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10 Bosons Being Their Own Antiparticles in Dirac Formulation ⋆

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Abstract. Using our earlier formalism of extending the idea of the Dirac sea (negative energy states) to also for Bosons, we construct a formalism for Bosons which are their own antiparticles. Since antiparticles in formalisms with a Dirac sea are at first formulated as holes, they are a priori formally a bit different from the particles themselves. To set up a formalism/a theory for Majorana fermions and for Bosons in which particles are their own antiparticles is thus at first non-trivial. We here develop this not totally trivial formalism for what one could call extending the name for the fermions, “Majorana-bosons”. Because in our earlier work had what we called “different sectors” we got there some formal extensions of the theory which did not even have positive definite metric. Although such unphysical sectors may a priori be of no physical interest one could hope that they could be helpful for some pedagogical deeper understanding so that also the formalism for particles in such unphysical sectors of their own antiparticles would be of some “academic” interest.

Povzetek. Avtorja razširita pojm Diracovega morja (stanj z negativno energijo) na bozone, ki so sami sebi antidelci. V kontekstu Diracovega morja so antidelci vrzeli v morju, tedaj formalno različni od delcev. Konstrukcija formalizma za Majoranine fermione in bozone, v kateri so delci enaki antidelcem, je zato vsaj na prvi pogled netrivialna. V tem prispevku avtorja napravita ta netrivialni korak in gledata fermione kot “Majoranine bozone”. V predhodnem delu avtorjev definirata “sektorje”, v katerih metrika ni pozitivno definitna. Taki sektorji sicer morda nimajo fizikalnega pomena, lahko pa pomagajo pedagoško globlje razumeti obravnavani formalizem.

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10.1 Introduction

Majorana [1] put forward the idea of fermions having the property of being their own antiparticles and such fermions are now usually called Majorana fermions or Majorana particles.

The Majorana field ψ_M is defined in general as real or hermitean $\psi_M^\dagger = \psi_M$.

Among the known bosons we have more commonly bosons, which are their own antiparticles, and which we could be tempted to call analogously "Majorana bosons" (a more usual name is "real neutral particles" [2]), such as the photon, Z^0, π^0, ... particles. Now the present authors extended [3,4] the Dirac sea idea [5] of having negative energy electron single particle states in the second quantized theory being already filled in vacuum, also to Bosons. This extension of the Dirac sea idea to Bosons has a couple of new features:

1) We had to introduce the concept of having a negative number of bosons in a single particle state. We described that by considering the analogy of a single particle state in which a variable number of bosons can be present to a harmonic oscillator, and then extend their wave functions from normalizable to only be analytical. The harmonic oscillator with wave functions allowed to be non-normalizable and only required to be analytical has indeed a spectrum of energies $E_n = (n + \frac{1}{2})\omega$ where now n can be all integers $n = \ldots, -3, -2, -1, 0, 1, 2, \ldots$. So it corresponds to that there can be a negative number of bosons in a single particle state.

2) It turns out though that these states - of say the "analytical wave function harmonic oscillator" corresponding to negative numbers of bosons have alternating norm square: For $n \geq 1$ we have as usual $\langle n|n\rangle = 1$ for $n \geq 0$ (by normalization) but for $n \leq -1$ we have instead $\langle n|n\rangle = c \cdot (-1)^n$ for n negative. (c is just a constant we put say $c = +1$.) This variation of norm square is needed to uphold the usual rules for the creation a^+ and annihilation a operators

$$\begin{aligned} a^+|n\rangle &= \sqrt{1+n}|n+1\rangle \\ a|n\rangle &= \sqrt{n}|n-1\rangle \end{aligned} \tag{10.1}$$

to be valid also for negative n.

3) With the relations (10.1) it is easily seen that there is a "barrier" between $n = -1$ and $n = 0$ in the sense that the creation and annihilation operators a^+, and a cannot bring you across from the space spanned by the $n = 0, 1, 2, \ldots$ states to the one spanned by the $n = -1, -2, \ldots$ one or opposite. It is indeed best to consider the usual space spanned by the $|n\rangle$'s with $n = 0, 1, 2, \ldots$ as one separate "sector" the "positive sector" and the one spanned by the $|n\rangle$ states with $n = -1, -2, \ldots$ as another "sector" called the "negative sector". Since in the harmonic oscillator with the wave functions only required to be analytical but not normalizable the states in the "positive sector" are not truly orthogonal to those in the "negative sector" but rather have divergent or ill-defined inner products with each other, it is best not even to (allow) consider inner products like say

$$\langle 0| - 1\rangle = \text{ill defined}$$
$$\langle n|p\rangle = \text{ill defined} \tag{10.2}$$

when $n \leq -1$ and $p \geq 0$ or opposite.
Basically we shall consider only <u>one sector</u> at a time.

4) The use of our formalism with negative number of particles to connect to the usual and physically correct description of bosons with some charge or at least an (at first) conserved particle number comes by constructing a "Dirac sea for bosons". That is to say one first notes that e.g. the free Klein-Gordon equation

$$\Box\phi = 0$$

has both positive and negative energy solutions, and that the inner product

$$\langle \varphi_1|\varphi_2\rangle = \int \varphi_1^* \overleftrightarrow{\partial}_0 \varphi_2 d^3\vec{X} \tag{10.3}$$

gives <u>negative norm square for negative energy eigenstates</u> and positive norm square for positive energy eigenstates.
Then the physical or true world is achieved by using for the negative energy single particle states the "negative sector" (see point 3) above) while one for the positive energy single particle states use the "positive sector". That is to say that in the physical world there is (already) a <u>negative number of bosons</u> in the negative energy single particle states. In the vacuum, for example, there is just -1 boson in each negative energy single particle state.
This is analogous to that for fermions there is in the Dirac sea just $+1$ fermion in each negative energy single particle state. For bosons – where we have -1 instead $+1$ particle– we just rather emptied Dirac sea by one boson in each single particle negative energy state being removed from a thought upon situation with with 0 particles everywhere. (Really it is not so nice to think on this removal because the "removal" cross the barrier from the "positive sector" to the "negative sector" and strictly speaking we should only look at one sector at a time (as mentioned in 3).)

5) It is rather remarkable that the case with the "emptied out Dirac sea" described in 4) – when we keep to positive sector for positive energy and negative sector for negative energy– we obtain a positive definite Fock space. This Fock space also has only positive energy of its excitations as possibilities. Indeed we hereby obtained exactly a Fock space for a theory with bosons, that are different from their antiparticles.

In the present paper we like to study how to present a theory for bosons which are their own antiparticles, Majorana bosons so to speak, in this formalism with the "emptied" Dirac sea.

Since in the Dirac sea formalisms – both for fermions and for bosons – an antiparticle is the removal of a particle from the Dirac sea, an antiparticle a priori is something quite different from a particle with say positive energy. Therefore to make a theory / a formalism for a theory with particle being identified with its antiparticles – as for Majorana fermions or for the photon, Z^0, π^0 – is in our or Dirac's Dirac sea formalisms a priori not trivial. Therefore this article. Of course it is at the end pretty trivial, but we think it has value for our understanding to develop the formalism of going from the Dirac sea type picture to the theories with particles being their own antiparticles ("Majorana theories").

One point that makes such a study more interesting is that we do not have to only consider the physical model in the boson case with using positive sector for positive single particle energy states and negative sector for negative energy single particle states. Rather we could – as a play– consider the sectors being chosen in a non-physical way. For example we could avoid "emptying" the Dirac sea in the boson-case and use the positive sector for both negative and positive energy single boson eigenstates. In this case the Fock space would not have positive norm square. Rather the states with an odd number of negative energy bosons would have negative norm square, and of course allowing a positive number of negative energy bosons leads to their being no bottom in the Hamiltonian for such a Fock space.

The main point of the present article is to set up a formalism for particles that are their own antiparticles (call them "Majorana") on the basis of a formalism for somehow charged particles further formulated with the Dirac sea. That is to say we consider as our main subject how to restrict the theory with the Dirac sea – and at first essentially charged particle – to a theory in which the particles and antiparticles move in the same way and are identified with each other.

For example to describe a one-particle state of a "Majorana" particle one would naturally think that one should use a state related to either the particle or the antiparticle for instance being a superposition of a particle and antiparticle state.

So the states of the Fock space H_{Maj} for describing the particles which are their own antipatricles shall be below identified with some corresponding states in the theory with Dirac sea. However, there are more degrees of freedom in a theory with charged particles (as the Dirac sea one) than in a corresponding theory for particles which are their own antiparticle. Thus the states in the with Dirac sea Fock space cannot all be transfered to the Fock space from "Majorana" particles. So only a certain subspace of the Fockspace for the with Dirac sea theory can be identified with states of some number of Majorana particles.

To develop our formalism for this transition from the Dirac sea theory to the one for Majorana particles, we therefore need a specification of which subspace is the one to be used to describe the "Majorana particles". Below we shall argue for that this subspace H_{Maj} becomes

$$H_{Maj} = \left\{ |\ \rangle \middle| \left(a(\vec{p}, E > 0) + a^{\dagger}(-\vec{p}, E < 0) \right) |\ \rangle = 0, \text{for all } \vec{p} \right\} \tag{10.4}$$

where we used the notation of $a(\vec{p}, E < 0)$ for the annihilation operator for a particle with momentum $\vec{p}$ and energy E corresponding to that being positive i.e. $E > 0 \Rightarrow E = \sqrt{m^2 + \vec{p}^2}$. Correspondingly the annihilation operator $a(\vec{p}, E < 0)$

annihilates a particle with energy $E = -\sqrt{m^2 + \vec{p}^2}$. The corresponding creation operators just have the dagger † attached to the annihilation operator, as usual. We define

$$r(\vec{p}) = \frac{1}{\sqrt{2}}\left(a(\vec{p}, E > 0) + a^\dagger(-\vec{p}, E < 0)\right) \tag{10.5}$$

This then shall mean, that we should identify a basis, the basis elements of which have a certain number of the "Majorana bosons", say, with some momenta – in a physical world we only expect conventional particles with positive energy – for the subspace H_{Maj} contained in the full space with Dirac sea.

We thus have to construct below creation $b^\dagger(\vec{p})$ and annihilation $b(\vec{p})$ operators for the particles which are their own antiparticles ("Majoranas"). These $b^\dagger(\vec{p})$ and $b(\vec{p})$ should now in our work be presented by formulas giving them in terms of the creation and annihilation operators for the theory with Dirac sea (and thus acting on the Fock space H of this "full" theory). In fact we shall argue for (below)

$$b^\dagger_n = \begin{cases} \frac{(a^\dagger\left(a^\dagger\vec{p}, E>0)-a(-\vec{p}, E<0)\right)}{\sqrt{2}} & \text{(on pos. sec for pos. } E\text{, neg sec for neg } E\text{)} \\ \frac{(a^\dagger\left(a^\dagger\vec{p}, E>0)+a(-\vec{p}, E<0)\right)}{\sqrt{2}} & \text{for both sectors} \\ \ldots \end{cases} \tag{10.6}$$

and then of course it has to be so that these $b^\dagger(\vec{p})$ and $b(\vec{p})$ do not bring a Hilbert vector out of the subspace H_{Maj} but let it stay there once it is there. It would be the easiest to realize such a keeping inside H_{Maj} by action with $b^\dagger(\vec{p})$ – and we shall have it that way – if we arrange the commutation rules

$$\begin{aligned} \left[r(\vec{p}), b^\dagger(\vec{p'})\right] &= 0 \\ \left[r(\vec{p}), b(\vec{p'})\right] &= 0 \end{aligned} \tag{10.7}$$

(Here the commutation for $\vec{p} \neq \vec{p'}$ is trivial because it then concerns different d.o.f. but the $\left[r(\vec{p}), b^\dagger(\vec{p'})\right] = 0$ and $\left[r(\vec{p}), b(\vec{p'})\right] = 0$ are the nontrivial relations to be arranged (below))

Indeed we shall find below

$$\begin{aligned} b^\dagger(\vec{p}) &= \frac{1}{\sqrt{2}}\left(a^\dagger(\vec{p}, E > 0) + a(-\vec{p}, E < 0)\right) \quad (\text{defined on both pos.}) \\ b(\vec{p}) &= \frac{1}{\sqrt{2}}\left(a(\vec{p}, E > 0) + a^\dagger(-\vec{p}, E < 0\right) \end{aligned} \tag{10.8}$$

It is then that we shall arrange that if we extrapolate to define also the $b^{(\dagger)}(\vec{p}, E < 0)$ and not only for positive energy $b^{(\dagger)}(\vec{p}, E > 0) = b^{(\dagger)}(\vec{p})$ we should obtain the formula usual in conventional description of Majorana particle theories

$$\begin{aligned} b^\dagger(\vec{p}) = b^\dagger(\vec{p}, E > 0) &= b(-\vec{p}, E < 0) \\ b(\vec{p}) = b(\vec{p}, E > 0) &= b^\dagger(-\vec{p}, E < 0) \end{aligned} \tag{10.9}$$

For fermions we simply do construct these $r(\vec{p})$ and $b(\vec{p})$ rather trivially and it must be known in some notation to everybody. For bosons, however almost nobody but ourselves work with Dirac sea at all, and therefore it must be a bit more new to get particles which are their own antiparticles into such a scheme. For bosons also we have already alluded to the phenomenon of different "sectors" (see 3) above) being called for due to our need for negative numbers of particles. We therefore in the present article as something also new have to see what becomes of the theory with bosons being their own antiparticles when we go to the unphysical sector-combinations. (The physical combination of sectors means as described in point 4) above, but if we e.g. have the positive sector both for negative and positive energy single particle states, this is a unphysical sector-combination.) This is a priori only a discussion though of academic interest, since the truly physical world corresponds to the physical combination described in point 4) with the Dirac sea "emptied out".

However, in our attempts to describe string field theory in a novel way we raised to a problem that seemed formally to have solution using such on unphysical sector-combination.

In the following section 10.2 we just, as a little warm up, discuss the introduction in the Majorana fermion theory on a subspace of the Fock space of a fermion theory in Dirac sea formulation.

In section 10.3 we then review with more formalism our "Dirac sea for bosons" theory.

Then in section 10.4 we introduce the formalism $r(\vec{p})$, $b^\dagger(\vec{p})$ and $b(\vec{p})$ relevant for the Majorana rather theory or for particles which are their own antiparticles. The operators $r(\vec{p})$ defined in (10.5) are the operators defined to be used for singling out the Majorana subspace, and $b^\dagger(\vec{p})$ and $b(\vec{p})$ are the creation and annihilation operators for "Majorana-bosons".

In section 10.5 we go to the unphysical sector combinations to study the presumably only of acdemic interest problems there.

In section 10.6 we bring conclusion and outlook.

10.2 Warming up by Fermion

10.2.1 Fermion Warm Up Introduction

As the warming up consider that we have a fermion theory at first described by making naively (as if nonrelativistically, but we consider relativity) creation $a^\dagger(n, \vec{p}, E > 0)$ and $a^\dagger(n, \vec{p}, E < 0)$ for respectively positive and negative energy E of the single particle state. Also we consider the corresponding annihilation operators $a(\sigma, \vec{p}, E > 0)$ and $a(\sigma, \vec{p}, E < 0)$

The physically relevant second quantized system takes its outset in the physical vacuum in which all the negative energy $E < 0$ single particle states are filled while the positive energy ones are empty.

$$| \ vac\ phys\rangle = \prod_{\sigma,\vec{p}} a^\dagger(\sigma, \vec{p}, E < 0) \mid 0\ totally\ empty\rangle \tag{10.10}$$

Of course in modern practice you may ignore the Dirac sea and just start from the physical vacuum $| \; vac \; phys\rangle$ and operate on that with creation and annihilation operators. If you want to say create on antiparticle with momentum $\vec{p}$ (and of course physically wanted positive energy) you operate on $| \; vac \; phys\rangle$ with

$$a^{\dagger}_{anti}(\sigma, \vec{p}, E > 0) = a(\sigma^1, -\vec{p}, E < 0) \tag{10.11}$$

i.e. the antiparticle creation operator $a^{\dagger}_{anti}(\sigma, \vec{p}, E > 0)$ is equal to the annihilation operator $a(\sigma^1, -\vec{p}, E < 0)$ with the "opposite" quantum numbers.

10.2.2 Constructing Majorana

Now the main interest of the present article is how to construct a theory of particles being their own antiparticle ("Majorana") from the theory with essentially charged particles – carrying at least a particle-number "charge"– by appropriate projection out of a sub-Fock space and by constructing creation and annihilation operators for the Majorana –in this section– fermions.

Let us remark that this problem is so simple, that we can do it for momentum value, and if we like to simplify this way we could decide to consider only one single value of the momentum $\vec{p}$ and spin. Then there would be only two creation and two annihilation operators to think about

$$\begin{aligned} a^{\dagger}(E > 0) &= a^{\dagger}(\sigma, \vec{p}, E > 0) \\ a^{\dagger}(E < 0) &= a^{\dagger}(\sigma, \vec{p}, E < 0) \end{aligned} \tag{10.12}$$

and thus the whole Fock space, we should play with would only have $2 \cdot 2 = 4$ states, defined by having filled or empty the two single particle states being the only ones considered in this simplifying description just denoted by "$E > 0$" and "$E < 0$".

In fact the construction of a full Majorana-formalism will namely be obtained by making the construction of the Majorana Fock (or Hilbert) space for each momentum $\vec{p}$ and spin and then take the Cartesian product of all the obtained Majorana-Fock spaces, a couple for each spin and momentum combination.

The four basis states in the Fock space after throwing away all but one momentum and one spin-state are:

$$\begin{aligned} | \; 1 \; antiparticle\rangle &= | \; vac \; totally \; empty\rangle \\ | \; vac \; phys\rangle &= a^{\dagger}(E < 0) \; | \; vac \; totally \; empty\rangle \\ | \; 1 \; fermion \; in \; phys\rangle &= a^{\dagger}(E > 0) a^{\dagger}(E < 0) \; | \; vac \; totally \; empty\rangle \\ | \; both \; particle \; and \; antip.\rangle &= a^{\dagger}(E > 0) \; | \; vac \; totally \; empty\rangle \end{aligned} \tag{10.13}$$

Considering the situation from the point of view of the physical vacuum

$$| \; vac \; phys\rangle = a^{\dagger}(E < 0) \; | \; vac \; totally \; empty\rangle \tag{10.14}$$

creating a Majorana particle should at least either a particle or an antiparticle or some superposition of the two (but not both).

So the one Majorana particle state shoule be a superpositon of

$$\mid 1\ fermion\ in\ phys\rangle = a^\dagger(E>0)a^\dagger(E<0)\mid vac\ totally\ empty\rangle \tag{10.15}$$

and

$$\mid 1\ antiferm\ in\ phys\rangle = \mid vac\ totally\ empty\rangle \tag{10.16}$$

The most symmetric state would natutally be to take with coefficients $\frac{1}{\sqrt{2}}$ these two states with equal amplitude:

$$\mid 1\ Majorana\rangle = \frac{1}{\sqrt{2}}\left(a^\dagger(E>0)a^\dagger(E<0)+1\right)\mid vac\ totally\ empty\rangle \tag{10.17}$$

We should then construct a creation operators $b^\dagger(\sigma,\vec{p})$ or just $b^\dagger$ so that

$$b^\dagger \mid vac\ phys\rangle = \mid 1\ Majorana\rangle \tag{10.18}$$

Indeed we see that

$$b^\dagger = \frac{1}{\sqrt{2}}\left(a^\dagger(E>0)+a(E<0)\right) \tag{10.19}$$

will do the job.

If we use $b^\dagger$ and

$$b = \frac{1}{\sqrt{2}}\left(a(E>0)+a^\dagger(E<0)\right) \tag{10.20}$$

it turns out that states needed are the – superpositoins of –

$$\mid vac\ phys\rangle = a^\dagger(E<0)\mid vac\ totally\ empty\rangle \tag{10.21}$$

and

$$b^\dagger \mid vac\ phys\rangle = \mid 1\ Majorana\rangle \tag{10.22}$$

This subspace which in our simplyfication of ignoring all but one momentum and spin state actually represents the whole space H_{Maj} used to describe the Majorana theory has only <u>2 dimensions</u> contrary to the full Hilbert space H which in our only one momentum and spin consideration has <u>4 dimensions</u>.

So it is a genuine subspace and we shall look for an operator $r = r(h,\vec{p})$ which gives zero when acting on H_{Maj} but not when it acts on the rest of H.

It is easily seen that

$$r = \frac{1}{\sqrt{2}}\left(a(E>0)-a^\dagger(E<0)\right) \tag{10.23}$$

will do the job. Thus we can claim that

$$H_{Maj} = \{|\rangle \mid r\,|\rangle = 0\} \tag{10.24}$$

Written for the full theory with all the momenta and spins we rather have

$$H_{Maj} = \{|\ \ \rangle \in H \mid \forall_h \vec{p}[r(\vec{p}, h)\ |\rangle = 0]\} \tag{10.25}$$

where

$$r(\vec{p}, h) = \frac{1}{\sqrt{2}} \left(a(\vec{p}, h, E > 0) - a^\dagger(-\vec{p}, h, E < 0) \right) \tag{10.26}$$

and $a(\vec{p}, h, E > 0)$ is the annihilation operator for a fermion with momentum $\vec{p}$ and eigenstate h of the normalized helicity

$$h \sim \vec{\Sigma} \cdot \vec{p} / \mid \vec{p} \mid \tag{10.27}$$

where $\vec{\Sigma}$ is the spin angular momentum, and the energy $E = +\sqrt{\vec{p}^2 + m^2}$.

The fully described creation opperator for a Majorana particle fermion with momentum $\vec{p}$ and helicity h,

$$b^\dagger(\vec{p}, h) = \frac{1}{\sqrt{2}} \left(a^\dagger(\vec{p}, h, E > 0) + a(-\vec{p}, h, E < 0) \right) \tag{10.28}$$

and the corresponding annihilation operator

$$b(\vec{p}, h) = \frac{1}{\sqrt{2}} \left(a(\vec{p}, h, E > 0) + a^\dagger(-\vec{p}, h, E < 0) \right) \tag{10.29}$$

One easily checks that the operation with these operators $b(\vec{p}, h)$ and $b^+(\vec{p}, h)$ map H_{Maj} on H_{Maj} because

$$\begin{aligned} \left\{ r(\vec{p'}, h'), b(\vec{p}, h) \right\}_+ &= 0 \\ \left\{ r(\vec{p'}, h'), b^\dagger(\vec{p}, h) \right\}_+ &= 0 \end{aligned} \tag{10.30}$$

10.3 Review of Dirac Sea for Bosons

Considering any relativistically invariant dispersion relation for a single particle it is, by analyticity or better by having a finite order differential equation, impossible to avoid that there will be both negative and positive energy (eigen) solutions. This is true no matter whether you think of integer or half integer spin or on bosons or fermions(the latter of course cannot matter at all for a single particle theory). In fact this unavoidability of also negative energy single particle states is what is behind the unavoidable CPT-theorem.

There is for each type of equation a corresponding inner product for single particle states, so that for instance the Klein-Gordon equation and the Dirac equation have respectively

$$\langle \varphi_1 \mid \varphi_2 \rangle = \int \varphi_1^* \frac{\overleftrightarrow{\partial}}{\partial t} \varphi_2 d^3\vec{X} \text{ (Klein Gordon)} \tag{10.31}$$

and

$$\langle \psi_1 \mid \psi_2 \rangle = \int \psi_1^\dagger \psi_2 d^3\vec{X} = \int \vec{\psi}_1 \gamma^0 \psi_2 d^3\vec{X} \text{ (for Dirac equation)} \tag{10.32}$$

(see e.g. [7])

At least in these examples –but it works more generally– the inner product of a single particle state with itself, the norm square, gets negative for integer spin and remains positive for the half integer spin particles, when going to the negative energy states.

For integer spin particles (according to spin statistics theorem taken to be bosons) as for example a scalar we thus have negative norm square for the negative energy single particle states. This means that for all the states for which we want to make an analogy to the filling of the Dirac sea, we have to have in mind, that we have this negative norm square.

That is to say, that thinking of second quantizing the norm square of a multiple particle state in the Fock space would a priori alternate depending on whether the number of particles (bosons) with negative energy is even or odd.

Physically we do not want such a Fock space, which has non-positive-definite norm –since for the purpose of getting positive probabilities we need a positive definite inner product –.

The resolution to this norm square problem in our "Dirac sea for bosons" –model is to compensate the negative norm square by another negative norm square which appears, when one puts into a single particle state a negative number of bosons.

This is then the major idea of our 'Dirac sea for bosons"-work, that we formally –realy of course our whole model in this work is a formal game – assume that it is possible to have a negative number of particles (bosons) in a single particle state. That is to say we extend the usual idea of the Fock space so as to not as usual have its basic vectors described by putting various non-negative numbers of bosons into each single particle state, but allow also to have a negative number of bosons.

Rather we allow also as Fock-space basis vector states corresponding to that there could be negative integer numbers of bosons. So altogether we can have any integer number of bosons in each of the single particle states (whether it has positive or negative energy at first does not matter, you can put any integer number of bosons in it anyway).

In our "Dirac-sea for Bosons" –paper [3] we present the development to include negative numbers of particles via the analogy with an harmonic oscillator. It is well-known that a single particle state with a non-negative number of bosons in it is in perfect correspondance with a usual harmonic oscillator[6] in which the number of excitations can be any positive number or zero. If one extend the harmonic oscillator to have in the full complex plan extending the position variable q (say)and the wave function $\psi(q)$ to be formally analytical wave function only, but give up requiring normalizability, it turns out that the number of excitations n extends to $n \in Z$, i.e. to n being any integer. This analogy to extend harmonic oscillator can be used to suggest how to build up a formalism withe creation $a^\dagger$ and annihilation operators a and an inner product for a single particle states in which one can have any integer number of bosons.

It is not necessary to use extended harmonic oscillator. In fact one could instead just write down the usual relations for creation and annihilation operators first for a single particle state say

$$a^\dagger(\vec{p}, E > 0) \mid k(\vec{p}, E > 0)\rangle = \sqrt{k(\vec{p}, E > 0) + 1} \mid k(\vec{p}, E > 0) + 1\rangle \tag{10.33}$$

and

$$a(\vec{p}, E > 0) \mid k(\vec{p}, E > 0)\rangle = \sqrt{k(\vec{p}, E > 0)} \mid k(\vec{p}, E > 0) - 1\rangle \tag{10.34}$$

or the analogous ones for a negative energy single particle state

$$a^\dagger(\vec{p}, E < 0) \mid k(\vec{p}, E >)\rangle = \sqrt{k(\vec{p}, E > 0) + 1} \mid k(\vec{p}, E > 0) + 1\rangle \tag{10.35}$$

and

$$a(\vec{p}, E < 0) \mid k(\vec{p}, E > 0)\rangle = \sqrt{k(\vec{p}, E < 0)} \mid k(\vec{p}, E < 0) - 1\rangle \tag{10.36}$$

and then extend them - formally by allowing the number $k(\vec{p}, E > 0)$ of bosons in say a positive energy single particle state with momentum $\vec{p}$ and (positive energy) to be also allowed to be negative. You shall also allow the numbers $k(\vec{p}, E < 0)$ in a negative energy single particle state with momentum $\vec{p}$ to be both positive or zero and negative.

Then there are a couple of very important consequences:

A) You see from these stepping formulas that there is a "barriere" between the number of bosons k being $k = -1$ and $k = 0$. Operating with the annihilation operator a on a state with $k = 0$ particles give zero

$$a \mid k = 0\rangle = 0 \tag{10.37}$$

and thus does not give the $\mid k = -1\rangle$ as expected from simple stepping. Similar one cannot with the creation operator $a^\dagger$ cross the barriere in the opposite direction, since

$$a^\dagger \mid k = -1\rangle = \sqrt{-1 + 1} \mid k = 0\rangle = 0 \tag{10.38}$$

Thus we have that the states describing the number of bosons k in a given single particle state are not connected by –a finite number of operations – of creation and annihilation operatiors.
Really this means that we make best by considering the positive sector of the space of positive or zero number of bosons and another sector formed from the $| k\rangle$ states with $k = -1, -2, -3, ...$ being a negative number of bosons. By ordinary creation and annihilation operators, as they would occur in some interaction Hamiltonian, one cannot cross the barriere. This means that if to beign with one has say a negative number of boson in a given single particle state, then an ordinary interaction cannot change that fact.
Thus we take it that one can choose once forever to put some single particles states in their positive sector and others in their negative sector, and they then will stay even under operation of an interaction Hamiltonian. If one for example make the ansatz that all the negative energy single particle states have a negative number of bosons while the positive energy states have zero or a positive number of bosons in them, then this ansatz can be kept forever.
This special choice we call the "physical choice" and we saw already[3] –and shall see very soon here – that this choice gives us a positive definite Fock space.

B) The norm square of the states $| k\rangle$ (with $k = -1, -2, \ldots$) i.e. with negative numbers k of bosons have to vary alternatingly with k even, k odd.
Using the writing of a negative k

$$| k\rangle \sim \qquad a^{|k|-1} \mid k = -1\rangle \tag{10.39}$$

We may evaluate $\langle k \mid k\rangle \sim < -1 \mid (a^\dagger)^{|k|-1} a^{|k|-1} \mid -1\rangle$ for $k \leq -1$.
Now using still the usual commutation rule

$$[a^\dagger, a] = -1 \tag{10.40}$$

you easily see that we normalize by putting

$$| k\rangle = \frac{1}{\sqrt{(|k|-1)!}} a^{|k|-1} \mid -1\rangle \tag{10.41}$$

and

$$\langle k \mid k\rangle = (-1)^{|k|} \tag{10.42}$$

say for $k \leq -1$ (having taken $\langle -1 \mid -1\rangle = -1$.) while of course for $k = 0, 1, 2, \ldots$ you have $\langle k \mid k\rangle = 1$.
The major success of our "Dirac sea for bosons" is that one can arrange the sign alternation with (10.42) with the total number of negative energy bosons to cancel against the sign from in (10.31) so as to achieve, if we choose the "physical sector", to get in total the Fock space, which has positive norm square.
This "physical sector" corresponds to that negative energy single particle states are in the negative sectors, while the positive energy single particle states are in the positive sector.

The basis vectors of the full Fock space for the physical sector are thus of the form

$$|\ldots, k(\vec{p}, E>0), \ldots; \ldots, k(\vec{p}, E<0), \ldots\rangle \tag{10.43}$$

where the dots ... denotes that we have one integer number for every momentum vector –value ($\vec{p}$ or $\vec{p'}$), but now the numbers $k(\vec{p}, E>0)$ of particles in a positive energy are– in the physical sector-combination– restricted to be non-negative while the numbers of bosons in the negative energy single particle states are restricted to be negative

$$\begin{aligned} k(\vec{p}, E>0) &= \quad 0, 1, 2, \ldots \\ k(\vec{p}, E<0) &= -1, -2, -3, \ldots \end{aligned} \tag{10.44}$$

These basis vectors (10.43) are all orthogonal to each other, and so the inner product is alone given by their norm squares

$$\begin{aligned} &\langle \ldots, k(\vec{p}, E>0), \ldots; \ldots, k(\vec{p'}, E<0), \ldots | \\ &| \ldots, k(\vec{p}, E>0), \ldots; \ldots, k(\vec{p'}, E<0), \ldots\rangle \\ &= (-1)^{\sharp(neg\ energy\ b)} \prod_{\vec{p'}} (-1)^{|k(\vec{p'}, E<0)|} = 1 \end{aligned} \tag{10.45}$$

Here $\sharp(neg\ energy\ b)$ means the total number of negative energy bosons i.e.

$$\sharp(neg\ energy\ b) = \sum_{\vec{p'}} k(\vec{p}, E<0) \tag{10.46}$$

(a negative number in our physical sector-combination). The factor

$$(-1)^{\sharp(neg\ energy\ b.)}$$

comes from (10.31) which gives negative norm square for single particle states with negative energy, because $\frac{\overleftrightarrow{\partial}}{\partial t}$ is essentially the energy. The other factor $\prod_{\vec{p'}} (-1)^{|k(\vec{p'}, E<0)|}$ comes from (10.42) one factor for each negative single particle energy state, i.e. each $\vec{p'}$. Had we here chosen another sector-combination, e.g. to take $k(\vec{p}, E<0)$ non-negative as well as $k(\vec{p}, E>0)$, then we would have instead

$$\left.\begin{aligned} k(\vec{p}, E>0) &= 0, 1, 2, \ldots \\ k(\vec{p'}, E<0) &= 0, 1, 2, \ldots \end{aligned}\right\} \text{(both pos sectors.)} \tag{10.47}$$

and the inner with themselves, norm squares product for the still mutually orthogonal basis vectors would be

$$\begin{aligned} &\langle \ldots, k(\vec{p}, E>0), \ldots; \ldots, k(\vec{p'}, E<0), \ldots \\ &| \ldots, k(\vec{p}, E>0), \ldots; \ldots, k(\vec{p'}, E<0), \ldots\rangle \\ &= (-1)^{\sharp(neg\ energy\ b)} \\ &\text{(for both positive sectors)} \end{aligned} \tag{10.48}$$

and that for this case ("sector combination"), the inner product is <u>not</u> positive definite.

Such strange sector combination is of course mainly of academical interest. But for instance this last mentioned "both positive sector" sector-combination, can have easily position eigenstate particles in the Fock space description. Normally positon is not possible to be well defined in relativistic theories.

As already mentioned above, we have a slightly complicated inner product in as far as we have sign-factors in the inner product coming from two different sides:

1)The inner product sign-factor from the single particle wave function coming from (10.31) gives a minus for negative energy particles, ending up being $(-1)^{\sharp(neg.\ energy\ b)}$ in (10.45).

2)The other inner product sign factor comes from (10.42).

In the above, we have used the dagger symbol " $\dagger$ " on $a^\dagger$ to denote the Hermitian conjugate w.r.t. only the inner product coming from (10.42), but have not included the factor from 1) meaning from (10.31). Thus we strictly speaking must consider also a full dagger (full $\dagger_f$) meaning hermitian conjugation corresponding the full inner product also including 1), i.e. the (10.31) extra minus for the negative energy states. So although we have not changed $a(\vec{p}, E > 0)$ nor $a(\vec{p}, E < 0)$ we have to distinguish two different $a^\dagger$'s namely $a^\dagger$ and $a^{\dagger_f}$. In fact we obtain with this notation of two different $\dagger(')$s.

$$a^{\dagger_f}(\vec{p}, E > 0) = a^\dagger(\vec{p}, E > 0) \tag{10.49}$$

but

$$a^{\dagger_f}(\vec{p}, E < 0) = -a^\dagger(\vec{p}, E < 0) \tag{10.50}$$

Since at the end, the physical/usual second quantized Boson-theory has as its inner product the full inner product one should, in the physical use, use the Hermitian conjugation $\dagger_f$. So the creation operators to be identified with creation operators are respectively:

For a <u>particle</u>;

$$a^\dagger_{usual}(\vec{p}) = a^{\dagger f}(\vec{p}, E > 0) \tag{10.51}$$

while for an <u>antiparticle</u> of momentum $\vec{p}$ it is;

$$a^\dagger_{usual\ anti}(\vec{p}) = a(-\vec{p}, E < 0) \tag{10.52}$$

Similarly:

$$\begin{aligned} a_{usual}(\vec{p}) &= a(\vec{p}, E > 0) \\ a_{usual\ anti}(\vec{p}) &= a^{\dagger_f}(-\vec{p}, E < 0) = -a^\dagger(-\vec{p}, E < 0) \end{aligned} \tag{10.53}$$

Using the extended commutation rules

$$[a(\vec{p}, \gtrless E), a^\dagger(\vec{p'}, \gtrless E)] = \delta_{\vec{p'}\vec{p}} \cdot \begin{cases} 1 \text{ for same } < \text{ or } > \\ 0 \text{ for different } < \text{ or } > \end{cases} \tag{10.54}$$

so that for instance

$$[a(\vec{p}, < E), a^{\dagger}(\vec{p'}, < E)] = \delta_{\vec{p}\vec{p'}} \tag{10.55}$$

We quickly derive the correspondingcommutation rules using the "full dagger"

$$[a(\vec{p}, E > 0), a^{\dagger_f}(\vec{p'}, E > 0)] = \delta_{\vec{p}\vec{p'}} \tag{10.56}$$

and

$$[a(\vec{p}, E < 0), a^{\dagger_f}(\vec{p'}, E < 0)] = \delta_{\vec{p}\vec{p'}} \tag{10.57}$$

10.4 "Majorana-bosons"

We shall now in this section analogously to what we did in sections 10.2 for Fermions as a warm up excercise from our Fock space defined in section 10.3 for e.g. the physical sector- combination extract a subspace H_{Maj} and on that find a description of now bosons which are their own antiparticles. There would be some meaning in analogy to the Fermion case to call such bosons which are their own antiparticles by the nickname "Majorana-bosons".

As for the fermions we shall expect a state with say $k_{Maj}(\vec{p})$ "Majorana-bosons"with momentum equal to $\vec{p}$ to be presented as a superposition of a number of the "essentially charged" bosons or antibosons of the type discussed in foregoing section. Here an antiparticle of course means that one has made the number of bosons in a negative energy single particle state one unit more negative. Typically since the physical vacuum has $k(\vec{p}, E < 0) = -1$ for all momenta and an antiparticle of momentum $\vec{p}$ would mean that $k(-\vec{p}, E < 0)$ gets decreased from -1 to -2. If you have several antiparticles l say in the same state with momentum $\vec{p}$ of course you decrease $k(-\vec{p}, E < 0)$ to $-1 - l, k(-\vec{p}, E < 0) = -1 - l$ (for l antiparticles).

In other words we expect a state with say l_{Maj} "Majorana-bosons" with momentum $\vec{p}$ to be a superposition of states in the Fock space with the number of antiparticles running from $l = 0$ to $l = l_{Maj}$ while correspondingly the number with momentum $\vec{p}$ is made to $l_{Maj} - l$ so that there are together in the representing state just equally many particles or antiparticles as the number of "Majorana-bosons" l_{Maj} wanted.

We actually hope –and we shall see we shall succeed– that we can construct a "Majorana-boson" creation operator for say a "Majorana-boson" with momentum $\vec{p}$,$b^{\dagger}(\vec{p})$ analogously to the expressions (10.19) and (10.20)
$b^{\dagger}(\vec{p}) = \frac{1}{\sqrt{2}}\left(a^{\dagger}(\vec{p}, E > 0) + a(-\vec{p}, E < 0)\right)$ and $b(\vec{p}) = \frac{1}{\sqrt{2}}\left(a(E > 0) + a^{\dagger}(E < 0)\right)$.

Since an extra phase on the basis states does not matter so much we could also choose for the boson the "Majorana boson" creation and annihilation operators to be

$$
\begin{aligned}
b^{\dagger_f}(\vec{p}) &= \frac{1}{\sqrt{2}}\left(a^{\dagger}(E>0) + a(-\vec{p}, E<0)\right) \\
&= \frac{1}{\sqrt{2}}\left(a^{\dagger_f}(\vec{p}, E>0) + a(-\vec{p}, E<0)\right) \\
b(\vec{p}) &= \frac{1}{\sqrt{2}}\left(a(\vec{p}, E>0) + a^{\dagger_f}(-\vec{p}, E<0)\right) \\
&= \frac{1}{\sqrt{2}}\left(a(\vec{p}, E>0) - a^{\dagger}(-\vec{p}, E<0)\right).
\end{aligned} \tag{10.58}
$$

One must of course then check –first on the physical sector-combination but later on others– that $b^{\dagger}(\vec{p})$ and $b(\vec{p})$ obey the usual commutation rules

$$
\begin{aligned}
\left[b(\vec{p}), b(\vec{p'})\right] &= 0 \\
\left[b^{\dagger_f}(\vec{p}), b^{\dagger_f}(\vec{p'})\right] &= 0 \\
\left[b(\vec{p}), b^{\dagger_f}(\vec{p'})\right] &= \delta_{\vec{p}\vec{p'}}.
\end{aligned} \tag{10.59}
$$

We also have to have a vacuum for the "Majorana-boson" theory, but for that we use in the physical sector-combination theory the same state in the Fock space as the one for the "essentially charged" boson system. This common physical vacuum state (in the Fock space) is characterized as the basis vector

$$
| \ldots, k(\vec{p}, E>0), \ldots; \ldots, k(\vec{p}, E>0), \ldots\rangle \tag{10.60}
$$

with

$$
k(\vec{p}, E>0) = 0 \text{ for all } \vec{p} \tag{10.61}
$$

and

$$
k(\vec{p}, E<0) = -1 \text{ for all } \vec{p} \tag{10.62}
$$

Indeed we also can check then of course that defining

$$
| \, vac \; phys\rangle = | \ldots, k(\vec{p}, E>0) = 0, \ldots; \ldots, k(\vec{p'}, E<0) = -1, \ldots\rangle \tag{10.63}
$$

we have

$$
b(\vec{p}) \;\; | \, vac \; phys\rangle = 0 \tag{10.64}
$$

On the other hand, we can also see that e.g.

$$
\begin{aligned}
&\frac{1}{\sqrt{l_{Maj}!}}\left(b^{\dagger_f}(\vec{p})\right)^{l_{Maj}} | \, vac \; phys\rangle \\
&= \frac{1}{2^{l_{Maj}/2}} \cdot \Sigma_l \binom{l_{Maj}}{l} \left(a^{\dagger}(\vec{p}, E>0)\right)^{l} a(-\vec{p}, E<0)^{l_{Maj}-l} \, | \, vac \; phys\rangle \\
&= \Sigma_l \binom{l_{Maj}}{l} \;\; | \ldots, k(\vec{p}, E>0) = l, \ldots; \ldots, k(-\vec{p}, E<0) = l_{Maj} - l, \ldots\rangle \\
&\cdot \sqrt{l!}\sqrt{(l_{Maj}-l)!} \cdot \frac{1}{\sqrt{l_{Maj}!}} \\
&= \frac{1}{2^{l_{Maj}/2}} \cdot \Sigma_l \sqrt{\binom{l_{Maj}}{l}} \, | \ldots, k(\vec{p}, E>0), \ldots; \ldots, k(-\vec{p}, E<0), \ldots\rangle
\end{aligned} \tag{10.65}
$$

If we only put the Majorana-boson particles into the momentum $\vec{p}$ state of course only $k(\vec{p}, E > o)$ and $k(-\vec{p}, E < 0)$ will be different from their $| \, phys \; vac\rangle$ values 0 and -1 respectively for $E > 0$ and $E < 0$. But really the extension to put "Majorana-bosons" in any number of momentum states is trivial.

We now have also to construct the analogous operator to the $r(\vec{p})$ for the fermions so that we can characterize the subspace H_{Maj} to be for the boson case

$$H_{Maj} = \{|\rangle \mid r(\vec{p}) \; |\rangle = 0\}. \tag{10.66}$$

We in fact shall see that the proposal

$$\begin{aligned} r(\vec{p}) &= \frac{1}{\sqrt{2}} \left(a(+\vec{p}, E > 0) + a\dagger(-\vec{p}, E < 0)\right) \\ &= \frac{1}{\sqrt{2}} (a(\vec{p}, E > 0) - a^{\dagger_f}(-\vec{p}, E < 0) \end{aligned} \tag{10.67}$$

does the job.
Now we check (using (10.58))

$$\begin{aligned} &\left[r(\vec{p}), b^{\dagger_f}(\vec{p})\right] = \\ &= \left[\frac{1}{\sqrt{2}} \left(a(\vec{p}), E > 0) + a^{\dagger}(-\vec{p}, E < 0)\right), \frac{1}{\sqrt{2}} \left(a^{\dagger}(\vec{p}), E > 0) + a(-\vec{p}, E < 0)\right)\right] \end{aligned}$$

or

$$\begin{aligned} &= \frac{1}{2} \left[\left(a(\vec{p}, E > 0) - a^{\dagger_f}(-\vec{p}, E < 0)\right), (a^{\dagger_f}(\vec{p}, E > 0) + a(-\vec{p}, E < 0))\right] \\ &= \frac{1}{2}(1 - 1) = 0 \end{aligned} \tag{10.68}$$

and also

$$\begin{aligned} &[r(\vec{p}), b(\vec{p})] = \\ &= \left[\frac{1}{\sqrt{2}} \left(a(\vec{p}), E > 0) - a^{\dagger_f}(-\vec{p}, E < 0)\right), \frac{1}{\sqrt{2}} \left(a(\vec{p}), E > 0) + a^{\dagger_f}(-\vec{p}, E < 0)\right)\right] \\ &= \left[\frac{1}{\sqrt{2}} \left(a(\vec{p}), E > 0) + a^{\dagger}(-\vec{p}, E < 0)\right), \frac{1}{\sqrt{2}} \left(a(\vec{p}), E > 0) - a^{\dagger}(-\vec{p}, E < 0)\right)\right] \\ &= 0 \end{aligned} \tag{10.69}$$

We should also check that the physical vacuum

$$| \; phys \; vac\rangle = | \ldots, k(\vec{p}, E > 0) = 0, \ldots; \ldots, k(\vec{p'}, E < 0) = -1, \ldots\rangle \tag{10.70}$$

in which there is in all negative energy (with momentum $\vec{p'}$ say) single particle states $k(\vec{p'}, E < 0) = -1$ bosons, and in all positive energy single particle states $k(\vec{p}, E > 0) = 0$ bosons is annihilated by the $r(\vec{p})$ operators.

Now indeed

$$\begin{aligned}
& r(\vec{p}) \mid phys\, vac\rangle \\
&= \frac{1}{2}\left(a(\vec{p}, E>0) - a^{\dagger f}(-\vec{p}, E, 0)\right) \mid phys\, vac\rangle \\
&= \frac{1}{2}\left(a(\vec{p}, E>0) + a^{\dagger}(-\vec{p}, E, 0)\right) \cdot \mid \ldots, k(\vec{p}, E>0) = 0, \ldots; \ldots, k(\vec{p}, E<0) \\
&= -1, \ldots\rangle \\
&= 0
\end{aligned} \tag{10.71}$$

basically because of the barriere, meaning the square roots in the formulas (10.34,10.35) became zero.

The result of this physical section for the most attractive formalism with $b(\vec{p})$ and $b^{\dagger f}(\vec{p})$ annihilating and creating operators for the Boson-type particle being its own antiparticle (=Majorana Boson) and to them corresponding useful state condition operator $r_f(\vec{p})$ is summarized as:

$$\begin{aligned}
b(\vec{p}) &= \frac{1}{\sqrt{2}}\left(a(\vec{p}, E>0) - a^{\dagger f}(-\vec{p}, E<0)\right) \\
b^{\dagger f}(\vec{p}) &= \frac{1}{\sqrt{2}}\left(a^{\dagger f}(\vec{p}, E>0) - a(-\vec{p}, E<0)\right) \\
\mid phys\, vac\rangle &= \mid \ldots, k(\text{all } \vec{p}, E>0) = 0, \ldots; \ldots, k(\text{all } \vec{p}, E<0) = -1, \ldots\rangle \\
r_f(\vec{p}) &= \frac{1}{\sqrt{2}}\left(a(\vec{p}, E>0) + a^{\dagger f}(-\vec{p}, E<0)\right)
\end{aligned} \tag{10.72}$$

the useful subspace for bosons being their own antiparticles being

$$H_{Maj\ f} = \left\{\mid\rangle \ \mid \forall_{\vec{p}} r_f(\vec{p}) \mid\rangle = 0\right\} \tag{10.73}$$

(One should note that whether one chooses our $r(\vec{p'})'s$ or the $r_f(\vec{p})'s$ to define makes no difference for the space $H_{Maj\ f}$ rather than H_f, since we actually have $r(\vec{p}) = r_f(\vec{p})$ the two expressions being just expressed in terms of different $a^{\dagger}(\vec{p}, E<0)$ and $a^{\dagger f}(\vec{p}, E<0)$ say)

We can easily check that our explicit state expressions (10.65) indeed are annihilated by $r(\vec{p})$ It were formally left out the$E>0$ or $E<0$ for the $b(\vec{p})$ and $b^{\dagger f}(\vec{p})$ it being understood that $E>0$, but formally we can extrapolate also to $E<0$ and it turns of $b(\vec{p}, E>0) = b^{\dagger f}(-\vec{p}, E<0)$?

10.4.1 Charge Conjugation Operation

Since we discuss so much bosons being their own antiparticles coming out of a formalism in which the bosons –at first– have antiparticles <u>different</u> from themselves, we should here define a charge conjugation operator **C** that transform a boson into its antiparticle:

That is to say we want this operator acting on the Fock space to have the commutation properties with our creation and annihilation operators

$$\mathbf{C}^{-1} a(\vec{p}, E > 0)\mathbf{C} = a^{\dagger_f}(-\vec{p}, E < 0) \tag{10.74}$$

and

$$\mathbf{C}^{-1} a^{\dagger_f}(\vec{p}, E > 0)\mathbf{C} = a(-\vec{p}, E < 0). \tag{10.75}$$

We also have

$$\mathbf{C}^{-1} a(\vec{p}, E < 0)\mathbf{C} = a^{\dagger_f}(-\vec{p}, E > 0) \tag{10.76}$$

and

$$\mathbf{C}^{-1} a^{\dagger_f}(\vec{p}, E < 0)\mathbf{C} = a(-\vec{p}, E > 0) \tag{10.77}$$

These requirements suggest that we on the basis of (10.43) for the Fock space have the operation

$$\begin{aligned} &\mathbf{C}\,|\ldots \tilde{k}\,(\vec{p}, E > 0), \ldots; \ldots, \tilde{k}\,(\vec{p'}, E < 0), \ldots\rangle \\ &=|\ldots, k(\vec{p}, E > 0) = -\,\tilde{k}\,(-\vec{p'}, E < 0) + 1, \ldots; \\ &\quad \ldots, k(\vec{p'}, E < 0) = -1-\,\tilde{k}\,(-\vec{p'}, E > 0), \ldots\rangle. \end{aligned} \tag{10.78}$$

Using the "full inner product" this **C** operation conserves the norm, and in fact it is unitary under the full inner product corresponding hermitean conjugation $\dagger_f$ i.e.

$$\mathbf{C}^{\dagger_f}\mathbf{C} = \mathbf{1} = \mathbf{C}\mathbf{C}^{\dagger_f} \tag{10.79}$$

But if we used the not full inner product, so that the norm squares for basis vector would be given by (10.81) and therefore corresponding hermitean conjugation $\dagger$, then if **C** acts on a state in which the difference of the number of positive and negative energy bosons is odd, the norm square would change sign under the operation with **C**.

So under $\dagger$ the charge conjugation operator could not possibly be unitary:

$$\mathbf{C}^{\dagger}\mathbf{C} \neq \mathbf{1} \neq \mathbf{C}\mathbf{C}^{\dagger} \tag{10.80}$$

10.5 "Majorana boson" in unphysical sector-combination

As an example of one of the unphysical sector-combination we could take what in our earlier work "Dirac sea for Bosons" were said to be based on the naive vacuum. This naive vacuum theory means a theory in which we do not make any emptying vacuum but rather let there be in both positive and negative single particle energy states a positive or zero number of particles. So in this naive vacuum attached sector combination we can completely ignore the extrapolated negative number of boson possibilities; we so to speak could use the analogue of the harmonic oscillator with normalized states only.

This means that the inner product excluding the negative single particle state normalization using (10.31) will be for this naive vacuum sector combination completely positive definite.

However, including the negative norm factor for the negative energy states from (10.31) so as to get the full inner product we do no longer have the positive definite Hilbert inner product on the Fock space. Now rather we have for basis vectors (10.43) instead of (10.45) that the norm squares

$$\langle \ldots, k(\vec{p}, E > 0), \ldots; \ldots, k(\vec{p'}, E < 0), \ldots \mid \ldots, k(\vec{p}, E > 0), \ldots; \ldots, k(\vec{p'}, E < 0), \ldots \rangle \\ = (-1)^{\sharp(neg.\ energy\ b.)} \tag{10.81}$$

This means that the norm square of a basis vector is positive when the number of negative energy bosons is even, but negative when the number of negative energy bosons is odd.

In the naive vacuum sector combination the vacuum analogue Fock space state is the "naive vacuum",

$$\mid naive\, vac.\rangle = \mid \ldots, k(\vec{p}, E > 0) = 0, \ldots; \ldots, k(\vec{p}, E < 0) = 0, \ldots \rangle. \tag{10.82}$$

In analogy to what we did in the foregoing section 10.4, we should then construct the states with various numbers of bosons of the Majorana type being their own antiparticles by means of some creation and annihilation operators $b^{\dagger_f}(\vec{p})$ and $b(\vec{p})$, but first one needs a vacuum that is its own "anti state" so to speak, meaning that the charge conjugation operator **C** acting on it gives it back. i.e. one need a vacuum $\mid vac?\rangle$ so that

$$\mathbf{C} \mid vac?\rangle = \mid vac?\rangle \tag{10.83}$$

But this is a trouble! The "naive vacuum" $\mid naive\, vac.\rangle$ in not left invariant under the charge conjugation operator **C** defined in the last subsection of Section 10.4 by (10.78).

Rather this naive vacuum is by **C** transformed into a quite different sector combination, namely in that sector combination, in which there is a negative number of bosons in both positive and negative energy single particle eigenstates. i.e. the charge conjugation operates between one sector combination and another one! But this then means, that we cannot make a representation of a theory with (only) bosons being their own antiparticles unless we use more than just the naive vacuum sector combination. i.e. we must include also the both number of particles being negative sector combination.

In spite of this need for having the two sector combinations –both the naive all positive particle number and the opposite all negative numbers of particles– in order that the charge conjugation operator should stay inside the system –Fock space, we should still have in mind that the creation and annihilation operators cannot pass the barriers and thus can not go from sectors, also the inner product between different sector combinations are divergent and ill defined (and we should either avoid such inner products or define them arbitrarily).

So if we construct "Majorana boson" creation and annihilation operators analogoulsy to the $b(\vec{p})$ and $b^{\dagger_f}(\vec{p})$ in foregoing section as a linear combination of

$a^{(\dagger)}(\vec{p}, \gtrless E)$ operators operating with such $b(\vec{p})$ and $b^{\dagger_f}(\vec{p'})$s will stay inside one sector combination. For instance such $b(\vec{p})$ and $b^{\dagger_f}(\vec{p})$ constructed analogously to the physical sector ones formally would operate arround staying inside the naive vacuum sector combination if one starts there, e.g. on the naive vacuum $|\, naive\ vac.\rangle$. In this –slightly cheating way– we could then effectively build up a formalism for bosons which are their own antiparticles inside just <u>one</u> sector combination. When we say that it is "slightly cheating" to make this construction on only one sector combination it is because we cannot have the true antiparticles if we keep to a sector combination only, which is not mapped into itself by the charge conjugation operator **C**. It namely then would mean that the true antiparticle cannot be in the same sector combination.

Nevertheless let us in this section 5 study precisely this "slightly cheating" formalism of keeping to the naive vacuum sector combination with positive numbers of particles only.

We then after all simply use the naive vacuum $|\, naive\ vac.\rangle$ defined by (10.82) as the "Majorana boson"-vacuum although it is not invariant under **C**, which we must ignore or redefine, if this shall be o.k.

We may e.g. build up a formalism for the slightly cheating Majorana bosons by starting from the $|\, naive\ vac.\rangle$ (10.82) and build up with $b^{\dagger_f}(\vec{p})$ taken to be the same as

$$b^{\dagger_f}(\vec{p}) = \frac{1}{\sqrt{2}}\left(a^{\dagger_f}(\vec{p}, E>0) + a(-\vec{p}, E<0)\right) \tag{10.84}$$

and

$$\begin{aligned} b(\vec{p}) &= \frac{1}{\sqrt{2}}\left(a(\vec{p}, E>0) + a^{\dagger_f}(-\vec{p}, E<0)\right) \\ &= \frac{1}{\sqrt{2}}\left(a(\vec{p}, E>0) - a^{\dagger}(-\vec{p}, E<0)\right) \end{aligned} \tag{10.85}$$

We have already checked that for all sector combinations we have

$$\left[b(\vec{p}, b^{\dagger_f}(\vec{p'})\right] = \delta_{\vec{p}\vec{p'}} \tag{10.86}$$

and of course

$$\begin{aligned} \left[b(\vec{p}, b(\vec{p'}\right] &= 0 \\ &= \left[b^{\dagger_f}(\vec{p}), b^{\dagger_f}(\vec{p'})\right] \end{aligned} \tag{10.87}$$

So we see that we can build up using $b(\vec{p})$ and $b^{\dagger_f}(\vec{p})$ a tower of states with any nonnegative number of what we can call the Majorana bosons for any momentum $\vec{p}$.

We can also in all the sector combinations use the already constructed

$$\begin{aligned} r(\vec{p}) &= \frac{1}{2}\left(a(\vec{p}, E>0) - a^{\dagger_f}(-\vec{p}, E<0)\right) \\ &= \frac{1}{2}\left(a(\vec{p}, E>0) + a^{\dagger}(-\vec{p}, E<0)\right) \end{aligned} \tag{10.88}$$

to fullfill the commutation conditions

$$\left[r(\vec{p}), b^{\dagger_f}(\vec{p})\right] = 0$$
$$\left[r(\vec{p}), b(\vec{p'})\right] = 0 \tag{10.89}$$

and we even have

$$r(\vec{p}) \mid naive\ vac.\rangle = 0 \tag{10.90}$$

So indeed we have gotten a seemingly full theory of "Majorana Bosons" inside the naive vacuum sector combination subspace

$$H_{Maj} = \{|\rangle \mid \forall_{\vec{p}}\ (r(\vec{p})\ |\rangle = 0)\} \tag{10.91}$$

but it is not kept under the **C** as expected.

But really what we ended up constructing were only a system of positive energy particle states since the creation with $b^{\dagger_f}(\vec{p}) = b^{\dagger}(\vec{p})$ starting from the naive vacuum only produces positive energy particles in as far as the $a(-\vec{p}, E < 0)$ contained in $b^{\dagger_f}(\vec{p})$ just gives zero on the naive vacuum.

So this a "bit cheating" formalism really just presented for us the "essentially charged" positive energy particles as "the Majorana-bosons".

That is to say this a bit cheating formalism suggests us to use in the naive vacuum sector combination the "essentially charged particles" as were they their own antiparticles.

If we similarly built a Majorana boson Fock space system of the

$$\begin{aligned}\mathbf{C} \mid naive\ vac.\rangle &= \mid vac.\ with\ both\ E > 0\ and\ E < 0\ emptied\ out\rangle \\ &= \mid \ldots, k(\vec{p}, E > 0) = -1, \ldots; \ldots, k(\vec{p}, E < 0) = -1, \ldots\rangle,\end{aligned} \tag{10.92}$$

we would obtain a series of essentially antiparticles (with positive energies) constructed in the "both numbers of bosons negative" sector combination.

What we truly should have done were to start from the superposition

$$\begin{aligned}\mid self\ copy\ vac.\rangle &\hat{=} \frac{1}{\sqrt{2}}\left(\mid naive\ vac.\rangle + \mathbf{C} \mid naive\ vac.\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(\mid \ldots, k(\vec{p}, E > 0) = 0, \ldots; \ldots, k(\vec{p}, E < 0) = 0, \ldots\rangle\right. \\ &\quad \left. + \mid \ldots, k(\vec{p}, E > 0) - 1, \ldots; \ldots, k(\vec{p}, E < 0) = -1, \ldots\rangle\right)\end{aligned} \tag{10.93}$$

and then as we would successively go up the latter with $b^{\dagger_f}(\vec{p})$ operators we would successively fill equally many positive energy particles into the $\mid naive\ vac.\rangle$ and positive energy antiparticles in $\mathbf{C} \mid naive\ vac.\rangle$. Note that analogously to the above called "a bit cheating" Majorana-boson construction using *only the positive energy single particle states* we obtain here only use of the positive energy states for the naive vacuum sector combination and only the negative energy single particle states for the Charge conjugation to the naive vacuum sector combination. Also it should not be misunderstood: The filling in is *not* running parallel in the sense that the sectors truly follow each other. Rather one has to look for if there is Majorana boson by looking into *both* sector-combination- projections.

So we see that what is the true Majorana boson theory built on the two unphysical sector combinations having respectively nonzero numbers of particles (the naive vacuum construction) and negative particles number in both positive and negative energies is the following:

A basis state with $n(\vec{p})$ Majorana bosons with momentum $\vec{p}$, -and as we always have for Majorana's positive energy- gets described as a superposition ot two states –one from each of the two sector combinations– with just $n(\vec{p})$ ordinary (positive energy) essentially charged bosons (of the original types of our construction created by $a^{\dagger}$..) and a corresponding Fock space state from the other sector, now with $n(\vec{p})$ antiparticles in the other sector combination (the one built from $\mathbf{C} \mid naive\, vac.\rangle$).

Both of these separate sector combinations have for the used states a positive definite Hilbert space.

As already stated the overlap between different sector combinaions vectors are divergent and illdefined.

We can check this rather simple way of getting the Majorana bosons described in our on the state $\frac{1}{2}\left(\mid naive\, vac.\rangle + \mathbf{C} \mid naive\, vac.\rangle\right)$ built system of states by noting what the condition $r(\vec{p})\mid\rangle = 0$ tells us the two sector combinations:

On a linear combination of basis vectors of the naive vacuum construction type

$$\begin{aligned} |\rangle = \Sigma \mid \ & k(\vec{p}, E > 0) \geq 0, \ldots; \ldots, k(\vec{p}, E < 0) \geq 0, \ldots\rangle \\ & C_{\ldots k(\vec{p}, E>0)\ldots;\ldots \tilde{k}(\dot{\vec{p}}, E<0)\ldots} \end{aligned} \tag{10.94}$$

the requirement

$$r(\vec{p}) \mid\rangle = 0 \tag{10.95}$$

relates coefficients which correspond to basis states being connected by $k(\vec{p}, E > 0)$ going one up while $k(-\vec{p}, E < 0)$ going one unit down or opposite. As we get the relation

$$\begin{aligned} &\sqrt{1 + k(\vec{p}, E > 0)}C_{\ldots k(\vec{p}, E>0)+1, \ldots;\ldots, k(-\vec{p}, E<0), \ldots} \\ &+ C_{\ldots, k(\vec{p'}, E>0), \ldots;\ldots, k(-\vec{p}, E<0)-1, \ldots} \sqrt{k(-\vec{p}, E < 0)} = 0 \end{aligned} \tag{10.96}$$

we can easily see that the states being annihilated are of the form

$$\sum (-1)^{k(\vec{p}, E>0)} \frac{\sqrt{k(-\vec{p}, E < 0)!}}{\sqrt{k(\vec{p}, E > 0)!}} \cdot \mid \ldots, k(\vec{p}, E > 0), \ldots; \ldots, k(-\vec{p}, E < 0), \ldots\rangle, \tag{10.97}$$

where we sum over $k(\vec{p}, E > 0)$ and the difference $d = k(\vec{p}, E > 0) - k(-\vec{p}, E < 0)$ is FIXED.

As a special case we might look at possibility that the difference

$$d = k(\vec{p}, E > 0) - k(-\vec{p}, E < 0) \tag{10.98}$$

were 0. In this case the $|\ naive\ vac.\rangle$ itself would be in the series. In this case the solution (10.97) reduces to

$$\sum_{k=0} (-1)^k \mid \ldots, k(\vec{p}, E > 0) = k, \ldots; \ldots, k(-\vec{p}, E < 0) = k, \ldots\rangle \tag{10.99}$$

But it is now the problem that this series does not converge. But for appropriate values of the difference d,

$$d \geq 2, \tag{10.100}$$

the series (10.97)converge.

For the convergent cases we can estimate the norm square of a state (10.97) to go proportional to

$$\| \,|\rangle\|^2 \propto \sum_{k=0}^{\infty} \frac{(k-d)!}{k!}(-1)^{k-d} \tag{10.101}$$

where the $(-1)^{k-d}$ now comes from the alternating "full" norm square due to the factor $(-1)^{\sharp(neg.\ energy\ b.)}$. This expression in turn is proportional to

$$\begin{aligned} \sum_{k=0}^{\infty} \binom{k-d}{-d} (-1)^{k-d} &= \sum_{n=-d} \binom{n}{-d} (-1)^n \qquad (n = k-d) \\ &= \frac{(-1)^{-d}}{(1-(-1))^{-d+1}} \end{aligned} \tag{10.102}$$

which is zero for $d - 1 \geq 1$.

So indeed it is seen that the basis states in H_{Maj} part inside the naive vacuum sector combination has zero norm. Since the states with different numbers of Majorana-bosons are represented by mutually orthogonal it means that the whole part of the naive vacuum sector combination used to represent the Majorana-bosons has totally zero inner product. Basically that means that the inner product transfered from the original theory with its "essentially charged bosons" to the for Majorana bosons in subspace H_{Maj} turns out to be zero.

This result means –extrapolating to suppose zero norm also in the divergent cases– that in the unphysical sector combination we get no non-trivial inner product for the Majorana-bosons.

If ones use the true Majorana boson description by as necessary combining two sector combinations, one could use the ambiguity (and divergence) of the inner product of states from different sectors to make up instead a non trivial inner product.

10.5.1 Overview of All four Sector Combinations

Strictly speaking we could make an infinite number of sector combinations, because we for every single particle state – meaning for every combination of a spin state and a momentum say $\vec{p}$ – could choose for just that single particle state to postulate the second quantized system considered to be started at such a side of the "barriers" that just this special single particle state had always a negative number of bosons in it. For another one we could instead choose to have only a non-negative numbe of bosons. Using all the choice possibilities of this type would lead us so to speak to the infinite number of sector combinations $2^{\#\text{"single particle states"}}$, where #"single particle states" means the number of single particle states. But most of these enormously many sector combinations would not be Lorentz invariant nor rotational invariant. Really, since the sector combination should presumably rather be considered a part of the initial state condition than of the laws of Nature, it might be o.k. that it be not Lorentz nor rotational invariant. Nevertheless we strongly suspect that it is the most important to consider the Lorentz and rotational invariant sector-combination-choices. Restricting to the latter we can only choose a seprate sector for the positive enegry states and for the negative energy sector, and then there would be only $2^2 = 4$ sector combinations.

Quite generally we have the usual rules for creation and annihilation operators,but you have to have in mind that we have two different hermitean conjugations denoted respectively by $\dagger$ and by $\dagger_f$, and that the creation operators constructed from the same annihilation operators are related

$$\begin{aligned} a^{\dagger_f}(\vec{p}, E > 0) &= a^{\dagger}(\vec{p}, E > 0) \\ a^{\dagger_f}(\vec{p}, E < 0) &= -a^{\dagger}(\vec{p}, E < 0) \end{aligned} \tag{10.103}$$

These "usual" relations are

$$\begin{aligned} \left[a(\vec{p}, E > 0), a^{\dagger}(\vec{p'}, E > 0)\right] &= \delta_{\vec{p}\vec{p'}} \\ \left[a(\vec{p}, E < 0), a^{\dagger}(\vec{p'}, E < 0)\right] &= \delta_{\vec{p}\vec{p'}} \\ \left[a(\vec{p}, E > 0), a^{\dagger_f}(\vec{p'}, E > 0)\right] &= \delta_{\vec{p}\vec{p'}} \\ \left[a(\vec{p}, E < 0), a^{\dagger_f}(\vec{p'}, E < 0)\right] &= -\delta_{\vec{p}\vec{p'}} \end{aligned} \tag{10.104}$$

while we have exact commutation for a with a or for $a^{\dagger}$ or $a^{\dagger_f}$ with $a^{\dagger}$ or $a^{\dagger_f}$. Each $a(\vec{p}, E \gtrless 0)$ or $a^{\dagger_f}$ or $a^{\dagger}$ act changing only the number of particle in just the single relevant single particle state, meaning it changes only $k(\vec{p}, E \gtrless 0)$; the rules are as seen analytical continuations generally

$$
\begin{aligned}
&a^{\dagger}(\vec{p}, E>0)\,|\ldots, k(\vec{p}, E>0), \ldots; \ldots, k(\vec{p}', E<0), \ldots\rangle \\
=&\sqrt{k(\vec{p}, E>0)+1}\,|\ldots, k(\vec{p}, E>0)+1, \ldots; \ldots, k(\vec{p}', E<0), \ldots\rangle \\
&a^{\dagger}(\vec{p}', E<0)\,|\ldots, k(\vec{p}, E>0), \ldots; \ldots, k(\vec{p}', E<0), \ldots\rangle \\
=&\sqrt{k(\vec{p}', E<0)+1}\,|\ldots, k(\vec{p}, E>0), \ldots; \ldots, k(\vec{p}', E<0)+1, \ldots\rangle \\
&a^{\dagger_f}(\vec{p}, E>0)\,|\ldots, k(\vec{p}, E>0), \ldots; \ldots, k(\vec{p}', E<0), \ldots\rangle \\
=&\sqrt{k(\vec{p}, E>0)+1}\,|\ldots, k(\vec{p}, E>0)+1, \ldots; \ldots, k(\vec{p}', E<0), \ldots\rangle \\
&a^{\dagger_f}(\vec{p}', E<0)\,|\ldots, k(\vec{p}, E>0), \ldots; \ldots, k(\vec{p}', E<0), \ldots\rangle \\
=&-\sqrt{k(\vec{p}', E<0)+1}\,|\ldots, k(\vec{p}, E>0), \ldots; \ldots, k(\vec{p}', E<0)+1, \ldots\rangle \\
&a(\vec{p}, E>0)\,|\ldots, k(\vec{p}, E>0), \ldots; \ldots, k(\vec{p}', E<0), \ldots\rangle \\
=&\sqrt{k(\vec{p}, E>0)}\,|\ldots, k(\vec{p}, E>0)-1, \ldots; \ldots, k(\vec{p}', E<0), \ldots\rangle \\
&a(\vec{p}', E<0)\,|\ldots, k(\vec{p}, E>0), \ldots; \ldots, k(\vec{p}', E<0), \ldots\rangle \\
=&\sqrt{k(\vec{p}', E<0)}\,|\ldots, k(\vec{p}, E>0)+1, \ldots; \ldots, k(\vec{p}', E<0)-1, \ldots\rangle
\end{aligned} \tag{10.105}
$$

The four sector combination with the same sector for the same sign of the energy E of the single particle states were called:

1)The "physical sector" has

$$
\begin{aligned}
k(\vec{p}, E>0) &= 0, 1, 2, \ldots \\
k(\vec{p}, E<0) &= -1, -2, -3\ldots
\end{aligned} \tag{10.106}
$$

2)The "sector-combination constructed from the naive vacuum" has

$$
\begin{aligned}
k(\vec{p}, E>0) &= 0, 1, 2, \ldots \\
k(\vec{p}, E<0) &= 0, 1, 2, \ldots
\end{aligned} \tag{10.107}
$$

3)The "both sectors with negative numbers" sector-combination has

$$
\begin{aligned}
k(\vec{p}, E>0) &= -1, -2, -3\ldots \\
k(\vec{p}, E<0) &= -1, -2, -3\ldots
\end{aligned} \tag{10.108}
$$

4)The "a positive number with negative energy and vise versa" has

$$
\begin{aligned}
k(\vec{p}, E>0) &= -1, -2, -3\ldots \\
k(\vec{p}, E<0) &= 0, 1, 2, \ldots
\end{aligned} \tag{10.109}
$$

In the physical sector combination the Fock space ends up having positive definite norm square and so this sector-combination is the one usual taken for being the in nature realized one.

10.5.2 Formulas for "Majorana particles"

The theory of Majorana fermions may be so well known that we had nothing to say, but it were written about it in section 2.

For the boson case we introduced for each (vectorial) value of the momentum an operator acting on the Fock space called $r(\vec{p})$ defined by (10.88)

$$\begin{aligned} r(\vec{p}) &= \frac{1}{\sqrt{2}}\left(a(\vec{p}, E > 0) - a^{\dagger_f}(-\vec{p}, E < 0)\right) \\ &= \frac{1}{\sqrt{2}}\left(a(\vec{p}, E > 0) + a^{\dagger}(-\vec{p}, E < 0)\right) \end{aligned} \tag{10.110}$$

with the properties

$$\begin{aligned} \left[r(\vec{p}), b(\vec{p'})\right] &= 0 \\ \left[r(\vec{p}), b^{\dagger}(\vec{p'})\right] &= 0 \\ \left[r(\vec{p}), b^{\dagger_f}(\vec{p'})\right] &= 0 \end{aligned} \tag{10.111}$$

where the creation $b^{\dagger_f}(\vec{p})$ $\left(= b^{\dagger_f}(\vec{p}\right)$ and annihilation $b(\vec{p})$ operators for the "Majorana bosons" (i.e. boson being its own antiparticle) were defined in terms of the a's as

$$b^{\dagger_f}(\vec{p}) = \frac{1}{\sqrt{2}}\left(a^{\dagger_f}(\vec{p}, E > 0) + a(-\vec{p}, E < 0)\right) \tag{10.112}$$

and

$$b(\vec{p}) = \frac{1}{\sqrt{2}}\left(a(\vec{p}, E > 0) + a^{\dagger_f}(-\vec{p}, E < 0)\right) \tag{10.113}$$

$$= \frac{1}{\sqrt{2}}\left(a(\vec{p}, E > 0) - a^{\dagger}(-\vec{p}, E < 0\right) \tag{10.114}$$

These operators obey (see(10.86) and (10.87))

$$\begin{aligned} \left[b(\vec{p}, b^{\dagger_f}(\vec{p'})\right] &= \delta_{\vec{p}\vec{p'}} \\ \left[b(\vec{p}, b(\vec{p'}\right] &= 0 \\ \left[b^{\dagger_f}(\vec{p}), b^{\dagger_f}(\vec{p'})\right] &= 0 \end{aligned} \tag{10.115}$$

and so these operators are suitable for creating and annihilation of particles, and indeed these particles are the "Majorana bosons". As a replacement for the in usual formalism for "Majorana bosons" say

$$b(\vec{p}, E) = b^{\dagger_f}(-\vec{p}, -E) \tag{10.116}$$

we have in our notation

$$b(-\vec{p}) \mid_{\substack{with \\ >\leftrightarrow<}} = b^{\dagger_f}(\vec{p}) \tag{10.117}$$

as is easily seen from (10.113) and (10.112) just above.

But now we need also a vacuum from which to start the creation of the "Majorana bosons" with $b^{\dagger_f}(\vec{p})$. In the two sector-combinations 1) the physical one and 4) "a positive number with negative energy and vice versa" there are the suitable vacua:

In 1)

$$| \, physical \, vac\rangle = | \ldots, k(\vec{p}, E > 0) = 0, \ldots ; \ldots, k(\vec{p}, E < 0) = -1, \ldots\rangle \tag{10.118}$$

and in 4)

$$| \begin{matrix} pos \text{ in } E < 0 \\ neg \text{ in } E > 0 \end{matrix} \rangle = 1 \ldots, k(\vec{p}, E > 0) = -1, \ldots ; \ldots, k(\vec{p}, E < 0) = 0, \ldots\rangle \tag{10.119}$$

In the sector-combinations 2) and 3), however, there are no charge conjugation symmetric states to use as the vacuum state for a "Majorana-boson" formalism. In this case the vacuum of 2) goes under charge conjugation **C** into that of 3).

10.6 Outlook on String Field Theory Motivation

One of our own motivations for developping the sort of boson Dirac sea theory for bosons being their own antiparticles, i.e. a theory with Dirac sea, were to use it in our own so called "novel string field theory"[8–11].

In this "novel string field theory" we sought to rewrite the whole of string theory[12–15,28,29] (see also modified cubic theory [24])- although we did not yet come to superstrings[25,17,26] although that should be relatively easy - into a formalism in which there seems a priori to be no strings. The strings only come out of our novel string field theory [18–23] by a rather complicated special way of looking at it. In fact our basic model in this novel string field theory is rather like a system of /a Fock space for massless scalar particles, which we call "objects" in our formulation, but they have much although not all properties similar to scalar massless particles. These particles/objects we think must be in an abstract way what we here called Majorana bosons. This means they should be their own antiparticles to the extend that they have antiparticles.

But their being put into cyclically ordered orientable chains may put a need for a deeper understanding of the Majorananess for these "objects".

The reason for the objects, that in our novel string field theory are a kind of constituents, for the strings being supposed to a nature reminiscent of the Majorana particles or being their own antiparticles, is that they carry in themselves no particle number or charge, except that they can have (26)-momentum. (For complete consistency of the bosonic string theory it is wellknown that 26 space time dimensions are required.) The bulk of the string (in string theory) can namely be shrunk or expanded ad libitum, and it is therefore not in itself charged, although it can carry some conserved quantum numbers such as the momentum densities.

We take this to mean that the string as just bulk string should be considered to be equal to its own antimaterial. If we think of splitting up the string into small pieces like Thorn[27], or we split the right and left mover parts separately like we did ourselves, one would in both cases say that the pieces of Thorn's or the objects of ours should be their own antiparticles. With our a bit joking notation: they should be Majorana. Thus we a priori could speculate that, if for some reason we should also like to think our objects as particles, then from the analytical properties of the single particle in relativistic theories must have both positive and negative energy states. Then a treatment of particles being their own antiparticles in the Dirac sea formulation could - at least superficially -look to be relevant.

One could then ask, what we learned above, that could be of any help suggesting, how to treat long series of "objects", if these objects are to be considered bosons that are their own antiparticles:

- 1. In the novel string field theory of ours it is important for the association to the strings, that one considers ring shaped chains of objects. We called such ring shaped chains of objects for "cyclically ordered chains". Now such ordering of our "objects" (as we call them), or of any type of particles, into chains in which each particle (or "object") can be assigned a number (although in our special model only a number modulo some large number N) is o.k. for particles with an *individuality*. However, if we have particles (or "objects") that are say *bosons*, then all particles are identical - or one could say any allowed state is a superposition of states in which all possible permutations on the particles have been performed and a superposition of the results of all these permutations with same amplitude only is presented as the final state -. But this then means that *one cannot order them*, because you cannot say, which is before which in the ordering, because you cannot *name* the single particle. You could only say, that some particle A is, say, just before some particle B in a (cyclic) ordering, if you characterize A as being the particle with a certain combination of coordinates (or other properties) and B as being the one with a certain other combination of coordinates (and other properties). Unless you somehow specify by e.g. some approximate coordinates (or other characteristic) which particle you think about, it has no meaning to express some relation involving the relative ordering, say, of two bosons.
- 2. The problem just mentioned in assigning order to bosons means, that the concept of "cyclically ordered chains" of objects - or for that matter building up any string from particle pieces like Thorn say - cannot be done once the particles or objects are bosons, but rather should be preferably formulated *before one symmetrize the wave function* under the particle permutation so as to implement that they are bosons. One shall so to speak go back in the "pedagogical" development of boson-theory and think in the way before the symmetry principle under permutations making the particles bosons were imposed. In this earlier stage of the description the cyclically ordered chains, or any type of ordering, which one might wish, makes sense. So here it looks

that going back and postponing the boson constraint is needed for ordering chains.

- 3. But seeking to go back prior to boson or fermion formulation makes a problem for the Dirac sea - in both boson and fermion cases -: If we want to consider the case of individual particles or objects fully, we have to imagine that we have given *names (or numbers) to all the particles in the Dirac sea*!
  For this problem we may think of a couple of solutions:
  - a. We could imagine an interaction that would organize the particles in the ground state (to be considered a replacement for the physical vacuum) or that some especially important state for the Fock space obtained by imposing some other principle is postulated to make up a kind of vacuum state. Then one could hope or arrange for the interaction or state-selecting principle chosen, that the vacuum state becomes such, that the objects (or particles) in the Dirac sea goes into such a state, that these objects have such positions or momenta, that it due to this state becomes possible to recognize such structure that their ordering in the wanted chain becomes obvious. If so, then the (cyclic) ordering can come to make sense.
    This solution to the problem may be attractive a priori, because we then in principle using the now somewhat complicated state of the vacuum can assign orderings to the whole Dirac sea, and thus in principle give an individuality even to the Dirac sea particles and missing particles / the holes can make sense, too. They so to speak can inherit their individuality from the particles missing, which before being removed were sitting in the chains of the vacuum. We have thus at least got allowance to talk about a chain ordering for pieces of chains for the holes. There is so to speak an ordering of the holes given by the ordering of the particles removed from the Dirac sea originating from the chain postulated to have appeared from the interaction or from some special selection principle for the vacuum state.
    A little technical worry about the "gauge choice" in our novel string field theory: In our novel string field theory we had made a gauge choice for the parametrization of the strings, that led to the objects having a special component of their momenta p^+, or in the language of our papers on this string field theory $J^+(I)$ for the Ith object in the chain *fixed* to a chosen small value $a\alpha'/2$. Since the argument for there having to be negative energy solutions(to say the Dirac equation) and thus a need for a Dirac sea at all is actually analyticity of the equation of motion, we would suppose that also for our objects one should keep "analyticity" in developing ones picture of the "negative energy states" and thereby of the Dirac sea. But then the p^+ or J^+, which is fixed to constant could hardly get continued to anything else than the same constant ? This sounds a bit unpleasant, if we imagine the p^+ be lightlike or timelike, because then we cannot find the negative energy state with the chosen gauge condition, and the whole reason for the Dirac sea seems to have disappeared. And thus the discussion of Majorana may also have lost its ground. But if we imagine the gauge choice fixed component to be spacelike, then we obtain, that

the gauge condition surface intersects the light cone in two disconnected pieces that are actually having respectively positive and negative energy. So assumming the gauge choice done with a space-like component we have indeed the possibility of the Majorananess discussion! And also in this case of a spacelike p or J component being fixed (by gauge choice) our construction of the Majorana bosons makes perfect sense.

Now it gets again severely complicated by the chains postulated in the vaccuum. In the space-like gauge fixing case it also becomes of course complicated, but the complication is due to the complicated state rather than to the gauge fixing alone.

Let us, however, stress again: To make a ordering of the objects in the Dirac sea into say cyclically ordered chains a much more complicated state in the Fock space is needed than the simple say physical vacuum.

To figure out how to think about such a situation with a "complicated" vacuum state replacing the, say, "physical vacuum" as discussed above, we might think about the analogous situation with the fermions. When one has a quantum field theory with fermions having interactions, it means that the interaction part of the Hamiltonian has caused that the ground state for the full Hamiltonian is no longer the state with just the Dirac sea fillied and the positive energy single particle states empty. Rather it is a "complicated" superposition of states in the Fock space, most of which would in the free theory have positive energy. These are states which can be described as states with some - infinite - number of positive energy fermions and some anti-fermions present (in addition to the vacuum with just the Dirac sea filled). The presence of anti-fermions (holes) means, that if one acts with a creation operators $b^\dagger(\vec{p}, E < 0)$ for inserting a fermion with a negative energy $(E < 0)$, then one shall not necessarily get 0 as in the free theory vacuum, because one has the possibility(chance) of hitting a single particle state in which there is a *hole*. The Fock-space state created by such an action will have higher full Hamiltonian energy than the "interaction vacuum", because the latter is by definition the lowest energy state, but one has anyway succeeded in inserting a fermion in a state which from the free theory counted has a negative energy. It should be absolutely possible that such an inserted in the just mentioned sense negative energy particle could be part of the construction of say a bound state or some composite object resonance or so. Similarly it could on top of a "complicated vacuum" (meaning a ground state e.g. for the full Hamiltonian but not for the free one) be possible to remove with an annihilation operator $a(\vec{p}, E > 0)$ a particle from a single particle state (having with the free Hamiltonian) positive energy $(E > 0)$. One could namely have the chance of hitting a positve energy single particle state, in which there already is a particle in the "complicated vacuum". Such a removal or hole in a positive single particle state is what we ought to call a "negative energy anti-particle".

We here sought to argue, that if one for some reason or another (because of interaction and taking the ground state, or because one has postuleted some "complicated vacuum" just to make ordering make sense) use a "complicated vacuum", then it becomes possible formally to add particles or anti-particles with *negative energy*.

Especially we want to stress the possibility that, if one wants to describe properly a resonance or a bound state composed or several particles (e.g. fermions) then one might need to assign some of the constituents negative energy in the sense just alluded to here.

Strictly speaking it comes to look in the "complicated vacuum" as if one has got *doubled the number of species of effective particle*, because one now by acting with e.g. $a^\dagger(\vec{p}, E > 0)$ both can risk to produce a positive energy particle, and can risk to fill in a hole in positive energy single particle state and thereby creating a negative energy anti-particle. So operating with the same operator we risk *two different results*, which may be interpreted as if one had effectively had two different types of operators and thereby doubly as many types of particles as we started with. We have so to speak - in the case of non-Majorana particles - gotten both positive and negative energy particles and also both positve and negative anti-particles effective on the "complicated vacuum".

If we go to make our particles Majorana, we reduce the number of species by a factor two (as expected in as far as Majorana means that particle and anti-particle gets identified.)

In the case of the "complicated vacuum" the transition to Majorana also reduce the number of species by a factor 2 and thus compensates for the effect of the "complicated vacuum". With Majorana the particles and anti-particles are no longer distinguished, but with the "complicated vacuum" we obtain both positive and negative energy (Majorana)particles. It essentially functions as if the particle were no more Majorana. The "complicated vacuum", so to speak, removed the Majorananess.

We hope in later publication to be able to check that the just delivered story of the interaction vacuum increasing the number of species effectively by the factor two, is found when using the Bethe-Salpeter equation to describe bound states. Then there ought according to the just said to be effectively both negative and positive states relevant for the "constituent" particles in the Bethe-Salpeter equation.

Applying the just put forward point of view on the objects in our novel string field theory we should imagine that in this formulation with the "complicated vacuum" being one with chains in it it is possible for some objects to have their energy negative. Nevertheless a whole chain formed from them might end up with positive energy by necessity.

Such a possibility of negative energy for single objects that can nevertheless be put onto the vacuum might be very important for complete annihilation of pieces of one chain put onto the vacuum with part on an other one also put onto that vacuum. If we did not have such possibility for both signs along the chains, then we could not arrange that two incomming cyclically ordered chains could partly annihilate, because energy conservation locally along the chains would prevent that.

At least in principle it must though be admitted, that such a picture based on an interaction or by some restriction of the state of the whole world makes a complicated vacuum is a bit complicated technically.
But physically it is wellknown, that the vacuum in quantum field theories is a very complicated state, and so we might also expect that in string theory a similarly complicated vacuum would be needed. And that should even be the case in our novel string field theory in spite of the statement, often stated about this theory, that it has no interaction properly; all the seeming interactions being fake. But we could circumvent the need for an interaction to produce the complicated vacuum, we seemingly need by claiming that we instead have a restriction on the *Fock space states* of the system of objects, that is allowed. Such a constraint could force the vacuum to be more complicated, and thus in succession lead to that it becomes allowed in the more complicated vacuum to have some of the objects having even negative energy, which in turn could allow a *complete annihilation* of objects from one cyclicaally ordered chain and another set up in the same state (built on the complicated vacuum)
  - b. We give up seeing any chain structure in the vacuum as a whole, but rather attempt to be satisfied with ordering the missing particles, (or may be the antiparticles?).
    Naturally we would start imagining that we can have a Majorana boson, if we wish, represented by -1 negative energy boson, because the Majorana boson is a superposiotion of a particle and an antiparticle, and the latter really can be considered -1 particle of negative energy.
    At first one might think that having two bound states or two strings, which would like to partially annihilate -as it seems that we need in our derivation of Venezianoamplitude in our novel string field theory - could be indeed achieved by having part of one of these composed structures treated or thought upon as consisting of antiparticles, since one would say that particle and anti-particle can annihilate. However, when anti-particle and particle both with positive energy annihilate, then at least some energy is in excess and they therefore cannot annihilate completely into nothing. Rather there would have to some emmitted material left over to take away the energy. If we therefore as it seems that we would to get the terms missing in our novel string field theory t get the correct three term Veneziano amplitude should have a total annihilaton without left over such positive energy particles and antiparticles are not sufficient. Therefore this b. alternative seems not to truly help us with the problem of our novel string field theory to reproduce the Veneziano model fully.
- 4. In our formalism above - taken in the physical vacuum - the "Majorana-boson" became a superposition of being a hole and a genuine positive enrgy particle. The hole meant it were in part of the superposition - i.e. with some probability 50 % - -1 particle with negative energy. So one would with significant probability be able to consider that the "Majoran-boson" were indeed a lack of a negative energy original particle. For calculating amplitudes of some

sort one would then imagine that we might even have to add up contributions from the holes and contributions from the positive energy particles.
For each object, say, we should think we should have both a contribution in which it is considered a particle (with positive energy) and one in which it is a hole.

- 5. From the construction of the creation and annihilation operators for the "Bosons being their own antiparticles " - the b's - being constructed as containing the quite analogous contributions from a hole part and a particle part, it looks that in building up states with many Majorana- bosons one gets an analogous built up for both the holes and the particles and with say the analogous momenta.
  Here analogous means that the holes are holes for states with opposite momentum, but since it is holes it becomes the same net momentum for the hole as from the particle analogous to it.
- 6. With any sort of even formal interaction one would think that a hole and a particle can annihilate as stuff annihilate anti-matter. But if you have a pair of positive energy particles or anti-particles, they can only annihilate into some other particles of some sort. They cannot just disappear together. That is however, possible, if you have a negative energy particle and a positive energy one of just opposite four(or 26) momenta.
- 7. If one would say choose a gauge so that the particles get as in our gauge choice in our papers on the novel string theory that a certain momentum component, p^+ say, is specified to be a fixed value $a\alpha'/2$ as we choose, then one would have to let the particle, the state of which is made the hole have its $p^+ = -a\alpha'/2$, i.e. the opposite value. (Then if one has negative numbers of such particles, of course they contribute a positive p^+ again.) If one has indeed completely opposite four momenta - including energy - then an anihilation without left over is possible, otherwise not. It is therefore it is so crucial with negative energy constituents, if any such total disappearance of a pair is needed/wanted.
  But if we have physically only the free simple vacuum in which one has just for bosons emptied the negative energy states and for fermions just filled the negative energy states and no more, then all modifications will even particle for particle have positive energy. It will either be a removal of a negative energy particle meaning an antiparticle created or an insertion of a positive energy particle. Both these modifications would mean insertion of positive energy and they could not annihilate with each other without leaving decay material. So to have a piece of a cyclically ordered chain annihilate *without decay material* with another piece, it is needed that we *do not just have the free theory vacuum*. We need instead something like a "complicated vacuum" such as can be gotten by the effect of either interactions, or from some more complicated postulate as to what the vacuum state should be.
  In our novel string field theory, in which it is claimed that there are no interactions in the object formulation, we cannot refer to interactions. Rather we must refer to making a postulate about what the "complicated vacuum state" should be. As already mentioned above we need in order that ordering

into the cyclically ordered chains can make sense to have as the (vacuum) state a state in which the various objects can have so different single particle states that we can use their single particle state characteristic to mark them so as to give them sufficient individuality. Really we should postulate such a "complicated vacuum state" that there would for each object be an effectively unique successor lying as neighbor for the first one. But such restrictions to somewhat welldefined positions relative to neighbors in a chain must mean that it cannot at all be so, that there are just, 1 for fermions, -1 for bosons, particels in the negative energy states and zero in the positive ones. Rather it means, that considering such a free vacuum as starting point the state with the chains organized into the "compicated vacuum" is strongly excited. So there are many both particles and anti particles present in this "complicated vacuum " needed to have chains inside the vacuum.
But as already said such "complicated vacuum" can give the possibility of having effectively negative energy constituents. Since our objects are essentially constituents, this also means that our objects in a complicated vacuum can get allowed to be of negative energy. We must arrange that by allowing them in our gauge choice to get the J^+ have both signs. If so we may enjoy the full annihilation without left over material.

- 8. To construct an operator creating a chain (or series) of Majorana particles - in our novel SFT we mean the objects - we strictly speaking should use a specific linear combination of the hole and the positive energy particle (or object) for every Majorana particle created along the chain, but if we project out at the end the constructed Fock space state into the subspace used for the Majorana boson description, it is not so important to use precisely the correct linear combination. We shall namely obtain the right linear combination, since in that case it comes out of such a projection automatically.
But trusting that projecting into the Majorana-describing sub-space will do the job, we can just choose at will whether we use a series of positive energy particle (or object) creation operator or instead the corresponding hole creating (destruction of negative energy) operator.

Since our objects are a priori Majorana ones, it may at the end due to the doubling of state-types mentioned get them rather described effective as non-Majorana, in the way that they can be in both positive and negative energy single particle states. This actually reminds us more about the "naive vacuum" sector combination. But now it is the result of the "complicated vacuum" and of the thereby associated "doubling of the number of species effectively".

10.6.1 The "Rough Dirac Sea" in General

Let us extract and stress the idea, which we suppose will be very important for our formulation of the scattering amplitude for strings in our novel string field theory, but which could also be imagined to deliver an approximation that could be useful especially for bound states with many constituents, "the (very) rough Dirac sea". This rough Dirac sea is really the same as what we called above the "complicated vacuum".

The picture of true rough sea (a rough sea is the opposite of a calm sea, and it means that there lots of high waves may actually) be a very good one to pedagogically promote the idea of the effects of the "complicated vacuum" or the "rough Dirac sea" leading to that we effectively get negative energy particles and antiparticles.

In this picture the "calm Dirac sea" means the free approximation vacuum, in which - in the physical choice of sector combination, which is what one normally will have in mind - the negative energy states are filled for the fermion case, while "emptied out" in the boson case. In any case this calm Dirac sea is the picture for the theory vacuum in the unperturbed approximation (the free vacuum). But in interacting quantum field theories the vacuum gets perturbed by the interaction and becomes a more complicated state "the complicated vacuum", and it is for this "complicated vacuum" that the analogy with the rough sea is very good. There should have been near the surface -at the average surface height - a region in heights, in which you find with some probability water and with some probability air. Just at the should-have-been surface (= average surface) one expects that the probability for finding water is 50 % and for finding air in a given point is 50 %.

Now imagine: we come with an extra water molecule (or may be just a tiny bit of water) and want to insert it into the sea or the air not too far from the "should-have-been surface". Now if there happen to be a wave of water present, where you want to or attempt to insert such an extra tiny bit of water, you will not succeed, and that is analogous to getting zero, when you want to create a particle with a creation operator into a state that is already filled (say, we think for simplicity on the fermion case). If, however, there happen to be a valley in the waves, you will succeed in inserting a tiny bit of water *even if it is under the average water height*! This corresponds to inserting a negative energy particle into the "rough Dirac sea" or the "complicated vacuum". You may also think about removing a droplet of water. That will of course only succeed, if there *is* some water in the point in space, wherein you want to do it. Again it is not guaranteed that you can remove a bit of water in the rough sea, even if you attempt to remove it deeper than the average water height, because there might be a valley among the waves. Also if you hit a wave you might be able to remove a bit of water from a height above the average height.

In this way we see that you can produce sometimes a hole in the water both with positive and negative height (analogous to the both positive and negative (single particle) energy). Similarly you may produce both above and below extra bubbles of water.

This means that we have got a kind of doubling: While in the calm Dirac sea you can only make droplets (particles) above the average surface and only holes (antiparticles) below, we now in *the rough sea can do all four combinations*.

10.6.2 Infinite Momentum Frame Wrong, in Rough Dirac Sea?

With "rough Dirac sea"-thinking we arrived at the idea, that one might describe for instance a bound state or resonance as composed of constituent particles not all having positive energy; but some of the constituents could have *negative* energy.

It must be legal to choose to describe a bound state or resonance state by a linear combination - weighted with what is essentially a wave function for the constituents in the bound state or resonance - of creation operators and annihilation operators (for describing the contained anti-particles among the constituent particles) and let it act on the vacuum. We might, say, think of an operator of the form

$$A^\dagger(\text{bound state}) = \tag{10.120}$$
$$= \int \Psi(\vec{p}_1, h_1, s_1; ...; \vec{p}_N, h_N, s_N)$$
$$* \prod_{h_1, s_1} (a^\dagger(\vec{p}_1, h_1, s_1) d^3\vec{p}_{1,h_1,s_1}) \cdots \prod_{h_N, s_N} (a^\dagger(\vec{p}_n, h_N, s_N) d^3\vec{p}_N); \tag{10.121}$$
$$|\text{bound state (Fock)state}\rangle = \tag{10.122}$$
$$= A^\dagger(\text{bound state})|\text{“complicated vacuum”}\rangle \tag{10.123}$$
$$= \int \Psi(\vec{p}_1, h_1, s_1; ...; \vec{p}_N, h_N, s_N) \tag{10.124}$$
$$\prod_{h_1, s_1} (a^\dagger(\vec{p}_1, h_1, s_1) d^3\vec{p}_{1,h_1,s_1}) \cdots \prod_{h_N, s_N} (a^\dagger(\vec{p}_n, h_N, s_N) d^3\vec{p}_N) \tag{10.125}$$
$$|\text{“complicated vacuum”}\rangle \tag{10.126}$$

where $\Psi(\vec{p}_1, h_1, s_1; ...; \vec{p}_N, h_N, s_N)$ is (essentially) the wave function for a bounds state of N constituents numbered from 1 to N. The momenta of the constituents are denoted by $\vec{p}_i$ with $i = 1, 2, ..., N$, while the internal quantum numbers are denoted h_i, and then *there is the symbol* s_i *that can be* s_i = *“positive”*$= (E > 0)$ *or* s_i =*“negative”* $= (E < 0)$, meaning that the single particle energy of the constituent here is allowed to be both positive and negative, it being denoted by s_i, which of these two possibilities is realized for constituent number i. In this expression (10.126) we took just N constituents, but it is trivial to write formally also the possibillity of the bound state being in a state, that is a superposition of states with different values of the number N of constituents:

$$A^\dagger(\text{bound state}) = \tag{10.127}$$
$$= \sum_{N=1,2,...} \int \Psi_N(\vec{p}_1, h_1, s_1; ...; \vec{p}_N, h_N, s_N)$$
$$* \prod_{h_1, s_1} (a^\dagger(\vec{p}_1, h_1, s_1) d^3\vec{p}_{1,h_1,s_1}) \cdots \prod_{h_N, s_N} (a^\dagger(\vec{p}_n, h_N, s_N) d^3\vec{p}_N). \tag{10.128}$$

In this way we could describe a (bound) state inserted on the background of the true (“complicated”) vacuum with a superposition of different numbers of constituents. In principle we could find a wave function set, $\Psi_N(\vec{p}_1, h_1, s_1; ...; \vec{p}_N, h_N, s_N)$ for $N = 1, 2, ...$, that could precisely produce the (bound) state or resonace in question. It might because of the allowance of both negative and positive energy constituents be possible to construct in this way a given state in more than one way. But one could well imagine, that if we would like to have the wave function reasonably smooth, then it would be hard to quite avoid the negative energy constituents contributions - they are of course only relevant by giving nonzero contributions to the state created provided the Diarc sea is rough - and thus it looks like being essentially needed to use wave functions with also negative energy constituents, unless one is willing to give up the accuracy in which the influence from the interaction on the vacuum must be included.

But if we thus accept a description with negative constituent energy, the usual thinking on the "infinite momentum frame"[31] seems wrong:

If we in fact have constituents with single particle state negative energy, then boosting such a state eversomuch in the longitudinal momentum direction cannot bring these negative energy constituents to get posive longitudinal and thereby positive Bjorken x. So the usual story that provided we boost enough all constituents obtain positive x cannot be kept in our rough Dirac sea scenario with its negative energy constituents!

This may be the reason for the trouble in our novel string field theory which triggered us into the present work. In this novel string field theory formulation we namely used infinite momentum frame and actually took it, that all the there called objects - which are essentially constituents - had their $J^+ = a\alpha'/2$. But now the 26-momentum, which is proportional to the J^μ, should then for all the objects have the $+$ component positve. But now the notation is so, that this $+$ component means the longitudinal momentum in the infinite momentum frame. So we assumed a gauge choice in our formulation of this novel string field theory which is inconsistent with the negative energy constituent story arising from rough Dirac sea.

This "mistake" is very likely to be the explanation for the strange fact, that we in deriving the Veneziano model from our novel string field theory formalism only got *one out of the three terms we would have expected.*

The suggested solution to our trouble would then be to allow also for constituents with the J^+ being negative. That would mean we could not keep to the simple gauge choice enforcing a positive value to J^+ but would have to allow also negative values for this J^+.

That in turn might then allow constituent pairs from say different bound states - or different strings as it would be in our formalism - to totally annihilate meaning without leaving any material after them, because no excess energy would have to be there after the annihilation. Negative energy and positive energy together have the chance of such total annihilation.

10.7 Conclusion and Outlook

The in many ways intuitively nice and appealing language of the Dirac sea, which we have in an earlier work extended also to be applicable for bosons, is at first not so well suited for particles –"Majorana particles"– which are identical to their own antiparticles. In the present article we have nevertheless developped precisely this question of how to describe particles –bosons or fermions– which are, as we call it, "Majorana". We use also this terminology "Majorana" even for bosons to mean that a particle is its own antiparticle. The fermion case is rather well known. So our main story was first to review, how it were at all possible to make (a free) theory for bosons based on a Dirac sea, and secondly the new features of this Dirac sea for boson theory as follows:

a)negative norm squares
b)negative number of particles.

The main point then became how to get what we call a Majorana-boson theory through these new features. This comes about by constructing in terms of the creation and annihilation operators $a^{\dagger_f}(\vec{p}, E > 0)$ and $a(\vec{p}, E < 0)$ for a type of boson that might have a charge, some creation and annihilation operators $b(\vec{p})$ and $b^{\dagger_f}(\vec{p})$ for the Majorana boson, which is really a superposition of a boson and an anti-boson of the type described by a and $a^\dagger$.

10.7.1 The old Dirac sea for bosons

The Dirac sea for boson theory is based on having a Fock space, for which a basis consists of states with a number of bosons $k(\vec{p}, E \gtrless 0)$, which can be both positive, zero and negative integer, in both positive and negative energy E single particle states for each $3 - momentum\ \vec{p}$,

$$|\ldots, k(\vec{p}, E > 0), \ldots; \ldots, k(\vec{p'}, E < 0), \ldots\rangle \in Fock\ space. \tag{10.129}$$

Because of the complication that the inner product

$$\langle \varphi_1 |_{(f)} \varphi_2\rangle = \int \varphi_1^* \frac{\overleftrightarrow{\partial}}{\partial_t} \varphi_2 d^3\vec{X} \tag{10.130}$$

for a (single particle) boson is not positive definite we have to distinguish two different inner products $|$ and $|_f$ say and thus also the two thereto responding hermitean conjugations $\dagger$ and $\dagger_f$, meaning respectively without and with the $\int \varphi_1^* \frac{\overleftrightarrow{\partial}}{\partial_t} \varphi_2 d^3\vec{X}$ included. In fact we have for the norm square for these two inner products

$$\begin{aligned}
&\langle \ldots, k(\vec{p}, E > 0), \ldots; \ldots, k(\vec{p'}, E < 0), \ldots |_f \\
&\quad \ldots, k(\vec{p}, E > 0), \ldots; \ldots, k(\vec{p'}, E < 0), \ldots\rangle \\
=&(-1)^{\sharp(neg.\ energy\ b.}\langle \ldots, k(\vec{p}, E > 0), \ldots; \ldots, k(\vec{p'}, E < 0), \ldots | \\
&\quad \ldots, k(\vec{p}, E > 0), \ldots; \ldots, k(\vec{p'}, E < 0), \ldots\rangle \\
=&(-1)^{\sharp(neg.\ energy\ b.}\cdot \prod_{(\vec{p}, E \gtrless 0) for\ which\ k \le -1} (-1)^{|k|}
\end{aligned} \tag{10.131}$$

10.7.2 Main Success of Our Previous Dirac Sea (also) for Bosons:

The remarkable feature of the sector with the emptied out Dirac sea for bosons - what we called the physical sector - is that one has arranged the sign alternation (10.42) with the total number of negative energy bosons to cancel the sign from (10.31) so as to achieve that the total Fock space has positive norm square. This "physical sector" corresponds to that negative energy single particle states are in the negative sectors, while the positive energy single particle states are in the positive sector.

Thus the basis vectors of the full Fock space for the physical sector are of the form

$$|\ldots, k(\vec{p}, E > 0), \ldots; \ldots, k(\vec{p}, E < 0), \ldots\rangle \tag{10.132}$$

where the dots ... denotes that we have one integer number for every momentum vector –value ($\vec{p}$ or $\vec{p'}$), but now the numbers $k(\vec{p}, E > 0)$ of particles in a positive energy are– in the physical sector-combination– restricted to be non-negative while the numbers of bosons in the negative energy single particle states are restricted to be negative

$$\begin{aligned} k(\vec{p}, E > 0) &= \quad 0, 1, 2, \ldots \\ k(\vec{p}, E < 0) &= -1, -2, -3, \ldots \end{aligned} \tag{10.133}$$

In this physical sector our Dirac Sea formalism is completely equivalent to the conventional formalism for quantizing Bosons with "charge" (i.e. Bosons that are not their own antiparticles), say e.g. π^+ and π^-.

But let us remind ourselves that this idea of using Dirac sea **allows one to not fill the Dirac sea**, if one should wish to think of such world. With our extension of the idea of the Dirac sea to also include Bosons one also gets allowed to not empty out to have -1 boson in each negative energy single particle state. But for bosons you have the further strange feature of the phantasy world with the Dirac sea not treated as it should be to get physical, that one even gets negative norm square states, in addition to like in the fermion case having lost the bottom in the energy.

10.7.3 Present Article Main Point were to Allow for Bosons being their own Antiparticles also in Dirac sea Formalism

We could construct a "Majorana-boson" creation operator for say a "Majorana-boson" with momentum $\vec{p}$, $b^\dagger(\vec{p})$ analogously to the expressions (10.19) and (10.20).

$b^\dagger(\vec{p}) = \frac{1}{\sqrt{2}}\left(a^\dagger(\vec{p}, E > 0) + a(-\vec{p}, E < 0)\right)$ and $b(\vec{p}) = \frac{1}{\sqrt{2}}\left(a(E > 0) + a^\dagger(E < 0)\right)$

Since an extra phase on the basis states does not matter so much we could also choose for the bosons the "Majorana boson" creation and annihilation operators to be

$$\begin{aligned} b^{\dagger_f}(\vec{p}) &= \frac{1}{\sqrt{2}}\left(a^\dagger(E > 0) + a(-\vec{p}, E < 0)\right) \\ &= \frac{1}{\sqrt{2}}\left(a^{\dagger_f}(\vec{p}, E > 0) + a(-\vec{p}, E < 0)\right) \\ b(\vec{p}) &= \frac{1}{\sqrt{2}}\left(a(\vec{p}, E > 0) + a^{\dagger_f}(-\vec{p}, E < 0)\right) \\ &= \frac{1}{\sqrt{2}}\left(a(\vec{p}, E > 0) - a^\dagger(-\vec{p}, E < 0)\right) \end{aligned} \tag{10.134}$$

Such creation operators $b^{\dagger_f}(\vec{p})$ and their corresponding annihilation operators $b^{(}\vec{p})$ make up the completely usual creation and annihilation operator algebra for Bosons that are their own antiparticles in the case of the "physical sector combination". This "physical sector combination" means that we emptied out the Dirac sea in the sense that in the "vacuum" put just -1 boson in each negative energy single particle state. This correspondence means that our formalism is for this "physical sector combination" completely equivalent to how one usually describes Bosons - naturally without charge - which are their own antiparticles. But our formalism is to put into the framework of starting with a priori "charged" Bosons which then quite analogously to fermions have the possibility of having negative energy (as single particles). We then treat the analogous problem(s) to the Dirac sea for Fermions, by "putting *minus* one boson in each of the negative energy single particle states. That a bit miraculously solves both the problem of negative norm squares and negative second quantized energy, and even we can on top of that *restrict the theory*, if we so wish,*to enforce the bosons to be identified with their own antiparticles.*

We saw above that

- 1. We obtain the Fock-space (Hilbert-space) for the Bosons being their own antiparticles by *restriction* to a subspace
$$H_{Maj} = \{|\rangle \mid r(\vec{p})\,|\rangle = 0\}, \tag{10.135}$$
where we have defined
$$\begin{aligned} r(\vec{p}) &= \frac{1}{\sqrt{2}}\left(a(+\vec{p}, E > 0) + a\dagger(-\vec{p}, E < 0)\right) \\ &= \frac{1}{\sqrt{2}}(a(\vec{p}, E > 0) - a^{\dagger_f}(-\vec{p}, E < 0). \end{aligned} \tag{10.136}$$
Of course when one forces in the original Dirac sea formalism the antiparticles differnt from the particles to behave the same way in detail it means a drastic reduction of the degrees of freedom for the second quantized system - the Fock space-, and thus it is of course quite natural that we only use the subspace H_{Maj} being of much less (but still infinite) dimension than the original one.
- 2. In our formalism - since we use to write the creation operator for the boson being its own antiparticle as a sum of creation of a particle and of a hole (10.134) - a "Majorana-boson"is physically described as statistically or in superposition being with some chanse a particle and with some chance a hole. Really it is obvious, that it is 50 % chance for each. So the physical picture is that the "Majorana-boson" is a superposition of a hole and an original positive energy particle in the "physical sector combination".
- 3. We could construct a charge conjugation operation **C** which on our Fock space with both negative and positive energy states present as possibilities obtained the definition:
$$\begin{aligned} &\mathbf{C}\,|\ldots \tilde{k}\,(\vec{p}, E > 0), \ldots; \ldots, \tilde{k}\,(\vec{p'}, E < 0), \ldots\rangle \\ &=|\ldots, k(\vec{p}, E > 0) = -\,\tilde{k}\,(-\vec{p'}, E < 0) + 1, \ldots; \\ &\quad \ldots, k(\vec{p'}, E < 0) = -1 -\,\tilde{k}\,(-\vec{p'}, E > 0), \ldots\rangle. \end{aligned} \tag{10.137}$$

Of course the state of the system of negative single particle comes to depend on that of the positive energy system after the charge conjugation and oppositely. With (10.58) or (10.134) one sees that on the whole system or Fock space of Bosons being their own antiparticles is left invariant under the charge conjugation operator **C**. This is as expected since these "Majorana Bosons" should be invariant under **C**.

10.7.4 The Unphysical Sector Combinations and Boson-theories therein with Bosons being their own antiparticles

As a curiosity - but perhaps the most new in the present article - we have not only the physical sector combination, which so successfully just gives the usual formalism for both "charged" bosons and for what we called Majorana-bososns (the ones of their own antiparticles) but three more "sector-combinations" meaning combinations of whether one allows only negative numbers of bosons, or only non-negative numbers for the positive and the negative single particle states. The reader should have in mind that there is what we called the barier, meaning that the creation and annihilation operators cannot cross from a negative number of particles in a single particle state to a positive one or opposite, and thus we can consider the theories in which a given single particle state has a positive or zero number of particles in it as a completely different theory from one in which one has a negative number of bosons in that single particle state. For simplicity we had chosen to only impose that we only considered that all single particle states with one sign of the single particle energy would have their number of particles being on the same side of the barrier. But even with this simplifying choice there remained $2^2 = 4$ different sector-combinations. One of these sector-combinations - and of course the most important one because it matches the usual and physical formalism - were the "physical sector combination" characterized by their being a non-negative number k of bosons in all the positive energy single particle states (i.e. for $E > 0$), while the number k of bosons in the negative energy single particle states (i.e. for $E < 0$) is restricted to be genuinely negative $-1, -2, -3, ...$

The sector combination possibility 4) in our enumeration above the Fock space gets negative definite instead of as the one for the physical sector combination which gets positive definite. But these sector combinations are analogous or isomorphic with the appropriate sign changes allowed. Also our charge conjugation operator **C** operates inside both the "physical sector combination" and inside the sector combination number 4), which is characterized as having just the opposite to those of the physical sector, meaning that in sector combination 4) one has a negative number of particles in each positive energy (single particle)state, while there is a positive or zero number in the negative energy states. Thus the construction of particles being their own antiparticles would be rather analogous to that in the physical sector combination.

Less trivial is it to think about the two sector combinations 2) and 3) because now the charge conjugation operator **C** *goes between them*:Acting with the charge conjugation operator **C** on a state in the section combination 2) which we called the "naive vacuum sector combination" one gets a result of the operation in the *different* *sector combination* namely 3). You can say that the charge conjugation operator does not respect the barrier, it is only the creation and annihilation operators which respect this barrier. A priori one would therefore now expect that one should construct the formalism for the boson being its own antiparticle for these sector combinations 2) and 3) based on a Fock space covering both parts of the sector combination 2) and part of 3). To realize that one gets eigenstates of the charge conjugation operator such a combination of the the two sector combinations is of course also needed. However, if one just wanted to realize an algebra of the creation and annihilation operators that could be interpreted as a formalism for the boson type being its own antiparticle, one might throw away one of the two sector combinations, say combination 3), and keep only the "naive vacuum sector combination" 2). Since the creation and annihilation operators can*not* cross the barrier from one sector combination into the other one, such a keeping to only one of the two sectors between the charge conjugation operator goes back and forth would not make much difference for the creation and annihilation operators. We did in fact develop such a formalism for bosons being their own antiparticles in this way in alone "the naive vacuum sector combination". Interestingly it now turned out that keeping to only one sector combination the whole Fock space constructed for the boson being its own antiparticle became of zero norm square. Really we should say Hilbert inner product became completely zero for the subsector of the Fock space - of this unphysical "naive vacuum sector combination" -. This is of course at least possible since the sector combinations 2)(=the naive vacuum one) and 3) have both positive and negative norm square states- so that no-zero Hilbert vectors can be formed as linear combinations of positive and negative normsquare Hilbert-vectors. (In the physical sector combination nor the sector combination 4) zero norm states cannot be found because the Hilbert innerproduct is respectively positively and negatively definite.).

10.7.5 Speculations Bound States, Rough Dirac Sea etc.

Then in the last section above we have some to the rest more weakly connected speculations meant to be of help for the original problem bringing us to the considerations in this article, namely our "novel string field theory". A major suggestion, that came out of these considerations were to have in mind that, when you have an interacting quantum field theory, the vacuum gets into a rather complicated superposition of Fock space states, that makes descriptions as the "rough Dirac sea" or "the complicated vacuum" appropriate. While in say the "physical vacuum" - descussed in the article - you can only remove particles from negative energy states and only add particles to the positive single particle states, one does *not* have this restriction in the interacting vacuum, or say the "rough Dirac sea" vacuum. This point of view suggests that to make a proper description of a bound state or a resonance by means of a wave function in a relativistic quantum field theory, describing how to add or remove constituents from the "rough Dirac sea"-vacuum one should include *also negative energy possibilities* for the particles or antiparticle constituents.

These considerations are also hoped to be helpful for the problems we have for the moment with obtaining the full Veneziano model amplitude from our novel string field theory.

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