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Hypercube Embeddings of Wythoffians

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Abstract

The Wythoff construction takes a d-dimensional polytope P, a subset S of $\{0, \dots, d\}$ and returns another d-dimensional polytope $P(S)$. If P is a regular polytope, then $P(S)$ is vertex-transitive. This construction builds a large part of the Archimedean polytopes and tilings in dimension 3 and 4.

We want to determine, which of those Wythoffians $P(S)$ with regular P have their skeleton or dual skeleton isometrically embeddable into the hypercubes H_m and half-cubes $\frac{1}{2}H_m$. We find six infinite series, which, we conjecture, cover all cases for dimension $d > 5$ and some sporadic cases in dimension 3 and 4 (see Tables 1 and 2).

Three out of those six infinite series are explained by a general result about the embedding of Wythoff construction for Coxeter groups. In the last section, we consider the Euclidean case; also, zonotopality of embeddable $P(S)$ are addressed throughout the text.

Keywords: Coxeter group, embedding.

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1 Wythoff kaleidoscope construction

A *flag* in a poset is an arbitrary completely ordered subset. We say that a connected poset $\mathcal{K}$ is a d*-dimensional complex* (or, simply, a d*-complex*) if every maximal flag in $\mathcal{K}$ has size $d+1$. In a d-complex $\mathcal{K}$ every element x can be uniquely assigned a number $\dim(x) \in \{0, \dots, d\}$, called the *dimension* of x, in such a way, that the minimal elements of $\mathcal{K}$ have dimension zero and $\dim(y) = \dim(x) + 1$ whenever $x < y$ and there is no z with $x < z < y$.

The elements of a complex $\mathcal{K}$ are called *faces*, or k*-faces* if the dimension of the face needs to be specified. Furthermore, 0-faces are called *vertices* and d-faces (maximal faces) are called *facets*. If we reverse the order on $\mathcal{K}$ then the resulting poset $\mathcal{K}^*$ is again a d-complex, called the *dual complex*. Clearly, the vertices of $\mathcal{K}^*$ are the facets of $\mathcal{K}$ and, more generally, the dimension function on $\mathcal{K}^*$ is given by $\dim^*(x) = d - \dim(x)$.

We will often use the customary geometric language. If $x < y$ and $\dim(x) = k$, we will say that x *is a* k*-face of* y.

A d-complex is a *polytope* if every submaximal flag (that is, a flag of size d) is contained in exactly two maximal flags. In the polytopal case, 1-faces are called *edges*, because each of them has exactly two vertices. Starting from the next section we will deal exclusively with polytopes. The *skeleton* of a polytope $\mathcal{K}$ is the graph formed by all vertices and edges of $\mathcal{K}$.

For a flag $F \subset \mathcal{K}$ define its *type* as the set $t(F) = \{\dim(x) | x \in F\}$. Clearly, $t(F)$ is a subset of $\Delta = \{0, \dots, d\}$ and, reversely, every subset of Δ is the type of some flag.

Let Ω be the set of all nonempty subsets of Δ and fix an arbitrary $V \in \Omega$. For two subsets $U, U' \in \Omega$ we say that U' *blocks* U (from V) if for all $u \in U$ and $v \in V$ there is a $u' \in U'$, such that $u \leq u' \leq v$ or $u \geq u' \geq v$. This defines a binary relation on Ω, which we will denote as $U' \leq U$. We also write $U' \sim U$ if $U' \leq U$ and $U \leq U'$, and we write $U' < U$ if $U' \leq U$ and $U \not\leq U'$.

It is easy to see that $\leq$ is reflexive and transitive, which implies that $\sim$ is an equivalence relation. Let $[U]$ denote the equivalence class containing U. It will be convenient for us to choose canonic representatives in equivalence classes. It can be shown that if $U \sim U'$ then $U \cap U' \sim U \sim U \cup U'$. This yields that every equivalence class X contains a unique smallest (under inclusion) subset $m(X)$ and unique largest subset $M(X)$. If $X = [U]$ then $m(X)$ and $M(X)$ can be specified as follows: $m(X)$ is the smallest subset of U that blocks U, while $M(X)$ is the largest subset of Δ that is blocked by U. The subsets $m(X)$ will be called the *essential* subsets of Δ (with respect to V). Let $E = E(V)$ be the set of all essential subsets of Δ. Clearly, the above relation $<$ is a partial order on E. Also, $V \in E$ and V is the smallest element of E with respect to $<$.

We are now ready to explain the *Wythoff construction*. Naturally, our description is equivalent to the one given in [4] and [5], that generalized the original paper [28]. Other relevant reference to the subject are [17] and [24]. Suppose $\mathcal{K}$ is a d-complex and let Δ, Ω, V, $\leq$, and E be as above. The *Wythoff complex* (or *Wythoffian*) $\mathcal{K}(V)$ consists of all flags F such that $t(F) \in E$. For two such flags F and F', we have $F' < F$ whenever $t(F') < t(F)$ and F' is *compatible* with F (that is, $F \cup F'$ is a flag). It can be shown that $\mathcal{K}(V)$ is again a d-complex and that $\dim(F) = d + 1 - |M([t(F)])|$. It can also be shown that if $\mathcal{K}$ is a d-polytope then $\mathcal{K}(V)$ is again a d-polytope for all V. For a concrete Euclidean realization of such polytopes, see [13].

Since there are $2^{d+1} - 1$ different subsets V, there are, in general, $2^{d+1} - 1$ different Wythoffians constructed from the same complex $\mathcal{K}$. It is easy to see that $\mathcal{K}(V) = \mathcal{K}^*(d - V)$, where $d - V = \{d - v | v \in V\}$. This means that the dual complex does not produce new Wythoffians. Furthermore, in the case of self-dual complexes (that is, where $\mathcal{K} \cong \mathcal{K}^*$), this reduces the number of potentially pairwise non-isomorphic Wythoffians to $2^d + 2^{\lceil \frac{d-1}{2} \rceil} - 1$.

Some of the Wythoffians are, in fact, familiar complexes. First of all, $\mathcal{K}(\{0\}) = \mathcal{K}$ and $\mathcal{K}(\{d\}) = \mathcal{K}^*$. Furthermore, $\mathcal{K}(\{1\})$ is also known as the *median* complex $\mathrm{Med}(\mathcal{K})$ of $\mathcal{K}$ and the dual of $\mathcal{K}(\Delta)$ is known as the *order complex* of $\mathcal{K}$ (see [26]). We will call $\mathcal{K}(\Delta)$ the *flag complex* of $\mathcal{K}$. Thus, the order complex is the dual of the flag complex.

Since in this paper we are going to deal with the skeletons of $\mathcal{K}(V)$ and $\mathcal{K}(V)^*$ (in the polytopal case), we need to understand elements of $\mathcal{K}(V)$ of types 0, 1, $d-1$, and d. Since V is the unique smallest essential subset, the vertices (0-faces) of $\mathcal{K}(V)$ are the flags of type V. For a flag F to be a 1-face of $\mathcal{K}(V)$, $U = t(F)$ must have the property that $M([U])$ misses just one dimension k from Δ. Clearly, k must be in V. Now $U = U_k$ can be readily computed. Namely, U_k is obtained from V by removing k and including instead the neighbors of k (that is, $k-1$ and/or $k+1$). Thus, $\mathcal{K}(V)$ has exactly $|V|$ types of 1-faces. Turning to the facets (d-faces), we see that, for F to be a facet of $\mathcal{K}(V)$, we need that $U = t(F)$ be an essential subset of size one, such that $M([U]) = U$. The latter condition can be restated as follows: U should block no other 1-element set. From this we easily obtain that the relevant sets $U = \{k\}$ are those for which $k = 0$ (unless $V = \{0\}$), $k = d$ (unless $V = \{d\}$), or $\min(V) < k < \max(V)$. Finally, if F is a $(d-1)$-face then $U = t(F)$ is essential of size one or two and $M([U])$ is of size exactly two. We will not try to make here a general statement about all such subsets U. However, in the concrete situations below, it will be easy to list them all.

2 Archimedean Wythoffians: $d = 3, 4$

In this section we start looking at particular examples of Wythoffians, namely, at the *Archimedean Wythoffians*. These polytopes come by the Wythoff construction from the regular convex polytopes. A complex (in particular, a polytope) is called *regular* if its group of symmetries acts transitively on the set of maximal flags. *Convex polytopes* are the ones derived from convex hulls H of finite sets of points in $\mathbb{R}^d$. (We assume that the initial set of points contains $d+1$ points in general position; equivalently, the interior of H is nonempty.) The faces of the polytope are the convex intersections of the boundary of H with proper affine subspaces of $\mathbb{R}^d$. In particular, the polytope is $(d-1)$-dimensional, rather than d-dimensional.

It is well-known that the regular convex polytopes fall into four infinite series: regular p-gon, simplices α_d, hyperoctahedra β_d, and hypercubes γ_d; and five sporadic examples: the icosahedron and dodecahedron for $d = 3$, and the 24-cell, 600-cell, and 120-cell for $d = 4$.

The *half-cube graph* $\frac{1}{2}H_m$ (respectively *Johnson graph* $J(m, n)$) is the graph formed by all $x \in \{0, 1\}^m$ such that $\sum_{i=1}^m x_i$ is even (respectively equal n) with two vertices adjacent if their Hamming distance is 2.

We are interested in the following

Main Question: *Which Archimedean Wythoffians have skeleton graph or dual skeleton graph isometrically embeddable, for a suitable m, in the hypercube graph H_m or half-cube graph $\frac{1}{2}H_m$?*

Recall ([1, 12, 9, 25, 8] and books [7, 6]) that a mapping ϕ from a graph Γ to a graph Γ' is an *isometric embedding* if $d_{\Gamma'}(\phi(u), \phi(v)) = d_\Gamma(u, v)$ for all $u, v \in \Gamma$. For brevity, we will often shorten "isometric embedding" to just "embedding". Notice that H_m is an isometric subgraph of $\frac{1}{2}H_{2m}$, which means that every graph isometrically embeddable in a hypercube is also embeddable in a half-cube. There is also an intermediate class of graphs—those that are embeddable in a Johnson graph $J(m, n)$. A graph is said to be *hypermetric* if its path-metric satisfies the inequality

$$\sum_{1\le i<j\le n} b_i b_j d_G(i,j) \le 0$$

for any vector $b \in \mathbb{Z}^n$ with $\sum_i b_i = 1$. In the special case, when b is a permutation of $(1,1,1,-1,-1,0,\dots,0)$, the above inequality is called 5-*gonal*. The validity of hypermetric inequalities is necessary for embeddability but not sufficient: an example of hypermetric, but not embeddable graph $K_7 - C_5$ (amongst those, given in Chapter 17 of [7]).

Below, when we state our results on embeddability of the skeleton graphs Γ, we will indicate the smallest class in the above hierarchy, containing Γ.

We remark that γ_d is dual to β_d, which means that they produce the same Wythoffians. Thus, one can skip the case $\mathcal{K} = \gamma_d$ altogether. Similarly, we can skip the cases where $\mathcal{K}$ is the dodecahedron, since the latter is dual to the icosahedron, and the 120-cell, since it is dual to the 600-cell.

In the remainder of this section we state the results of a computer calculation carried out in the computer algebra system `GAP` [11].

We start with the case $d = 3$. In this case $\mathcal{K}$ is 2-dimensional, that is, $\mathcal{K}$ is a *map* (and so, we switch to the notation $\mathcal{M} = \mathcal{K}$). It is easy to see that $\mathcal{M}(V)$ with V=$\{0\}$, $\{0,1\}$, $\{0,1,2\}$, $\{0,2\}$, $\{1\}$, $\{1,2\}$, and $\{2\}$ correspond, respectively, to the following maps: original map $\mathcal{M}$, truncated $\mathcal{M}$, truncated $\mathrm{Med}(\mathcal{M})$, $\mathrm{Med}(\mathrm{Med}(\mathcal{M}))$, $\mathrm{Med}(\mathcal{M})$, truncated $\mathcal{M}^*$ and $\mathcal{M}^*$.

In Table 1 we give a complete answer to our Main Question in the case $d = 3$. The table lists all Archimedean Wythoffians and dual Wythoffians, whose skeleton graph is embeddable. The details of the embedding, such as the dimension of the embedding and whether or not it is equicut, are also provided. Recall that an embedding of a graph Γ in a hypercube is called *equicut* if each cut on Γ, produced by a coordinate of the hypercube, splits Γ in half. An embedding is called q-*balanced* if each coordinate cut on Γ has parts of sizes q and $|\Gamma| - q$. We will indicate in the table whether the embedding is equicut, q-balanced, or neither. Finally, for brevity, we truncated in the table the word “truncated” to just “tr”.

A striking property of this table is that it contains all possible Wythoffians (all five regular polytopes and 11 of the 13 Archimedean polytopes; missing are the Snub Cube and Snub Dodecahedron, which are not Wythoffian). Furthermore, for each of these polytopes, exactly one of the skeleton and the dual skeleton is embeddable.

This nice picture does not extend to the case $d = 4$, where far fewer embeddings exist. Our Table 2 gives a complete answer to the Main Question.

Notice that the total number of Archimedean Wythoffians for $d = 4$ is 45, see [3, 18, 19]. Thus, Table 2 indicates that the embeddable cases become more rare as d grows, and that, likely, there are only finitely many infinite series of embeddings. Furthermore, Tables 1 and 2 lead us to a number of concrete conjectures about possible infinite series of embeddings. In the next section we resolve those conjectures in affirmative by constructing the series and verifying the embedding properties.

We conclude this section with some further remarks about the embeddings in Tables 1 and 2. The majority of these embeddings are unique. The only exception is the Tetrahedron α_3, whose skeleton, the complete graph K_4, has two isometric embeddings. We also checked that all the skeleton graphs for $d = 3$, that turn out to be non-embeddable, violate, moreover, the so-called 5-*gonal inequality*.

Embeddable Wythoffian	n	embedding	equicut?
Tetrahedron= $\alpha_3(\{0\}) = \alpha_3(\{2\})$	4	$= J(4,1); = \frac{1}{2}H_3$	$q = 1$; yes
Octahedron= $\beta_3(\{0\}) = \alpha_3(\{1\})$	6	$= J(4,2)$	yes
Cube= $\beta_3(\{2\}) = \beta_3(\{0\})^*$	8	$= H_3$	yes
Icosahedron= $Ico(\{0\})$	12	$\frac{1}{2}H_6$	yes
Dodecahedron= $Ico(\{2\})$	20	$\frac{1}{2}H_{10}$	yes
(tr Tetrahedron)* $= \alpha_3(\{0,1\})^* = \alpha_3(\{1,2\})^*$	8	$\frac{1}{2}H_7$	no
(Cuboctahedron)* $= \beta_3(\{1\})^* = \alpha_3(\{0,2\})^*$	14	H_4	yes
(tr Cube)* $= \beta_3(\{1,2\})^*$	14	$J(12,6)$	no
Rhombicuboctahedron= $\beta_3(\{0,2\})$	24	$J(10,5)$	yes
tr Octahedron= $\beta_3(\{0,1\}) = \alpha_3(\{0,1,2\})$	24	H_6	yes
(tr Icosahedron)* $= Ico(\{0,1\})^*$	32	$\frac{1}{2}H_{10}$	yes
(Icosidodecahedron)* $= Ico(\{1\})^*$	32	H_6	yes
(tr Dodecahedron)* $= Ico(\{1,2\})^*$	32	$\frac{1}{2}H_{26}$	no
tr Cuboctahedron= $\beta_3(\{0,1,2\})$	48	H_9	yes
Rhombicosidodecahedron= $Ico(\{0,2\})$	60	$\frac{1}{2}H_{16}$	yes
tr Icosidodecahedron= $Ico(\{0,1,2\})$	120	H_{15}	yes

Table 1: Embeddable Archimedean Wythoffians for $d = 3$.

3 Relation with Coxeter groups

A group W is a *Coxeter group* if it is a group generated by a set $S = \{s_0, \dots, s_{d-1}\}$ with the elements s_i satisfying to the relations $s_i^2 = 1$ and $(s_i s_j)^{m_{ij}} = 1$ with $m_{ij} \geq 2$. We refer to [14] for all facts used in this Section.

One can encode the matrix W by a *Coxeter graph* on vertices $\{0, \dots, d-1\}$ with two edges being adjacent if $m_{ij} \geq 3$.

The group is called *irreducible* if the Coxeter graph is connected. The irreducible finite Coxeter groups are classified and denoted by A_d, B_d, D_d, E_6, E_7, E_8, F_4, H_3, H_4, $\mathsf{I}_2(p)$ (see [14, page 34]). A Coxeter group is the symmetry group of a regular polytope if and only if its Coxeter graph is a path. A finite Coxeter group W can be represented as a group of isometries of $\mathbb{R}^d$ with the s_i being reflexions and their hyperplanes H_i delimiting a fundamental domain $\mathcal{S}$.

We now define the Wythoff construction for Coxeter groups. An algebraic way is explained in [15] but we choose an easier and more geometric way following [20]. Take a finite Coxeter group W with a fundamental simplex $\mathcal{S}$. If $T \subseteq \Delta = \{0, \dots, d-1\}$, then we take a point $v \in \mathcal{S}$ such that $v \in H_i$ if and only if $i \notin T$. The Wythoff construction $\mathsf{W}(T)$ is then defined as the convex hull of the orbit $\mathsf{W}(v)$. The orbit $\mathsf{W}(v)$ is on the sphere and in [20] it is proved, using Delaunay polytopes, that the combinatorics of $\mathsf{W}(T)$ depends only on W and T.

If $T = \{0, \dots, d-1\}$, then $\mathsf{W}(T)$ is the flag complex of W. Its skeleton is the *Cayley graph* $Cay(\mathsf{W}, S)$ of W obtained from the canonical generating set $S = \{s_0, \dots, s_{d-1}\}$.

Given a Coxeter group W, denote by T the set of elements, which are conjugate to an element of S.

Proposition 1. *If* W *is a finite Coxeter group,* S *is its canonical generating set, then the Cayley graph* $Cay(\mathsf{W}, S)$ *is isometrically embeddable into* $H_{|T|}$.

Proof. Any finite Coxeter group can be realized as a group of isometries of a space $\mathbb{R}^d$. The

Embeddable Wythoffian	n	embedding	equicut?
$\alpha_4 = \alpha_4(\{0\}) = \alpha_4(\{3\})$	5	$= J(5,1)$	$q = 1$
$\beta_4 = \beta_4(\{0\})$	8	$= \frac{1}{2}H_4$	yes
$\gamma_4 = \beta_4(\{3\}) = \beta_4(\{0\})^*$	16	$= H_4$	yes
$\alpha_4(\{1\}) = \alpha_4(\{2\})$	10	$= J(5,2)$	$q = 4$
$\alpha_4(\{0,3\})^*$	30	H_5	yes
$\beta_4(\{0,3\})$	64	$\frac{1}{2}H_{12}$	yes
$\alpha_4(\{0,1,2,3\})$	120	H_{10}	yes
$\beta_4(\{0,1,2\}) = 24-cell(\{0,1\}) = 24-cell(\{2,3\})$	192	H_{12}	yes
$\beta_4(\{0,1,2,3\})$	384	H_{16}	yes
$24-cell(\{0,1,2,3\})$	1152	H_{24}	yes
$600-cell(\{0,1,2,3\})$	14400	H_{60}	yes

Table 2: Embeddable Archimedean Wythoffians for $d = 4$.

W	$\|T\|$	regular polytope
A_d	$d(d+1)/2$	α_d
B_d	d^2	γ_d or β_d
D_d	$d(d-1)$	none
E_6	36	none
E_7	63	none
E_8	120	none
F_4	24	24-cell
H_3	15	Dodecahedron or Icosahedron
H_4	60	120-cell or 600-cell
$\mathsf{I}_2(p)$	p	regular p-gon

Table 3: Embeddings of flag complexes of finite irreducible Coxeter groups.

elements of T are realized as reflections and the corresponding hyperplanes realize a plane arrangement and W acts simply transitively on its cells. By a theorem of Eppstein ([10]), the graph formed by the cells with two cells adjacent if they share a $(d-1)$-dimensional face, is isometrically embeddable into the hypercube $\{0,1\}^M$ with M being the number of hyperplanes, i.e. $|T|$. □

See in Table 3 the dimensions of the hypercube of embeddings with the corresponding regular polytopes if existing.

This proposition is, certainly, folklore, but we could not find an appropriate reference. For example, [2] address the 3-dimensional case. Also, in [21], an embedding of finitely generated Coxeter group into cube complexes is given. But this is a topological embedding, while our embeddings are isometric.

The above setting can be generalized to affine Coxeter groups $\widetilde{\mathsf{W}}$. An *affine Coxeter group* is a group $\widetilde{\mathsf{W}}$ obtained by adding a reflection s_d along an hyperplane not passing though 0 to a finite Coxeter group W. The group $\widetilde{\mathsf{W}}$ then has a fundamental domain delimited by $d+1$ hyperplanes $H_0, \dots, H_d$. If $\widetilde{T}$ denotes the set of classes of parallel hyperplanes of $\widetilde{\mathsf{W}}$, then its Cayley graph $Cay(\mathsf{W}, S)$ embeds into $\mathbb{Z}^{|\widetilde{T}|}$ and one has $|\widetilde{T}| = |T|$ according to the nomenclature in [14].

Note also that (see [23, 16]), given a finite Coxeter group of root system $R = \{r_1, \dots,$

$r_N\}$, the zonotopal polytope $[-r_1, r_1]+\cdots+[-r_N, r_N]$ is, actually, the Wythoff construction of W on S. It is easy to see that this zonotopal embedding is isometric embedding into $H_{|T|}$ given in Proposition 1.

4 Infinite series of embeddings

Since in this section we are only interested in the infinite series of embeddings, we restrict ourselves to the cases $\mathcal{K} = \alpha_d$ and β_d. These polytopes can be described in combinatorial terms as follows: The faces of α_d are all proper nonempty subsets of the set $\{1, \ldots, d+1\}$. The order on α_d is given by containment, and the dimension of the face X is $|X|-1$. Clearly, α_d has $\binom{d+1}{k+1}$ faces of dimension k. The faces of β_d are the sets $\{\pm i_1, \ldots, \pm i_k\}$, where the signs are arbitrary and $\{i_1, \ldots, i_k\}$ is a nonempty subset of $\{1, \ldots, d\}$. Again, the order is defined by containment and the dimension of a face X is $|X| - 1$. Thus, β_d has $2^{k+1}\binom{d}{k+1}$ faces of dimension k.

Two infinite series of embeddable skeletons are well-known:

1. The skeleton of $\alpha_d(\{0\}) = \alpha_d(\{d-1\})$ is the complete graph K_{d+1}, which coincides with $J(d+1, 1)$.
2. The skeleton of $\beta_d(\{d-1\}) = \beta_d(\{0\})^* = \gamma_d$ coincides with the hypercube graph H_d.

The first of these embeddings can be generalized as follows.

Proposition 2. *If $k \in \{0, \ldots, d-1\}$ then the skeleton of $\alpha_d(\{k\})$ coincides with $J(d+1, k+1)$.*

Proof. We refer to the discussion of the vertices and edges at the end of Section 1. According to that discussion, the vertices of $\alpha_d(\{k\})$ are the k-faces, that is, the subsets of $\{1, \ldots, d+1\}$ of size $k+1$. Furthermore, the only types, leading to edges, are $k-1$ (if $k > 1$) and $k+1$. This means that two vertices are on an edge if and only if their symmetric difference, as sets, has size two. □

Note that the above isomorphism is not quite new, see for example [22, page 18] and [27, page 8-9].

The above result explains a number of entries in Tables 1 and 2. We now turn to the series showing up in line 14 of Table 1 and in line 8 of Table 2.

Proposition 3. *The skeleton of $\alpha_d(\{0, d-1\})^*$ coincides with H_{d+1} with two antipodal vertices removed. It is an isometric subgraph of H_{d+1}.*

Proof. Again we refer to the discussion in Section 1. When $V = \{0, d-1\}$, every one-element subset U of $\Delta = \{0, \ldots, d-1\}$ has the property that $M([U]) = U$. This means that all elements of $\alpha_d(V)$ (that is, all nonempty proper subsets of $\{1, \ldots, d+1\}$) are vertices of $\alpha_d(V)^*$. This also means that the types corresponding to edges necessarily have size two. If $U = \{a, b\} \subset \Delta$ and $a < b$ then $M([U]) = \{k \mid a \leq k \leq b\}$. Therefore, edges of $\alpha_d(V)^*$ are flags, whose type is of the form $\{k, k+1\}$. Thus, we come to the following description of the skeleton Γ of $\alpha_d(V)^*$: Its vertices are all nonempty proper subsets of $\{1, \ldots, d+1\}$; two subsets are adjacent when one of them lies in the other and their sizes differ by one. This matches the well-known definition of H_{d+1} as the graph on the set of all subsets of $\{1,\dots,d+1\}$. In fact, Γ is H_{d+1} with two antipodal vertices ($\emptyset$ and the entire $\{1,\dots,d+1\}$) removed. The last claim is clear. $\square$

From Proposition 1, we know that $\alpha_d(\{0,\dots,d-1\})$ is embeddable into $H_{\binom{d+1}{2}}$, which explains line 16 of Table 1 and line 4 of Table 2. Also $\beta_d(\{0,\dots,d-1\})$ is embeddable into H_{d^2}, which explains line 6 of Table 1 and line 5 of Table 2. There is an easy connection between those two embeddings, that is the skeleton $\alpha_{d-1}(\{0,\dots,d-2\})$ is a subcomplex of the complex $\beta_d(\{0,\dots,d-1\})$.

The dual Wythoff polytope in this proposition is, in fact, the zonotopal Voronoi polytope of the root lattice A_d. Note that the polytope $\alpha_d(\{0,\dots,d-1\})$ is known as the *permutahedron*. It is the zonotopal Voronoi polytope of the dual root lattice A_d^*. Remark also that $\beta_d(\{0,\dots,d-1\})$ is not the Voronoi polytope of a lattice, since its number of vertices, $2^d d!$, is greater than $(d+1)!$.

The following embedding series leaves trace in Tables 1 and 2 in lines 16 and 7, respectively.

Proposition 4. *The skeleton of $\beta_d(\{0,\dots,d-2\})$ is isomorphic to the skeleton of $\mathsf{D}_d(\{0,\dots,d-1\})$, which is isometrically embeddable into $H_{d(d-1)}$.*

Proof. We define the following linear functions:

$$f_i(x) = x_{i+1} - x_{i+2}, \text{ for } 0 \le i \le d-2, \text{ and } f_{d-1}(x) = x_d.$$

Denote by H_i the hyperplane defined by $f_i(x) = 0$ and by s_i the reflection along the hyperplane H_i. Denote by $\mathcal{S}$ the simplex defined by the inequalities $f_i(x) \ge 0$. The reflections s_i generate the Coxeter group B_d of fundamental domain $\mathcal{S}$, whose corresponding regular polytope is β_d.

One way to compute $\beta_d(\{0,\dots,d-2\})$ is to take a vertex $v \in \mathcal{S} \cap H_{d-1}$ with $v \notin H_i$ for $i \le d-2$. The polytope obtained as convex hull of the orbit $\mathsf{B}_d(v)$ is then $\beta_d(\{0,\dots,d-2\})$. Now, the key argument is that $\mathcal{S} \cup s_{d-1}(\mathcal{S})$ is also a simplex $\mathcal{S}'$ defined by the inequalities

$$f'_i(x) = f_i(x) \ge 0, \text{ for } 0 \le i \le d-2, \text{ and } f'_{d-1}(x) = x_{d-1} + x_d \ge 0.$$

Those inequalities define hyperplanes which themselves define orthogonal reflections and so, a finite Coxeter group named D_d. The point v lies inside of $\mathcal{S}'$ so, the skeleton of $\beta_d(\{0,\dots d-2\})$ is, actually, the skeleton of $\mathsf{D}_d(\{0,\dots,d-1\})$.

The isometric embedding follows from Proposition 1 and Table 3. $\square$

Another way to see the embedding $\beta_d(\{0,\dots,d-2\})$ from $\beta_d(\{0,\dots,d-1\})$ by *projection*, that is removing dimensions and considering the obtained graph. The idea here is simply to remove the coordinates corresponding to the planes $x_i = 0$.

For $d = 3$, the polytope $\beta_d(\{0,\dots,d-2\})$ is the zonotopal Voronoi polytope of the lattice A_3^*. For $d = 4$ and higher, it is a zonotope but it is not the Voronoi polytope of a lattice, since the number of vertices, $2^{d-1}d!$, is greater than $(d+1)!$.

The examples in lines 8 and 9 of Tables 1 and 2, respectively, suggest that the skeleton of $\beta_d(\{0,d-1\})$ might be embeddable in a half-cube for all d. The next proposition demonstrates that the actual situation is somewhat more complicated. We first need to recall some further concepts.

Suppose Γ is a graph and ϕ is a mapping from Γ to a hypercube H_m. We say that ϕ is an *embedding with scale* λ if for all vertices $x, y \in \Gamma$ the distance in H_m between $\phi(x)$ and $\phi(y)$ (the Hamming distance) coincides with $\lambda d_\Gamma(x, y)$. Clearly, isometric embeddings in a hypercube are scale 1 embeddings, while isometric embeddings in a half-cube are scale 2 embeddings. A graph is an *ℓ_1-graph* if it has a scale λ embedding into a hypercube for some λ. A finite rational-valued metric embeds isometrically into some space ℓ_1^k if and only if it is scale λ embeddable into H_m for some λ and m. See [1, 25, 8, 7, 6] for details on ℓ_1-embedding.

Proposition 5. *The skeleton of $\beta_d(\{0, d-1\})$ is an ℓ_1-graph for all d. However, if $d > 4$, it is not an isometric subgraph of a half-cube.*

Proof. Let Γ be the above skeleton graph. The vertices of this graph can be identified with all tuples of the form $(k; \varepsilon_1, \dots, \varepsilon_d)$, where $1 \le k \le d$ and the signs ε_i are arbitrary. Thus, there are $d2^d$ vertices. The edges of Γ arise in two ways: (1) We have that $(i; \varepsilon_1, \dots, \varepsilon_d)$ is adjacent to $(j; \varepsilon_1, \dots, \varepsilon_d)$ for all $i \neq j$. (2) We also have that $(i; \varepsilon_1, \dots, \varepsilon_d)$ is adjacent to $(i; \varepsilon_1, \dots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1} \dots, \varepsilon_d)$, again for all $i \neq j$.

Let Γ_1 be the graph whose vertices are all tuples $(\varepsilon_1, \dots, \varepsilon_d)$, $\varepsilon_i = \pm 1$, and where two tuples are adjacent whenever they differ in just one entry. Let Γ_2 be the graph whose vertices are $\pm k$, $1 \le k \le d$, and where vertices s and t are adjacent whenever $|s| \neq |t|$. It is clear that Γ_1 is isomorphic to the hypercube H_d, while Γ_2 is isomorphic to the hyperoctahedron graph $K_{d\times 2}$ (complete multipartite graph with d parts of size two; also known as the cocktail-party graph). Mapping the vertex $(k; \varepsilon_1, \dots, \varepsilon_d)$ of Γ to the ordered pair $((\varepsilon_1, \dots, \varepsilon_d), \varepsilon_k k)$ defines an embedding ϕ of Γ into the Cartesian product graph $\Gamma_1 \times \Gamma_2$. It is easy to see that this embedding is isometric. Since Γ projects surjectively onto both Γ_1 and Γ_2, we can now determine the Graham-Winkler Cartesian product graph for Γ (cf. [12]). Namely, that Cartesian product graph has d complete graphs of size two and the cocktail-party graph Γ_2 as its factors. (We assume that $d > 2$.) It follows from [25] that Γ has a scale λ embedding in a hypercube if and only if every factor has. In our case, every factor is an ℓ_1-graph, hence Γ is an ℓ_1-graph, too. Furthermore, the cocktail-party graph $K_{d\times 2}$ with $d > 4$ requires $\lambda > 2$, which proves the second claim of the proposition. □

The infinite series exhibited in this section explain a majority of the examples from Tables 1 and 2, including, in fact, *all* examples from Table 2. This allows us to make the following conjecture.

Conjecture 6. *If Γ is the skeleton of the Wythoffian $\mathcal{K}(V)$ or of the dual Wythoffian $\mathcal{K}(V)^*$, where $\mathcal{K}$ is a regular convex polytope, and Γ is isometrically embeddable in a half-cube, then Γ can be found either in Table 1, Table 2, or in one of the infinite series discussed in this section.*

5 From the hypercubes to the cubic lattices?

The above conjecture shows one direction of possible further research. Another possibility is extending the results of this paper to cover the case of infinite regular polytopes, that is, regular partitions of the Euclidean and hyperbolic space. In this section we briefly discuss what is known about the easier Euclidean case.

In the infinite case, instead of embedding the skeleton graphs up to scale into hypercubes H_m, we embed them into the m-dimensional cubic lattice Z_m (including $m = \infty$) taken with its metric ℓ_1. Notice that this is a true generalization, because, according to [1], a finite metric that can be embedded into a cubic lattice, can also be embedded into a hypercube.

All regular partitions of Euclidean d-space (d finite) are known [5]. They consist of one infinite series $\delta_d = \delta_d^*$, which is the partition into the regular d-dimensional cubes, two 2-dimensional ones, (3^6) (partition into regular triangles) and $(6^3) = (3^6)^*$ (partition into regular 6-gons), and two 4-dimensional ones, $h\delta_4$ (partition into 4-dimensional hyperoctahedra) and $h\delta_4^*$ (partition 24-cells). Notice that the latter two partitions are the Delaunay and Voronoi partitions associated with the lattice D_4. In particular, below we use the notation $Vo(\mathsf{D}_4)$ in place of $h\delta_4^*$.

In the following table we give a complete list of Wythoffians of regular partitions of the Euclidean plane. We use the classical notation for the vertex-transitive partition of the Euclidean plane; namely, each partition is identified by its *type*, listing clock-wise the sizes of the faces containing a fixed vertex. In particular, the regular partitions of the Euclidean plane are $(4^4) = \delta_2$, (3^6), and $(6^3) = (3^6)^*$. In the second column we indicate the embedding. We put Z_m for an embedding with scale one and $\frac{1}{2}Z_m$ for an embedding with scale two.

Wythoffian	embedding
$\delta_2 = \delta_2(\{0\}) = \delta_2(\{1\}) = \delta_2(\{2\}) = \delta_2(\{0,2\})$	Z_2
$(3^6) = (3^6)(\{0\})$	$\frac{1}{2}Z_3$
$(6^3) = (3^6)(\{2\}) = (3^6)(\{0,1\})$	Z_3
$(4.8^2) = \delta_2(\{0,1\}) = \delta_2(\{1,2\} = \delta_2(\{0,1,2\}$	Z_4
$(4.6.12) = (3^6)(\{0,1,2\})$	Z_6
$(3.4.6.4) = (3^6)(\{0,2\})$	$\frac{1}{2}Z_3$
$(3.6.3.6)^* = (3^6)(\{1\})^*$	Z_3
$(3.12^2)^* = (3^6)(\{1,2\})^*$	$\frac{1}{2}Z_\infty$

Table 4: Embeddable Wythoffian cases for plane partitions.

All Archimedean Wythoffians or their dual, which are not mentioned in Table 4, are non-embeddable and, moreover, they do not satisfy the 5-gonal inequality.

In this table we separated the three regular plane partitions from the *Archimedean* (*i.e.*, vertex- but not face-transitive) ones. Notice that, out of the eight Archimedean partitions, five are Wythoffians. Missing are partitions $(3^2.4.3.4)$, $(3^3.4^2)$ and $(3^4.6)$. It turns out that for all regular and Archimedean plane partitions (and in particular, for all our Wythoffians) exactly one out of itself and its dual is embeddable. In this respect the situation here repeats the situation for the Archimedean polyhedra for $d = 3$, see Section 2 and Tables 9.1 and 4.1–4.2 in [6].

We now turn to the next dimension, $d = 3$. Here we identify the Wythoffians as particular partitions of the Euclidean 3-space in two ways. First, in column 2 we give the number of that partition in the list of 28 regular and Archimedean partitions of the 3-space from [6]. Secondly, we identify in column 3 the tiles of the partition. Here, as before, β_3 and γ_3 are the Octahedron and the Cube, respectively. Also, "Cbt" stands for the Cuboctahedron and "Rcbt" stands for the Rhombicuboctahedron. Clearly, "tr" stands for "truncated" and Prism_8 is the regular faced 8-gonal prism. In some cases we also indicate the chemical names of the corresponding partitions. In column 4 we give the details of the embedding. If the particular Wythoffian is non-embeddable, we put "non 5-gonal" in that column to indicate that it fails the 5-gonal inequality. The information in column 4 is taken from Table 10.1 from [6].

Wythoffian	no	tiles	embedding
$\delta_3 = \delta_3(\{0\}) = \delta_3(\{3\}) = \delta_3(\{0,3\})$	1	γ_3	Z_3
$\delta_3(\{1,2\}) = Vo(\mathsf{A}_3^*)$	2	tr β_3	Z_6
$\delta_3(\{0,1,2\}) = \delta_3(\{1,2,3\})$=zeolit Linde A	16	γ_3, tr β_3, tr Cbt	Z_9
$\delta_3(\{0,1,2,3\})$=zeolit ρ	9	Prism$_8$, tr Cbt	Z_9
$\delta_3(\{1\}) = \delta_3(\{2\}) = De(J-complex)$	8	β_3, Cbt	non 5-gonal
$\delta_3(\{0,1\}) = \delta_3(\{2,3\})$=boride CaB_6	7	β_3, tr γ_3	non 5-gonal
$\delta_3(\{0,2\}) = \delta_3(\{1,3\})$	18	γ_3, Cbt, Rcbt	non 5-gonal
$\delta_3(\{0,1,3\}) = \delta_3(\{0,2,3\})$=selenide $Pd_{17}Se15$	23	γ_3, Prism$_8$, tr γ_3, Rbct	non 5-gonal

Table 5: Wythoffians of regular partitions of the 3-space.

As Table 5 indicates, only eight out of 28 regular and Archimedean partitions of the 3-space arise as the Wythoffians of the cubic partition δ_3.

Finally, in Table 6 we collected some information about the dimensions $d \geq 4$.

Wythoffian	tiles	embedding
$\delta_d = \delta_d(\{0\}) = \delta_d(\{d\}) = \delta_d(\{0,d\})$	γ_d	Z_d
$\delta_d(\{0,1\})$=tr δ_d	β_d, tr γ_d	non 5-gonal
$Vo(\mathsf{D}_4) = Vo(\mathsf{D}_4)(\{0\}$	$24-cell$	non 5-gonal
$Vo(\mathsf{D}_4)^* = Vo(\mathsf{D}_4)(\{4\})$	β_4	non 5-gonal
$Vo(\mathsf{D}_4)(\{1\} = Med(Vo(\mathsf{D}_4))$	γ_4, $Med(24-cell)$	non 5-gonal
$Vo(\mathsf{D}_4)(\{0,1\}$=tr $Vo(\mathsf{D}_4)$	γ_4, tr $24-cell$	Z_{12}

Table 6: Some Wythoffians of regular partitions of the d-space, $d \geq 4$.

Notice that again, as in Table 2, few Wythoffians for $d = 3$ possess embeddings. This gives hope that there is only a small number of infinite series of embeddings in the Euclidean case. One of the infinite series is shown in line 1 of Table 6.

Proposition 7. *(i) The skeleton graph of the Wythoffian flag complex $\delta_d(\{0,\dots,d\})$ is isometrically embeddable in Z_{d^2}.*

(ii) The skeleton graph of the Wythoffian $\delta_d(\{0,\dots,d-1\}) = \delta_d(\{1,\dots,d\})$ is also isometrically embeddable in Z_{d^2}.

Proof. The situation is completely analogous to the one for $\beta_d(\{0,\dots,d-1\})$ and $\beta_d(\{0,\dots,d-2\})$ of Proposition 4. We simply add the function $f_d(x) = 1 - x_1$ and the Coxeter groups B_d and D_d are replaced, respectively, by $\widetilde{\mathsf{C}}_d$ and $\widetilde{\mathsf{B}}_d$ (see [14, page 96]). The number of classes of parallel hyperplanes in $\widetilde{G}$ is equal to the number of hyperplanes in G. The result follows from Table 3. □

We think that those tilings are zonotopal but we did not check it.

It appears (see line 4 of Table 4 and line 2 of Table 5) that $\delta_d(\{1,2\})$ may have an embedding for all d. In this case, however, we are reluctant to formulate an exact conjecture. Perhaps, the situation will be more clear when the case $d = 4$ is completed.

We have already pointed out that only eight out of 28 regular and Archimedean partitions of the Euclidean 3-space are Wythoffians of δ_3. This indicates that, maybe, one needs to derive Wythoffians from a larger class of Euclidean partitions. The obvious candidates are the Delaunay and Voronoi partitions of interesting Euclidean lattices, in particular, the root lattices. For the case of such lattices themselves (that is, for $V = \{0\}$ or $\{d\}$), see Chapter 11 of [6]. The zonotopal embeddings of $Vo(\mathsf{A}_d)$ in Z_{d+1} and $Vo(\mathsf{A}_d^*)$ in $Z_{\binom{d+1}{2}}$, correspond to the zonotopal embeddings of the corresponding tiles from Proposition 1.

It is, of course, also very interesting to consider the Wythoffians of the regular partitions of the *hyperbolic* d-space. In fact, in [9] (see also Chapter 3 of [6]), the embeddability was decided for any regular tiling P of the d-sphere, Euclidean d-space, hyperbolic d-space or Coxeter's regular hyperbolic honeycomb (with infinite or star-shaped cells or vertex figures). The large program will be to generalize it for all Wythoffians of such general P.

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