N}\mathcal{A}_e$. Since $\mathcal{N}\mathcal{A}_e$ is the stabilizer of a point under the action of $\mathcal{A}$ on the space G/N of $\mathcal{N}$-orbits, we know that $\mathcal{N}^+$ is the kernel of the action of $\mathcal{A}$ on G/N, so $\mathcal{A}/\mathcal{N}^+$ is a group of colour-preserving automorphisms of $\mathrm{Cay}(G/N; \overline{S})$, where $\overline{S}$ is the image of S in G/N.. Therefore, by induction on $|\mathcal{A}|$, we know that $\widehat{G}\mathcal{N}^+/\mathcal{N}^+$ is normal in $\mathcal{A}/\mathcal{N}^+$, so $\widehat{G}\mathcal{N}^+$ is normal in $\mathcal{A}$. Then we may assume $\widehat{G}\mathcal{N}^+ = \mathcal{A}$, for otherwise, by induction on $|\mathcal{A}|$, we would know $\widehat{G} \blacktriangleleft \widehat{G}\mathcal{N}^+$, so $\widehat{G} \triangleleft \mathcal{A}$, as desired. Since $|G|$ is odd, this implies that $\mathcal{N}^+$ contains a Sylow 2-subgroup of $\mathcal{A}$. In fact, since $\mathcal{N}^+$ is normal and all Sylow 2-subgroups are conjugate, this implies that $\mathcal{N}^+$ contains every Sylow 2-subgroup. In particular, it contains $\mathcal{A}_e$. Therefore $\mathcal{N}^+ = \mathcal{N}\mathcal{A}_e$, so

$$\mathcal{N}\mathcal{A}_e \triangleleft \mathcal{A}.$$

This means that $\mathcal{A}_e$ acts trivially on G/N, so, for every $s \in S \smallsetminus N$, $\mathcal{A}_e$ preserves the orientation of every s-edge. (This uses the fact that, since $|s|$ is odd, $s \not\equiv s^{-1} \pmod{N}$ if $s \notin N$.) This implies:

$$\text{for } \varphi \in \mathcal{A}_e,\ g \in G, \text{ and } x \in \langle S \smallsetminus N \rangle, \text{ we have } \varphi(gx) = \varphi(g)\,x. \tag{6.9}$$

Let $(S \cap N)^{\langle S \smallsetminus N \rangle} = \{\, gsg^{-1} \mid s \in S \cap N,\ g \in \langle S \smallsetminus N \rangle \,\}$. Now, suppose $t \in (S \cap N)^{\langle S \smallsetminus N \rangle}$ and $h \in N$. There exists $s \in S \cap N$ and $x \in \langle S \smallsetminus N \rangle$, such that $xsx^{-1} = t$. From (6.9) and the fact that φ is colour-preserving, we see that

$$\varphi(h\,t) = \varphi(h\,xsx^{-1}) = \varphi(h)\,x\,s^{\pm 1}x^{-1} = \varphi(h)\,t^{\pm 1}.$$

Hence, $\varphi|_N$ is a colour-preserving automorphism of

$$\mathrm{Cay}\Big(N; (S \cap N)^{\langle S \smallsetminus N \rangle} \cup \big(\langle S \smallsetminus N \rangle \cap N\big)\Big).$$

Since S generates G, it is easy to see that this Cayley graph is connected.

Note that $C_{\mathcal{A}_e}(\mathcal{N})$ is normalized by both $\mathcal{N}$ and $\mathcal{A}_e$, so it is a normal subgroup of $\mathcal{N}\mathcal{A}_e$. Therefore, it must be trivial (since the largest normal 2-subgroup of $\mathcal{N}\mathcal{A}_e$ is characteristic, and is therefore normal in $\mathcal{A}$, but the stabilizer $\mathcal{A}_e$ does not contain any nontrivial normal subgroups of $\mathcal{A}$). So

$$\mathcal{A}_e \text{ acts faithfully by conjugation on } \mathcal{N}. \tag{6.10}$$

Also, we know that $\varphi|_N$ is an automorphism of N (by Remark 6.2, since φ normalizes $\mathcal{N} = \widehat{N}$). Since, being a colour-preserving automorphism, φ either centralizes or inverts every element of the generating set of N, this implies that $\varphi^2|_N$ is trivial. Since this is true for every $\varphi \in \mathcal{A}_e$, we conclude that $\mathcal{A}_e$ acts on N via an elementary abelian 2-group. From (6.10), we conclude that $\mathcal{A}_e$ is elementary abelian.

We can think of $\mathcal{N}$ as a vector space over $\mathbb{Z}_p$, and, for each homomorphism $\gamma\colon \mathcal{A} \to \{\pm 1\}$, let

$$\mathcal{N}_\gamma = \{\, n \in \mathcal{N} \mid ana^{-1} = \gamma(a)\,n \text{ for all } a \in \mathcal{A}_e \,\}.$$

(This is called the "weight space" associated to γ.) Since every linear transformation satisfying $T^2 = I$ is diagonalizable, and $\mathcal{A}_e$ is commutative, the elements of $\mathcal{A}_e$ can be simultaneously diagonalized. This means that if we let $\Gamma = \big\{\, \gamma \mid \mathcal{N}_\gamma \neq \{0\} \,\big\}$, then, since eigenspaces for different eigenvalues are always linearly independent, we have $\mathcal{N} = \bigoplus_{\gamma \in \Gamma} \mathcal{N}_\gamma$. This direct-sum decomposition is canonically defined from the action of $\mathcal{A}_e$ on $\mathcal{N}$. Since $\widehat{G}$ acts on $\mathcal{N}\mathcal{A}_e$ (by conjugation), we conclude that the action of $\widehat{G}$ on $\mathcal{N}$ by conjugation must permute the weight spaces. More precisely, there is an action of G on Γ, such that $\widehat{g}\mathcal{N}_\gamma\widehat{g}^{-1} = \mathcal{N}_{g\gamma}$ for all $g \in G$. Since N is abelian, this factors through to a well-defined action of G/N on Γ.

If the G-action on Γ is trivial, then every weight space is G-invariant, which implies that the action of $\widehat{G}$ on $\mathcal{N}$ commutes with the action of $\mathcal{A}_e$. Since $\mathcal{A}_e$ acts faithfully, we conclude that $\widehat{G}$ centralizes $\mathcal{A}_e\mathcal{N}/\mathcal{N}$; that is, $[\widehat{G}, \mathcal{A}_e] \subseteq \mathcal{N} \subseteq \widehat{G}$. So $\mathcal{A}_e$ normalizes $\widehat{G}$, as desired.

We may now assume that the G-action is nontrivial, so there is some $g \in G$ with an orbit of some length $n > 1$ on Γ. Let γ_0 be an element of this orbit, so $\widehat{g}^n$ normalizes $\mathcal{N}_{\gamma_0}$. Since $S \setminus N$ generates G/N, we may assume $g \in S \setminus N$, so (6.9) tells us that $\langle \widehat{g} \rangle \cap \mathcal{N}$ is centralized by $\mathcal{A}_e$. However, the minimality of $\mathcal{N}$ implies that $C_{\mathcal{N}}(\mathcal{A}_e) = \mathcal{N} \cap Z(\mathcal{N}\mathcal{A}_e)$ is trivial. Therefore, $\langle \mathcal{N}, \widehat{g} \rangle = \mathcal{N} \rtimes \langle \widehat{g} \rangle$ is a semidirect product. So

$$\langle \mathcal{N}_{\gamma_0}, \widehat{g} \rangle = \left(\bigoplus\nolimits_{\gamma \in \langle g \rangle \gamma_0} \mathcal{N}_\gamma \right) \rtimes \langle \widehat{g} \rangle.$$

Then modding out $C_{\langle \widehat{g} \rangle}(\mathcal{N}_{\gamma_0})$ yields a section of $\widehat{G}$ that is isomorphic to $\mathcal{N}_{\gamma_0} \wr_\alpha \mathbb{Z}_n$, where α is the automorphism of $\mathcal{N}_{\gamma_0}$ induced by the conjugation action of $\widehat{g}^n$. So G has a semi-wreathed section, as described in (1). This completes the proof of this case.

Case 2. *Assume $\mathcal{N} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_r$, where each $\mathcal{L}_i$ is a nonabelian simple group, and $\mathcal{L}_i \cong \mathcal{L}_1$ for all i.* We know that $\mathcal{A} = \widetilde{G}\mathcal{A}_e$, $\mathcal{A}_e$ is a 2-group, and $|G|$ is odd, so $\widehat{G}$ is a 2-complement in $\mathcal{A}$. (By definition, this means that $|\widehat{G}|$ is odd and $|\mathcal{A} : \widehat{G}|$ is a power of 2 [6, p. 88].) So $\mathcal{L}_1$ is a nonabelian simple group that has a 2-complement (namely, $\widehat{G} \cap \mathcal{L}_1$). By using the Classification of Finite Simple Groups, it can be shown that this implies $\mathcal{L}_1 \cong \mathrm{PSL}(2, p)$, for some Mersenne prime $p \geq 7$ (see [8, Thm. 1.3]).

Note that $\mathcal{A}_e \cap \mathcal{L}_i$ is a Sylow 2-subgroup of $\mathrm{PSL}(2, p)$. Therefore, it is dihedral [4, Lem. 15.1.1(iii)] and has order $p + 1$ (because p is a Mersenne prime). Let

- $\mathcal{C}_i$ be the unique cyclic subgroup of order $(p + 1)/2$ in $\mathcal{A}_e \cap \mathcal{L}_i$,
- $\mathcal{C}_i^2$ be the unique subgroup of index 2 in $\mathcal{C}_i$, and
- $\mathcal{C}^2 = \mathcal{C}_1^2 \times \cdots \mathcal{C}_r^2 \subset \mathcal{A}_e \cap (\mathcal{L}_1 \times \cdots \times \mathcal{L}_r)$.

Since every element of $\mathcal{C}_i$ is a colour-preserving automorphism, it either fixes or inverts each element of S, so we know that $\mathcal{C}_i^2$ fixes every element of S. Since stabilizers are conjugate, this implies $\widehat{s}^{-1}\mathcal{C}^2\widehat{s} \subseteq \mathcal{A}_e$, for every $s \in S$. We must have $p > 7$, for otherwise $\widehat{G} \cap \mathcal{L}_i$, being the 2-complement of $\mathrm{PSL}(2, 7)$, would be the nonabelian group of order 21, as in (2). This implies that $\mathcal{C}_i^2$ is the unique cyclic subgroup of order $(p + 1)/4$ in the dihedral group $\mathcal{A}_e \cap \mathcal{L}_i$, so we must have $\widehat{s}^{-1}\mathcal{C}^2\widehat{s} = \mathcal{C}^2$, which means that $\widehat{s}$ normalizes $\mathcal{C}^2$. Since this holds for every s in the generating set S, we conclude that $\widehat{G}$ normalizes $\mathcal{C}^2$.

Note that $\mathcal{A}_e$ normalizes $\mathcal{A}_e \cap \mathcal{N}$, and that $\mathcal{C}^2 \blacktriangleleft \mathcal{A}_e \cap \mathcal{N}$ (since, as was mentioned above, $\mathcal{C}_i$ is the unique cyclic subgroup of its order in $A_e \cap \mathcal{L}_i$). Therefore, $\mathcal{C}^2 \triangleleft \mathcal{A}_e$. We conclude that $\mathcal{C}^2$ is normal in $\widehat{G}\mathcal{A}_e = \mathcal{A}$. So $\mathcal{C}_1^2 = \mathcal{C}^2 \cap \mathcal{L}_1$ is normal in $\mathcal{L}_1$, contradicting the fact that L_1 is simple. □

Lemma 6.11. *To prove a group G is strongly CCA (or CCA), it suffices to consider only the connected Cayley graphs* $\mathrm{Cay}(G; S)$*, such that every element of S has prime-power order.*

Proof. Suppose φ is a colour-permuting automorphism of some connected Cayley graph $\mathrm{Cay}(G; S)$. There is a permutation π of S, such that $\varphi(gs) = \varphi(g)\,\pi(s)^{\pm 1}$, for all $g \in G$ and $s \in S$. (Furthermore, if φ is colour-preserving, then π can be taken to be the identity permutation.) By induction on k, this implies $\varphi(gs^k) = \varphi(g)\,\pi(s)^{\pm k}$, for all $k \in \mathbb{Z}$. Hence, if we let $S^* = \{\, s^k \mid s \in S,\ k \in \mathbb{Z} \,\}$, then φ is a colour-permuting automorphism of $\mathrm{Cay}(G; S^*)$. Now, let

$$S_0 = \{\, t \in S^* \mid |t| \text{ is a prime-power} \,\}.$$

Then φ is a colour-permuting automorphism of $\mathrm{Cay}(G; S_0)$, and S_0 generates G, since every element s of the generating set S can be written as a product of elements of $\langle s \rangle$ that have prime-power order [4, Thm. 1.3.1(iii), p. 9], and therefore belong to S_0. (Furthermore, φ is colour-preserving if the permutation π is the identity permutation.) □

Lemma 6.12. *Suppose*

- *C is a cyclic, normal subgroup of a group H,*
- *$|C|$ is relatively prime to $|H : C|$,*
- *no element of $H \setminus C$ centralizes C, and*
- *α is any automorphism of H.*

Then $\alpha(h) \in hC$, for every $h \in H$.

Proof. Since no other subgroup of H has the same order as C, we know that $\alpha|_C$ is an automorphism of C, so there exists $r \in \mathbb{Z}$, such that $\alpha(c) = c^r$, for every $c \in C$. Then, for any $h \in H$ and $c \in C$, we have

$$\alpha(h)\, c^r\, \alpha(h)^{-1} = \alpha(h)\, \alpha(c)\, \alpha(h)^{-1} = \alpha(hch^{-1}) = (hch^{-1})^r = hc^r h^{-1},$$

so $h^{-1}\,\alpha(h)$ centralizes C. By assumption, this implies $h^{-1}\,\alpha(h) \in C$, as desired. □

Corollary 6.13. *The following are equivalent:*

1. *There is a group of order n that is not CCA.*
2. *There is a group of order n that is not strongly CCA.*
3. *$n \geq 8$, and n is divisible by either 4, 21, or a number of the form $p^q \cdot q$, where p and q are primes (not necessarily distinct) and p is odd.*

Proof. We prove ($1 \Leftrightarrow 3$) and ($1 \Leftrightarrow 2$).

($3 \Rightarrow 1$) If n is divisible by 4, then there is a generalized dicyclic group of order n, which is not CCA (see Corollary 2.8(4)). The nonabelian group of order 21 and the wreath product $\mathbb{Z}_p \wr \mathbb{Z}_q$ (which is of order $p^q \cdot q$) are not CCA (see Examples 2.3 and 2.4). Taking an appropriate direct product yields a non-CCA group whose order is any multiple of these (see Proposition 3.1).

($1 \Rightarrow 3$) Assume there is a group G of order n that is not CCA, but n is not divisible by 4, 21, or a number of the form $p^q \cdot q$. From Theorem 6.8, we see that n is even. (Otherwise, $n = |G|$ is divisible by the order of a semi-wreathed product $|A \wr_\alpha \mathbb{Z}_k|$. If we let p and q be prime divisors of $|A|$ and k, respectively, then $|A \wr_\alpha \mathbb{Z}_k| = |A|^k \cdot k$ is a multiple of $p^q \cdot q$.) Furthermore, n must be square-free, for otherwise it is a multiple of either 4 or $p^2 \cdot 2$, for some prime p. Therefore, G is a semidirect product $\mathbb{Z}_k \rtimes \mathbb{Z}_\ell$.

We may assume the centre of G is trivial, for otherwise we can write G as a nontrivial direct product, so Proposition 3.2 (and induction on n) implies that G is CCA. Therefore, k is odd (so ℓ is even), so we may write $G = \mathbb{Z}_k \rtimes (\mathbb{Z}_m \times \mathbb{Z}_2)$, and $\mathbb{Z}_m \times \mathbb{Z}_2$ acts faithfully on $\mathbb{Z}_k$. Let $H = \mathbb{Z}_k \rtimes \mathbb{Z}_m$, so $|H| = km$ is odd, and H is the (unique) subgroup of index 2 in G.

Let φ be a colour-preserving automorphism of a connected Cayley graph $\mathrm{Cay}(G; S)$. (We wish to show that φ is affine.) There is no harm in assuming that every element of S has prime order (see Lemma 6.11). Fix some $t \in S$ with $|t| = 2$.

We claim we may assume that t is the only element of order 2 in S, and that $H = \langle S \setminus \{t\} \rangle$. To see this, let

- T be the set of all elements of order 2 in S, and
- $S' = \{t\} \cup \{\, uv \mid u, v \in T, u \neq v \,\} \cup (S \setminus T)$.

It is easy to see that φ is a colour-preserving automorphism of the connected Cayley graph $\operatorname{Cay}(G; S')$, and that $G = \langle S \setminus \{t\} \rangle \langle t \rangle$. This establishes the claims.

From Theorem 6.8 (and the fact that $|H|$ is odd), we know that $\varphi|_H$ is affine. By composing with a left translation, we may assume that φ fixes e. Then $\varphi|_H$ is a group automorphism. By composing with an automorphism of $\mathbb{Z}_k \rtimes (\mathbb{Z}_m \times \mathbb{Z}_2)$ of the form $(x, y, z) \mapsto (x^r, y, z)$, we may assume $\varphi|_{\mathbb{Z}_k}$ is the identity map. Also, since $\varphi(s) \in \{s^{\pm 1}\}$ for every $s \in S$, and $|H/\mathbb{Z}_k| = m$ is odd, Lemma 6.12 implies that φ also fixes every element of $(S \cap H) \setminus \mathbb{Z}_k$. Hence, $\varphi|_H$ is an automorphism that fixes every element of a generating set, so $\varphi(h) = h$ for every $h \in H$. Since $\varphi(ht) = \varphi(h)\, t = ht$, for all $h \in H$ (because φ is colour-preserving and $t = t^{-1}$), we conclude that φ fixes every element of G, and is therefore affine, as desired.

($1 \Rightarrow 2$) Obvious.

($2 \Rightarrow 1$) Assume there is a group G of order n that is not strongly CCA, but n is not divisible by 4, 21, or a number of the form $p^q \cdot q$. Let φ be a colour-permuting automorphism of some connected Cayley graph $\operatorname{Cay}(G; S)$, such that $\varphi(e) = e$.

As in the proof of ($1 \Rightarrow 3$) above, we see that we may assume $|G|$ is square-free, and we may write $G = \mathbb{Z}_k \rtimes (\mathbb{Z}_m \times \mathbb{Z}_2)$, where $\mathbb{Z}_m \times \mathbb{Z}_2$ acts faithfully on $\mathbb{Z}_k$. Let $H = \mathbb{Z}_k \rtimes \mathbb{Z}_m$ be the (unique) subgroup of index 2 in G. From ($1 \Rightarrow 3$) above, we know that G is CCA, so $\widehat{G} \triangleleft \mathcal{A}^0$. Hence, $\widehat{H} \blacktriangleleft \mathcal{A}_0$ (since it is the unique largest normal subgroup of odd order), so φ normalizes $\widehat{H}$. This implies that the restriction of φ to H is an automorphism of H.

For each $s \in S$, let $\widetilde{s} = \varphi(s) \in S$. To prove that φ is affine, it suffices to show $\varphi(xs) = \varphi(x)\, \widetilde{s}$ for all $x \in G$ and $s \in S$ (see Remark 1.6(4)). If this is not the case, then, since φ is colour-permuting, there must be some x, such that $\varphi(xs) = \varphi(x)\, \widetilde{s}^{-1}$ (and $\widetilde{s}^{-1} \neq \widetilde{s}$, which means $|s| \neq 2$). This will lead to a contradiction.

We may assume every element of S has prime order (see Lemma 6.11). Since $|s| \neq 2$, this implies $s \in H$. Then, since $\varphi|_H$ is an automorphism, but

$$\varphi(xs) = \varphi(x)\, \widetilde{s}^{-1} = \varphi(x)\, \varphi(s)^{-1} \neq \varphi(x)\, \varphi(s),$$

we must have $x \notin H$. Since H has only two cosets, and there must be some element of S that is not in H, this implies that we may assume $x \in S$, after multiplying on the left by an appropriate element of H (and using the fact that φ normalizes $\widehat{H}$). Note that, since $x \notin H$, and every element of S has prime order, this implies $|x| = 2$. So the order of $\widetilde{x}$ is also 2, which implies $\widetilde{x} \notin H$ (since $|H|$ is odd).

Since φ is colour-permuting, we have

$$\varphi(xs) = \varphi({}^{x}s\, x) = \varphi({}^{x}s)\widetilde{x}.$$

Also, by the choice of x and s, we have

$$\varphi(xs) = \varphi(x)\, \widetilde{s}^{-1} = \widetilde{x}\, \widetilde{s}^{-1}.$$

Therefore

$$\varphi({}^{x}s) = {}^{\widetilde{x}}\widetilde{s}^{-1}.$$

Since $\mathbb{Z}_m$ acts faithfully on $\mathbb{Z}_k$, we have $\alpha(h) \equiv h \pmod{\mathbb{Z}_k}$, for every automorphism α of H (see Lemma 6.12). Since φ and conjugation by x are automorphisms of H, this implies $s \equiv s^{-1} \pmod{\mathbb{Z}_k}$. Since $|s|$ is odd, we conclude that $s \in \mathbb{Z}_k$.

Then, since the automorphism group of a cyclic group is abelian, we have

$$\varphi({}^{x}s) = {}^{x}\varphi(s) = {}^{x}\widetilde{s},$$

so $x^{-1}\widetilde{x}$ must invert $\widetilde{s}$. But this is impossible, because, as was mentioned above, x and $\widetilde{x}$, being of order 2, cannot be in H, so they are both in the other coset of H, so $x^{-1}\widetilde{x} \in H$ has odd order. This contradiction completes the proof that φ is affine. □

Remark 6.14. It is not necessary to assume p is odd in the statement of Corollary 6.13(3), because $2^q \cdot q$ is divisible by 4, which is already in the list of divisors.

Theorem 6.8 implies that very few small groups of odd order fail to be strongly CCA:

Corollary 6.15. *Let $G_{21} = \mathbb{Z}_7 \rtimes \mathbb{Z}_3$ be the* (*unique*) *nonabelian group of order* 21*. Then the only groups of odd order less than* 100 *that are not strongly CCA are G_{21}, $G_{21} \times \mathbb{Z}_3$, and $\mathbb{Z}_3 \wr \mathbb{Z}_3$.*

Proof. Suppose G is a group of odd order, such that G is not strongly CCA and $|G| < 100$. From Corollary 6.13, we see that $|G|$ is divisible by either 21 or $3^3 \cdot 3 = 81$. Since $|G| < 100$, this implies that $|G|$ is either 21, $21 \times 3 = 63$, or $3^3 \cdot 3 = 81$. Also, G must be nonabelian (see Corollary 4.2).

- The nonabelian group G_{21} of order 21 is not CCA (see Example 2.3).
- There are two nonabelian groups of order 63. One of them, the direct product $G_{21} \times \mathbb{Z}_3$, is not CCA (see Proposition 3.1).
- Theorem 6.8 implies that $\mathbb{Z}_3 \wr \mathbb{Z}_3$ is the only non-CCA group of order 81 (see also Example 2.4).

To complete the proof, we sketch a verification that the following group of order 63 is CCA:

$$G = \mathbb{Z}_7 \rtimes \mathbb{Z}_9 = \langle\, x, a \mid x^7 = a^9 = e,\ a^{-1}xa = x^2 \,\rangle.$$

Let φ be a colour-preserving automorphism of a connected Cayley graph $\mathrm{Cay}(G; S)$, such that $\varphi(e) = e$. We may assume S is either $\{a^{\pm 1}, x^{\pm 1}\}$ or $\{a^{\pm 1}, (ax)^{\pm 1}\}$, after discarding redundant generators, applying an automorphism of G, and replacing some elements by appropriate powers (cf. the proof of Lemma 6.11).

If $S = \{a^{\pm 1}, x^{\pm 1}\}$, then we may assume $\varphi(x) = x$, by composing with an automorphism of G. Also, since φ is colour-preserving, it must pass to a well-defined automorphism of the cycle $\mathrm{Cay}\big(G/\langle x\rangle; \{a^{\pm 1}\}\big)$, so there exists $\epsilon \in \{\pm 1\}$, such that $\varphi(ga) = \varphi(g)\, a^{\epsilon}$ for all $g \in G$. Then, since $(1, 1)$ is the only pair $(\epsilon, \delta) \in \{\pm 1\}^2$ that satisfies $a^{-\epsilon} x^{\delta} a^{\epsilon} = x^2$, we see that $\varphi(x^i a^j) = x^i a^j$ for all i and j, so φ is the identity map, which is certainly affine.

Assume, now, that $S = \{a^{\pm 1}, (ax)^{\pm 1}\}$. Let $a_1 = a$ and $a_2 = xa$. For any $g \in G$ and $\epsilon \in \{\pm 1\}$, if $\varphi(g\, a_1) = \varphi(g)\, a_1^{\epsilon}$, then, since $a_1^3 = a_2^3$ (and φ is colour-preserving), we have $\varphi(g\, s^m) = \varphi(g)\, s^{\epsilon m}$, for all m and all $s \in S$. Since S generates G, this implies $\varphi(gs) = \varphi(g)\, s^{\epsilon}$ for all g and all $s \in S$. So φ is affine. □

7 Groups of small order

In this section, we briefly explain which groups of order less than 32 are CCA (or strongly CCA). First, note that almost all of the abelian ones are strongly CCA:

Proposition 7.1 (cf. Proposition 4.1)**.** *An abelian group of order less than* 32 *is not strongly CCA if and only if it is either*

- $\mathbb{Z}_2 \times \mathbb{Z}_4$ (*of order* 8),
- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ (*of order* 16)*, or*
- $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$ (*of order* 24).

None of these are CCA.

Also note that almost all of the groups whose order is not divisible by 4 are CCA:

Proposition 7.2. *The only groups that are not strongly CCA, and whose order is* < 32 *and not divisible by* 4 *are:*

- *the wreath product* $\mathbb{Z}_3 \wr \mathbb{Z}_2$*, which is isomorphic to* $D_6 \times \mathbb{Z}_3$ *and has order* 18*, and*
- *the nonabelian group of order* 21.

Neither of these is CCA.

Proof. For the groups of odd order, the conclusion is immediate from Theorem 6.8 and Example 2.3 (see Corollary 6.15 for a stronger result). Proposition 3.2 deals with the groups $D_6 \times \mathbb{Z}_5$ and $D_{10} \times \mathbb{Z}_3$ of order 30. For all of the other groups of even order, it suffices to note that if m is odd, then every generalized dihedral group of order $2m$ is strongly CCA (see Proposition 5.6). □

So it is surprising that very few of the remaining groups are strongly CCA:

Proposition 7.3. *The only nonabelian groups that are strongly CCA and whose order is* < 32 *and divisible by* 4 *are:*

- *the dihedral groups of order* 8, 16, *and* 24,
- *the alternating group* A_4*, which is of order* 12,
- *another group of order 16, namely, the semidirect product* $\mathbb{Z}_8 \rtimes \mathbb{Z}_2$ *in which* $a^{-1}xa = x^5$ *for* $x \in \mathbb{Z}_8$ *and* $\langle a \rangle = \mathbb{Z}_2$*, and*
- *three additional groups groups of order 24, namely,* $D_8 \times \mathbb{Z}_3$, $A_4 \times \mathbb{Z}_2$*, and the semidirect product* $\mathbb{Z}_3 \rtimes \mathbb{Z}_8$ *in which* $\mathbb{Z}_8$ *inverts* $\mathbb{Z}_3$.

Furthermore, the only groups of order < 32 *that are CCA, but not strongly CCA, are:*

- *the dihedral groups* D_{12}, D_{20}*, and* D_{28}*, and*
- *the group* $D_{12} \times \mathbb{Z}_2$*, which is a generalized dihedral group of order 24.*

Sketch of proof. The result can be verified by an exhaustive computer search, but we summarize a case-by-case analysis that can be carried out by hand, using the classification of groups of order less than 32. Each group of such small order can be specified by its "GAP Id," which is an ordered pair $[n, k]$, where n is the order of the group, and k is the id number that has been assigned to that particular group (see [5], for example).

Assume G is nonabelian, $|G| < 32$, and $|G|$ is divisible by 4. We may assume that G is neither generalized dicyclic, semidihedral, nor generalized dihedral, for otherwise Corollary 2.8(4,5) and Propositions 5.3 and 5.6 determine whether G is CCA or strongly CCA. By inspection of the list of groups of each order, we see that this leaves only thirteen possibilities for G, and we consider each of these GAP Ids separately. In most cases, Proposition 2.5 implies that G is not CCA.

$[12,3]$ $= A_4$. This group is strongly CCA (see Example 7.5 below).

$[16,3]$ $= \langle\, a,b,c \mid a^4 = b^2 = c^2 = e, ab = ba, bc = cb, cac = ab\,\rangle$. Proposition 2.5 applies with $S = \{a^{\pm 1}, c\}$, $T = \{a^{\pm 1}\}$, and $\tau = a^2 \in Z(G)$.

$[16,6]$ $= \langle\, a,x \mid a^8 = x^2 = e, xax = a^5\,\rangle = \langle a\rangle \rtimes \langle x\rangle = \mathbb{Z}_8 \rtimes \mathbb{Z}_2$. This group is strongly CCA (see Example 7.5 below).

$[16,13]$ $= \langle\, a,x,y \mid a^4 = x^2 = e, a^2 = y^2, xax = a^{-1}, ay = ya, xy = yx\,\rangle$. Proposition 2.5 applies with $S = \{a^{\pm 1}, x, y^{\pm 1}\}$, $T = \{a^{\pm 1}, y^{\pm 1}\}$, and $\tau = a^2 \in Z(G)$.

$[20,3]$ $= \langle\, a,b \mid a^5 = b^4 = e,\ bab^{-1} = a^2\,\rangle$. Proposition 2.5 applies with $S = \{a^{\pm 1}, b^{\pm 1}\}$, $T = \{b^{\pm 1}\}$, and $\tau = b^2$ (which inverts a).

$[24,1]$ $= \mathbb{Z}_3 \rtimes \mathbb{Z}_8$, where $\mathbb{Z}_8$ inverts $\mathbb{Z}_3$. This group is strongly CCA (see Example 7.5 below).

$[24,3]$ $= \mathrm{SL}(2,3) \cong Q_8 \rtimes \mathbb{Z}_3 = \langle i,j\rangle \rtimes \langle a\rangle$, where $aia^{-1} = j$ and $a^{-1}ia = ij$. Proposition 2.5 applies with $S = \{i^{\pm 1}, a^{\pm 1}\}$, $T = \{i^{\pm 1}\}$, and $\tau = i^2 \in Z(G)$.

$[24,5]$ $= S_3 \times \mathbb{Z}_4$. Proposition 2.5 applies with $T = \{(1,2)\} \times \{\pm 1\}$, $S = \big\{\big((2,3),0\big)\big\} \cup T$, and $\tau = (e,2) \in Z(G)$.

$[24,8]$ $= \mathbb{Z}_3 \rtimes D_8 = \langle\, a,b,c \mid a^3 = b^4 = c^2 = e,\ bab^{-1} = a^{-1},\ ac = ca, cbc^{-1} = b^{-1}\,\rangle$. Proposition 2.5 applies with $S = \{(ab)^{\pm 1}, b^{\pm 1}, c\}$, $T = \{(ab)^{\pm 1}, b^{\pm 1}\}$, and $\tau = b^2 \in Z(G)$.

$[24,10]$ $= D_8 \times \mathbb{Z}_3$. Since D_8 is strongly CCA (see Proposition 5.6), the same is true for this group (see Proposition 3.2).

$[24,11]$ $= Q_8 \times \mathbb{Z}_3$. This is not CCA, since Q_8 is not CCA (see Corollary 2.8(3) and Proposition 3.1).

$[24,12]$ $= S_4$. Let $a = (1,2,3,4)$ and $b = (1,2,4,3)$, so Proposition 2.5 applies, with $S = \{a^{\pm 1}, b^{\pm 1}\}$, $T = \{a^{\pm 1}\}$, and $\tau = a^2 = (1,3)(2,4)$, which inverts b.

$[24,13]$ $= A_4 \times \mathbb{Z}_2$. This group is strongly CCA (see Example 7.5 below). □

The following simple observation plays a key role in the proof of Example 7.5.

Lemma 7.4. *Let*

- φ *be a colour-permuting automorphism of a Cayley graph* $\mathrm{Cay}(G;S)$*, such that* $\varphi(e) = e$*,*

- $\widetilde{a} = \varphi(a)$ *and* $\widetilde{b} = \varphi(b)$*, for some* $a, b \in S$*,*
- $\tau(v) \in \{\pm 1\}$*, such that* $\varphi(va) = \varphi(v)\, \widetilde{a}^{\tau(v)}$*, for all* $v \in G$*, and*
- $k_1, k_2, \ldots, k_{2r} \in \mathbb{Z} \smallsetminus \{0\}$*, such that* $a^{k_1} b^{k_2} a^{k_3} \cdots b^{k_{2r}} = e$ *(and* $r \geq 2$*).*

If $\epsilon_1 = \epsilon_3$ *and* $\epsilon_2 = \epsilon_4$*, for all* $\epsilon_1, \ldots, \epsilon_{2r} \in \{\pm 1\}$*, such that* $\widetilde{a}^{\epsilon_1 k_1} \widetilde{b}^{\epsilon_2 k_2} \cdots \widetilde{b}^{\epsilon_{2r} k_{2r}} = e$*, then* $\varphi(va) = \varphi(v)\, \widetilde{a}$ *and* $\varphi(vb) = \varphi(v)\, \widetilde{b}$*, for all* $v \in \langle a^{k_3}, b^{k_2} \rangle$*.*

Proof. Since φ is colour-permuting, there exist $\sigma, \tau \colon G \to \{\pm 1\}$, such that

$$\varphi(va) = \varphi(v)\, \widetilde{a}^{\sigma(v)} \text{ and } \varphi(va) = \varphi(v)\, \widetilde{b}^{\tau(v)} \text{ for all } v \in G.$$

We wish to show $\sigma(v) = \tau(v) = 1$ for all $v \in \langle a^{k_3}, b^{k_2} \rangle$. Since $\sigma(e) = \tau(e) = 1$, it suffices to show that $\sigma(vb^{k_2}) = \tau(va^{k_3}) = \tau(v)$ for all $v \in G$.

The two parts of the proof are very similar, so we show only that $\sigma(vb^{k_2}) = \sigma(v)$. The relation $a^{k_1} b^{k_2} a^{k_3} \cdots b^{k_{2r}} = e$ represents a closed walk starting at v (or at any other desired vertex). Applying φ yields a closed walk starting at $\varphi(v)$. Since φ is colour-permuting, this closed walk corresponds to a relation of the form $\widetilde{a}^{\epsilon_1 k_1} \widetilde{b}^{\epsilon_2 k_2} \cdots \widetilde{b}^{\epsilon_{2r} k_{2r}} = e$, with $\epsilon_i \in \{\pm 1\}$. By assumption, we must have $\epsilon_1 = \epsilon_3$. Therefore

$$\sigma(a^{k_1} b^{k_2}) = \epsilon_3 = \epsilon_1 = \sigma(v).$$

This establishes the desired conclusion, since $\sigma(v) = \sigma(va^{k_1})$, and va^{k_1} is an arbitrary element of G. □

Example 7.5. The groups $[12, 3]$, $[16, 6]$, $[24, 1]$, and $[24, 13]$ from the proof of Proposition 7.3 are strongly CCA.

Proof. We consider each of the four groups individually; for convenience, let G be the group under consideration. Suppose φ is a colour-permuting automorphism of a connected Cayley graph $\mathrm{Cay}(G; S)$, such that $\varphi(e) = e$, and let $\widetilde{s} = \varphi(s)$, for each $s \in S$. We wish to show $\varphi \in \mathrm{Aut}\, G$.

Assume $G = [12, 3]$. Let $a \in S$ with $|a| = 3$, and let N be the (unique) subgroup of order 4 in G.

Assume, for the moment, that there exists $b \in S \cap N$ (so $|b| = 2$). Then $(ab)^3 = e$. Suppose $i, j, k \in \{\pm 1\}$, with

$$e = \widetilde{a}^i\, \widetilde{b}\, \widetilde{a}^j\, \widetilde{b}\, \widetilde{a}^k\, \widetilde{b} \equiv \widetilde{a}^{i+j+k} \pmod{N},$$

so $i+j+k \equiv 0 \pmod 3$. Since $i, j, k \in \{\pm 1\}$, this implies $i = j = k$. We conclude from Lemma 7.4 that $\varphi(vs) = \varphi(v)\, \widetilde{s}$, for all $v \in \langle a, b \rangle = G$ and $s \in \{a^{\pm 1}, b\}$, so $\varphi \in \mathrm{Aut}\, G$.

We may now assume $|s| = 3$ for all $s \in S$. Let $b \in S \smallsetminus \langle a \rangle$. We may assume $a \equiv b \pmod{N}$, by replacing b with its inverse if necessary. Write $\widetilde{b} = \widetilde{a}^r x$, with $r \in \{\pm 1\}$ and $x \in N$. Note that $(a^{-1}b)^2 = e$. Suppose $i, j, k, \ell \in \{\pm 1\}$, with

$$e = \widetilde{a}^{-i}\, \widetilde{b}^j\, \widetilde{a}^{-k}\, \widetilde{b}^\ell = \widetilde{a}^{-i+rj-k+r\ell} \cdot \begin{cases} (\widetilde{a}^{k-r\ell} x \widetilde{a}^{-k+r\ell})x & \text{if } j = \ell = 1, \\ (\widetilde{a}^k x \widetilde{a}^{-k})x & \text{if } j = -1 \text{ and } \ell = 1, \\ \widetilde{a}^{-r\ell}\, (\widetilde{a}^k x \widetilde{a}^{-k}) x\, \widetilde{a}^{r\ell} & \text{if } j = 1 \text{ and } \ell = -1, \\ \widetilde{a}^{-r\ell}\, (\widetilde{a}^{k-rj} x \widetilde{a}^{-k+rj}) x\, \widetilde{a}^{r\ell} & \text{if } j = \ell = -1. \end{cases}$$

Since the component in N must be trivial, and no nontrivial power of $\widetilde{a}$ centralizes x, we see that we must have $j = \ell$ and $k = rj = r\ell$. Then, since the exponent of $\widetilde{a}$ must be 0, this implies $i = k$. We conclude from Lemma 7.4 that $\varphi(vs) = \varphi(v)\,\widetilde{s}$, for all $v \in \langle a, b\rangle = G$ and $s \in \{a^{\pm 1}, b^{\pm 1}\}$, so $\varphi \in \operatorname{Aut} G$.

Assume $G = [16, 6]$. Let $a \in S$ with $|a| = 8$. Let $b \in S \setminus \langle a\rangle$.

Assume, for the moment, that $|b| = 8$. Write $b^2 = a^{2r}$, for some odd r. Then we must have $\widetilde{b}^2 = \widetilde{a}^{2r}$. This implies that if $i, j \in \{\pm 1\}$, such that

$$e = \widetilde{b}^{2i}\,\widetilde{a}^{-2rj},$$

then $i = j$ (since $|\widetilde{b}^2| = |b^2| > 2$). We conclude (much as in Lemma 7.4) that $\varphi(vs) = \varphi(v)\,\widetilde{s}$, for all $v \in \langle a, b\rangle = G$ and $s \in \{a^{\pm 1}, b^{\pm 1}\}$, so $\varphi \in \operatorname{Aut} G$.

We may now assume $|b| \in \{2, 4\}$, so $b^2 \in \langle a^4\rangle$. Note that, since $b \notin \langle a\rangle$, we have $bab^{-1}a^3 = e$. Suppose $i, j, k, \ell \in \{\pm 1\}$, with

$$e = \widetilde{b}^i\,\widetilde{a}^j\,\widetilde{b}^{-k}\,\widetilde{a}^{3\ell} = \widetilde{b}^{i-k}\,\widetilde{a}^{5j+3\ell} \equiv a^{j-\ell} \pmod{\langle \widetilde{a}^4\rangle},$$

so $j = \ell$. Then we must also have $i = k$. We conclude from Lemma 7.4 that $\varphi(vs) = \varphi(v)\,\widetilde{s}$, for all $v \in \langle a, b\rangle = G$ and $s \in \{a^{\pm 1}, b^{\pm 1}\}$, so $\varphi \in \operatorname{Aut} G$.

Assume $G = [24, 1]$. Let $a \in S$ with $|a| = 8$, and let $b \in S$, such that $b \notin \langle a\rangle$. Write $\widetilde{b} = \widetilde{a}^r x$, where $\langle x\rangle = \mathbb{Z}_3$. We may assume $\widetilde{b}$ has prime-power order (see Lemma 6.11), and we know that $\widetilde{a}^2$ centralizes x, so either r is odd or $r = 0$.

Assume, for the moment, that $r = 0$, which means $\langle \widetilde{b}\rangle = \mathbb{Z}_3 = \langle b\rangle$. Then a inverts b, so $aba^{-1}b = e$. Suppose $i, j, k, \ell \in \{\pm 1\}$, with

$$e = \widetilde{a}^i\,\widetilde{b}^j\,\widetilde{a}^{-k}\,\widetilde{b}^\ell = \widetilde{a}^{i-k}\,\widetilde{b}^{-j+\ell}.$$

Since the exponents of $\widetilde{a}$ and $\widetilde{b}$ must be 0, we have $i = k$ and $j = \ell$. We conclude from Lemma 7.4 that $\varphi(vs) = \varphi(v)\,\widetilde{s}$, for all $v \in \langle a, b\rangle = G$ and $s \in \{a^{\pm 1}, b^{\pm 1}\}$, so $\varphi \in \operatorname{Aut} G$.

We may now assume that r is odd. The proof of Lemma 6.11 shows there is no harm in replacing b with a power that is relatively prime to 8, so we may assume $r = 1$. Since $a^2 \in Z(G)$, we have $a^2ba^{-2}b^{-1} = e$. Suppose $i, j, k, \ell \in \{\pm 1\}$, with

$$e = \widetilde{a}^{2i}\,\widetilde{b}^j\,\widetilde{a}^{-2k}\,\widetilde{b}^{-\ell} = \widetilde{a}^{2i-2k}\,\widetilde{b}^{j-\ell} \equiv \widetilde{a}^{j-\ell} \pmod{\langle \widetilde{a}^4, x\rangle}.$$

Then $j = \ell$. Therefore $\widetilde{a}^{2i-2k} = e$, so $i = k$. For $v \in G$ with $\varphi(va) = \varphi(v)\,\widetilde{a}$, we conclude from the proof of Lemma 7.4 that $\varphi(vba) = \varphi(vb)\,\widetilde{a}$. In addition, interchanging the roles of a and b tells us that if $\varphi(vb) = \varphi(v)\,\widetilde{b}$, then $\varphi(vab) = \varphi(va)\,\widetilde{b}$. We conclude that $\varphi(vs) = \varphi(v)\,\widetilde{s}$, for all $v \in \langle a, b\rangle = G$ and $s \in \{a^{\pm 1}, b^{\pm 1}\}$, so $\varphi \in \operatorname{Aut} G$.

Assume $G = [24, 13]$. We may assume $|s| \in \{2, 3\}$, for all $s \in S$ (see Lemma 6.11). Let $a \in S$ with $|a| = 3$. Choose $b \in S$, such that $b \notin A_4$. Since every element of order 3 is contained in A_4, we must have $|b| = 2$.

Assume, for the moment, that $\langle a, b\rangle = G$. Note that $(aba^{-1}b)^2 = e$, and, for convenience, let $\widetilde{b}_m = \widetilde{a}^{-m}\,\widetilde{b}\,\widetilde{a}^m$ for $m \in \mathbb{Z}$. Suppose $i, j, k, \ell \in \{\pm 1\}$, with

$$e = \widetilde{a}^i\,\widetilde{b}\,\widetilde{a}^{-j}\,\widetilde{b}\,\widetilde{a}^k\,\widetilde{b}\,\widetilde{a}^{-\ell}\,\widetilde{b} = \widetilde{a}^{i-j+k-\ell} \cdot \widetilde{b}_{-j+k-\ell}\,\widetilde{b}_{k-\ell}\,\widetilde{b}_{-\ell}\,\widetilde{b}.$$

This implies $k = \ell$, for otherwise 0, $-\ell$, and $k-\ell$ are all distinct modulo 3, so $\widetilde{b}_{k-\ell}\,\widetilde{b}_{-\ell}\,\widetilde{b} \equiv \widetilde{b}_1\,\widetilde{b}_{-1}\,\widetilde{b} \equiv e \pmod{\mathbb{Z}_2}$, but $b_{-j+k-\ell}$ is obviously nontrivial $(\mathrm{mod}\ \mathbb{Z}_2)$. (Then, since the exponent of $\widetilde{a}$ is 0, we must also have $i = j$.) We conclude from Lemma 7.4 that $\varphi(vs) = \varphi(v)\,\widetilde{s}$, for all $v \in \langle a, b\rangle = G$ and $s \in \{a^{\pm 1}, b\}$, so $\varphi \in \operatorname{Aut} G$.

We may now assume $\langle a, s\rangle \neq G$, for all $s \in S$. Then, since $b \notin A_4$ (and b is an element of order 2 in S), we see that $b \in Z(G)$. Since $Z(G)$ has only one nontrivial element, this implies that $S = (S \cap A_4) \cup \{b\}$, and that $\widetilde{b} = b$ (since only b-edges make 4-cycles with the edges of every other colour). Therefore

$$\operatorname{Cay}(G; S) \cong \operatorname{Cay}(A_4; S \cap A_4) \times \operatorname{Cay}\bigl(\mathbb{Z}_2; \{b\}\bigr),$$

and $\varphi(b) = b$. Since A_4 is strongly CCA, it is now easy to see that $\varphi \in \operatorname{Aut} G$. $\square$

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Geometric point-circle pentagonal geometries from Moore graphs

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Abstract

We construct isometric point-circle configurations on surfaces from uniform maps. This gives one geometric realisation in terms of points and circles of the Desargues configuration in the real projective plane, and three distinct geometric realisations of the pentagonal geometry with seven points on each line and seven lines through each point on three distinct dianalytic surfaces of genus 57. We also give a geometric realisation of the latter pentagonal geometry in terms of points and hyperspheres in 24 dimensional Euclidean space. From these, we also obtain geometric realisations in terms of points and circles (or hyperspheres) of pentagonal geometries with k circles (hyperspheres) through each point and $k-1$ points on each circle (hypersphere).

Keywords: Uniform map, equivelar map, dessin d'enfants, configuration of points and circles

Math. Subj. Class.: 05B30, 05B45, 14H57, 14N20, 30F10, 30F50, 51E26

1 Introduction

A compact Klein surface S is a surface (possibly with boundary and non-orientable) endowed with a dianalytic structure, that is, the transition maps are holomorphic or antiholomorphic (the conjugation $z \to \overline{z}$ is allowed). If the surface S admits analytic structure and is closed, then the surface is a Riemann surface. By the uniformization theorem each Klein surface is a quotient $S = U/G$, where U is either the Riemann sphere, the complex Euclidean plane or the hyperbolic plane, and G is a group without elliptic elements. In the case of surfaces without boundary the group G is torsion-free.

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The surface inherits the geometry of its universal covering space U through this quotient. Incidences between lines and circles in S follow the same axioms as in the covering space and geodesics on the surface come from lines in the covering space. In what follows, terms like line and circle will refer to such geometric objects, if not defined otherwise.

A *map* is a drawing of a graph on a surface such that the complement of the drawing is a disjoint union of topological discs called faces. So a map consists of a set of vertices, a set of edges and a set of faces. The genus of a map is the genus of the surface in which the graph is embedded, and can be calculated through the Euler characteristic using a generalization of Euler's polyhedron formula. Given a map with $|V|$ points, $|E|$ edges and $|F|$ faces, the Euler characteristic is $\chi = |V| - |E| + |F|$. The genus g of an orientable surface satisfies $\chi = 2 - 2g$ and the genus h of a non-orientable surface satisfies $\chi = 2 - h$. By considering the map as the lifting of the segment $[0, 1]$ in $\mathbb{C}$, the map determines the structure of the dianalytic surface. In general, a given surface allows different maps, and a given graph can be embedded as a map on different surfaces [18, 5, 4]. However, among the different maps of a graph there is one which has the largest Euler characteristic, then called the Euler characteristic of the graph. This map will have the smallest orientable or non-orientable genus of all maps of this graph, depending on whether it is orientable or not.

It makes sense to consider both the smallest orientable and the smallest non-orientable genus of the graph. For example, the orientable genus of a planar graph is the genus of the sphere, which is 0. This is also the orientable genus of the 1-skeleta of the Platonic surfaces. The Petersen graph is not planar and so it has orientable genus at least 1. Since it can be drawn without crossings on the torus, it has orientable genus exactly 1. The hemi-dodecahedron is the abstract polyhedron obtained by identifying antipodal points in the dodecahedron. The 1-skeleton of this polyhedron is the Petersen graph, defining a map of the Petersen graph in the real projective plane, so the non-orientable genus of the Petersen graph is 1.

The study of configurations in projective real or complex plane is a classical subject in geometry. Configurations appear naturally as arrangements of lines, planes or circles in a geometric plane or space. In contrast with the situation when graphs are realised as maps on surfaces, the requirement that there should be no crossings on the surface other than the incidences defined by the configuration is typically relaxed (although not always). For example, Hilbert and Cohn-Vossen [16] define a planar point-line configuration as follows.

"A plane configuration is a system of v points and b straight lines arranged in a plane in such a way that every point is incident with r lines and every line is incident with k points."

Note that it is not required that the b lines should meet only in the v points, only the incidences in the distinguished points are important. However, extra incidences *on* these points are often regarded as an anomaly.

For example, consider Desargues' Theorem, which is a theorem regarding the realisation of the configuration in Figure 1 in projective planes. In a projective plane every pair of lines intersect, therefore every pair of the 10 lines in the configuration in Figure 1 must meet at some point. Some of these points do not belong to the configuration. Similarly, there is of course a line between each pair of points, but some of these lines do not belong to the configuration. What makes it a configuration is the fact that in any of the 10 points there are $r = 3$ of the 10 lines intersecting, and on any of the 10 lines there are $k = 3$ of the 10 points. However, if it was drawn so that a 4th of the 10 points accidently were on an

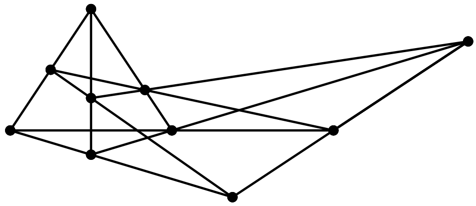

Figure 1: Desargues' Theorem: two triangles are perspective from a point if and only if they are perspective from a line.

extra line, then some lines would have 3 points and others would have 4 points, making the configuration degenerate. An (r, k)-*combinatorial configuration* is a set of incidences between two sets of v and b elements called points and lines respectively, defined in analogy with the planar and linear definition above, but without considering realisability in some geometric space; see for example [13, 22]. A combinatorial configuration is called *linear* if each pair of lines meet at most once. Linear combinatorial configurations are often simply called combinatorial configurations. Combinatorial configuration with $k = 2$ or $r = 2$ are graphs or their duals are graphs, respectively. Therefore it is typically required that $r \geq 3$ and $k \geq 3$.

A *pentagonal geometry* is a (linear) combinatorial configuration in which, for any point p, all points that are not collinear with p are on a single line, which is called the opposite line of p. A pentagonal geometry has *order* (k, r) if there are r lines through each point and k points on each line [1]. There are two classes of lines in a pentagonal geometry, lines that are the opposite line of some point, and lines that are not. A pentagonal geometry with no non-opposite lines is self-polar by the polarity that associates each point with its opposite line. The deficiency graph (P, E) of a configuration is a graph with vertex set P, consisting of the points of the configuration, and edge set E, consisting of all the pairs (p, q) such that the points $p, q \in P$ are not collinear. In a pentagonal geometry, for each $p \in P$ the opposite line of p is formed by the points that are neighbours to p in the deficiency graph. When $r = k$, the number of points equals the number of lines, so there are no non-opposite lines, and all lines are defined by the neighbourhood of some point in the deficiency graph. Given the deficiency graph it is then possible to construct the pentagonal geometry by drawing a (combinatorial) line through the neighbourhood of each point in the deficiency graph. This construction of pentagonal geometries was described in [1], where it also was proved that pentagonal geometries with $r = k$ are exactly the ones with a Moore graph (of diameter two) as deficiency graph.

There are only three known Moore graphs of diameter 2: the cycle graph of length 5, the Petersen graph, and the Hoffman-Singleton graph. These graphs have valency 2, 3 and 7, respectively. They are unique for their valencies [17]. The existence of a Moore graph of valency 57 is still an open question. The pentagonal geometries obtained from these graphs are, respectively, the ordinary pentagon, the Desargues configuration (Figure 1) and a pentagonal geometry with parameters $(7, 7)$ and with 50 points and 50 lines. In [1], it was also proved that all pentagonal configurations of order $(k, k + 1)$ can be constructed from pentagonal geometries of order $(k+1, k+1)$ through the removal of one point and its opposite line. There are therefore at most three such pentagonal geometries, with $k = 2, 6$

and maybe 56.

The construction of pentagonal geometries from a graph with a combinatorial line through the neighbourhood of each vertex can also be used to construct other configurations. Indeed, the same construction works for any graph with the property that any two vertices have at most one common neighbour. In other words, the graph should be without cycles of length 4. This construction seems to appear first in an article by Lefèvre-Percsy, Percsy and Leemans, as the neighbourhood geometry (of rank 2) of a graph [20], and later, in the context of geometric realisations of configurations in articles by Gévay and Pisanski [9, 10]. If the graph can be drawn in the real Euclidean plane in such a way that a circle can be traced through the neighbours of each point, then the drawing and the construction together give rise to a geometric point-circle configuration in the real Euclidean plane [10]. For example, any 3-regular graph has this property, defining a point-circle $(3, 3)$-configuration in the real Euclidean plane. Also unit-distance graphs can be used for the same purpose. Indeed, a circle is defined as the collection of points at a given distance from the center of the circle. As an extra feature, a unit-distance graph gives an isometric point-circle configuration, in which all circles have the same radius.

As was observed in [9], a point-plane configuration in real Euclidean 3-space, constructed through a similar construction from the 1-skeleton of a 3-polytope, defines a point-circle configuration in the real Euclidean plane through stereographic projection whenever the points in each plane are concyclic. In particular, it was proved in [9] that any Platonic or Archimedean solid gives a point-circle configuration on the Riemann sphere, and that the circle-preserving property of the stereographic projection implies that any point-circle configuration drawn on the sphere can also be drawn in the real Euclidean plane.

In this article we will generalize this construction on the sphere to surfaces in general. This construction is motivated by the study of geometric realisations of pentagonal geometries.

2 Constructing configurations of points and isometric circles on surfaces

The geometric construction of Gévay and Pisanski described above does not require an embedding without crossings of the graph. Rather, what the construction requires from the graph embedding is that the neighbours of each vertex are concyclic [10]. On the sphere, any circle is a planar section, so any point-circle configuration gives a point-plane configuration in 3-space. Since more than 3 points in a plane are not necessarily concyclic, the converse is not true in general when $k > 3$.

A nice way of making neighbours concyclic is to mimic the idea of using a map of the 1-skeleton of a convex polytope. A regular tiling (p, q) of the universal covering space U of a Riemann surface is a collection of congruent polygons which partitions and fills up the entire space, in such a way that p q-gons meet at each vertex. The stabilizer of this tiling is a subgroup of a triangle group $\Gamma(p, 2, q)$. Since the polygons are congruent, the neighbours of each vertex are concyclic on isometric circles. The distance is the spherical, the Euclidean or the hyperbolic distance respectively.

Definition 2.1. A *uniform map of type* (p, q) on a Riemann surface with universal covering space U is the quotient of a regular tiling of U of type (p, q) by the action of a torsion-free group $G \subseteq \Gamma(p, 2, q)$.

This terminology comes from the theory of dessin d'enfants [12, 18, 26], where also the term *uniform dessin d'enfants* is used. In the theory of tilings and polytopes the word *uniform map* instead refers to a map with an automorphism group acting transitively on the vertices. In the literature of tilings and polytopes, our uniform maps are instead known as *equivelar maps*. In particular, our uniform maps are not necessarily vertex transitive.

In a uniform map of type (p, q) the vertices have valency p, the edges have valency 2, and the faces have valency q. Any map with this property is a uniform map of type (p, q). A map is *regular* if its automorphism group acts transitively on triples of incident vertices, edges and faces, that is, on the flags. This implies that a regular map is always uniform.

Isometric circles through the neighbours of each vertex of a regular tiling of U will be mapped to isometric circles through the neighbours of each vertex of the corresponding uniform map on U/G. Since each circle contains p points and p circles goes through each point, this construction gives a configuration of points and circles on the surface, and we have proved the following.

Theorem 2.2. *A uniform map on a surface produces a configuration of points and isometric circles on the same surface.*

On the sphere, this construction gives a configuration of points and isometric circles which can be taken to a configuration of points and non-isometric circles on the Euclidean plane through stereographic projection from a suitable point.

The uniform maps on the sphere are regular. Consequently, there are two infinite families of uniform maps of the sphere, the hosohedra of type $(n, 2)$ for $n \geq 1$, consisting of n digons meeting at two antipodal vertices, and the dihedra of type $(2, n)$ for $n \geq 1$, consisting of two n-gons meeting at n vertices along a meridian. The result from applying Theorem 2.2 to a hosohedron is a degenerate configuration consisting of two points and two circles of radius zero, each point occuring with multiplicity n on one of the circles. By instead using a dihedron one obtains a configuration of n points and n circles with two points on each circle. This configuration is connected if n is odd, otherwise the configuration consist of two disconnected components.

A part from the two infinite families just described, which result in configurations of limited interest, there are only five more uniform maps on the sphere, corresponding to the Platonic solids. Of the resulting configurations, there is only one which is linear when regarded as a combinatorial configuration.

Theorem 2.3. *The only linear point-circle configuration (with $r > 2$ and $k > 2$) coming from a uniform map on the Riemann sphere is the $(20_3, 20_3)$-configuration on Figures 1, 2, 3 in [10], obtained from the dodecahedron projected on the sphere. In the real projective plane, the only linear point-circle configuration coming from a uniform map is the Desargues configuration, obtained from the hemidodecahedron.*

Proof. The uniform maps of type (p, q) on the sphere satisfying $p > 2$ and $q > 2$ are the Platonic solids. Gévay and Pisanski constructed point-plane configurations from all Platonic (and Archimedean) solids except the octahedron in [10]. The octahedron has the property that the planes through the neighbours of two antipodal vertices coincide. They also proved that their construction gives a combinatorial point-line configuration (i.e. in which any two combinatorial lines share at most one point) only if the graph does not have cycles of length 4. The only Platonic solid graph without cycles of length 4 is the dodecahedron graph. The uniform maps in the real projective plane are obtained from the uniform maps on the sphere by identifying antipodal points. As we pointed out in the introduction, the hemi-dodecahedron is obtained from the dodecahedron in this way. □

The sphere has finite area, implying that each uniform tiling has a finite number of tiles. Hence the automorphism group of the tiling is finite and has a finite number of subgroups. Therefore the finite number of regular tilings with $p, q \geq 3$ of the sphere gives a finite number of uniform maps. The situation is different in the Euclidean and the hyperbolic plane. The Euclidean plane has a finite number of regular tilings (of types $(6, 3)$, $(3, 6)$ and $(4, 4)$), but here the area is infinite, resulting in infinitely many uniform maps. The hyperbolic plane has infinite area and there are infinitely many regular tilings, and consequently infinitely many uniform maps.

In comparison with the situation in the Euclidean plane, where it is possible to construct isometric point-circle configurations without starting with a planar unit-distance embedding of the graph, it is clear that in general it is not necessary to require the graph to be embedded as a uniform map on the surface. Isometric point-circle configurations can in some cases be obtained using other embeddings (non-uniform, non-congruent, with crossings) of the graph on the surface. However, in this article we focus on point-circle configurations coming from uniform maps, more precisely, on those coming from uniform pentagonal maps of Moore graphs of diameter 2.

3 Geometric pentagonal geometries

Here (in three subsections) we discuss different geometric realisations of pentagonal geometries, with focus on embeddings in Riemann surfaces.

3.1 The ordinary pentagon

The ordinary pentagon is the smallest non-degenerate pentagonal geometry. Its deficiency graph is the cycle graph on 5 vertices. This graph can also be seen as a point-line realisation of the configuration itself. The ordinary pentagon can also be constructed as a point-circle configuration with two points on each circle from its deficiency graph using the geometric construction by Gévay and Pisanski. So it can be argued that any point-line realisation of the ordinary pentagon produces a point-circle realisation of the same.

The cycle graph on 5 vertices has diameter 2 and girth 5, as do all Moore graphs (of diameter 2). The smallest number of edges in any face of an embedding of this graph on a surface is therefore 5. For example, it can be embedded in the Riemann sphere as a pentagonal cycle along one of the geodesics. This map has 5 vertices, 5 edges and 2 faces and so the orientable genus is 0. We call it a pentagonal map, meaning simply that all faces have 5 vertices. By introducing one new vertex on the midpoint of each edge of this pentagonal spherical map, and identifying antipodal points in the resulting decagonal map one obtains a non-orientable pentagonal map with 1 face in the real projective plane, so the non-orientable genus is 1. So the ordinary pentagon can be realised as a configuration of points and circles on the Riemann sphere (and consequently in the Euclidean plane), and in the real projective plane.

3.2 The Desargues configuration

The Desargues configuration is a $(3, 3)$-configuration on 10 points and 10 lines. It is the pentagonal geometry with the Petersen graph, the 3-regular Moore graph, as deficiency graph. The polarity of the Desargues configuration is known as the von Staudt polarity [28](cf. [8]). Figure 1 shows a classical drawing of Desargues configuration in the real plane as the 10 points and 10 lines of Desargues' Theorem. There are plenty of geometric realisations of the Desargues configuration in terms of incidences of points and lines. Indeed, finite projective planes over finite fields are called Desarguesian since they admit the Desargues configuration as points and lines.

The automorphism group of the (combinatorial) Desargues configuration is S_5, the symmetric group acting on a set of five elements. When a configuration is realised geometrically, the automorphism group of the realisation is a subgroup of the automorphism group of the combinatorial configuration. Geometric realisations of the Desargues configuration were studied by Coxeter in [8], where he showed how to realise subgroups of S_5 as collineations of certain embeddings of the Desargues configuration in some geometric space. Among his collection of geometric realisations of the Desargues configuration, there are two which have the full automorphism group S_5. The first is due to Edge, who proved that in $PG(2,5)$, the interior points of a conic, together with the lines that are neither tangents nor secants to the same conic, form a Desargues configuration. The second is an embedding of the Desargues configuration on a non-orientable surface of Euler characteristic -5. This embedding arises from a regular map of the Menger graph (collinearity graph) of the configuration on the surface, with automorphism group S_5. Coxeter observed that the 30 edges in this regular map are situated on 10 geodesics of the surface in such a way that the vertices of the map together with the 10 geodesics form a Desargues configuration of points and lines on the surface, which also has automorphism group S_5.

We saw in the introduction that any $(3,3)$-configuration can be realised as a configuration of points and circles in the Euclidean plane using Gévay and Pisanski's geometric spherical construction and an embedding of some 3-regular graph [10]. In particular this is true for the Desargues configuration, using an embedding of the Petersen graph. Gévay and Pisanski also showed how to make the circles isometric. Unit-distance embeddings of the graph always produce isometric circles, but some embeddings with edges of different lengths also work. They provided two examples of the Desargues configuration as a configuration of points and isometric circles in the real Euclidean plane, coming from a unit-distance and a non-unit-distance embedding of the Petersen graph, respectively. The automorphism group of these realisations are the cyclic group C_5 and the dihedral group D_5 [10].

We show now that the Desargues configuration also can be drawn as a point-circle configuration in the real projective plane from a pentagonal map of the Petersen graph. Indeed, the (Riemann) spherical $(3,3)$-configuration on 20 points and 20 circles constructed from the dodecahedron in [10] is the double cover of a $(3,3)$-configuration on 10 points and 10 circles in the real projective plane which can be constructed analogously from the hemi-dodecahedron. Since the 1-skeleton of the hemi-dodecahedron is the Petersen graph, it is easy to see that this configuration on 10 points and 10 circles is a point-circle realisation of the Desargues configuration. Figure 2 shows this point-circle configuration constructed in this way from the Petersen graph embedded as the 1-skeleton of the hemi-dodecahedron. Be aware that incidences outside the vertices may be accidental. The automorphism group of this realisation is the symmetric group S_5.

3.3 The pentagonal geometry with the Hoffman-Singleton graph as deficiency graph

The third and last pentagonal geometry that we will discuss in this article is the $(7,7)$ pentagonal geometry which has the Hoffman-Singleton graph as deficiency graph. The Hoffman-Singleton graph was first constructed by Hoffman and Singleton in 1960 [17]. It

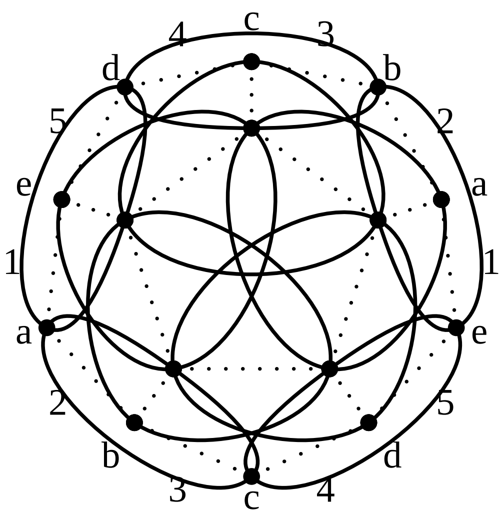

Figure 2: The Desargues configuration (black lines) obtained from the Petersen graph embedded in the real projective plane as the hemi-dodecahedron (dotted lines). Points are identified according to letters, and edges are identified according to numbers.

is a symmetric graph with automorphism group $P\Sigma U(3,5) = PSU(3,5) \rtimes C_2$, which has order 252000.

The group $PSU(3,5)$ is the automorphism group of the Hermitian curve over $\mathbb{F}_{25}$. The first geometric construction of the Hoffman-Singleton graph in this curve was described by Benson and Losey [2] in 1971. Recently Shimada presented a unified construction of the Hoffman-Singleton graph, the Higman-Sims graph and the McLaughlin graph in this curve [24].

In a classical construction by Robertson [23](cf. [15]) the Hoffman-Singleton graph is obtained after connecting the vertices of 5 pentagons and 5 pentagrams. Later this construction was interpreted in terms of affine geometry over $\mathbb{F}_5$ by Hafner [15].

There is also the following construction of the Hoffman-Singleton graph due to Haemers [14]. Take as vertices the union of the points v_p and the lines v_l of $PG(3,2)$. Put an edge between a point vertex v_p and a line vertex v_l if p is a point on l. This gives each point vertex valency 7 and each line vertex valency 3. Also put an edge between two line vertices v_l and $v_{l'}$ if $l \cap l' = \emptyset$. This makes the graph 7-regular. To see that this is the Hoffman-Singleton graph, observe that the girth is 5 and that there are 50 vertices.

Other geometric constructions of the Hoffman-Singleton graph are described for example in [3].

In all the constructions described above, except in the first two ([2, 24]), it is required that the vertex set be partitioned into two parts, and then the vertices in the different parts are represented by geometric objects of different types. We argue that in these cases what is dealt with are not geometric *realisations* of the Hoffman-Singleton graph, but geometric *constructions*.

Our interest in this article is focused on geometric realisations of the $(7,7)$ pentagonal geometry in the classical sense. That means realisations of the configuration in terms of points and lines, or points and circles, in the plane or on some other two-dimensional surface. We are also interested in higher dimensional generalizations, hyperplanes instead of lines and hyperspheres instead of circles. In particular, all geometric realisations of the $(7,7)$ pentagonal geometry will be circular or spherical.

The construction of the $(7,7)$-pentagonal geometry from the Hoffman-Singleton graph associates the points of the pentagonal geometry with the vertices of the graph. Therefore we are interested in geometric realisations of the Hoffman-Singleton graph in which all the vertices are represented by geometric objects of the same type.

3.3.1 The (7,7) pentagonal geometry as a point-circle configuration on a surface of characteristic -55

Just as for the smaller Moore graphs of diameter 2, in a drawing without crossings of the Hoffman-Singleton graph on some surface, all faces will have at least 5 vertices. It can be seen from Eulers polyhedron formula that a map with only pentagonal faces will have the smallest possible genus. Indeed, since the Hoffman-Singleton graph has 50 vertices and 175 edges, the Euler characteristic of the map is $\chi = |V| - |E| + |F| = -125 + |F|$, where $|F|$ is at most $350/5 = 70$, so χ reaches its largest value of -55 if all faces are pentagons. In that case the surface has non-orientable genus 57. It can be proved that such a map does exist, but cannot be a regular map [6]. Consequently, the automorphism group of the map will not be the full automorphism group of the graph. More precisely, there exist maps representing the Hoffman-Singleton graph which have as automorphism group the cyclic groups C_7, C_5 and the trivial group. These maps sit on non-orientable surfaces of the form $S = \mathbb{H}/G$, where G is a torsion-free non-normal subgroup of $\Gamma(7,2,5)$.

Remark 3.1. All these maps can be taken with congruent pentagons, and from Theorem 2.2 we obtain configurations of points and circles on the surfaces, in which the circles are isometric in terms of a quotient of the hyperbolic distance.

Figures 3, 4 and 5 show examples of these three distinct geometric realisations of the combinatorial pentagonal geometry of order $(7,7)$ in terms of points and circles, represented in the Poincaré disk. In the case of the realisation coming from the map with automorphism group C_7, the group action divides the vertex set of the map into seven orbits, each of length seven and one additional fixed point. In Figure 5 the map and the configuration is represented so that the automorphism of order seven is visible as a rotation around the fixed point. It is much harder to visualize the automorphisms of the geometric realisation with automorphism group C_5. The group action does not fix any vertex, edge nor face of the map.

Proposition 3.2. *The dianalytic surfaces of Euler characteristic* 55 *that admit geometric realisations of the* $(7,7)$ *pentagonal geometry as a point-circle configuration with different automorphism groups are different.*

Proof. Consider the non-orientable Riemann surfaces $S_i = \mathbb{H}/G_i$ admitting the maps representing the Hoffman-Singleton graph, where G_i are torsion-free non-normal subgroups of $\Gamma(7,2,5)$. Note that $\Gamma(7,2,5)$ is a non-arithmetic triangle group [27], and that it is maximal with respect to inclusion [25]. By Theorem 1 in [11], two groups G and G', contained

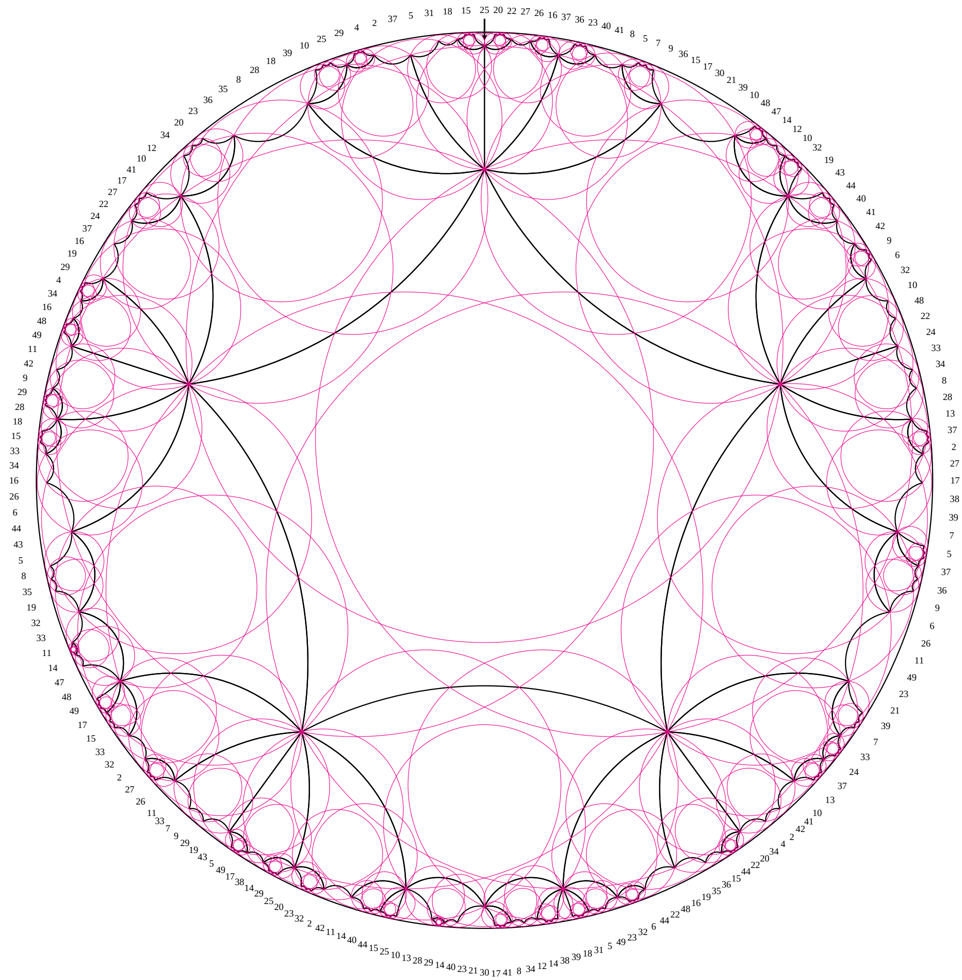

Figure 3: A drawing in a non-orientable surface of genus 57 (identification along the border according to the labelling of the points). The edges colored magenta give the pentagonal geometry of order $(7, 7)$. The edges colored black give the pentagonal map of the Hoffman-Singleton graph. The automorphism group of this realisation is the trivial group.

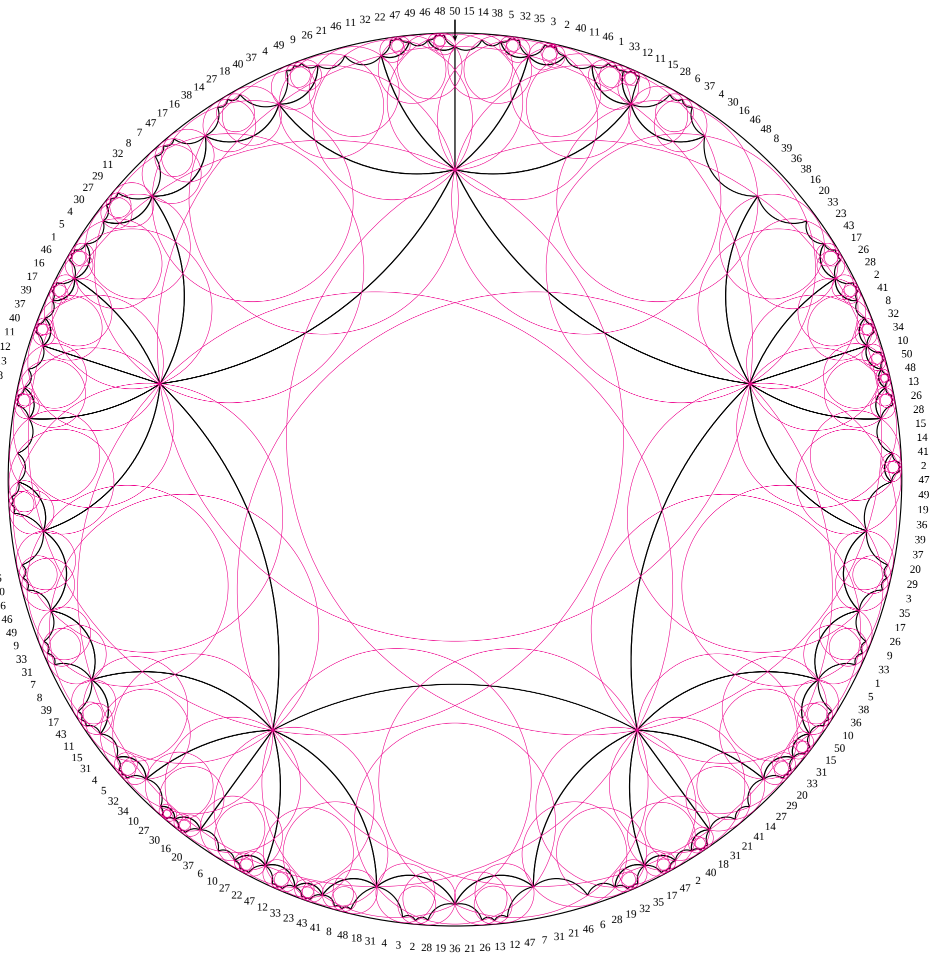

Figure 4: A drawing in a non-orientable surface of genus 57 (identification along the border according to the labelling of the points). The edges colored magenta give the pentagonal geometry of order $(7, 7)$. The edges colored black give the pentagonal map of the Hoffman-Singleton graph. The automorphism group of this realisation is the cyclic group of order five.

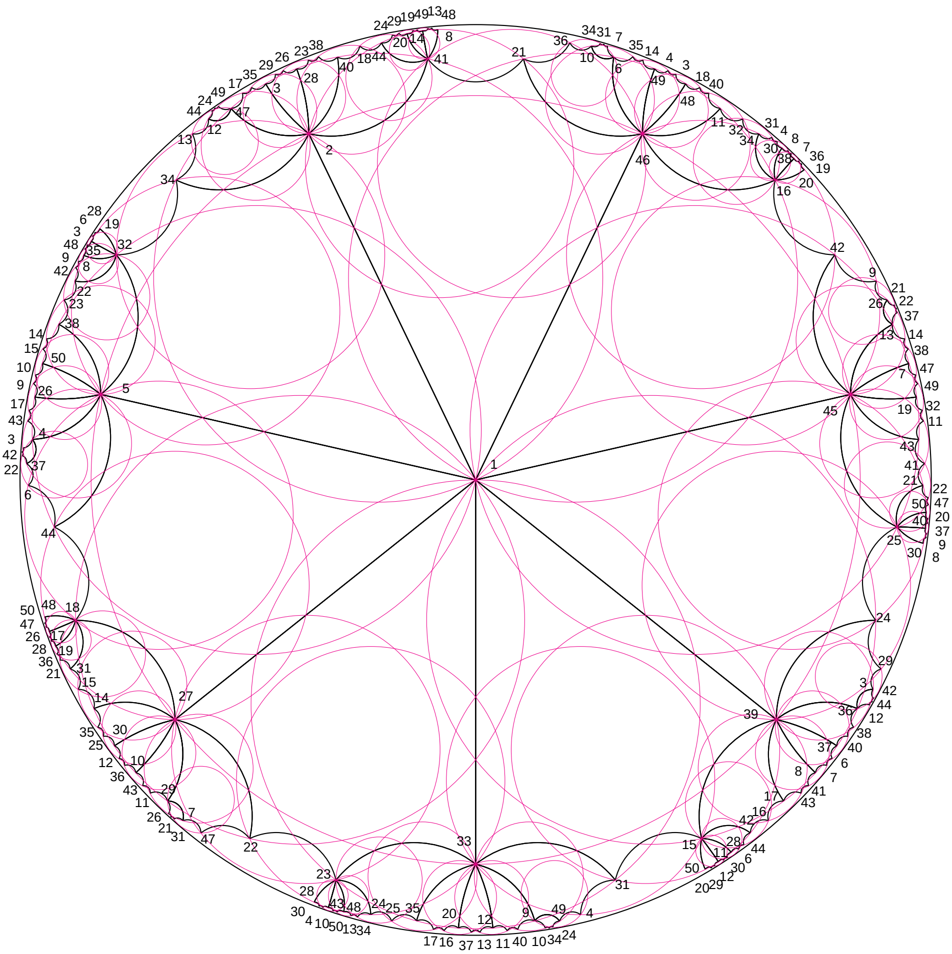

Figure 5: A drawing in a non-orientable surface of genus 57 (identification along the border according to the labelling of the points). The edges colored magenta give the pentagonal geometry of order $(7, 7)$. The edges colored black give the pentagonal map of the Hoffman-Singleton graph. The automorphism group of this realisation is the cyclic group of order 7.

in a non-arithmetic Fuchsian triangle group $\Gamma(p,r,q)$, are the uniformizing groups of two dianalytically equivalent surfaces if and only if they are conjugate in a maximal Fuchsian triangle group extending $\Gamma(p,r,q)$, and so the result follows. □

Note that Coxeter realised Desargues configuration by embedding its Menger graph (collinearity graph) as a map on a surface of Euler characteristic -5. The edges of the Menger graph nicely line up along the geoedesics. A similar geometric realisation of the $(7,7)$ pentagonal geometry is impossible. Indeed, in this case there are 7 points on each line and 7 lines through each point, so that at each vertex in the Menger graph there are 49 edges. If a combinatorial line with 7 points is represented by a geodesic on this surface, then all the edges between the vertices representing these points in the embedded Menger graph must be on this geodesic. Therefore the geometric representations of these edges would partially overlap, but this is never the case in a map.

3.3.2 The (7,7)-pentagonal geometry as a point-hypersphere configuration in 24 dimensional Euclidean space

The Leech lattice is a Euclidean unimodular lattice in 24 dimensions with extraordinary properties. It can be constructed from the Golay code and provides the optimal kissing configuration of unit balls (hyperspheres) in 24 dimensions and the densest lattice ball packing in $\mathbb{E}^{24}$. Each unit ball touches 196560 other unit balls. The Leech lattice was found in 1967 by Leech [19].

There is a construction of the Higman-Sims graph in the Leech lattice [7]. Start with three lattice points forming the vertices of a triangle with sides of length 2, $\sqrt{6}$ and $\sqrt{6}$. The number of lattice points at distance 2 from at least one of the vertices of the triangle is exactly 100. Construct a graph with these 100 points as vertices and with an edge between two points whenever the distance between them is $\sqrt{6}$. Then this graph is the Higman-Sims graph with automorphism group the Higman-Sims sporadic simple group HS. It is well-known that the vertex set of the Higman-Sims graph can be partitioned into two copies of the Hoffman-Singleton graph. The automorphism groups of these two copies of the Hoffman-Singleton graph in the Leech lattice are two conjugate subgroups of HS, each isomorphic to $PSU(3,5) \rtimes C_2$, the automorphism group of the combinatorial Hoffman-Singleton graph.

Note that the edges in one of these embeddings of the Hoffman-Singleton graph all have length $\sqrt{6}$. Hence there is a hypersphere centered at each vertex of radius $\sqrt{6}$, such that the graph vertices contained in each hypersphere are exactly those that are adjacent to the vertex in the center. Indeed, this is (more or less) Theorem 2.2 for the Euclidean plane generalized to higher dimensions. The result is a geometric realisation of the $(7,7)$-pentagonal geometry as a point-hypersphere configuration in 24 dimensions. The automorphism group of this embedding of the Hoffman-Singleton graph is $PSU(3,5) \rtimes C_2$. Since this is the automorphism group of the combinatorial object, this is the largest possible.

The embedding of the Hoffman-Singleton graph in the Leech lattice is not unit-distance, but it is isometric, as required by the construction in Theorem 2.2. It was proved by Maehara and Rödl that any graph of maximum valency d can be embedded as a unit-distance graph in $\mathbb{E}^{2d}$ [21], however this does not say anything about the symmetry group of the embedding.

3.4 The pentagonal geometries of order $(k, k + 1)$

Since all pentagonal geometries of order $(k, k+1)$ can be constructed from pentagonal geometries of order $(k+1, k+1)$ through the removal of one point and its opposite line, there are at most three (connected) pentagonal geometries of order $(k, k + 1)$, with $k = 2, 6$ and maybe 56. The automorphism group of these combinatorial pentagonal geometries is the point-stabilizer of the automorphism group of the corresponding combinatorial pentagonal geometry of order $(k + 1, k + 1)$. The pentagonal geometry of order $(2, 3)$ is constructed from the Desargues configuration and has automorphism group $S_3 \times C_2$. The pentagonal geometry of order $(6, 7)$ has automorphism group S_7.

As a consequence of the construction of pentagonal geometries of order $(k, k+1)$ from those of order $(k+1, k+1)$, any geometric realisation of a $(k+1, k+1)$ pentagonal geometry gives rise to a geometric realisation of the corresponding pentagonal geometry of order $(k, k+1)$, by simply removing from the realisation a point and the geometric realisation of its opposite combinatorial line. In the case of point-circle (point-hypersphere) realisations, the geometric realisation of a combinatorial line is a circle (hypersphere). Therefore any geometric realisation, in terms of points and circles (or hyperspheres), of a pentagonal geometry of order $(k+1, k+1)$, described previously in this article, gives rise to a geometric realisation in terms of points and circles (or hyperspheres) of the corresponding pentagonal geometry of order $(k, k + 1)$.

The automorphism group of the geometric realisation of the pentagonal geometry of order $(k, k + 1)$ is the intersection of the point-stabilizer of the combinatorial pentagonal geometry of order $(k+1, k+1)$ and the automorphism group of its corresponding geometric realisation.

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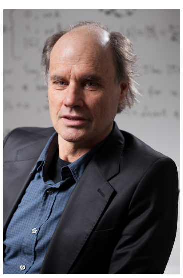


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