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Q1

We just learned that *Ars Mathematica Contemporanea* has been ranked 60th among the 312 mathematical journals in the ISI's Journal Citation Report for the year 2015. This makes it the first ever scientific journal published in Slovenia that has been placed in the upper quartile: Q1.

We understand very well that a high score is only a necessary condition for excellence. We are sure that a new player in the elite company of Q1 journals will be met by skepticism from many mathematicians who value tradition and prestige. And since AMC is a 21st century journal less than 10 years old, and published in a small country of only two million people, it may never be able to compete with established journals having a long tradition dating a century or more. Nevertheless, we will do our best to continue to publish high quality mathematics while keeping the journal free of charge for both authors and readers.

We would like to thank our authors, who believe in our journal and entrust their best work to the AMC! Sincere thanks also to the referees and editors who ensure the quality of published papers!

Dragan Marušič and Tomaž Pisanski

Editors In Chief

GEMS 2013

This issue of *Ars Mathematica Contemporanea* offers a collection of papers presented at the Sixth Workshop 'Graph Embeddings and Maps on Surfaces' (GEMS), which took place in Smolenice, Slovakia, the week 14–19 July 2013.

The GEMS workshop series began with the idea of a small conference in Slovakia that would bring together researchers interested in various aspects of graphs embedded in surfaces. The first GEMS workshop was held in Donovaly the week 21–26 August 1994, and was attended by 33 participants from 14 countries. The topics covered by the workshop included combinatorial and topological properties of embedded graphs, construction of graph embeddings in surfaces, symmetries of embedded graphs, regular maps and hypermaps, group actions on graphs and surfaces, and convex polytopes.

The Donovaly workshop was followed by similar workshops in Banská Bystrica (1997), Bratislava (2001), Stará Lesná (2005), Tále (2009) and Smolenice (2013). The GEMS workshop is now held regularly every four years, organised by the leaders of the Slovak topological graph theory school: Roman Nedela, Jozef Širáň, and Martin Škoviera. These workshops have become very well known for their informal atmosphere, allowing time for discussion of research problems and exchange of information between both individual researchers and international research teams.

The venue for the most recent GEMS workshop was Smolenice Castle, the very same place where an event considered the world's first truly international graph theory meeting was held fifty years earlier. Together with the the Seventh Czech-Slovak International Symposium on Graph Theory, Combinatorics, Algorithms and Applications (held in Košice a week before the GEMS 2013 workshop), it constituted one of the highlights of celebrations to commemorate the 50th anniversary of this unique scientific event.

We believe that readers will find the selected papers from GEMS 2013 both interesting and inspirational for further research. We also hope that after a similar special issue devoted to GEMS 2009, this issue will be followed by subsequent collections of papers, based on lectures delivered at GEMS workshops in 2017 and beyond.

Jozef Širáň and Martin Škoviera
Guest Editors

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Algorithmic enumeration of regular maps*

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Abstract

Given a finite group G, we describe an algorithm that enumerates the regular maps having G as rotational subgroup, using the knowledge of its table of ordinary characters and its subgroup lattice. To show the efficiency of our algorithm, we use it to compute that, up to isomorphism, there are 796,772 regular maps whose rotational subgroup is the sporadic simple group of O'Nan and Sims.

Keywords: Regular map, O'Nan sporadic simple group, subgroup lattice, character table.

Math. Subj. Class.: 05E18, 52B10, 20D08

1 Introduction

According to Coxeter (see [9], Chapter 8), systematic enumeration of orientable regular maps began in the 1920s by fixing a genus g and enumerating all maps embeddable on surfaces of genus g. Genus 2 was the first case considered by Errera and finished by Threlfall. Since then, a lot of work has been done on the subject, culminating in the enumeration of all orientable maps on surfaces of genus up to 301 by Conder (see [5, 4] and Conder's website for the latest results[1]).

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Another way to enumerate orientable maps is to fix a group G (or a family of groups) and count how many regular maps have G acting as rotational subgroup of the full automorphism group. In other words, we want to determine, for a given group G, the number of pairs of elements $[R, S] \in G^2$ such that

$$o(RS) = 2, o(R) = p, o(S) = q \text{ and } \langle R, S\rangle = G \tag{1.1}$$

where p and q are arbitrary orders of elements in G. The second type of enumeration can be done using a formula due to Frobenius [12] (see Section 2.2) based on character theory. Frobenius' formula has been used by Sah (see [20], Section 2) to obtain some enumeration results for the first group of Janko and the small Ree groups ${}^2G_2(q)$ with $q = 3^{2e+1}$, among other things. Conder et al. [6] extracted an enumeration result for all regular hypermaps of a given type with automorphism group isomorphic to $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$ from the latter reference. Their result does not make use of character theory.

Jones and Singerman [16] set up the theoretical framework that links the study of maps to that of Riemann surfaces, showing among others that every map $\mathcal{M}$ is isomorphic to some canonical map $\overline{\mathcal{M}}$ on a Riemann surface. In [11], Downs and Jones set up the theoretical framework to determine the number of orientable maps of type $\{3, p\}$ with automorphism group a group $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$. Jones and Silver showed in [15] that the Suzuki groups $\mathrm{Sz}(q)$ are automorphism groups of regular maps of type $\{4, 5\}$. They also enumerated these maps: they used character theory and techniques developed by Philip Hall in [13] using Möbius inversion to show that there is at least one pair $[R, S]$ as above in each $\mathrm{Sz}(q)$. Then they used the fact that each element of order 4 is not conjugate to its inverse in $\mathrm{Aut}(\mathrm{Sz}(q))$ to conclude that every such map has to be chiral. For more results of that kind, we refer to [15, 14]. Mazurov and Timofeenko also used similar techniques to find those sporadic groups that can be generated by triples of involutions, two of which commute (see [18, 21]), therefore determining which sporadic groups are full automorphism groups of non-orientable regular maps.

Given a pair $[R, S] \in G^2$ satisfying (1.1), we can construct a regular map $\mathcal{M}$ of type $\{p, q\}$ from it with G being the orientation-preserving subgroup of the full automorphism group of $\mathcal{M}$. Frobenius' formula therefore gives us the number of regular maps that have G as such subgroup. The idea of the present paper is to use this formula in a systematic way to determine for a given group G what are the possible types for a map $\mathcal{M}$ with G being either the orientation-preserving subgroup of $\mathrm{Aut}(\mathcal{M})$ or G being the full automorphism group of $\mathcal{M}$ in the non-orientable case.

In this paper, we design an algorithm to compute up to isomorphism the number of regular maps (reflexible or chiral) having a given group G as group of orientation-preserving automorphisms, based on the character tables of G and its subgroups and on the subgroup lattice of G. To show the efficiency of our algorithm, we implemented it in MAGMA [2] and used it on the O'Nan sporadic simple group $\mathrm{O'N}$. The choice of $\mathrm{O'N}$ is motivated by the fact that this is one of the most mysterious sporadic groups. Its smallest permutation representation is on 122,760 points and its subgroup lattice is relatively small.

The motivation of the paper first came from abstract regular polytopes. A recent paper by the authors and Mark Mixer [8] classifies all abstract regular polytopes of rank at least four for the O'Nan group. Hence rank three remains open. For a simple group G, a non-orientable regular map $\mathcal{M}$ whose full automorphism group is G is also an abstract regular polyhedron while a chiral map is a chiral polyhedron. Hence, getting to know which types are possible for G is also interesting in the study of abstract polyhedra whose automorphism group is G.

There is most likely a very large number of pairwise non-isomorphic abstract polyhedra having the O'Nan group as automorphism group. For instance, as shown in [17], the third Conway group, whose order is comparable, has $21,118$ abstract regular polyhedra up to isomorphism. Here, we derive the possible types $\{p,q\}$ for maps having $O'N$ as automorphism group. Our results for the O'Nan group may be summarized as follows.

Theorem 1.1. *Let G be the O'Nan sporadic simple group and let*

$$P := \{3,4,5,6,7,8,10,11,12,14,15,16,19,20,28,31\}.$$

1. *There exist two elements $R,S \in G$ such*

$$o(R) = p, o(S) = q, o(RS) = 2, \langle R,S\rangle = G$$

   for every $p \leq q \in P$ except for $\{p,q\} = \{3,3\}$, $\{3,4\}$, $\{3,5\}$, $\{3,6\}$, $\{3,7\}$, $\{3,12\}$ and $\{4,4\}$.
2. *There are 796,772 orbits of such pairs $\{R,S\}$ under the action of $\mathrm{Aut}(\mathrm{O'N}) = \mathrm{O'N} : C_2$.*
3. *Orientably-regular but chiral maps $\mathcal{M}$ with $\mathrm{Aut}(\mathcal{M}) = G$ exist for all pairs $\{p,q\}$ of (1) except $\{3,15\}$ (that is 128 possible types).*
4. *Non-orientable regular maps $\mathcal{M}$ with $\mathrm{Aut}(\mathcal{M}) = G$ exist for all pairs $\{p,q\}$ of (1) except $\{20,q\}$, $\{31,q\}$ (with $q \in P$), $\{3,10\}$, $\{4,5\}$ and $\{4,6\}$ (that is 95 possible types).*
5. *Reflexible maps $\mathcal{M}$ with $\mathrm{Aut}(\mathcal{M}) = \mathrm{Aut}(G)$ exist for all pairs $\{p,q\}$ of (1) except $\{8,q\}$, $\{16,q\}$ (with $q \in P$) (that is 98 possible types).*

The paper is organized as follows. In Section 2, we introduce the theoretical background needed to understand this paper. In Section 3, we describe our algorithm. In Section 4, we summarize the results obtained on the O'Nan sporadic simple group and obtain (1) and (2) of Theorem 1.1. In Section 5, we determine the types of maps that exist for the O'Nan group, deriving (3), (4) and (5) of Theorem 1.1. In Section 6, we give an algorithm to generate efficiently all maps of type $\{p,q\}$ for a fixed p. Finally, in Section 7, we conclude our paper with some remarks.

2 Theoretical background

2.1 Regular maps

In this paper, a map is a 2-cell embedding of a connected graph into a closed surface without boundary. Such a map $\mathcal{M}$ has a vertex-set $V := V(\mathcal{M})$, an edge-set $E := E(\mathcal{M})$ and a set of faces $F := F(\mathcal{M})$. We call $V \cup E \cup F$ the set of *elements* of $\mathcal{M}$. A triple $T := \{v,e,f\}$ where $v \in V$, $e \in E$ and $f \in F$ is called a *flag* if each element of T is incident with the other elements of T. The map is called *orientable* if the underlying surface on which the graph is embedded is orientable. Otherwise, it is called *non-orientable*. Faces of $\mathcal{M}$ are simply-connected components of the space obtained by removing the embedded graph from the surface. An *automorphism* of a map is a permutation of its elements preserving the sets V, E and F and incidence between the elements. Automorphisms form a group under composition called the *automorphism group* of the map and denoted by $\mathrm{Aut}(\mathcal{M})$.

If there exist a face f and two automorphisms R and S such that R cyclically permutes the consecutive edges of f and S cyclically permutes the consecutive edges incident to some vertex v of f, then $\mathcal{M}$ is called a *regular map* in the sense of Brahana [3]. In this case, the group $\mathrm{Aut}(\mathcal{M})$ acts transitively on the vertices, on the edges and on the faces. All faces are thus bordered by the same number of edges, say p and all the vertices have same degree, say q. The pair $\{p, q\}$ is known as the *type* of $\mathcal{M}$. Observe that the topological *dual* of $\mathcal{M}$, denoted by $\mathcal{M}^*$ is obtained by switching vertices and faces (that is $V(\mathcal{M}^*) := F(\mathcal{M})$, $E(\mathcal{M}^*) := E(\mathcal{M})$, $F(\mathcal{M}^*) := V(\mathcal{M})$). It is also regular and its type is $\{q, p\}$.

Note that R and S may be assumed to be such that RS interchanges v with one of its neighbors along an edge e on the border of f, interchanging f with the other face containing e. The three automorphisms R, S and RS then satisfy the following relations.

$$R^p = S^q = (RS)^2 = 1 \tag{2.1}$$

If a regular map $\mathcal{M}$ also has an automorphism a which flips the edge e but preserves f, then we say that $\mathcal{M}$ is *reflexible*. In that case, $\mathrm{Aut}(\mathcal{M})$ has a unique orbit on the set of flags. Moreover, $\mathrm{Aut}(\mathcal{M})$ is generated by the three automorphisms a, $b := aR$ and $c := bS$ that satisfy the following relations: $a^2 = b^2 = c^2 = (ab)^p = (ac)^2 = (bc)^q$.

If the map $\mathcal{M}$ is orientable, then the elements $R = ab$ and $S = bc$ generate a normal subgroup of $\mathrm{Aut}(\mathcal{M})$ of index 2, consisting of all elements expressible as words of even length in $\{a, b, c\}$. This subgroup is called the *rotational subgroup* and denoted by $\mathrm{Aut}^+(\mathcal{M})$. All elements of $\mathrm{Aut}^+(\mathcal{M})$ are precisely those preserving the orientation of the underlying surface while all other elements of $\mathrm{Aut}(\mathcal{M})$ reverse the orientation. In the non-orientable case, each of a, b and c can be expressed as a word in $\{R, S\}$ and hence, $\mathrm{Aut}(\mathcal{M}) = \langle R, S\rangle$.

If there is no automorphism a which flips the edge e but preserves f, then we say that the map $\mathcal{M}$ is *chiral*. Its automorphism group can be generated by the rotations R and S and $\mathcal{M}$ is necessarily orientable. Moreover, chiral maps occur in opposite pairs, each member of which is obtainable from the other by reflection.

2.2 Frobenius' formula

The search for maps having $G := \langle R, S\rangle$ as an automorphism group is equivalent to the search for triples of elements $x, y, z \in G$ satisfying (1.1) by posing $x = (RS)^{-1} = RS$, $y = R$ and $z = S$. Let G be a finite group and let

$$\Pi_G(\{p, q\}) := \{[x, y, z] \in G^3 | o(x) = 2, o(y) = p, o(z) = q, o(xyz) = 1\}.$$

In order to determine the cardinality $\pi_G(\{p, q\})$ of $\Pi_G(\{p, q\})$, we use the following result, due to Frobenius (see [12], section 4, equation 2).

Theorem 2.1. *If C_i, C_j and C_k denote conjugacy classes of elements in a finite group G, the number of solutions of $g_i g_j g_k = 1$ in G, with each $g_x \in C_x$ is*

$$\lambda_{i,j,k} = \frac{|C_i| \cdot |C_j| \cdot |C_k|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g_i)\chi(g_j)\chi(g_k)}{\chi(1)} \tag{2.2}$$

where $\mathrm{Irr}(G)$ is the set of irreducible characters of G.

This theorem gives us an easy way to compute $\pi_G(\{p, q\})$.

Corollary 2.2. *Let G be a group. Let $C_1, \dots, C_r$ be the conjugacy classes of elements of G. Let $K_n := \{i \in \{1, \dots, r\} \mid o(x) = n$ for some $x \in C_i\}$. Then*

$$\pi_G(\{p,q\}) = \sum_{i \in K_p} \sum_{j \in K_q} \sum_{k \in K_2} \frac{|C_i| \cdot |C_j| \cdot |C_k|}{|G|} \sum_{\chi \in Irr(G)} \frac{\chi(g_i)\chi(g_j)\chi(g_k)}{\chi(1)}. \tag{2.3}$$

Proof. Straightforward. □

Let $\Gamma_G(\{p,q\}) := \{[x,y,z] \in \Pi_G(\{p,q\}) \mid \langle x,y,z \rangle = G\}$ and let $\gamma_G(\{p,q\}) := |\Gamma_G(\{p,q\})|$. The following lemma is the basis of our algorithm.

Lemma 2.3. *For a given group G and two integers $p, q > 1$, we have*

$$\gamma_G(\{p,q\}) = \pi_G(\{p,q\}) - \sum_{H<G} \gamma_H(\{p,q\})$$

or equivalently

$$\gamma_G(\{p,q\}) = \pi_G(\{p,q\}) - \sum_{H \in \mathcal{C}} \gamma_H(\{p,q\}) \times [G : N_G(H)]$$

where $\mathcal{C}$ is a set containing one representative of each conjugacy class of nontrivial proper subgroups of G.

Proof. Straightforward. □

As observed by Hall in [13], page 135, the number $n_G(\{p,q\})$ of pairwise non-isomorphic triples satisfying (1.1) is then obtained by dividing $\gamma_G(\{p,q\})$ by the order of the automorphism group of G. In other words,

$$n_G(\{p,q\}) = \frac{\gamma_G(\{p,q\})}{|\mathrm{Aut}(G)|}. \tag{2.4}$$

Following Lemma 2.3, we readily see that, in order to compute $\gamma_G(\{p,q\})$, it suffices to get one representative H of each conjugacy class of subgroups of G, and for each such H, to compute its normalizer and $\gamma_H(\{p,q\})$.

3 An algorithm to compute $\gamma_G(\{p,q\})$

Let G be a finite group. We detail an algorithm that determines $\pi_G(\{p,q\})$ and $\gamma_G(\{p,q\})$ for given values of p and q. In view of the developments of Section 2.2, $\pi_G(\{p,q\})$ can be computed using only the table of ordinary characters of G. Assuming that the character table of G is available (as it is the case for many simple groups in MAGMA [2] for instance) or easily computable, this is straightforward. Trickier is the computation of $\gamma_G(\{p,q\})$. Since

$$\gamma_G(\{p,q\}) = \pi_G(\{p,q\}) - \sum_{H<G} \gamma_H(\{p,q\})$$

we observe that there is a natural recursive way to compute $\gamma_G(\{p,q\})$. It only requires the knowledge of the subgroup lattice of G.

```
Input : G a permutation group
        p, q two positive integers

Output : γ_G := γ_G({p,q})

Compute the subgroup lattice Λ(G) of G.
  The subgroup lattice Λ(G) can be seen as an ordered list
  with least element G and greatest element the trivial
  group. If a subgroup of class i contains a subgroup of
  class j, then i < j.

For each conjugacy class C of subgroups of G,
  Take a representative H of C.
  If the order of H is divisible by pq/GCD(p,q)
    Compute δ : H → H̃ an isomorphism that reduces
      the permutation degree of H.
    Compute the subgroup lattice Λ(H̃) of H̃ and
      compute π_H̃({p,q}) using equation (2.3).

Now, read through Λ(G) starting from the trivial subgroup.
  At each step i, let H be a subgroup of the i^th conjugacy
  class that is considered and compute γ_H̃({p,q}).
    This computation requires the knowledge of Λ_H̃ and
    γ_Ĩ({p,q}) for all I < H. Note however that it is
    guaranteed that γ_Ĩ({p,q}) has been already computed at
    this stage since the lattice Λ(G) is endowed
    with a suitable ordering as mentioned earlier.

When all steps above have been done, γ_G has been computed.
Return γ_G
```

Figure 1: An algorithm to compute $\gamma_G(\{p,q\})$

The algorithm given in Figure 1 makes use of the obvious recursive way of computing $\gamma_G(\{p,q\})$ but it is not a recursive algorithm. Indeed, it carefully avoids multiple computations, for instance by computing only once the subgroup lattice and character table of one representative of each conjugacy class of subgroups of G. It also tries to reduce the permutation degree of each subgroup before dealing with it which speeds up computations of the subgroup lattice and the character table of the subgroups.

Indeed, our algorithm computes (or at least yields) the Möbius function of G; this could be useful in many other contexts, e.g. in enumerating quotients isomorphic to G in other finitely generated groups.

4 An application: the O'Nan sporadic simple group

In order to illustrate the efficiency of our algorithm, we implemented it in MAGMA [2] and we ran it on O$'$N, the sporadic simple group of O'Nan, of order $460,815,505,920$ and smallest permutation representation degree $122,760$. Observe that $|\mathrm{Out}(\mathrm{O'N})| = 2$. In MAGMA, the function `SubgroupLattice` computes the subgroup lattice of a given finite group. In the case of O$'$N however, `SubgroupLattice` is not able to compute this lattice[2]. Fortunately, an algorithm to compute subgroup lattices of groups like O$'$N is made available in [7].

We computed the numbers $\pi_{\mathrm{O'N}}(\{p,q\})$, $\gamma_{\mathrm{O'N}}\{p,q\})$ and $n_{\mathrm{O'N}}(\{p,q\})$ for all possible values of p and q. Recall that, by Formula (2.4), $n_{\mathrm{O'N}}(\{p,q\})$ is obtained by dividing $\gamma_{\mathrm{O'N}}(\{p,q\})$ by the order of $\mathrm{Aut}(\mathrm{O'N})$ which is $2 \cdot |\mathrm{O'N}|$. There are 17 distinct orders of elements in O$'$N. One of them is 2 and if p or q is 2, then $\gamma_G(\{p,q\})$ is null as O$'$N is a simple group. Hence, in total, there are 16*15/2 + 16 = 136 possible unordered pairs $\{p,q\}$. Out of these, five give obviously 0, namely those pairs that give groups which are solvable or isomorphic to A_5, that is $\{3,3\}$, $\{3,4\}$, $\{3,5\}$, $\{3,6\}$ and $\{4,4\}$. Therefore, there remain 131 of them to compute. We give in Table 1 the values $n_G(\{p,q\})$ for O$'$N. The 131 cases have been spread on several processors. Each case took on average 5 days to finish. Point (1) of Theorem 1.1 is then obtained by collecting the nonzero entries of Table 1. The sum of all the numbers appearing in that table gives point (2) of Theorem 1.1.

Note that Woldar had already shown in [22] that O$'$N is not a Hurwitz group, meaning that $\gamma_{\{3,7\}} = 0$. He also showed that $\gamma_{\{3,11\}} \neq 0$. Moreover, in [10], Darafsheh, Ashrafi and Moghani showed that $\gamma_{\{p,q\}} \neq 0$ for the following twelve pairs: $\{3,19\}$, $\{3,31\}$, $\{5,7\}$, $\{5,11\}$, $\{5,19\}$, $\{5,31\}$, $\{7,11\}$, $\{7,19\}$, $\{7,31\}$, $\{11,19\}$, $\{11,31\}$ and $\{19,31\}$. Very recently, Al-Khadi [1] showed that $\gamma_{\{3,12\}} = 0$ and $\gamma_{\{3,q\}} \neq 0$ for $q \in \{8,10,12,14,16,20,28\}$.

5 Regular maps for $O'N$

By Table 1, we know exactly how many pairs of generating elements $\{R,S\}$ satisfying (2.1) exist up to isomorphism for any given pair $\{o(R),o(S)\}$. For instance, there are 7 such pairs $\{R,S\}$ with $\{o(R),o(S)\} = \{3,10\}$.

If there is no automorphism of $G := \langle R,S\rangle$ that inverts R and S, G is the full automorphism group of an orientably-regular but chiral map of type $\{p,q\}$ (and its dual of type $\{q,p\}$). The pair $\{R,S\}$ is then called *chiral*.

On the other hand, if there exists an automorphism $\theta \in \mathrm{Aut}(G)$ such that $\theta([R,S]) = [R^{-1},S^{-1}]$, then θ is an involution and the pair $\{R,S\}$ is called *reflexible*. In this case, the group generated by R, S and θ is the full automorphism group of a reflexible map $\mathcal{M}$ of type $\{p,q\}$, with $G \cong \mathrm{Aut}^+(\mathcal{M})$, the orientation-preserving subgroup (of index 2) and $\mathrm{Aut}(\mathcal{M}) \cong G : C_2$ where ':' denotes a semi-direct product. This semi-direct product is sometimes a direct product, namely when the automorphism θ is an inner automorphism of G. In that case, G is also the full automorphism group of a non-orientable map $\mathcal{N}$ of type $\{p,q\}$ (and its dual of type $\{q,p\}$). Moreover, $\mathcal{M}$ is then an orientable double cover of $\mathcal{N}$, and $\mathrm{Aut}(\mathcal{M}) \cong G \times C_2$.

The following lemma gives point (3) of Theorem 1.1.

[2] at least up to version 2.19-3 of MAGMA

Lemma 5.1. *Let G be the O'Nan sporadic simple group. For every pair $\{p, q\}$ of Theorem 1.1.(1) except $\{3, 15\}$, there exists at least one chiral map $\mathcal{M}$ of type $\{p, q\}$ with* $\mathrm{Aut}(\mathcal{M}) \cong G$.

Proof. A non-exhaustive computer search with MAGMA produced chiral maps of all possible types except $\{3, 15\}$ in a few days. By Table 1, there are 6 non-isomorphic pairs of type $\{3, 15\}$. Using MAGMA, we produced the 6 non-isomorphic pairs $\{R, S\}$ and checked that for each of them, there exists $\theta \in \mathrm{Aut}(G)$ that inverts R and S. For four of them, $\theta \in \mathrm{Inn}(G)$ and for two of them, θ is an outer automorphism. □

Since $\mathrm{O'N}$ is simple, a non-chiral regular map $\mathcal{M}$ with $\mathrm{Aut}(\mathcal{M}) = \mathrm{O'N}$ is necessarily non-orientable.

Lemma 5.2. *Let G be the O'Nan sporadic simple group. Non-orientable regular maps of type $\{p, q\}$ with G as full automorphism group do not exist for pairs $\{p, q\}$ with p or q equal to* 20 *or* 31.

Proof. It suffices to observe that all elements of order 20 and 31 are not conjugate to their inverse. Hence, an automorphism that would reverse R and S in this case is necessarily an outer automorphism. □

The above lemma combined with those values $\gamma_{\{p,q\}}$ equal to 0 gives at most 98 possible types for non-orientable maps having $\mathrm{O'N}$ as full automorphism group. A non-exhaustive brute force search gave in a few days examples of such maps for 92 types. For the remaining 6 types, we did exhaustive searches and here is a summary of what we found.

- $\{3, 10\}$: an exhaustive search found 6 chiral maps and 1 pair $\{R, S\}$ with θ an outer automorphism;
- $\{4, 5\}$: an exhaustive search found 16 chiral maps and 2 pairs $\{R, S\}$ with θ an outer automorphism;
- $\{4, 6\}$: an exhaustive search found 42 chiral maps and 1 pair $\{R, S\}$ with θ an outer automorphism;
- $\{5, 5\}$: an exhaustive search found 22 chiral maps, 2 non-orientable maps and 2 pairs $\{R, S\}$ with θ an outer automorphism;
- $\{5, 7\}$ and $\{7, 7\}$: we found at least one non-orientable map for each type.

The above results are summarized in point (4) of Theorem 1.1.

Lemma 5.3. *Let G be the O'Nan sporadic simple group. There is no reflexible map $\mathcal{M}$ of type $\{p, q\}$ such that* $\mathrm{Aut}(\mathcal{M}) = \mathrm{Aut}(G)$ *for any pair $\{p, q\}$ with p or q equal to* 8 *or* 16.

Proof. All elements of order 8 and 16 are conjugate to their inverse. Moreover, there is no outer automorphism mapping such an element to its inverse. □

The above lemma combined with those values $\gamma_{\{p,q\}}$ equal to 0 give at most 98 possible types for reflexible maps $\mathcal{M}$ with $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(G)$. A brute force search gave us 95 types for which such pairs exist in a couple of days. We dealt separately with the three types that the search did not find. Below is a summary of what we found.

- $\{4, 5\}$: an exhaustive search found 16 chiral maps and 2 pairs $[R, S]$ with θ an outer automorphism;
- $\{5, 5\}$: an exhaustive search found 22 chiral maps, 2 non-orientable regular maps and 2 pairs $[R, S]$ with θ an outer automorphism;
- $\{6, 12\}$: at least one pairs $[R, S]$ with θ an outer automorphism was found.

The above results are summarized in point (5) of Theorem 1.1.

6 Generating all maps for the O'Nan group

The O'Nan group has a unique conjugacy class of involutions and the centralizer of an involution is a group $4 \cdot L_3(4) : 2$ of order $161, 280$. It is the largest centralizer of an element of order at least 2. This suggests an algorithm to construct all of the 796,772 pairs (R, S) for the O'Nan group to study the prevalence of chirality over regularity for this group.

To generate all pairs $\{R, S\}$ with S an element of order p and R an element of order $\geq p$, we construct a permutation representation of $\mathrm{O'N}$ on its involutions. This is done by constructing the coset space of $\mathrm{O'N}$ on $C_{\mathrm{O'N}}(\rho)$ for an arbitrary involution $\rho \in \mathrm{O'N}$.

Let P be a sequence. We will use P to store pairs of elements of $\mathrm{O'N}$. Let G be the permutation representation on the cosets of $C_{\mathrm{O'N}}(\rho)$ and let $\phi : \mathrm{O'N} \to G$ be an isomorphism between $\mathrm{O'N}$ in its natural permutation representation and G. Let S be a sequence containing one representative of each conjugacy class of elements of order p in $\mathrm{O'N}$. For $s \in S$, let $\mathcal{O}$ be the set of orbits of $\phi(s)$. For each $o \in \mathcal{O}$, let x be a representative of o and let $\phi^{-1}(G_x)$ be the centralizer of an involution in $\mathrm{O'N}$ that correspond to the fixed point x. Let τ be the involution centralized by $\phi^{-1}(G_x)$. Let $R := \tau * S^{-1}$. Then $\{R, S\}$ is a pair with $RS = \tau$ an involution. If $\langle R, S \rangle = \mathrm{O'N}$ and there is no pair $\{R', S'\}$ in P isomorphic to $\{R, S\}$, append $\{R, S\}$ to P. When a new pair $\{R, S\}$ is found, we can determine whether it gives an orientably-regular but chiral map or a non-orientable map whose full automorphism group is $\mathrm{O'N}$. In the process, we use the results of Section 5 to shorten the computations: we keep track of how many pairs of each type have been generated so that, once we get the total number for a given type, we do not have to consider that type anymore.

Each chiral map (respectively non-orientable map) whose full automorphism group is $\mathrm{O'N}$ is also an abstract chiral polyhedron (respectively abstract regular polyhedron). Therefore, the algorithm described above permits in theory to construct all chiral and regular polyhedra for the O'Nan group.

7 Concluding remarks

In practice, to generate all the 284 pairs of type $\{3, q\}$, it took less than 4 hours on a computer with a processor running at 2.9Ghz. We needed 11 days to generate all 5176 pairs of type $\{4, q\}$ and 28 days for the 7738 pairs of type $\{5, q\}$. Experiments with other types gave an average time of more than five minutes per map.

Out of the 284 pairs of type $\{3, q\}$, 230 give a chiral map and 39 a non-orientable map with full automorphism group $\mathrm{O'N}$. Out of the 5176 pairs of type $\{4, q\}$, 4906 give a chiral map and 114 a non-orientable map with full automorphism group $\mathrm{O'N}$. Out of the 7738 pairs of type $\{5, q\}$, 7340 give a chiral map and 188 a non-orientable map with full automorphism group $\mathrm{O'N}$. The tendency of maps of chiral type being more prevalent seems confirmed by the partial results we obtained on maps of type $\{p, q\}$ with $q \geq p \geq 6$.

For all these maps, answering questions like "what are the exponents[3] of $\mathcal{M}$, is it self-dual, etc." is possible.

8 Acknowledgements

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[3]See [19] for a definition.

	3	4	5	6	7	8	10	11	12	14	15	16	19	20	28	31
3	0	0	0	0	0	10	7	37	0	10	6	68	57	44	20	25
4		0	18	43	102	284	285	503	120	166	234	1292	846	554	370	359
5			26	98	150	470	365	718	211	290	340	1966	1242	800	560	502
6				165	354	1122	953	1776	474	597	874	4700	2943	1948	1370	1268
7					648	1848	1506	2687	815	1054	1284	7448	4725	2916	2214	1995
8						5424	4460	8096	2370	3056	3960	22040	13926	8880	6276	5752
10							3613	6532	1969	2526	3072	17822	11262	7224	4992	4632
11								11839	3583	4601	5814	32488	20493	13094	9202	8325
12									1072	1391	1710	9764	6126	3984	2796	2577
14										1796	2330	12504	7899	5024	3616	3266
15											2834	15808	10020	6424	4496	4062
16												88784	56052	35644	25316	23048
19													35442	22572	15978	14553
20														14238	10292	7246
28															7246	6658
31																5999

Table 1: Values of $n_G(\{p,q\})$ with $G \cong \mathrm{O'N}$

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Isospectral genus two graphs are isomorphic

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Abstract

By a graph we mean a finite connected multigraph without bridges. The genus of a graph is the dimension of its homology group. Two graphs are isospectral is they share the same Laplacian spectrum. We prove that two genus two graphs are isospectral if and only if they are isomorphic. Also, we present two isospectral bridgeless genus three graphs that are not isomorphic.

The paper is motivated by the following open problem posed by Peter Buser: are isospectral Riemann surfaces of genus two isometric?

Keywords: Graph, Laplacian spectrum, isospectral graphs, Laplacian polynomial, spanning tree.

Math. Subj. Class.: 05C50, 15A18, 58J53

1 Introduction

Over the last decade, a few discrete versions of the theory of Riemann surfaces were created ([1, 18, 2, 8, 11]). In these theories, the role of Riemann surfaces is played by graphs. The genus of a graph is the dimension of its homology group. Under these assumptions, the theory of Jacobi manifolds is constructed and analogues of the Riemann-Hurwitz and Riemann-Roch theorems were proved. Counterparts of many other theorems from the classical theory of Riemann surfaces were derived in the discrete case ([9, 10, 16]).

Since the classical paper by Mark Kac [14], the question of what geometric properties of a manifold are determined by its Laplace operator has inspired many intriguing results. One class of manifolds whose spectral theory has been studied with many beautiful results is the class of compact Riemann surfaces with the canonical constant curvature metric. Wolpert [19] showed that a generic Riemann surface is determined by its Laplace spectrum. Nevertheless, pairs of isospectral non-isometric Riemann surfaces in every genus ≥ 4 are known. See papers by Buser [7], Brooks and Tse [5], and others. There are also examples of

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isospectral non-isometric surfaces of genus two and three with variable curvature ([5, 3]). At the same time, isospectral genus one Riemann surfaces (flat tori) are isometric [4]. Similar results are also known for graphs ([12, 13]).

Peter Buser [6] posed an interesting problem: are two isospectral Riemann surfaces of genus two isometric? Up to our knowledge the problem is still open but, quite likely, can be solved positively. The aim of this paper is to give a positive solution of an analogous problem for bridgeless graphs of genus two (Theorem 3.1). Also, we show that there are two isospectral bridgeless graphs of genus three that are not isomorphic (Figure 5). Because of the intrinsic link between Riemann surfaces and graphs we hope that our result will be helpful to make a progress in solution of the Buser problem.

2 Preliminary results

2.1 Laplacian matrix and Laplacian spectrum

The Laplacian matrix of a graph and its eigenvalues can be used in several areas of mathematical research and have a physical interpretation in various physical and chemical theories. The related adjacency matrix of a graph and its eigenvalues were much more investigated in the past than the Laplacian matrix. At the same time, the Laplacian spectrum is much more natural and more important than the adjacency matrix spectrum because of it numerous application in mathematical physics, chemistry and financial mathematics.

Graphs in this paper are finite and undirected, but they may have loops and multiple edges. Denote by $V(G)$ and $E(G)$, respectively, the number of vertices and edges of a graph G. Following [2] we denote by $g(G) = E(G) - V(G) + 1$ the $genus$ of G. This is the dimension of the first homology group of G. In graph theory, the term "genus" is traditionally used for a different concept, namely, the smallest genus of any surface in which the graph can be embedded, and the integer $g = g(G)$ is called the cyclomatic or the Betti number of G. We call g the genus of G in order to highlight the analogy with Riemann surfaces.

A *bridge* is an edge of a graph G whose deletion increases the number of connected components. Equivalently, an edge is a bridge if and only if it is not contained in any cycle. A graph is said to be *bridgeless* if it contains no bridges.

Let G be a graph. Denote by $\mathcal{V}(G)$ and $\mathcal{E}(G)$ the set of vertices and edges of a graph G respectively. For each $u, v \in \mathcal{V}(G)$, we set a_{uv} to be equal to the number of edges between u and v. The matrix $A = A(G) = [a_{uv}]_{u,v\in\mathcal{V}(G)}$, is called the *adjacency matrix* of the graph G.

Let $d(v)$ denote the valency of $v \in \mathcal{V}(G)$, $d(v) = \sum_u a_{uv}$, and let $D = D(G)$ be the diagonal matrix indexed by $\mathcal{V}(G)$ and with $d_{vv} = d(v)$. The matrix $L = L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G. It should be noted that loops have no influence on $L(G)$. Throughout the paper we shall denote by $\mu(G, x)$ the characteristic polynomial of $L(G)$. For brevity, we will call $\mu(G, x)$ the *Laplacian polynomial* of G. Its roots will be called the Laplacian eigenvalues (or sometimes just eigenvalues) of G. They will be denoted by $\mu_1(G) \le \mu_2(G) \le \ldots \le \mu_n(G)$, $(n = V(G))$, always enumerated in increasing order and repeated according to their multiplicity. Recall [17] that for connected graph G we always have $\mu_1(G) = 0$ and $\mu_2(G) > 0$.

Two graphs G and H are called *Laplacian isospectral* (or *isospectral*) if their Laplacian polynomials coincide: $\mu(G, x) = \mu(H, x)$.

The matrix $L(G)$ is sometimes called the *Kirchhoff matrix* of G due to its role in the well-known Matrix-Tree Theorem which is usually attributed to Kirchhoff. A generalization of the Matrix-Tree-Theorem was obtained in 1967 by A. K. Kel'mans who gave a combinatorial interpretation to all the coefficients of $\mu(X, x)$ in terms of the numbers of certain subforests of a graph X; see [15] and [17] for references and history of question. We present the result by Kel'mans in the following form.

Theorem 2.1. *[15] If $\mu(X, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^i c_i x^{n-i} + \ldots + (-1)^{n-1} c_{n-1} x$ then*

$$c_i = \sum_{S \subset V, |S| = n-i} T(X_S),$$

where $T(H)$ is the number of spanning trees of H, and X_S is obtained from X by identifying all points of S to a single point.

2.2 Theta graphs

Let u and v are two (not necessary distinct) vertices. Denote by $\Theta(k, l, m)$ the graph consisting of three internally disjoint paths joining u to v with lengths $k, l, m \geq 0$ (see Fig. 1). We set $\sigma_1 = \sigma_1(k, l, m) = k + l + m, \sigma_2 = \sigma_2(k, l, m) = k\,l + l\,m + k\,m$, and $\sigma_3 = \sigma_3(k, l, m) = k\,l\,m$. It is easy to see that two graphs $\Theta(k, l, m)$ and $\Theta(k', l', m')$ are isomorphic if and only if the unordered triples $\{k, l, m\}$ and $\{k', l', m'\}$ coincide; equivalently, $\sigma_1 = \sigma_1', \sigma_2 = \sigma_2'$ and $\sigma_3 = \sigma_3'$, where $\sigma_1' = \sigma_1(k', l', m'), \sigma_2' = \sigma_2(k', l', m')$, and $\sigma_3' = \sigma_1(k', l', m')$.

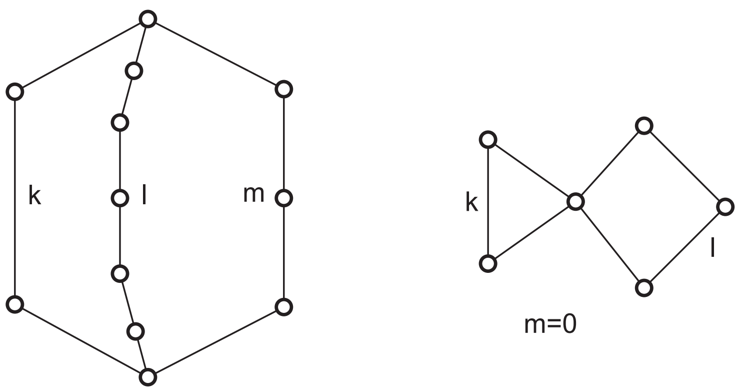

Figure 1: Theta graph $\Theta(k, l, m)$.

We make the following useful observations:

(i) If $\sigma_2 > 0$, then $\Theta(k, l, m)$ is a graph of genus two. In this case at least two of numbers $\{k, l, m\}$ are positive.

(ii) If $\sigma_1 > 0, \sigma_2 = 0$, then $\Theta(k, l, m)$ is a graph of genus one. Then exactly one of numbers $\{k, l, m\}$ is positive and the other two are zero. Moreover, $\Theta(k, l, m) = C_{k+l+m}$ is a cyclic graph with $k + l + m$ edges.

(iii) If $\sigma_1 = 0$, then $k = l = m = 0$ and $\Theta(k, l, m)$ is a graph of genus zero. More precisely, $\Theta(k, l, m) = \Theta(0, 0, 0)$ consists of one vertex.

Lemma 2.2. *Let G be an arbitrary bridgeless graph of genus two. Then G is isomorphic to $\Theta(k, l, m)$ for some k, l, m with $\sigma_2 = k\,l + l\,m + k\,m > 0$.*

Proof. Since the graph G is bridgeless it has no vertices of valency one. Denote by H the graph obtained from G by deleting of all vertices of valency two. Suppose that H has V vertices of valences $n_1, n_2, \ldots, n_V$ and E edges. Since the valency of each vertex of H is at least three we have $n_i \geq 3, i = 1, 2, \ldots, V$. Note that deleting of a vertex of valency two decreases the number of vertices and the number of edges of a graph by one. So, it does not affect the genus and H is still a graph of genus two. Thus $g(H) = 1 - V + E = 2$ and $E = V + 1$. Counting the sum of valences of H through vertices and through edges we obtain

$$n_1 + n_2 + \ldots + n_V = 2E.$$

Hence

$$3V \leq n_1 + n_2 + \ldots + n_V = 2E = 2V + 2,$$

or $V \leq 2$.

If $V = 1$ then $n_1 = 4$ and H is the figure eight graph consisting of one vertex and two loops. Putting back the vertices of valency two on the graph H we obtain the graph G isomorphic to $\Theta(k, l, 0)$ for some positive k and l. In particular, $\sigma_2 = k\,l > 0$.

If $V = 2$ then $n_1 = n_2 = 3$ and H is the theta graph consisting of two vertices and three edges. The graph G is obtained from H by adding the vertices of valency two. Hence, G is isomorphic to $\Theta(k, l, m)$ for some positive k, l, m. □

3 Main results

3.1 The main theorem and lemmas

The main result of the paper is the following theorem.

Theorem 3.1. *Two genus two bridgeless graphs are Laplacian isospectral if and only if they are isomorphic.*

The proof of the theorem is based on the following three lemmas.

Lemma 3.2. *Let $G = \Theta(k, l, m)$ be a theta graph and let $\mu(G, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^{n-1} c_{n-1} x$ be its Laplacian polynomial. Then $n = k + l + m - 1, c_1 = 2(k + l + m)$ and $c_{n-1} = (k\,l + l\,m + k\,m)(k + l + m - 1)$.*

Proof. The number of vertices, edges and spanning trees of graph G are given by

$$V(G) = k + l + m - 1,\ E(G) = k + l + m,\ T(G) = k\,l + l\,m + k\,m.$$

Then by ([15], formulas 2.15 and 2.16) we have $n = V(G) = k + l + m - 1$, $c_1 = 2E(G) = 2(k+l+m)$ and $c_{n-1} = V(G) \cdot T(G) = (k\,l+l\,m+k\,m)(k+l+m-1)$. □

Lemma 3.3. *Let $G = \Theta(k, l, m)$ be a theta graph and let $\mu(G, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^{n-1} c_{n-1} x$ be its Laplacian polynomial. Then*

$$c_{n-2} = A(\sigma_1,\, \sigma_2) + B(\sigma_1,\, \sigma_2)\sigma_3$$

where $A(s,t) = (4t - 3st - 2s^2t + s^3t + 4t^2 - st^2)/12$, $B(s,t) = (3 - 4s + s^2 - 3t)/12$, $\sigma_1 = k + l + m$, $\sigma_2 = k\,l + l\,m + k\,m$, and $\sigma_3 = k\,l\,m$.

Proof. By Theorem 2.1

$$c_{n-2} = \sum_{S \subset V,\, |S|=2} T(X_S), \tag{3.1}$$

where X_S runs through all graphs obtained from $G = \Theta(k, l, m)$ by gluing two vertices. There are exactly four types of such graphs G_1, G_2, G_3, and G_4 shown in the Fig. 2. We will enumerate the spanning trees of each type separately.

Type G_1. Glue two 3-valent vertices of graph G. As a result we obtain the graph G_1 shown on Fig. 2. The number of spanning trees of this graph is $T_1 = T(C_k) \cdot T(C_l) \cdot T(C_m) = k\,l\,m$.

Type G_2. Glue one 3-valent and one 2-valent vertices of graph G. The graph of type G_2 shown in Fig. 2 is obtained by gluing the upper 3-valent of graph G and a 2-valent vertex on the path of G labelled by k. For given i, $1 \leq i \leq k-1$ the number of spanning trees for a graph of type G_2 is equal to $T(C_i) \cdot T(\Theta(k-i,\, l,\, m)) = i\sigma_2(k-i,\, l,\, m)$. We set $F(k,\, l,\, m) = \sum\limits_{i=1}^{k-1} i\sigma_2(k-i,\, l,\, m)$. Then the total number of spanning trees for graphs of type G_2 is

$$T_2 = 2(F(k,\, l,\, m) + F(l,\, m,\, k) + F(m,\, k,\, l)).$$

The multiple 2 is needed since the graph $\Theta(k, l, m)$ has two 3-valent vertices.

Type G_3. Glue two 2-valent vertices of graph G lying on different paths. We choose one of them on the path labelled by k and the second on the path labbeled by l. Fix i, $1 \leq i \leq k-1$ and j, $1 \leq j \leq l-1$ and consider a graph of type G_3 shown in Fig. 2. This is a graph of genus three. To create a spanning tree on this graph we have to delete three edges. There are two different ways to do this. Firstly, we delete edges on three of the four paths labeled by $i, j, k-i$ and $l-j$. This be done in $\sigma_3(i, j, k-i, l-j)$ ways, where $\sigma_3(x, y, z, t) = xyz + xyt + xzt + yzt$. Secondly, if we delete an edge from the path labeled by m (in m possible ways) then we have to remove one edge from the pair of paths i, j and one edge from the pair $k-i, k-j$. Then we have $m((i+j)(k-i+l-j))$ possibilities to obtain a tree. As the result graph under consideration has $G_3(i,\, j,\, k,\, l,\, m) = \sigma_3(i, j, k-i, l-j) + m((i+j)(k-i+l-j)$ spanning trees. We set

$$J(k,\, l,\, m) = \sum_{i=1}^{k-1} \sum_{j=1}^{l-1} G_3(i,\, j,\, k,\, l,\, m).$$

Then the total number of spanning trees for graphs of type G_3 is

$$T_3 = J(k,\, l,\, m) + J(l,\, m,\, k) + J(m,\, k,\, l).$$

Type G_4. Glue two 2-valent vertices lying on the same path of graph G. Choose the path labelled by k. Let us fix i and j such that $1 \leq i < j \leq k-1$. Then the number of spanning trees for a given graph of type G_4 is $T(C_{j-i})T(\Theta(k+i-j,l,m)) = (j-i)\sigma_2(k+i-j,l,m)$. We set

$$H(k,\,l,\,m) = \sum_{i=1}^{k-2}\sum_{j=i+1}^{k-1}(j-i)\sigma_2(k+i-j,l,m).$$

As a result, the number of spanning trees of the given type is

$$T_4 = H(k,\,l,\,m) + H(l,\,m,\,k) + H(m,\,k,\,l).$$

Putting the obtained formulas in Mathematica 8 by (3.1) we get

$$c_{n-2} = T_1 + T_2 + T_3 + T_4 = A(\sigma_1,\sigma_2) + B(\sigma_1,\sigma_2)\sigma_3.$$

□

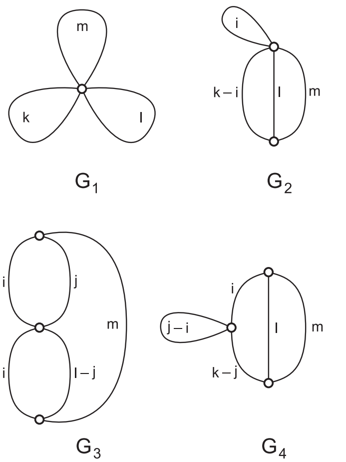

Figure 2: The graphs obtained from $\Theta(k,l,m)$ by gluing two vertices

Lemma 3.4. *Let $G = \Theta(k,l,m)$ be a theta graph and let*

$$\mu(G,x) = x^n - c_1x^{n-1} + \ldots + (-1)^{n-1}c_{n-1}x$$

be its Laplacian polynomial. Then

$$c_{n-3} = C(\sigma_1,\sigma_2) + D(\sigma_1,\sigma_2)\sigma_3 + E(\sigma_1,\sigma_2)\sigma_3^2,$$

where

$$\begin{aligned}C(s,t) &= (-34t + 21st + 25s^2t - 10s^3t - 3s^4t + s^5t - 50t^2 + 10st^2\\&\qquad + 12s^2t^2 - 2s^3t^2 - 16t^3 + st^3)/360,\\D(s,t) &= (-45 + 50s + 5s^2 - 12s^3 + 2s^4 + 24st - 9s^2t + 15t^2)/360,\\E(s,t) &= -3(-8+3s)/360.\end{aligned}$$

Proof. By Theorem 2.1

$$c_{n-3} = \sum_{S \subset V,\, |S|=3} T(X_S), \tag{3.2}$$

where X_S runs through all graphs obtained from $G = \Theta(k, l, m)$ by gluing three vertices. There are six types of such graphs W_1, W_2, W_3, W_4, W_5, and W_6 shown on the Fig. 3. We examine the spanning trees of each type separately.

Type W_1. To create a graph of type W_1 we identify two 3-valent vertices of graph G and one 2-valent vertex of G (say on the path labelled by k). The obtained graph is shown in the Fig. 3, has $i(k-i)l\,m$ spanning trees. Consider the sum $F^w(k,\, l,\, m) = \sum_{i=1}^{k-1} i(k-i)l\,m$. Find the total number of spanning trees for graphs of type W_1 by the formula

$$T_1^w = F^w(k,\, l,\, m) + F^w(l,\, m,\, k) + F^w(m,\, k,\, l).$$

Type W_2. Glue one 3-valent vertices of graph G and two 2-valent vertices lying on different paths of G (say on the paths labelled by k an l), obtaining a graph in Fig. 3. For given i and j, $1 \le i \le k-1$, $1 \le j \le l-1$, the number of spanning trees for graph of type W_2 is $ij\sigma_2(k-i, l-j, m)$. We set $H^w(k,\, l,\, m) = \sum_{i=1}^{k-1}\sum_{j=1}^{l-1} ij\sigma_2(k-i, l-j, m)$. Taking into account that graph $\Theta(k, l, m)$ has two 3-valent vertices we obtain the following formula the number of spanning trees for graphs of type W_2 :

$$T_2^w = 2(H^w(k,\, l,\, m) + H^w(l,\, m,\, k) + H^w(m,\, k,\, l)).$$

Type W_3. Glue one 3-valent vertices and two 2-valent vertices lying on the same path of G. For fixed i and j, $1 \le i < j \le k-1$, we have $i(j-i)\sigma_2(k-j, l, m)$ spanning trees for graph of type W_3. Summing over i and j we get

$$J^w(k,\, l,\, m) = \sum_{i=1}^{k-2}\sum_{j=i+1}^{k-1} i(j-i)\sigma_2(k-j, l, m).$$

Finally, the number of spanning trees for graphs of type W_3 is given by

$$T_3^w = 2(J^w(k,\, l,\, m) + J^w(l,\, m,\, k) + J^w(m,\, k,\, l)).$$

Type W_4. Glue three 2-valent vertices all lying on different paths of G. Fix i, j and s, $1 \le i \le k-1$, $1 \le j \le l-1$, $1 \le s \le m-1$. Then the number of spanning trees for a given graph of type W_4 is equal to $\sigma_2(i, j, s)\sigma_2(k-i, l-j, m-s)$. Summing over i, j and s we obtain the total number of spanning trees for graphs of type W_4 :

$$T_4^w = \sum_{i=1}^{k-1}\sum_{j=1}^{l-1}\sum_{s=1}^{m-1} \sigma_2(i, j, s)\sigma_2(k-i, l-j, m-s).$$

Type W_5. Glue two 2-valent vertices lying on a path and one 2-valent vertex lying on the other path of G. Denote by $G_3(i, j, k, l, m)$ the graph of type G_3 shown in Fig. 2. From the proof of previous Lemma we have $T(G_3(i, j, k, l, m)) = \sigma_3(i, j, k-i, l-j) + m((i+$

$j)(k-i+l-j))$. Fix i, j and s, $1 \leq i < j \leq k-1$, $1 \leq s \leq l-1$. Then the number of spanning trees for a graph of type W_5 in Fig. 3 is equal to

$$T(C_{j-i})T(G_3(i,s,k+i-j,l,m)) = (j-i)T(G_3(i,s,k+i-j,l,m)).$$

Consider the sum

$$K^w(k,\,l,\,m) = \sum_{i=1}^{k-2}\sum_{j=i+1}^{k-1}\sum_{s=1}^{l-1}(j-i)T(G_3(i,s,k+i-j,l,m)).$$

Then the number of spanning trees for graphs of type W_3 is given by

$$\begin{aligned} T_5^w = K^w(k,\,l,\,m) + K^w(l,\,m,\,k) + K^w(m,\,k,\,l) \\ + K^w(k,\,m,\,l) + K^w(l,\,k,\,m) + K^w(m,\,l,\,k). \end{aligned} \tag{3.3}$$

Type W_6. Glue three 2-valent vertices on the same path of G. Fixed i, j and s such that $1 \leq s < i < j \leq k-1$. Then the number of spanning trees for a given graph of type W_6 is equal to

$$T(C_{i-s})T(C_{j-i})T(\Theta(k-j+s,l,m)) = (i-s)(j-i)\sigma_2(k-j+s,l,m).$$

Summing over i, j and s we obtain

$$L^w(k,l,m) = \sum_{1\leq s<i<j\leq k-1}(i-s)(j-i)\sigma_2(k-j+s,l,m).$$

The total number of spanning trees in this case

$$T_6^w = L^w(k,\,l,\,m) + L^w(l,\,m,\,k) + L^w(m,\,k,\,l).$$

By (3.4) and straightforward calculation in Mathematica.8 we obtain

$$\begin{aligned} c_{n-3} &= T_1^w + T_2^w + T_3^w + T_4^w + T_5^w + T_6^w \\ &= C(\sigma_1,\sigma_2) + D(\sigma_1,\sigma_2)\sigma_3 + E(\sigma_1,\sigma_2)\sigma_3^2. \end{aligned}$$

□

3.2 Proof of the main theorem

Proof. Let G and G' be two bridgeless graphs of genus two. Then by Lemma 1 for suitable $\{k,l,m\}$ and $\{k',l',m'\}$ we have

$$G = \Theta(k,l,m) \text{ and } G' = \Theta(k',l',m').$$

Denote by $\mu(G,\,x) = x^n - c_1x^{n-1} + \ldots + (-1)^{n-1}c_{n-1}x$ and

$$\mu(G',\,x) = x^{n'} - c_1x^{n'-1} + \ldots + (-1)^{n'-1}c_{n'-1}x$$

their Laplacian polynomials.

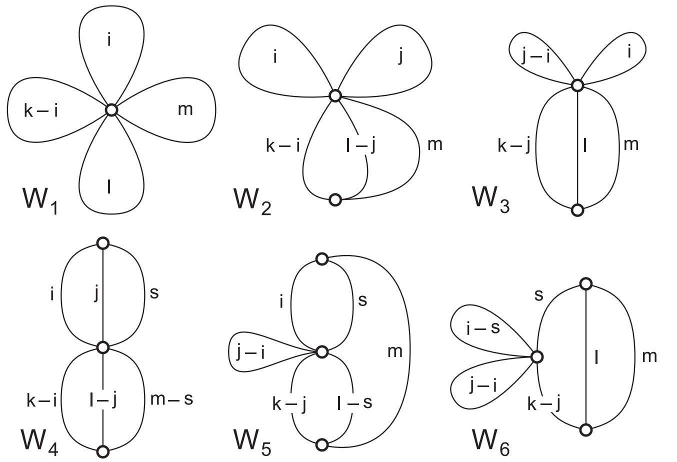

Figure 3: The graphs obtained from $\Theta(k,l,m)$ by gluing three vertices

Suppose that the graphs G and G' are isospectral. Then $n'=n, c_1'=c_1,\ldots,c_{n-1}'=c_{n-1}$. From the second and the last equalities by Lemma 2 we obtain

$$2\sigma_1 = 2\sigma_1' \text{ and } \sigma_2(\sigma_1-1)=\sigma_2'(\sigma_1'-1). \tag{3.4}$$

Since both graphs are of genus 2 we have $\sigma_1>1$ and $\sigma_1'>1$. Then the system of equations (3.4) gives $\sigma_1=\sigma_1'$ and $\sigma_2=\sigma_2'$. The theorem will be proved if we show that $\sigma_3=\sigma_3'$. We will do this in two steps. First of all, we note that by [13] isospectral graphs with $n\le 5$ vertices are isomorphic. So, we can assume that $n=k+l+m-1>5$, that is, $\sigma_1=k+l+m>6$.

By Lemma 3,

$$c_{n-2}=A(\sigma_1,\sigma_2)+B(\sigma_1,\sigma_2)\sigma_3, \tag{3.5}$$

where $A(s,t)=(4t-3st-2s^2t+s^3t+4t^2-st^2)/12$ and $B(s,t)=(3-4s+s^2-3t)/12$.

Step 1. $B(\sigma_1,\sigma_2)\neq 0$. Since $c_{n-2}'=c_{n-2}, \sigma_1=\sigma_1'$ and $\sigma_2=\sigma_2'$ from (3.5) we obtain

$$B(\sigma_1,\sigma_2)\sigma_3'=B(\sigma_1,\sigma_2)\sigma_3. \tag{3.6}$$

Hence $\sigma_3=\sigma_3'$ and the theorem is proved.

Step 2. $B(\sigma_1,\sigma_2)=0$. Then by Lemma 3

$$c_{n-3}=C(\sigma_1,\sigma_2)+D(\sigma_1,\sigma_2)\sigma_3+E(\sigma_1,\sigma_2)\sigma_3^2, \tag{3.7}$$

where

$$\begin{aligned}
C(s,t)&=(-34t+21st+25s^2t-10s^3t-3s^4t+s^5t-50t^2+10st^2\\
&\qquad +12s^2t^2-2s^3t^2-16t^3+st^3)/360,\\
D(s,t)&=(-45+50s+5s^2-12s^3+2s^4+24st-9s^2t+15t^2)/360,\\
E(s,t)&=-3(-8+3s)/360.
\end{aligned}$$

Since $c'_{n-3} = c_{n-3}, \sigma_1 = \sigma'_1$ and $\sigma_2 = \sigma'_2$ from (3.7) we obtain

$$D(\sigma_1, \sigma_2)\sigma'_3 + E(\sigma_1, \sigma_2){\sigma'_3}^2 = D(\sigma_1, \sigma_2)\sigma_3 + E(\sigma_1, \sigma_2)\sigma_3^2. \tag{3.8}$$

We note that $E(\sigma_1, \sigma_2) \neq 0$ for any integer σ_1. Then the above equation has two solutions with respect to σ'_3. The first solution is $\sigma'_3 = \sigma_3$ and the second one is

$$\sigma'_3 = -\frac{D(\sigma_1, \sigma_2)}{E(\sigma_1, \sigma_2)} - \sigma_3. \tag{3.9}$$

In the first case the theorem is proved. So we assume that σ'_3 is given by equation (3.9). Recall that $B(\sigma_1, \sigma_2) = 0$. Then $\sigma_2 = (3 - 4\sigma_1 + \sigma_1^2)/3$ and equation (3.9) can be rewritten in the form

$$\sigma'_3 = \frac{1}{729}(2(425 - 357\sigma_1 - 144\sigma_1^2 + 27\sigma_1^3) - \frac{490}{-8 + 3\sigma_1}) - \sigma_3. \tag{3.10}$$

Since σ_3 and σ'_3 are integers the number

$$N = 2(425 - 357\sigma_1 - 144\sigma_1^2 + 27\sigma_1^3) - \frac{490}{-8 + 3\sigma_1}$$

is an integer divisible by 729. Moreover, $-8 + 3\sigma_1$ is a divisor of 490 and the number $\sigma_2 = (3 - 4\sigma_1 + \sigma_1^2)/3$ is a positive integer. There are a finite number possibilities of a positive integer σ_1 to satisfy these three conditions, namely, $\sigma_1 \in \{6, 19, 166\}$. The case $\sigma_1 = 6$ can be excluded since we suggested that $\sigma_1 > 6$. Another way to exclude $\sigma_1 = 6$ is to check that in this case $\sigma'_3 = -3 - \sigma_3$ is negative.

Consider the remaining cases $\sigma_1 = 19$ and $\sigma_1 = 166$. By (3.10) in these cases we have $\sigma'_3 = 348 - \sigma_3$ and $\sigma'_3 = 327789 - \sigma_3$ respectively. The respective values of σ_2 are 96 and 8965.

Let $\sigma_1 = 19$. We have the following system of equations to find positive integer parameters k, l, m, σ_3 of the graph $G = \Theta(k, l, m)$:

$$k + l + m = 19, \; k\,l + l\,m + m\,k = 96, \; k\,l\,m = \sigma_3.$$

This system has only one solution $\{k, l, m\} = \{3, 4, 12\}, \sigma_3 = 144$.

Now we are able to find parameters k', l', m', σ'_3 of the graph $G' = \Theta(k', l', m')$. First of all, $\sigma'_3 = 348 - \sigma_3 = 204$. Then we have

$$k' + l' + m' = 19, k'l' + l'm' + m'k = 96, k'l'm' = 204.$$

The latter system has no integer solutions. So the case $\sigma_1 = 19$ is impossible.

Let $\sigma_1 = 166$. We have the following system k, l, m, σ_3.

$$k + l + m = 166, \; k\,l + l\,m + m\,k = 8965, \; k\,l\,m = \sigma_3.$$

This system has only one solution $\{k, l, m\} = \{39, 59, 68\}, \sigma_3 = 39 \cdot 59 \cdot 68$.

Find parameters k', l', m', σ'_3 of the graph $G' = \Theta(k', l', m')$. Now, $\sigma'_3 = 327789 - \sigma_3 = 171321$. Then we have

$$k' + l' + m' = 166, \; k'l' + l'm' + m'k' = 8965, \; k'l'm' = 171321.$$

The system has no integer solutions. The case $\sigma_1 = 166$ is also impossible.

This completes the proof. $\square$

4 Final remarks

1. The main Theorem 3.1 is not valid for genus two graphs with bridges. Indeed, the following two graphs (see Fig. 4) constructed in [12] are isospectral. They share the Laplacian polynomial

$$-72x + 192x^2 - 176x^3 + 73x^4 - 14x^5 + x^6.$$

The first of these graphs is bridgeless, while the second one has a bridge.

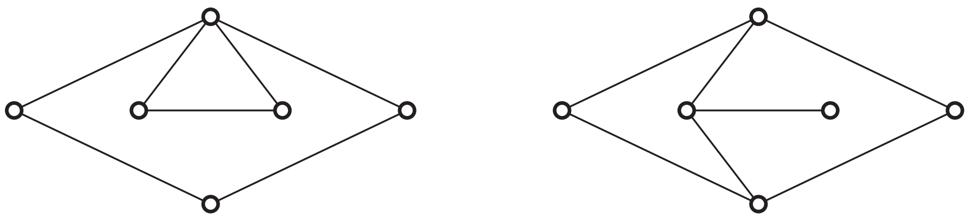

Figure 4: Isospectral graphs of genus two. The second graph has a bridge.

2. There are isospectral bridgeless graphs of genus three which are not isomorphic (see Fig. 5). These two graphs were constructed in [13].They share the Laplacian polynomial

$$-384x + 1520x^2 - 2288x^3 + 1715x^4 - 708x^5 + 164x^6 - 20x^7 + x^8.$$

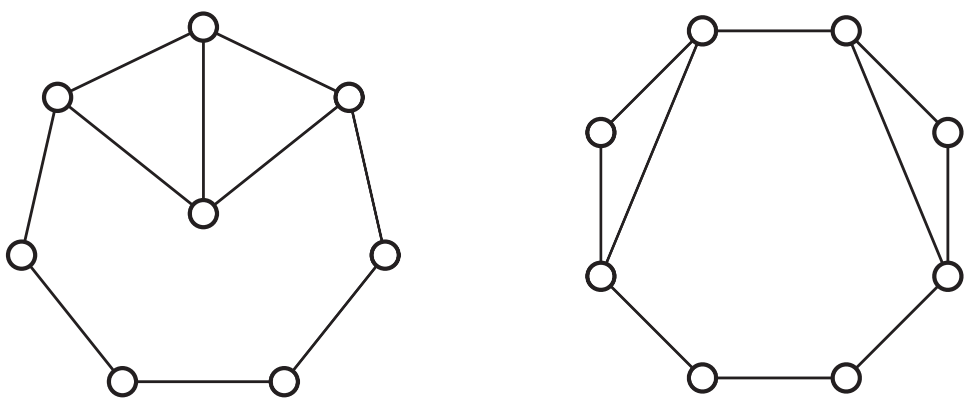

Figure 5: Isospectral graphs of genus three.

3. Any bridgeless graph of genus one is isomorphic to a cyclic graph C_n for some $n \geq 1$. If two cyclic graphs C_m and C_n are isospectral then their Laplace polynomials are of the same degree $m = n$. Hence, the graphs are isomorphic.

At the same time, there are isospectral unicycle graphs [20]. For example, the two genus one graphs shown on Fig. 6 share the Laplacian polynomial

$$28x - 146x^2 + 250x^3 - 194x^4 + 75x^5 - 14x^6 + x^7.$$

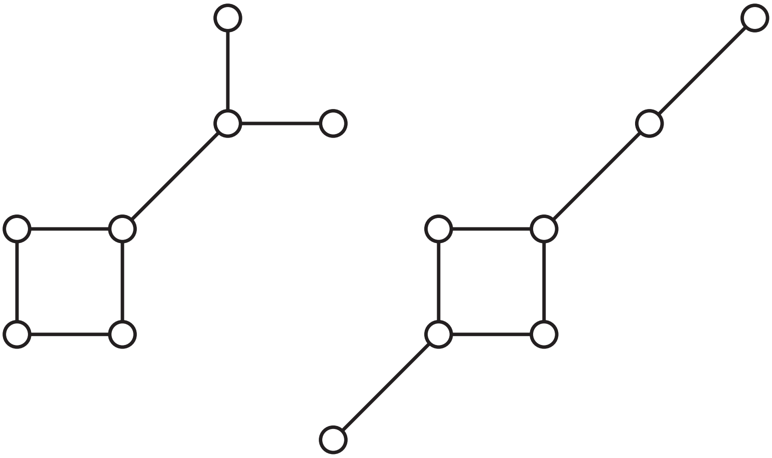

Figure 6: Isospectral graphs of genus one.

4. One can hear the genus of a graph. That is, the genus of a graph G is completely determined by its Laplace spectrum. Indeed, $g(G) = 1 - V(G) + E(G)$. Let $\mu(G, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^{n-1} c_{n-1} x$ be the Laplacian polynomial of G. By the arguments from the proof of Lemma 3.2 we have $n = V(G)$ and $c_1 = 2E(G)$. Thus $V(G)$ and $E(G)$, as well as the genus, are uniquely determined by the Laplacian polynomial.

   It follows from this observation, the previous remark, and the main result of the paper that the bridgeless graphs of genera one and two are recognisable by their Laplacian spectra among all bridgeless graphs.

5. One cannot hear a bridge of a graph. Indeed, the two graphs in Fig. 4 are isospectral. We are not able to recognise the existence of a bridge of the second graph by its spectrum.

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Combinatorial categories and permutation groups

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Abstract

The regular objects in various categories, such as maps, hypermaps or covering spaces, can be identified with the normal subgroups N of a given group Γ, with automorphism group isomorphic to Γ/N. It is shown how to enumerate such objects with a given finite automorphism group G, how to represent them all as quotients of a single regular object $\mathcal{U}(G)$, and how the outer automorphism group of Γ acts on them. Examples constructed include kaleidoscopic maps with trinity symmetry.

Keywords: Regular map, regular hypermap, covering space, permutation group, category.

Math. Subj. Class.: 20B25, 05A15, 05C10, 05E18, 20J99, 57M10

1 Introduction

In certain categories $\mathfrak{C}$, the objects $\mathcal{O}$ can be identified with the permutation representations of a particular group $\Gamma = \Gamma_{\mathfrak{C}}$ on sets $\Phi = \Phi_{\mathcal{O}}$, and the morphisms $\mathcal{O} \to \mathcal{O}'$ correspond to the functions $\Phi_{\mathcal{O}} \to \Phi_{\mathcal{O}'}$ commuting with the actions of Γ. In the case of maps on surfaces one takes Γ to be the free product $V_4 * C_2$ acting on flags, or $C_\infty * C_2$ acting on directed edges of oriented maps. The corresponding groups for hypermaps are $C_2 * C_2 * C_2$ and the free group $F_2 = C_\infty * C_\infty$ of rank 2. For abstract polytopes of a given type one can use the corresponding string Coxeter group, again acting on flags, though here one has to restrict attention to quotient groups satisfying the intersection property. In the case of coverings of a path-connected space X one uses the fundamental group $\pi_1 X$, acting on sheets or more precisely on the fibre over a base-point.

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In such a case we will call $\mathfrak{C}$ a permutational category, with parent group Γ. Each object $\mathcal{O}$ in such a category $\mathfrak{C}$ is a disjoint union of connected subobjects, corresponding to the orbits of Γ on Φ; one usually restricts attention to the connected objects, as we shall here, so that Φ can be identified with the set of cosets in Γ of a point-stabiliser $M = \Gamma_\phi$, where $\phi \in \Phi$. The permutation group induced by G on Φ is the monodromy group $G = \mathrm{Mon}\,\mathcal{O} = \mathrm{Mon}_{\mathfrak{C}}\mathcal{O}$ of $\mathcal{O}$, a subgroup of the symmetric group $\mathrm{Sym}\,\Phi$ on Φ. The automorphism group $A = \mathrm{Aut}\,\mathcal{O} = \mathrm{Aut}_{\mathfrak{C}}\mathcal{O}$ of $\mathcal{O}$, regarded as an object in $\mathfrak{C}$, is the centraliser of G in $\mathrm{Sym}\,\Phi$; since G is transitive on Φ, A acts semiregularly on Φ, and

$$A \cong N_\Gamma(M)/M \cong N_G(G_\phi)/G_\phi.$$

The most symmetric objects in $\mathfrak{C}$ are the regular objects, those for which A acts transitively (and hence regularly) on Φ. This is equivalent to M being a normal subgroup of Γ, in which case

$$A \cong \Gamma/M \cong G.$$

Indeed, in this case A and G can be identified with the left and right regular representation of the same group. In principle, understanding regular objects is sufficient for an understanding of all objects in $\mathfrak{C}$, since each object $\mathcal{O} \in \mathfrak{C}$ is the quotient of some regular object $\tilde{\mathcal{O}} \in \mathfrak{C}$, corresponding to the core N of M in Γ, by a group M/N of automorphisms of $\tilde{\mathcal{O}}$; moreover, $\tilde{\mathcal{O}}$ is finite if and only if $\mathcal{O}$ is finite, since N has finite index in Γ if and only if M has finite index. We shall therefore concentrate, for the remainder of this paper, on the regular objects in various categories $\mathfrak{C}$. In particular, we will study the set $\mathcal{R}(G) = \mathcal{R}_{\mathfrak{C}}(G)$ of regular objects $\mathcal{O} \in \mathfrak{C}$ with $\mathrm{Aut}\,\mathcal{O}$ isomorphic to a given group G. If Γ is finitely generated and G is finite then $r(G) := |\mathcal{R}(G)|$ is finite. We will consider how to calculate $r(G)$ in this case, how to represent the objects in $\mathcal{R}(G)$ as quotients of a single regular object $\mathcal{U}(G)$ in $\mathfrak{C}$, and how the outer automorphism group $\mathrm{Out}\,\Gamma$ of Γ acts on $\mathcal{R}(G)$. Examples will be given, in which the objects are maps, hypermaps or surface coverings, some of them relevant to recent work by Archdeacon, Conder and Širáň on kaleidoscopic maps with trinity symmetry [1].

2 Examples of permutational categories

Let us call a category $\mathfrak{C}$ a *permutational category* if it is equivalent to the category of permutation representations of some group Γ, called the *parent group* of $\mathfrak{C}$. This means that there are functors from each category to the other, so that their composition, in either order, is naturally equivalent to the identity. There are some well-known examples in the literature, though the equivalences are rarely expressed in terms of categories. We will summarise them briefly here; for further details, see, for example, [33] for maps, and [26] for hypermaps.

The category $\mathfrak{M}$ of maps on surfaces, with branched coverings of maps as its morphisms, is a permutational category, with parent group

$$\Gamma = \Gamma_{\mathfrak{M}} = \langle R_0, R_1, R_2 \mid R_i^2 = (R_0R_2)^2 = 1\rangle. \tag{2.1}$$

Here each R_i acts on the set Φ of vertex-edge-face flags of a map $\mathcal{O} \in \mathfrak{M}$ by changing, in the only way possible, the i-dimensional component of each flag while preserving its j-dimensional component for each $j \neq i$. (A boundary flag is fixed by R_i if no such change is possible.) This group, which is a free product

$$\langle R_0, R_2\rangle * \langle R_1\rangle \cong V_4 * C_2$$

of a Klein four-group and a cyclic group of order 2, can be regarded as the extended triangle group $\Delta[\infty, 2, \infty]$ of type $(\infty, 2, \infty)$, generated by reflections in the sides of a hyperbolic triangle with angles $0, \pi/2, 0$. This gives a functor from maps to permutation representations of Γ. Conversely, given a permutation representation of Γ on a set Φ, one can take a set of triangles in bijective correspondence with Φ, each with vertices labelled $0, 1, 2$, and use the cycles of R_i on Φ to join pairs of triangles across edges jk $(j, k \neq i)$; the result is the barycentric subdivision of a map $\mathcal{O} \in \mathfrak{M}$, with the vertices of $\mathcal{O}$ labelled 0 and its edges formed by edges of triangles labelled 01, so that midpoints of edges and faces of $\mathcal{O}$ are labelled 1 and 2. Branched coverings between maps $\mathcal{O}$ correspond to Γ-equivariant functions between sets Φ, so we obtain functors $\mathcal{O} \mapsto \Phi$ and $\Phi \mapsto \mathcal{O}$ which give the required equivalence of categories.

Other triangle groups act as parent groups for related categories. For the category $\mathfrak{M}_k$ of maps with all vertex-valencies dividing k we add the relation $(R_1R_2)^k = 1$ to the presentation (2.1), giving the parent group

$$\Gamma_{\mathfrak{M}_k} = \langle R_0, R_1, R_2 \mid R_i^2 = (R_0R_2)^2 = (R_1R_2)^k = 1 \rangle = \Delta[k, 2, \infty]. \tag{2.2}$$

Similarly, the isomorphic group $\Delta[\infty, 2, k]$ is the parent group for the dual maps, with all face-valencies dividing k. For the category $\mathfrak{H}$ of hypermaps, where hyperedges may be incident with any number of hypervertices and hyperfaces, we delete the relation $(R_0R_2)^2 = 1$ from (2.1), giving the parent group

$$\Gamma_{\mathfrak{H}} = \langle R_0, R_1, R_2 \mid R_i^2 = 1 \rangle = \Delta[\infty, \infty, \infty] \cong C_2 * C_2 * C_2 \tag{2.3}$$

again permuting flags. Similarly, the extended triangle group

$$\Delta[l, m, n] = \langle R_0, R_1, R_2 \mid R_i^2 = (R_1R_2)^l = (R_0R_2)^m = (R_0R_1)^n = 1 \rangle$$

is the parent group for hypermaps of type dividing (l, m, n), that is, of type (l', m', n') where l', m' and n' divide l, m and n.

For the corresponding categories $\mathfrak{M}^+$, $\mathfrak{M}_k^+$ and $\mathfrak{H}^+$ of oriented maps and hypermaps we take the orientation-preserving subgroups of index 2 in these groups, generated by the elements $X = R_1R_0$, $Y = R_0R_2$ and $Z = R_2R_1$ satisfying $XYZ = 1$. These are the triangle groups

$$\Gamma_{\mathfrak{M}^+} = \langle X, Y, Z \mid Y^2 = XYZ = 1 \rangle = \Delta(\infty, 2, \infty) \cong C_\infty * C_2, \tag{2.4}$$

$$\Gamma_{\mathfrak{M}_k^+} = \langle X, Y, Z \mid X^k = Y^2 = XYZ = 1 \rangle = \Delta(k, 2, \infty) \cong C_k * C_2 \tag{2.5}$$

and

$$\Gamma_{\mathfrak{H}^+} = \langle X, Y, Z \mid XYZ = 1 \rangle = \Delta(\infty, \infty, \infty) \cong C_\infty * C_\infty \cong F_2. \tag{2.6}$$

Similarly, the triangle group

$$\Delta(l, m, n) = \langle X, Y, Z \mid X^l = Y^m = Z^n = XYZ = 1 \rangle$$

is the parent group for oriented hypermaps of type dividing (l, m, n).

In the case of oriented hypermaps, the Walsh map [51] represents a hypermap as a bipartite map, with black and white vertices representing hypervertices and hyperedges, and edges representing their incidence; then X and Y permute the set Φ of edges by following the local orientation around their incident black and white vertices. For oriented maps, X rotates directed edges around their target vertices, while Y reverses them; equivalently, one can convert a map into the Walsh map of a hypermap by adding a white vertex at the centre of each edge, so that new edges correspond to directed edges of the original map.

If X is a path connected, locally path connected, and semilocally simply connected topological space [43, Ch. 13]), the unbranched coverings $\beta : Y \to X$ of X form a permutational category $\mathfrak{C}$ with the fundamental group $\Gamma = \pi_1 X$ as parent group, using unique path-lifting to permute the fibre $\Phi = \beta^{-1}(x_0) \subset Y$ of β over a chosen base-point $x_0 \in X$. The regular coverings β correspond to the normal subgroups N of Γ, with covering group $\mathrm{Aut}\,\beta \cong \Gamma/N$. If X is also a compact Hausdorff space (for instance, a compact manifold or orbifold), then Γ is finitely generated [43, p. 500].

The categories of maps and hypermaps described above can be regarded as obtained in the above way from suitable orbifolds X, such as a triangle with angles $\pi/l, \pi/m, \pi/n$ for hypermaps of type dividing (l, m, n), or a sphere with three cone-points of orders l, m, n in the oriented case. Similarly, Grothendieck's dessins d'enfants [21, 22] are the finite coverings of a sphere minus three points, so their parent group is its fundamental group $\Gamma = F_2$, with generators X, Y and Z inducing the monodromy permutations at the three punctures.

For the rest of this paper, $\mathfrak{C}$ will denote a permutations category with a finitely generated parent group Γ.

3 Counting regular objects

For each group G, there is a natural bijection between the set $\mathcal{R}(G) = \mathcal{R}_{\mathfrak{C}}(G)$ of (isomorphism classes of) regular objects $\mathcal{O} \in \mathfrak{C}$ with $\mathrm{Aut}\,\mathcal{O} \cong G$ and the set $\mathcal{N}(G) = \mathcal{N}_\Gamma(G)$ of normal subgroups N of Γ with $\Gamma/N \cong G$. These normal subgroups are the kernels of the epimorphisms $\Gamma \to G$. Two such epimorphisms have the same kernel if and only if they differ by an automorphism of G, so there is a bijection between $\mathcal{N}(G)$ and the set of orbits of $\mathrm{Aut}\,G$, acting by composition on the set $\mathrm{Epi}(\Gamma, G)$ of epimorphisms $\Gamma \to G$. This action of $\mathrm{Aut}\,G$ is semiregular, since only the identity automorphism of G fixes an epimorphism.

If G is finite then so is $\mathrm{Epi}(\Gamma, G)$, since each epimorphism $\Gamma \to G$ is uniquely determined by the images in G of a finite set of generators of Γ. In this case the sets $\mathcal{R}(G)$ and $\mathcal{N}(G)$ have the same finite cardinality

$$r(G) = r_{\mathfrak{C}}(G) = |\mathcal{R}(G)| = n(G) = n_\Gamma(G) = |\mathcal{N}(G)| = \frac{|\mathrm{Epi}(\Gamma, G)|}{|\mathrm{Aut}\,G|}. \tag{3.1}$$

In [24], Hall developed a method for counting epimorphisms onto G by first counting homomorphisms (generally an easier task) to subgroups of G, and then using Möbius inversion in the lattice $\Lambda(G)$ of subgroups of G.

Let σ and ϕ be functions from isomorphism classes of finite groups to $\mathbb{C}$ such that

$$\sigma(G) = \sum_{H \le G} \phi(H) \tag{3.2}$$

for all finite groups G. Then a simple calculation gives the Möbius inversion formula for G, namely

$$\phi(G) = \sum_{H \le G} \mu_G(H)\sigma(H) \tag{3.3}$$

where μ_G is the Möbius function on $\Lambda(G)$, defined recursively by

$$\sum_{K \geq H} \mu_G(K) = \delta_{H,G}, \tag{3.4}$$

with $\delta_{H,G} = 1$ if $H = G$ and 0 otherwise. (One can view this as a group-theoretic analogue of the inclusion-exclusion principle, which applies to the lattice of all subsets of G; in that situation, by replacing the condition $K \geq H$ in (3.4) with $K \supseteq H$ one assigns the value $(-1)^{|G \setminus H|}$ to $\mu_G(H)$ for each subset H of G.)

Each homomorphism $\Gamma \to G$ is an epimorphism onto a unique subgroup $H \leq G$, so one can take $\sigma(G)$ and $\phi(G)$ to be the numbers of homomorphisms and epimorphisms from Γ to G (or possibly those satisfying some extra condition, such as being smooth, i.e. having a forsion-free kernel). Thus

$$|\mathrm{Hom}(\Gamma, G)| = \sum_{H \leq G} |\mathrm{Epi}(\Gamma, H)|, \tag{3.5}$$

so Möbius inversion gives

$$|\mathrm{Epi}(\Gamma, G)| = \sum_{H \leq G} \mu_G(H) |\mathrm{Hom}(\Gamma, H)|. \tag{3.6}$$

This proves the first part of the following theorem; the second follows easily.

Theorem 3.1. *If $\mathfrak{C}$ is a permutational category with a finitely generated parent group Γ, and G is a finite group, then the number $r(G)$ of isomorphism classes of regular objects $\mathcal{O} \in \mathfrak{C}$ with $\mathrm{Aut}\,\mathcal{O} \cong G$ is given by*

$$r(G) = \frac{1}{|\mathrm{Aut}\,G|} \sum_{H \leq G} \mu_G(H) |\mathrm{Hom}(\Gamma, H)|. \tag{3.7}$$

The number $m(G)$ of isomorphism classes of objects $\mathcal{O} \in \mathfrak{C}$ with $\mathrm{Mon}\,\mathcal{O} \cong G$ is given by

$$m(G) = r(G)c(G), \tag{3.8}$$

where $c(G)$ is the number of conjugacy classes of subgroups of G with trivial core.

Applying equation (3.7) to a specific pair $\mathfrak{C}$ and G requires three ingredients: one must know $|\mathrm{Aut}\,G|$, $\mu_G(H)$ for each $H \leq G$, and $|\mathrm{Hom}(\Gamma, H)|$ for each $H \leq G$ such that $\mu_G(H) \neq 0$. The first is usually the easiest to deal with: for instance $|\mathrm{Aut}\,C_n|$ is given by Euler's function $\phi(n)$, while the automorphism groups of the finite simple groups are all known and can be found in sources such as [5, 52]. Finding the other two ingredients is generally more troublesome, and this has been achieved only in special cases.

4 Evaluating the Möbius function

Evaluating the Möbius function μ_G requires detailed knowledge of the subgroup lattice of G. It has been achieved for several infinite classes of groups G, and of course for specific groups which are not too large one can use systems such as GAP or MAGMA. In this context, the database of subgroup lattices described by Connor and Leemans in [4], and available at [3], is a valuable resource.

Example 4.1 A finite cyclic group $G = C_n$ of order n has a unique subgroup $H \cong C_m$ for each m dividing n, and no other subgroups. Hall [24] showed that $\mu_G(H) = \mu(n/m)$, where μ is the Möbius function of elementary number theory, given by $\mu(n) = (-1)^k$ if n is a product of k distinct primes, and $\mu(n) = 0$ otherwise. Indeed, here μ can be regarded as the Möbius function on the lattice of subgroups of finite index in the infinite cyclic group $\mathbb{Z}$.

Example 4.2 Similarly, it is an easy exercise to compute the Möbius function for a finite dihedral group; see [28].

Example 4.3 An elementary abelian group G of order p^d can be regarded as a vector space of dimension d over the field $\mathbb{F}_p$, and its subgroups H as the linear subspaces. The number of these of each codimension $k = 0, 1, \ldots, d$ is equal to the Gaussian binomial coefficient

$$\binom{d}{k}_p = \frac{(p^d - 1)(p^{d-1} - 1) \ldots (p^{d-k+1} - 1)}{(p^k - 1)(p^{k-1} - 1) \ldots (p - 1)},$$

and Hall [24] showed that they satisfy

$$\mu_G(H) = (-1)^k p^{k(k-1)/2}.$$

Hall showed that in any finite group G, if $\mu_G(H) \neq 0$ then H must be the intersection of a set of maximal subgroups of G, so in particular H must contain the Frattini subgroup $\Phi(G)$ of G, the intersection of all its maximal subgroups.

Example 4.4 If G is a d-generator finite p-group then $\Phi(G)$ is the subgroup $G'G^p$ generated by the commutators and p-th powers in G, and $G/\Phi(G)$ is an elementary abelian p-group of order p^d. The subgroups $H \leq G$ with $\mu_G(H) \neq 0$ all contain $\Phi(G)$, and correspond to the subgroups of $G/\Phi(G)$, with $\mu_G(H)$ given by the preceding example.

If $G = G_1 \times G_2$ where G_1 and G_2 are finite groups of coprime orders, each subgroup $H \leq G$ has the unique form $H = H_1 \times H_2$ where $H_i \leq G_i$. In this case Hall showed that $\mu_G(H) = \mu_{G_1}(H_1)\mu_{G_2}(H_2)$.

Example 4.5 Each nilpotent finite group G is a direct product of its Sylow subgroups, which are p-groups for the different primes p dividing $|G|$, so the preceding examples show how to compute μ_G.

Example 4.6 Dickson described the subgroups of the groups $L_2(q) = PSL_2(q)$ in [7, Ch. XII]. Using this, Hall [24] calculated the Möbius function μ_G for the simple groups $G = L_2(p)$ for primes $p \geq 5$. Equation (3.3) takes the form

$$\phi(G) = \sigma(G) - (p+1)\sigma(G_\infty) - \frac{p(p-1)}{2}\sigma(D_{\frac{p+1}{2}})$$
$$-\frac{p(p+1)}{2}D_{\frac{p-1}{2}} + p(p+1)\sigma(C_{\frac{p-1}{2}}) + |G|S,$$

where G_∞ is the subgroup of index $p+1$ fixing ∞, and S depends on the congruence classes of p mod (5) and mod (8), which determine the existence of proper subgroups $H \cong A_5$ or S_4. For example, if $p = 5$, or if $p \equiv \pm 2 \bmod (5)$ and $p \equiv \pm 3 \bmod (8)$, so that there are no such subgroups, then

$$S = -\frac{1}{12}\sigma(A_4) + \frac{1}{4}\sigma(V_4) + \frac{1}{3}\sigma(C_3) + \frac{1}{2}\sigma(C_2) - \sigma(1);$$

there are similar formulae in the other cases. In [10] Downs extended Hall's calculation of μ_G to $L_2(q)$ and $PGL_2(q)$ for all prime powers q; see [11] for a proof for $L_2(2^e)$ and a statement of results for $L_2(q)$ where q is odd, and [13] for some combinatorial applications by Downs and the author.

Example 4.7 The Suzuki groups $G = Sz(q)$ are a family of non-abelian finite simple groups, with $q = 2^e$ for some odd $e > 1$; see [5, 50, 52] for their properties, which are similar to those of the groups $L_2(2^e)$. Downs calculated μ_G for these groups in [12]; see [14] for a statement of the results and some applications.

5 Counting homomorphisms

In order to apply equation (3.7) to a group G, one needs to evaluate $|\mathrm{Hom}(\Gamma, H)|$ for those subgroups $H \leq G$ with $\mu_G(H) \neq 0$. If Γ has a presentation with generators X_i and defining relations R_j, this is equivalent to counting the solutions (x_i) in H of the equations $R_j(x_i) = 1$.

Example 5.1 If Γ is a free product $C_{m_1} * \cdots * C_{m_k}$ of cyclic groups of orders $m_i \in \mathbb{N} \cup \{\infty\}$, then

$$|\mathrm{Hom}(\Gamma, H)| = \prod_{i=1}^{k} \sum_{m|m_i} |H|_m$$

where $|H|_m$ denotes the number of elements of H of order m, and we regard all orders m as dividing ∞, so that $\sum_{m|\infty} |H|_m = |H|$. For instance, if Γ is a free group F_k of rank k then $|\mathrm{Hom}(\Gamma, H)| = |H|^k$. Similarly, the torsion theorem for free products [36, Theorem IV.1.6] implies that a homomorphism $\Gamma \to H$ is smooth if and only if it embeds each finite factor C_{m_i}, so the number of such homomorphisms can be found by multiplying k factors equal to $|H|_{m_i}$ or $|H|$ as m_i is finite or infinite.

For certain groups Γ, the character table of H gives $|\mathrm{Hom}(\Gamma, H)|$.

Example 5.2 If Γ is a polygonal group

$$\Delta(m_1, \ldots, m_k) = \langle X_1, \ldots, X_k \mid X_1^{m_1} = \ldots = X_k^{m_k} = X_1 \ldots X_k = 1 \rangle$$

of type $(m_1, \ldots, m_k)$ for some integers m_i, then $|\mathrm{Hom}(\Gamma, H)|$ can be found by summing the following formula (5.1) of Frobenius [18] over all choices of k-tuples of conjugacy classes $\mathcal{C}_i$ of elements of orders dividing m_i.

Theorem 5.1. *Let $\mathcal{C}_i$ $(i = 1, \ldots, k)$ be conjugacy classes in a finite group H. Then the number of solutions of the equation $x_1 \ldots x_k = 1$ in H, with $x_i \in \mathcal{C}_i$ for $i = 1, \ldots, k$, is*

$$\frac{|\mathcal{C}_1| \ldots |\mathcal{C}_k|}{|H|} \sum_{\chi} \frac{\chi(x_1) \ldots \chi(x_k)}{\chi(1)^{k-2}} \tag{5.1}$$

where $x_i \in \mathcal{C}_i$ and χ ranges over the irreducible complex characters of H.

Similarly, the number of smooth homomorphisms $\Gamma \to H$ can be found by restricting the summation to classes of elements of order equal to m_i. The case $k = 3$ of this theorem, where Γ is a triangle group, has often been used in connection with oriented maps and hypermaps: see [27] and [32], for instance.

Example 5.3 If Γ is an orientable surface group Π_g, that is, the fundamental group

$$\Pi_g = \pi_1 \mathcal{S}_g = \langle A_i, B_i \ (i = 1, \ldots, g) \mid \prod_{i=1}^{g} [A_i, B_i] = 1 \rangle$$

of a compact orientable surface $\mathcal{S}_g$ of genus $g \geq 1$, one can use the following theorem of Frobenius [18] and Mednykh [42], which counts solutions of the equation $\prod_{i=1}^{g} [a_i, b_i] = 1$:

Theorem 5.2. *If H is any finite group then*

$$|\mathrm{Hom}(\Pi_g, H)| = |H|^{2g-1} \sum_{\chi} \chi(1)^{2-2g}, \tag{5.2}$$

where χ ranges over the irreducible complex characters of H.

Example 5.4 If Γ is a non-orientable surface group

$$\Pi_g^- = \langle A_i \ (i = 1, \ldots, g) \mid \prod_{i=1}^{g} A_i^2 = 1 \rangle$$

of genus $g \geq 1$, one can use the following result of Frobenius and Schur [19]:

Theorem 5.3. *If H is a finite group then*

$$|\mathrm{Hom}(\Pi_g^-, H)| = |H|^{g-1} \sum_{\chi} c_{\chi}^{g} \chi(1)^{2-g}, \tag{5.3}$$

where χ ranges over the irreducible complex characters of H.

Here c_χ is the Frobenius-Schur indicator $|H|^{-1} \sum_{h \in H} \chi(h^2)$ of χ, equal to $1, -1$ or 0 as χ is the character of a real representation, the real character of a non-real representation, or a non-real character. See [28] for applications of these two theorems, and [48, Ch. 7] for several generalisations of them.

6 Enumerations

Using Theorem 3.1 one can now enumerate, for a given finite group G, the regular objects in $\mathfrak{C}$ with automorphism group G, and also the objects in $\mathfrak{C}$ with monodromy group G.

Example 6.1 It follows from a result of Hall [24] that if $G = L_2(p)$ for some prime $p \geq 5$ and $\mathfrak{C} = \mathfrak{H}^+$, so that $\Gamma = F_2$, then

$$r(G) = \frac{1}{4}(p+1)(p^2 - 2p - 1) - \epsilon,$$

where $\epsilon = 49, 40, 11$ or 2 as $p \equiv \pm 1 \bmod (5)$ and $\pm 1 \bmod (8)$, or $\pm 1 \bmod (5)$ and $\pm 3 \bmod (8)$, or $\pm 2 \bmod (5)$ and $\pm 1 \bmod (8)$, or $\pm 2 \bmod (5)$ and $\pm 3 \bmod (8)$. We also take $\epsilon = 2$ when $p = 5$, so that $r(G) = 19$ in this case; the 19 regular oriented hypermaps associated with the icosahedral group $G = L_2(5) \cong A_5$ have been described by Breda and the author in [2]. Since this group G has eight conjugacy classes of proper subgroups, all with trivial core since G is simple, it follows from equation (3.8) that there are $19 \times 8 = 152$ oriented hypermaps with monodromy group G, namely the quotients $\mathcal{O}/H$ where $\mathcal{O} \in \mathcal{R}(G)$ and $H < G$.

Example 6.2 In [10], Downs considered the categories $\mathfrak{H}$, $\mathfrak{H}^+$, $\mathfrak{M}$, $\mathfrak{M}^+$, $\mathfrak{M}_3$ and $\mathfrak{M}_3^+$, and gave formulae for $r(G)$ where $G = L_2(q)$ or $PGL_2(q)$ for any prime power q. The results for $G = L_2(2^e)$ are given in [13]. Typical results for odd q are:

$$r_{\mathfrak{M}}(L_2(p^e)) = \frac{1}{8e} \sum_{f|e} \mu\left(\frac{e}{f}\right) p^f(p^f - a)$$

for all $p > 2$ and odd $e > 1$, where $a = 2$ or 4 as $p \equiv 1$ or -1 mod (4), and

$$r_{\mathfrak{M}_3}(PGL_2(p^e)) = \frac{3}{4e} \sum_{f} \mu\left(\frac{e}{f}\right) (p^f - 1)$$

for $p > 3$ and $e > 1$, where the sum is over all factors f of e with e/f odd.

Example 6.3 Using Downs's calculation of the Möbius function for $G = Sz(2^e)$ in [12], he and the author have enumerated various combinatorial objects with automorphism group G in [14]. Typical results are that

$$r_{\mathfrak{H}^+}(G) = \frac{1}{e} \sum_{f|e} \mu\left(\frac{e}{f}\right) 2^f(2^{4f} - 2^{3f} - 9)$$

and

$$r_{\mathfrak{M}}(G) = \frac{1}{e} \sum_{f|e} \mu\left(\frac{e}{f}\right) (2^f - 1)(2^f - 2).$$

The second formula, which also gives the number of reflexible maps in $\mathcal{R}_{\mathfrak{M}^+}(G)$, has been obtained by more direct means by Hubard and Leemans in [25].

Example 6.4 If G is infinite then $\mathcal{R}(G)$ could be finite or infinite. For instance, if $\mathfrak{C} = \mathfrak{H}^+$, so that $\Gamma = F_2$, then $r(\mathbb{Z}^2) = 1$ whereas $r(\mathbb{Z}) = \aleph_0$.

7 Universal covers

For any group G, and any $\mathfrak{C}$, let

$$K(G) = K_{\mathfrak{C}}(G) = \bigcap_{N \in \mathcal{N}(G)} N. \tag{7.1}$$

This is a normal subgroup of Γ, so it corresponds to a regular object

$$\mathcal{U}(G) = \mathcal{U}_{\mathfrak{C}}(G) = \bigvee_{\mathcal{O} \in \mathcal{R}(G)} \mathcal{O} \tag{7.2}$$

which we will call the *universal cover* for G, the smallest object in $\mathfrak{C}$ covering each $\mathcal{O} \in \mathcal{R}(G)$. This has automorphism group

$$\overline{G} := \mathrm{Aut}\,\mathcal{U}(G) \cong \Gamma/K(G). \tag{7.3}$$

If Γ has generators X_i $(i \in I)$ then one can realise $\overline{G}$ as the subgroup of the cartesian power $G^{\mathcal{R}(G)}$ of G generated by the elements $(x_{i1}, x_{i2}, \ldots)$ for $i \in I$, where x_{ik} is the image of X_i in $G = \operatorname{Aut} \mathcal{O}_k$ for some numbering $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of the objects $\mathcal{O}_k \in \mathcal{R}(G)$. In particular, $\overline{G}$ has the same number of generators as Γ, and it satisfies all the identical relations satisfied by G: for instance, if G is nilpotent of class c, is solvable of derived length d, or has exponent e, then the same applies to $\overline{G}$. Finally, if G is finite, as we will assume from now on, then so are $\mathcal{U}(G)$ and $\overline{G}$, with $|\overline{G}|$ dividing $|G|^r$ where $r = r(G)$.

Example 7.1 Let $\mathfrak{C} = \mathfrak{H}^+$, so that $\Gamma = F_2$. If $G = C_n$ then $K(G) = \Gamma'\Gamma^n$, so

$$\overline{G} = \Gamma/\Gamma'\Gamma^n \cong C_n \times C_n.$$

Represented as a bipartite map, the hypermap $\mathcal{U}(G)$ is a regular embedding of the complete bipartite graph $K_{n,n}$ in a surface of genus $(n-1)(n-2)/2$. In fact, we obtain the same universal cover $\mathcal{U}(G)$ and group $\overline{G}$ whenever G is a 2-generator abelian group of exponent n.

This example shows that $\overline{G}$ can be a rather small subgroup of G^r, since $\overline{G} \cong G^2$ whereas $r > n$. However, if G is a non-abelian finite simple group, then the following result shows that $\overline{G} = G^r$ for any category $\mathfrak{C}$; see [29] for a proof.

Lemma 7.1. *Let $N_1, \ldots, N_r$ be distinct normal subgroups of a group Γ, with each $G_i := \Gamma/N_i$ non-abelian and simple. If $K = N_1 \cap \cdots \cap N_r$ then*

$$\Gamma/K \cong G_1 \times \cdots \times G_r.$$

Taking $\{N_1, \ldots, N_r\} = \mathcal{N}_\Gamma(G)$, so $G_i \cong G$ for $i = 1, \ldots, r$, gives the result.

Example 7.2 Let $\mathfrak{C} = \mathfrak{H}^+$ again, and let $G = L_2(5) \cong A_5$. By Example 6.1 we have $r(G) = 19$, so $\overline{G} = G^{19}$, of order

$$60^{19} = 609359740010496 \times 10^{17} \approx 6.1 \times 10^{31}.$$

Guralnick and Kantor [23] have shown that if G is a non-abelian finite simple group then each non-identity element of G is a member of a generating pair. If such a group G has exponent e then it follows that $\mathcal{U}_{\mathfrak{H}^+}(G)$ has type (e, e, e), so by the Riemann-Hurwitz formula it has genus

$$g = 1 + \frac{e-3}{2e}|G|^r.$$

In Example 7.2, for instance, G has exponent 30, so $\mathcal{U}_{\mathfrak{H}^+}(G)$ has genus

$$1 + \frac{9}{20} \times 60^{19} = 27421883004723200000000000000001 \approx 2.742 \times 10^{31}.$$

For any finite group G we have $|\mathrm{Epi}(F_2, G)| \leq |G|^2$, so

$$r_{\mathfrak{H}^+}(G) \leq \frac{|G|^2}{|\operatorname{Aut} G|} = \frac{|G|.|Z(G)|}{|\operatorname{Out} G|}$$

where $\operatorname{Out} G$ is the outer automorphism group $\operatorname{Aut} G/\operatorname{Inn} G$ of G. In particular, if G has trivial centre then

$$r_{\mathfrak{H}^+}(G) \leq \frac{|G|}{|\operatorname{Out} G|}. \tag{7.4}$$

If G is a non-abelian finite simple group, then a randomly-chosen pair of elements generate G with probability approaching 1 as $|G| \to \infty$: this was proved by Dixon [8] for the alternating groups, Kantor and Lubotzky [35] for the classical groups of Lie type, and Liebeck and Shalev [36] for the exceptional groups of Lie type. Moreover, convergence is quite rapid. It follows that for such groups the upper bound in (7.4) is asymptotically sharp, that is,

$$r_{\mathfrak{H}^+}(G) \sim \frac{|G|}{|\mathrm{Out}\, G|} \quad \text{as} \quad |G| \to \infty.$$

The information in [5, 52] shows that for each of the infinite families of non-abelian finite simple groups, $|\mathrm{Out}\, G|$ grows much more slowly than $|G|$, so that $r_{\mathfrak{H}^+}(G)$ grows almost as quickly as $|G|$. For instance, $r_{\mathfrak{H}^+}(A_n) \sim n!/4$ as $n \to \infty$. (See [9, 41] for more precise results concerning generating pairs for A_n.)

Example 7.3 If G is the Monster, the largest sporadic simple group, then

$$|G| = 2^{46}.3^{20}.5^9.7^6.11^2.13^3.17.19.23.29.31.41.47.59.71$$

$$= 808017424794512875886459904961710757005754368000000000$$

$$\approx 8.080 \times 10^{53}.$$

Since $|\mathrm{Out}\, G| = 1$ we have $r := r_{\mathfrak{H}^+}(G) \approx |G|$, so

$$|\overline{G}| = |G|^r \approx |G|^{|G|} \approx (8.080 \times 10^{53})^{8.080 \times 10^{53}} \approx 10^{10^{55.639}}.$$

Since G has exponent

$$e = 2^5.3^3.5^2.7.11.13.17.19.23.29.31.41.47.59.71$$

$$= 1165654792878376600800 \approx 1.166 \times 10^{21},$$

the universal cover $\mathcal{U}_{\mathfrak{H}^+}(G)$ has type (e, e, e) and genus approximately $|\overline{G}|/2$.

Similar considerations apply to other categorises $\mathfrak{C}$, though the universal covers $\mathcal{U}(G)$ and their automorphism groups $\overline{G}$ are usually rather smaller.

Example 7.4 If $\mathfrak{C} = \mathfrak{M}^+$ and $G = A_5$ then $r(G) = 3$: the orientably regular maps in $\mathcal{R}(G)$ are the icosahedron, the dodecahedron and the great dodecahedron, of types $\{3, 5\}$, $\{5, 3\}$ and $\{5, 5\}$, and of genera 0, 0 and 4. It follows that $\overline{G} = G^3$, of order 216000, and that $\mathcal{U}_{\mathfrak{M}^+}(G)$ is a map of type $\{15, 15\}$ and genus

$$g = 1 + \frac{11}{60} \times 60^3 = 39601.$$

Similarly $r(G) = 3$ if $\mathfrak{C} = \mathfrak{M}$: the three regular maps in $\mathcal{R}_{\mathfrak{M}}(G)$ are the non-orientable antipodal quotients of those in $\mathcal{R}_{\mathfrak{M}^+}(G)$, and the same applies to the universal covers $\mathcal{U}(G)$ in these two categories.

Example 7.5 It follows from Theorems 3.1 and 5.2 that there are 2016 regular coverings of an orientable surface of genus 2 with covering group $G = A_5$ [28]. They have genus 61, while $\mathcal{U}(G)$ has genus $1 + 60^{2016}$ and covering group G^{2016}.

8 Operations on categories

The automorphisms of the parent group Γ of $\mathfrak{C}$ permute the subgroups of Γ. Since inner automorphisms leave invariant each conjugacy class of subgroups, there is an induced action of the outer automorphism group

$$\Omega = \Omega_{\mathfrak{C}} := \mathrm{Out}\,\Gamma = \mathrm{Aut}\,\Gamma / \mathrm{Inn}\,\Gamma$$

of Γ on isomorphism classes of objects in $\mathfrak{C}$. Since Ω preserves normality and quotient groups, it leaves $\mathcal{N}(G)$ and hence $\mathcal{R}(G)$ invariant for each group G. Here we will consider, for various categories $\mathfrak{C}$, the isomorphic actions of $\Omega_{\mathfrak{C}}$ on these pairs of sets. In some cases, Γ decomposes as a free product, possibly with amalgamation, in which case the structure theorems for such groups [36, §7.2] often allow $\Omega_{\mathfrak{C}}$ to be determined explicitly. The case $\mathfrak{C} = \mathfrak{M}$, with $\Gamma = V_4 * C_2$, was dealt with by Thornton and the author in [33]; other cases considered here are similar, so proofs are omitted.

8.1 Operations on oriented hypermaps

In the case $\mathfrak{C} = \mathfrak{H}^+$, with $\Gamma = F_2$, James [26] interpreted Ω as the group of all operations on oriented hypermaps. For any integer $n \geq 1$, the automorphism group of F_n is generated by the elementary Nielsen transformations: permuting the free generators, inverting one of them, and multiplying one of them by another [40, Theorem 3.2]. When $n = 2$ one can identify $\Omega = \mathrm{Out}\,\Gamma$ with $GL_2(\mathbb{Z})$ through its faithful induced action on the abelianisation $\Gamma^{\mathrm{ab}} = \Gamma/\Gamma' \cong \mathbb{Z}^2$ of Γ [38, Ch. I, Prop. 4.5].

This group Ω can be decomposed as a free product with amalgamation as follows (see [6, §7.2] for presentations of Ω). If we take the images of X and Y as a basis for Γ^{ab}, then there is a subgroup $\Sigma \cong S_3 \cong D_3$ of Ω, generated by the matrices

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

of order 2 and 3; this group, which simply permutes the three vertex colours of an oriented hypermap, regarded as a tripartite map by stellating its Walsh map, was introduced by Machì in [39]. The central involution $-I$ of Ω reverses the orientation of each hypermap and, together with Σ, generates a subgroup

$$\Omega_1 = \Sigma \times \langle -I \rangle \cong S_3 \times C_2 \cong D_6$$

of Ω which preserves the genus of each hypermap and permutes the periods in its type. If a hypermap is represented as a bipartite map, then the matrices

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

reverse the cyclic order of edges around each black or white vertex, while preserving the order around those of the other colour; they are sometimes called Petrie operations, since they preserve the embedded bipartite graph but replace faces with Petrie polygons (closed zig-zag paths), so the genus may be changed. These two matrices, together with E, generate a subgroup $\Omega_2 \cong D_4$ such that

$$\Omega_0 := \Omega_1 \cap \Omega_2 = \langle E, -I \rangle \cong V_4 \cong D_2$$

and

$$\Omega = \Omega_1 *_{\Omega_0} \Omega_2 \cong D_6 *_{D_2} D_4.$$

The torsion theorem for free products with amalgamation [36, Theorem IV.2.7] shows that the operations of finite order are the conjugates of the elements of $\Omega_1 \cup \Omega_2$, described by Pinto and the author in [30].

For any 2-generator group G, the orbits of Ω on $\mathcal{R}(G)$ correspond to the T_2-systems in G, that is, the orbits of $\mathrm{Aut}\, F_2 \times \mathrm{Aut}\, G$ acting by composition on $\mathrm{Epi}(F_2, G)$ and hence on generating pairs for G. It is known [15, 45] that this action is transitive if G is abelian, whereas Garion and Shalev [20] have shown that if G is a non-abelian finite simple group then the number of orbits tends to ∞ as $|G| \to \infty$.

Example 8.1 It follows from work of Neumann and Neumann [45] that the 19 hypermaps in $\mathcal{R}(A_5)$ form two orbits of lengths 9 and 10 under Ω, which acts as $S_9 \times S_{10}$ on them. Those hypermaps whose type is a permutation of $(2,5,5)$, $(3,3,5)$ or $(3,5,5)^-$ form the first orbit, while those of type a permutation of $(2,3,5)$, $(3,5,5)^+$ or $(5,5,5)$ form the other; here the superscript $+$ or $-$ indicates that the generators of order 5 are or are not conjugate in A_5.

This example illustrates a useful result of Nielsen [46], that when $\Gamma = F_2$ the order of the commutator $[x,y]$ is an invariant of the action of Ω on $\mathcal{R}(G)$ for any group G: here the order is 3 or 5 for the hypermaps in the two orbits.

8.2 Operations on all hypermaps

When $\mathfrak{C} = \mathfrak{H}$ we have $\Gamma = C_2 * C_2 * C_2$, containing F_2 as a characteristic subgroup of index 2. As shown by James [26] there is again an action of $GL_2(\mathbb{Z})$ on hypermaps, as described above, but now extended to all hypermaps. In this case $-I$, induced by conjugation by R_1, is in the kernel of the action (since any orientation is now ignored), and there is a faithful action on $\mathfrak{H}$ of the group

$$\mathrm{Out}\,\Gamma \cong GL_2(\mathbb{Z})/\langle -I\rangle \cong PGL_2(\mathbb{Z}) \cong S_3 *_{C_2} V_4.$$

8.3 Operations on oriented maps

When $\mathfrak{C} = \mathfrak{M}^+$ we have $\Gamma = C_\infty * C_2$, with $\Omega = \mathrm{Out}\,\Gamma \cong V_4$. This group Ω is generated by vertex-face duality, induced by the automorphism of Γ transposing X and Z, and orientation-reversal, induced by inverting X and fixing Y. These two involutions commute, modulo conjugation by Y.

If we restrict to the category $\mathfrak{M}_k^+$ of oriented maps of valency dividing k, then $\Gamma = C_k * C_2$, with Ω isomorphic to the multiplicative group $\mathbb{Z}_k^*$ of units mod (k) provided $k > 2$. The elements of Ω are the operations H_j defined by Wilson in [53], raising the rotation of edges around each vertex to its jth power, and induced by automorphisms fixing Y and sending X to X^j for $j \in \mathbb{Z}_k^*$. These operations H_j, studied by Nedela and Škoviera in [44], preserve the embedded graph, but can change the surface. When $k = 5$, for instance, H_2 transposes the icosahedron and the great dodecahedron.

8.4 Operations on all maps

When $\mathfrak{C} = \mathfrak{M}$ we have $\Gamma = V_4 * C_2$, with $\Omega = \mathrm{Out}\,\Gamma \cong S_3$ induced by the automorphism group of the free factor $\langle R_0, R_2\rangle \cong V_4$ permuting its three involutions R_0, R_2 and R_0R_2.

As shown by Thornton and the author [33], this group Ω is simply an algebraic reinterpretation of the group of operations on regular maps introduced by Wilson in [53] (see also [37]). It is generated by the classical duality of maps, which transposes vertices and faces by transposing R_0 and R_2, and the Petrie duality, which transposes faces and Petrie polygons by transposing R_0 and R_0R_2; these two operations have a product of order 3 which acts as a triality operation, cyclically permuting the sets of vertices, faces and Petrie polygons of each map. As noted by Wilson, maps admitting trialities but not dualities seem to be rather rare: Poulton and the author have given some infinite families of examples in [31].

If we restrict to the category $\mathfrak{M}_k$ of maps of valency dividing k, then

$$\Gamma = \Delta[k,2,\infty] = \langle R_0, R_1\rangle *_{\langle R_0\rangle} \langle R_0, R_2\rangle \cong D_k *_{C_2} D_2,$$

where the amalgamated subgroup C_2 is generated by a reflection R_0 in each factor. If $k > 2$ the automorphisms of D_k fixing R_0 form a group isomorphic to $\mathbb{Z}_k^*$, sending R_0R_1 to $(R_0R_1)^j$ for any $j \in \mathbb{Z}_k^*$, while those of D_2 fixing R_0 simply permute R_2 and R_0R_2. These extend to automorphisms of Γ which generate a subgroup $\mathbb{Z}_k^* \times C_2$ of $\mathrm{Aut}\,\Gamma$: the first factor induces Wilson's operations H_j, and the second factor induces Petrie duality. The structure theorems for free products with amalgamation [36, §7.2] show that this subgroup maps onto $\mathrm{Out}\,\Gamma$. Since H_{-1} is induced by conjugation by R_0 we find that

$$\Omega \cong (\mathbb{Z}_k^*/\{\pm 1\}) \times C_2.$$

When $k = 3$, with $\Omega \cong C_2$, we obtain the outer automorphism of the extended modular group $\Gamma = PGL_2(\mathbb{Z})$ studied by Thornton and the author in [34].

8.5 Operations on surface coverings

If $\mathcal{S}_g$ is an orientable surface of genus $g \geq 1$, and $\Gamma = \pi_1\mathcal{S}_g$, then by the Baer-Dehn-Nielsen Theorem the group $\Omega = \mathrm{Out}\,\Gamma$ is isomorphic to the extended mapping class group $\mathrm{Mod}^{\pm}(\mathcal{S}_g)$ of $\mathcal{S}_g$, that is, the group of isotopy classes of self-homeomorphisms of $\mathcal{S}_g$ (see [17, Ch. 8]). The mapping class group $\mathrm{Mod}\,(\mathcal{S}_g)$ is the subgroup of index 2 corresponding to the orientation-preserving self-homeomorphisms; both groups are finitely presented, with $\mathrm{Mod}\,\mathcal{S}_g$ generated by the Dehn twists [17, Ch. 3]. The induced action of $\mathrm{Mod}^{\pm}(\mathcal{S}_g)$ on coverings of $\mathcal{S}_g$ corresponds to the action of Ω on permutation representations of Γ.

Example 8.2 If $g = 1$ then $\Gamma \cong \mathbb{Z}^2$ and $\Omega \cong \mathrm{Mod}^{\pm}(\mathcal{S}_1) \cong GL_2(\mathbb{Z})$, with $\mathrm{Mod}\,(\mathcal{S}_1)$ corresponding to $SL_2(\mathbb{Z})$. This is generated by the Dehn twists corresponding to the elementary matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

9 Invariance under operations

Although it is natural to regard the regular objects in $\mathfrak{C}$ as its most symmetric objects, some of these may have additional 'external' symmetries, in the sense that they are invariant (up to isomorphism) under some or all of the operations in Ω. Self-dual maps, such as the tetrahedron, are obvious examples. For any $\mathfrak{C}$ and G the group $K(G)$ defined in (7.1) is a characteristic subgroup of Γ, so the corresponding regular object $\mathcal{U}(G)$ is invariant under

Ω. This shows that each object $\mathcal{O} \in \mathfrak{C}$, regular or not, is covered by an Ω-invariant regular object $\mathcal{U}(G) \in \mathfrak{C}$, which is finite if and only if $\mathcal{O}$ is, and which has automorphism group $\overline{G}$ where $G = \mathrm{Mon}\,\mathcal{O}$. The smallest Ω-invariant regular object covering $\mathcal{O}$ can be obtained by restricting the normal subgroups N in (7.1) to those in the appropriate orbit of Ω on $\mathcal{N}_\Gamma(G)$.

Richter, Širáň and Wang [47] have shown that for infinitely many k there are regular k-valent maps which are invariant under the group of operations

$$\Omega_1 := \Omega_{\mathfrak{M}} \cong S_3$$

(see also [33, Theorem 3]), while Archdeacon, Conder and Širáň [1] have recently constructed infinite families of k-valent orientably regular maps invariant under both Ω_1 and the group

$$\Omega_2 := \Omega_{\mathfrak{M}_k^+} \cong \mathbb{Z}_k^*.$$

They call these 'kaleidoscopic maps with trinity symmetry'. In both cases, examples of such maps can be constructed as maps $\mathcal{U}_{\mathfrak{M}}(G)$ for finite groups G: for instance, the map denoted by M_n in [1, Theorem 2.2] has this form where G is a dihedral group of order $4n$, with $K(G) = \Gamma''(\Gamma')^n$ in $\Gamma = \Gamma_{\mathfrak{M}} \cong V_4 * C_2$.

The connection is as follows. For orientably regular maps, invariance under the operation $H_{-1} \in \Omega_2$ is equivalent to reflexibility, so one needs to find normal subgroups of Γ which are invariant under the actions of $\Omega_1 = \mathrm{Out}\,\Gamma$ (i.e. which are characteristic subgroups of Γ) and (for kaleidoscopic maps) of $\Omega_2 \cong \mathbb{Z}_k^*$, where k is the valency of the corresponding map. For any quotient G of Γ, these two groups Ω_i act by permuting the subgroups in $\mathcal{N}_\Gamma(G)$, so they leave invariant their intersection $K(G)$; the map $\mathcal{U}(G)$ corresponding to $K(G)$ is therefore kaleidoscopic with trinity symmetry.

Example 9.1 Let $G = A_5$, so that the three maps $\mathcal{M}_i$ $(i = 1, 2, 3)$ in $\mathcal{R}(G)$ are the antipodal quotients of the icosahedron, the dodecahedron and the great dodecahedron (see Example 7.4); these have types $\{3,5\}_5$, $\{5,3\}_5$ and $\{5,5\}_3$ where the subscript denotes Petrie length, as in [6, §8.6]. Their join $\mathcal{U}(G)$ is a non-orientable regular map of type $\{15,15\}_{15}$ and genus 39602, with automorphism group $\overline{G} \cong A_5^3$. The groups Ω_1 and Ω_2 permute the three maps $\mathcal{M}_i$ (Ω_1 transitively, while $\Omega_2 \cong \mathbb{Z}_{15}^* \cong C_2 \times C_4$ has orbits $\{\mathcal{M}_2\}$ and $\{\mathcal{M}_1, \mathcal{M}_3\}$), so $\mathcal{U}(G)$ is kaleidoscopic with trinity symmetry. (This is the non-orientable example constructed by a different method in [1, §7].)

Example 9.2 For an orientable example, we can take $G = A_5 \times C_2$, so $\mathcal{U}(G)$ is the join of $\mathcal{U}(A_5)$, described in the preceding example, and $\mathcal{U}(C_2)$, a reflexible map of type $\{2,2\}_2$ on the sphere corresponding to the derived group $K(C_2) = \Gamma'$ of Γ. This gives an orientable map of type $\{30,30\}_{30}$ and genus 187201, which is kaleidoscopic with trinity symmetry and has automorphism group $(A_5 \times C_2)^3$.

More generally, if G is a non-abelian finite simple group which is a quotient of Γ (the only ones which are not are $L_3(q)$, $U_3(q)$, $L_4(2^e)$, $U_4(2^e)$, A_6, A_7, M_{11}, M_{22}, M_{23} and McL, according to [49, Theorem 4.16]), these constructions yield a pair of non-orientable and orientable kaleidoscopic maps which have trinity symmetry and have automorphism groups G^r and $G^r \times C_2^3$, where $r = r_{\mathfrak{M}}(G)$.

Example 9.3 If G is the Suzuki group $Sz(8)$, of order $2^6.5.7.13 = 29120$, then $r = 14$ by Example 6.3; the resulting maps have types $\{k,k\}_k$ and $\{2k,2k\}_{2k}$ where $k = 455$, the least common multiple of the valencies $5, 7$ and 13 of the vertices in the 14 maps in $\mathcal{R}(G)$ (see [14]). The orientable map has genus

$$1 + \frac{29120^{14} \times 2^3}{4} \times \frac{453}{910} = 1 + 29120^{13} \times 28992.$$

If only trinity symmetry is required, as in [47], then smaller examples of this type can generally be found, with r dividing 6, by replacing $\mathcal{U}(G)$ with the join of an orbit of Ω_1 on $\mathcal{R}_{\mathfrak{M}}(G)$. For instance, if $G = L_2(p)$ for some prime $p \equiv \pm 1 \bmod (24)$ one can take $r = 1$.

10 Finiteness

Throughout this paper, we have generally assumed that the group G is finite. If it is not, then not only can $\mathcal{R}_{\mathfrak{C}}(G)$ be infinite, it can even split into infinitely many orbits under the action of $\Omega_{\mathfrak{C}}$.

Example 10.1 Let $\mathfrak{C} = \mathfrak{H}^+$, so that $\Gamma = F_2$, and let $G = \langle\, x, y \mid x^3 = y^2 \,\rangle$, the group $\pi_1(S^3 \setminus K)$ of the trefoil knot K. This group, isomorphic to the three-string braid group $B_3 = \langle a, b \mid aba = bab\rangle$ with $x = ab$ and $y = ab^2$, has centre $Z(G) = \langle x^3 \rangle \cong C_\infty$, with $G/Z(G) \cong C_3 * C_2 \cong PSL_2(\mathbb{Z})$. Dunwoody and Pietrowski [16] have shown that the pairs $x_i = x^{3i+1}$, $y_i = y^{2i+1}$ $(i \in \mathbb{Z})$ all generate G and lie in different T_2-systems. The corresponding normal subgroups $N \in \mathcal{N}_\Gamma(G)$, the kernels of the epimorphisms $\Gamma \to G$ given by $X \mapsto x_i, Y \mapsto y_i$, therefore all lie in different orbits of the group $\Omega = \mathrm{Out}\,\Gamma \cong GL_2(\mathbb{Z})$, as do the corresponding hypermaps in $\mathcal{R}_{\mathfrak{C}}(G)$.

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Iterated claws have real-rooted genus polynomials

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Abstract

We prove that the genus polynomials of the graphs called *iterated claws* are real-rooted. This continues our work directed toward the 25-year-old conjecture that the genus distribution of every graph is log-concave. We have previously established log-concavity for sequences of graphs constructed by iterative vertex-amalgamation or iterative edge-amalgamation of graphs that satisfy a commonly observable condition on their partitioned genus distributions, even though it had been proved previously that iterative amalgamation does not always preserve real-rootedness of the genus polynomial of the iterated graph. In this paper, the iterated topological operation is adding a claw, rather than vertex- or edge-amalgamation. Our analysis here illustrates some advantages of employing a matrix representation of the transposition of a set of productions.

Keywords: Topological graph theory, graph genus polynomials, log-concavity, real-rootedness.

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1 Introduction

Graphs are implicitly taken to be connected. Our ***graph embeddings*** are cellular and orientable. For general background in topological graph theory, see [1, 9]. Prior acquaintance with the concepts of *partitioned genus distribution* (abbreviated here as ***pgd***) and *production* (e.g., see [5, 11]) is prerequisite to reading this paper. Subject to this prerequisite, the exposition here is intended to be accessible both to graph theorists and to combinatorialists.

The ***genus distribution*** of a graph G is the sequence $g_0(G)$, $g_1(G)$, $g_2(G)$, ..., where $g_i(G)$ is the number of combinatorially distinct embeddings of G in the orientable surface of genus i. A genus distribution contains only finitely many positive numbers, and there are no zeros between the first and last positive numbers. The ***genus polynomial*** is the polynomial

$$\Gamma_G(z) \;=\; g_0(G) + g_1(G)z + g_2(G)z^2 + \dots$$

We say that a sequence $A = (a_k)_{k=0}^n$ is ***nonnegative*** if $a_k \geq 0$ for all k. An element a_k is said to be an ***internal zero*** of A if there exist indices i and j with $i < k < j$, such that $a_i a_j \neq 0$ and $a_k = 0$. If $a_{k-1}a_{k+1} \leq a_k^2$ for all k, then A is said to be ***log-concave***. If there exists an index h with $0 \leq h \leq n$ such that

$$a_0 \;\leq\; a_1 \;\leq\; \cdots \;\leq\; a_{h-1} \;\leq\; a_h \;\geq\; a_{h+1} \;\geq\; \cdots \;\geq\; a_n,$$

then A is said to be ***unimodal***. It is well-known that any nonnegative log-concave sequence without internal zeros is unimodal, and that any nonnegative unimodal sequence has no internal zeros. A prior paper [7] by the present authors provides additional contextual information regarding log-concavity and genus distributions.

1.1 The LCGD Conjecture and Real-Rootedness Problems

For convenience, we sometimes abbreviate the phrase "log-concave genus distribution" as ***LCGD***. Proofs that closed-end ladders and doubled paths have LCGDs [2] were based on closed formulas for their genus distributions. Proof that bouquets have LCGDs [8] was based on a recursion. The following conjecture was formulated in [8]:

LCGD Conjecture: Every graph has a log-concave genus distribution.

Stahl [12] used the term "H-linear" to describe chains of graphs obtained by amalgamating copies of a fixed graph H. He conjectured that a number of "H-linear" families of graphs have genus polynomials with nonpositive real roots, which implies the log-concavity of their sequences of coefficients, by Newton's theorem. (Since all the coefficients of a genus polynomial are non-negative, it follows that all the roots are non-positive.) Although it was shown [14] that the genus polynomials of some such families do indeed have real roots, Stahl's conjecture of real-rootedness for W_4-linear graphs (where W_4 is the 4-wheel) was disproved by Liu and Wang [10].

Our previous paper [7] proves, nonetheless, that the genus distribution of every graph in the W_4-linear sequence is log-concave. Thus, even though Stahl's proposed approach to log-concavity via roots of genus polynomials is sometimes infeasible, [7] does support Stahl's expectation that chains of copies of a graph are a relatively accessible aspect of the general LCGD problem. Moreover, Wagner [14] has proved the real-rootedness of the genus polynomials for a number of graph families for which Stahl made specific conjectures of real-rootedness.

This leads to a couple of research problems that are subordinate to the LCGD Conjecture, as follows:

> ***Real-rootedness Problem***: Characterize the graphs whose genus polynomials are not real-rooted.
> ***Real-rootedness Chain Problem***: Characterize the graphs H whose genus polynomials are real-rooted but whose H-linear chains contain graphs whose genus polynomials are not real-rooted.

Furthermore, we shall see here that Stahl's method of representing what we have elsewhere ([4, 6]) presented as a transposition of a *production system* for a surgical operation on graph embeddings as a matrix of polynomials can simplify a proof that a family of graphs has log-concave genus distributions.

1.2 Interlacing Roots in a Genus Polynomial Sequence

The earliest proofs [2, 8] of the log-concavity of the genus polynomials for a sequence of graphs appealed directly to the condition $a_{j-1}a_{j+1} \leq a_j^2$. The need for more powerful techniques motivated the development of the linear combination techniques of [7]. Here, to prove the log-concavity of the genus polynomials for the sequence of iterated claws, we combine Newton's theorem that a real-rooted polynomial is log-concave (Theorem 4.1) with a focus on interlacing of roots of consecutive genus polynomials for the graphs in the sequence to prove their log-concavity.

2 The Sequence of Iterated Claws

Let the rooted graph (Y_0, u_0) be isomorphic to the dipole D_3, and let the root u_0 be either vertex of D_3. For $n = 1, 2, \ldots$, we define the ***iterated claw*** (Y_n, u_n) to be the graph obtained the following surgical operation:

> ***Newclaw***: Subdivide each of the three edges incident on the root vertex u_{n-1} of the iterated claw (Y_{n-1}, u_{n-1}), and then join the three new vertices obtained thereby to a new root vertex u_n.

Figure 1 illustrates the graph (Y_3, u_3).

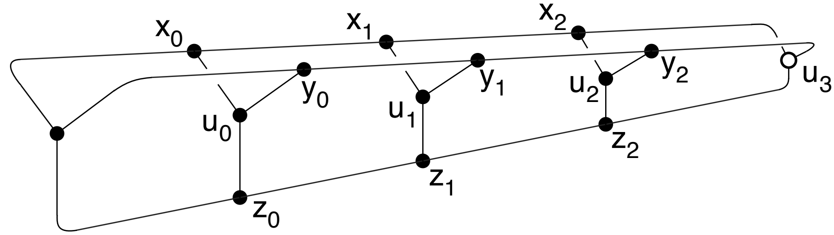

Figure 1: The rooted graph (Y_3, u_3).

The graph $K_{1,3}$ is commonly called a ***claw graph***, which accounts for our name *iterated claw*. The notation Y_n reflects the fact that a claw graph looks like the letter Y. We observe that $Y_1 \cong K_{3,3}$. A recursion for the genus distribution of the iterated claw graphs is derived in [6]. We observe that, whereas all of Stahl's examples [12] of graphs with log-concave genus distributions are planar, the sequence of iterated claws has rising minimum genus. (Example 3.2 of [7] is another sequence of rising minimum genus. However, the graphs in that sequence have cutpoints, unlike the iterated claws.)

We have seen in previous studies of genus distribution (especially [3]) that the number of productions and simultaneous recursions rises rapidly with the number of roots and the valences of the roots. The surgical operation newclaw is designed to circumvent this problem.

For a single-rooted iterated claw (Y_n, u_n), we can define three ***partial genus distributions***, also called ***partials***. Let

$$\begin{aligned} a_{n,i} &= \text{the number of embeddings } Y_n \to S_i \text{ such that} \\ &\quad\ \text{three different fb-walks are incident on the root } u_n; \\ b_{n,i} &= \text{the number of embeddings } Y_n \to S_i \text{ such that exactly} \\ &\quad\ \text{two different fb-walks are incident on the root } u_n; \\ c_{n,i} &= \text{the number of embeddings } Y_n \to S_i \text{ such that} \\ &\quad\ \text{one fb-walk is incident three times on the root } u_n. \end{aligned}$$

We also define ***partial genus polynomials*** to be the generating functions

$$\begin{aligned} A_n(z) &= \sum_{i=0}^{\infty} a_{n,i} z^i \\ B_n(z) &= \sum_{i=0}^{\infty} b_{n,i} z^i \\ C_n(z) &= \sum_{i=0}^{\infty} c_{n,i} z^i. \end{aligned}$$

Clearly, the full genus distribution is the sum of the partials. That is, for $i = 0, 1, 2, \ldots$, we have

$$g_i(Y_n) = a_{n,i} + b_{n,i} + c_{n,i}$$

and

$$\Gamma_{Y_n}(z) = A_n(z) + B_n(z) + C_n(z).$$

We define $g_{n,i} = g_i(Y_n)$.

Remark 2.1. Partitioned genus distributions and recursion systems for pgds were first used by Furst, Gross, and Statman [2]. Stahl [12] was first to employ a matrix equivalent of a production system to investigate log-concavity.

Theorem 2.2. *For $n > 1$, the effect on the pgd of applying the operation newclaw to the iterated claw (Y_{n-1}, u_{n-1}) corresponds to the following system of three productions:*

$$a_i \longrightarrow 12b_{i+1} + 4c_{i+2} \tag{2.1}$$

$$b_i \longrightarrow 2a_i + 12b_{i+1} + 2c_{i+1} \tag{2.2}$$

$$c_i \longrightarrow 8a_i + 8c_{i+1} \tag{2.3}$$

Proof. This is Theorem 4.5 of [6]. □

Corollary 2.3. *For $n > 1$, the effect on the pgd of applying the operation newclaw to the iterated claw (Y_{n-1}, u_{n-1}) corresponds to the following recurrence relations:*

$$a_{n,i} = 2b_{n-1,i} + 8c_{n-1,i} \tag{2.4}$$

$$b_{n,i} = 12a_{n-1,i-1} + 12b_{n-1,i-1} \tag{2.5}$$

$$c_{n,i} = 4a_{n-1,i-2} + 2b_{n-1,i-1} + 8c_{n-1,i-1} \tag{2.6}$$

Proof. The recurrence system (2.4), (2.5), (2.6) is induced by the production system (2.1), (2.2), (2.3). □

It is convenient to express such a recurrence system in matrix form:

$$V(Y_n) = M(z) \cdot V(Y_{n-1}) \tag{2.7}$$

with the ***production matrix***

$$M(z) = \begin{bmatrix} 0 & 2 & 8 \\ 12z & 12z & 0 \\ 4z^2 & 2z & 8z \end{bmatrix}. \tag{2.8}$$

Since the initial graph Y_0 in the sequence of iterated claws is isomorphic to the dipole D_3, the initial column vector for the sequence $V(Y_n)$ is

$$V(Y_0) = \begin{bmatrix} A_0(z) \\ B_0(z) \\ C_0(z) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2z \end{bmatrix} \tag{2.9}$$

Proposition 2.4. *The column vector $V(Y_n)$ is the product of the matrix power $M^n(z)$ with the column vector $V(Y_0)$.*

Corollary 2.5. *The column vector $V(Y_n)$ is the product of the matrix power $M^{n+1}(z)$ with the (artificially labeled) column vector*

$$V(Y_{-1}) = \begin{pmatrix} 0 \\ 0 \\ 1/4 \end{pmatrix}$$

Corollary 2.6. *To prove that every iterated claw has an LCGD, it is sufficient to prove that the sum of the third column of the matrix $M^n(z)$ is a log-concave polynomial.*

3 Characterizing Genus Polynomials for Iterated Claws

In this section, we investigate some properties of the genus polynomials of iterated claws. Corollary 2.6 leads us to focus on the sum of the third column of the matrix $M^n(z)$, which is expressible as $(1, 1, 1)M^n(z)(4V(Y_{-1}))$, which implies that it equals 4 times the genus polynomial of the iterated claw Y_{n-1}. Theorem 3.1 formulates a generating function $f(z, t)$ for this sequence of sums, and Theorem 3.2 uses the generating function to construct an expression for the genus polynomials from which we establish interlacing of roots in Section 4.

Theorem 3.1. *The generating function $f(z,t) = \sum_{n\geq 0}(1,1,1)M^n(z)(4V(Y_{-1}))t^n$ for the sequence of sums of the third column of $M^n(z)$ has the closed form*

$$f(z,t) \;=\; \frac{1+(8-12z)t-24zt^2}{1-20zt+8z(8z-3)t^2+384z^3t^3}. \tag{3.1}$$

Proof. Let $(p_n, q_n, r_n) = (1,1,1)M^n(z)$ for all $n \geq 0$. Then

$$\begin{aligned}(p_{n+1}, q_{n+1}, r_{n+1}) &= (p_n, q_n, r_n)M(z) \\ &= (12zq_n + 4z^2 r_n,\ 2p_n + 12zq_n + 2zr_n,\ 8p_n + 8zr_n).\end{aligned} \tag{3.2}$$

The third coordinate of Equation (3.2) implies that

$$p_n \;=\; \frac{1}{8}(r_{n+1} - 8zr_n). \tag{3.3}$$

By combining (3.3) with the first coordinate of (3.2) we obtain

$$q_n \;=\; \frac{1}{96z}(r_{n+2} - 8zr_{n+1} - 32z^2 r_n). \tag{3.4}$$

The second coordinate of (3.2) yields

$$q_{n+1} \;=\; 2p_n + 12zq_n + 2zr_n \tag{3.5}$$

Substituting (3.3) and (3.4) (twice) into (3.5) leads to the recurrence relation

$$r_n \;=\; 20zr_{n-1} + 8z(3-8z)r_{n-2} - 384z^3 r_{n-3} \tag{3.6}$$

with

$$\begin{aligned} r_0 &= 1, \\ r_1 &= 8+8z, \\ r_2 &= 160z + 96z^2. \end{aligned} \tag{3.7}$$

By multiplying Recurrence (3.6) by t^n and summing over all $n \geq 0$, we obtain Generating Function (3.1). □

It is easy to see that $\Gamma_{Y_n}(z) = r_{n+1}/4$, where r_n is defined in the proof of Theorem 3.1. In terms of $\Gamma_{Y_n}(z)$, the recurrence relation (3.6) becomes

$$\Gamma_{Y_n}(z) \;=\; 20z\Gamma_{Y_{n-1}}(z) + 8z(3-8z)\Gamma_{Y_{n-2}}(z) - 384z^3\Gamma_{Y_{n-3}}(z). \tag{3.8}$$

Theorem 3.2 provides an explicit expression for the genus polynomial $\Gamma_{Y_n}(z)$, a result is of independent interest. It is not used here toward proof of log-concavity.

Theorem 3.2. *The genus polynomial of the iterated claw Y_n is given by*

$$(1,1,1)M^{n+1}(z)V(Y_{-1}) \;=\; 2^{n-1}(h_{n+1}(z) + 2(2-3z)h_n(z) - 6zh_{n-1}(z)),$$

where

$$h_n(z) = \sum_{2j+i_1+i_2+i_3=n} \binom{j+i_1}{i_1}\binom{j+i_2}{i_2}\binom{j+i_3}{i_3}(1+\sqrt{3})^{i_2}(1-\sqrt{3})^{i_3}3^{j+i_1}(2z)^{n-j}.$$

Proof. By Theorem 3.1, we have

$$f(z,t) \;=\; \sum_{n\geq 0}(1,1,1)M^n(4V(Y_0))t^n \;=\; \frac{1+(8-12z)t-24zt^2}{1-20zt+8z(8z-3)t^2+384z^3t^3}.$$

Thus,

$$\begin{aligned} f(z/2,t/2) \;&=\; \frac{1+(4-3z)t-3zt^2}{1-5zt+z(4z-3)t^2+6z^3t^3} \\ &=\; \frac{1+(4-3z)t-3zt^2}{(1-2zt-2z^2t^2)(1-3zt)-3zt^2} \\ &=\; \sum_{j\geq 0}\frac{(1+(4-3z)t-3zt^2)3^jz^jt^{2j}}{(1-3zt)^{j+1}(1+\sqrt{3}zt)^{j+1}(1-\sqrt{3}zt)^{j+1}}. \end{aligned}$$

Using the combinatorial identity $(1-at)^{-m} = \sum_{j\geq 0}\binom{m-1+j}{j}a^jt^j$, and then finding the coefficient of t^n, we derive the equation

$$(1,1,1)M^n(z/2)V(Y_0) \;=\; 2^{n-2}(h_n(z)+2(2-3z)h_{n-1}(z)-6zh_{n-2}(z)),$$

which, by Corollary 2.5, completes the proof. □

Now let $g_{n,i}$ be the coefficient of z^i in $\Gamma_{Y_n}(z)$. The following table of values of $g_{n,i}$ for $n \leq 4$ is derived in [6].

$g_{n,i}$	$i=0$	1	2	3	4	5
$n=0$	2	2	0	0	0	0
1	0	40	24	0	0	0
2	0	48	720	256	0	0
3	0	0	1920	11648	2816	0
4	0	0	1152	52608	177664	30720

Denote by $\mathcal{P}_{s,t}$ the set of polynomials of the form $\sum_{k=s}^{t} a_kz^k$, where a_k is a positive integer for any $s \leq k \leq t$. The above table suggests that $\Gamma_{Y_n}(z) \in \mathcal{P}_{\lfloor (n+1)/2\rfloor,\,n+1}$ for $n \leq 4$. Theorem 3.3 shows that it holds true in general. Like Theorem 3.2, this enumerative result is of independent interest and is not used toward proof of log-concavity.

Theorem 3.3. *For all $n \geq 0$, the polynomial $\Gamma_{Y_n}(z) \in \mathcal{P}_{\lfloor (n+1)/2\rfloor,\,n+1}$. Moreover, we have the leading coefficient*

$$g_{n,n+1} \;=\; 4^n \sum_{k=0}^{\lfloor (n+1)/2\rfloor}\binom{n+2}{2k+1}3^k, \tag{3.9}$$

and, for any number i such that $\lfloor (n+1)/2\rfloor + 1 \;\leq\; i \leq n$, we have

$$g_{n,i} \;>\; 11g_{n-1,i-1}. \tag{3.10}$$

Proof. We see in the table above, for $n \leq 4$, that $\gamma_{\min}(Y_n) = \lfloor (n+1)/2 \rfloor$ and that $\gamma_{\max}(Y_n) = n+1$, or equivalently, that $\Gamma_{Y_n}(z) \in \mathcal{P}_{\lfloor (n+1)/2 \rfloor, n+1}$. We see also, for $n \leq 4$, that Equation (3.9) and Inequality (3.10) are true. Now suppose that $n \geq 5$. For convenience, let $g_{k,i} = 0$ for all $i < 0$. We can also take $g_{k,i} = 0$ for $i > k+1$, by induction using (3.8), for $k < n$. From Recurrence (3.8) and the induction hypothesis, we have

$$g_{n,i} = 20 g_{n-1,i-1} + 24 g_{n-2,i-1} - 64 g_{n-2,i-2} - 384 g_{n-3,i-3}, \qquad n \geq 3. \tag{3.11}$$

For $i > n+1$, the induction hypothesis implies that each of the four terms on the right side of Recurrence (3.11) is zero-valued. So the degree of $\Gamma_{Y_n}(z)$ is at most $n+1$. Let $s_i = g_{i,i+1}$. Taking $i = n+1$ in (3.11), we get

$$s_n = 20 s_{n-1} - 64 s_{n-2} - 384 s_{n-3}, \tag{3.12}$$

with the initial values $s_0 = 2$, $s_1 = 24$, $s_2 = 256$. The above recurrence can be solved by a standard generating function method, see [15, p.8]. In practice, we use the command `rsolve` in the software Maple and get the explicit formula directly as

$$s_n = 4^n \sum_{k \geq 0} \binom{n+2}{2k+1} 3^k.$$

It follows that $g_{n,n+1} > 0$. Hence the degree of $\Gamma_{Y_n}(z)$ is exactly $n+1$.

Similarly, for $i < \lfloor (n+1)/2 \rfloor$, the four terms on the right side of (3.11) are zero-valued, so the minimum genus of Y_n is at least $\lfloor (n+1)/2 \rfloor$. Moreover, applying (3.11) with $i = \lfloor (n+1)/2 \rfloor$ and using the induction hypothesis $g_{k,i} = 0$ for all $i < \lfloor (k+1)/2 \rfloor$ with $k < n$, we find the first term is positive for n odd and zero for n even, the second term is always positive, and the third and fourth terms are always zero. In other words,

$$g_{n,\lfloor (n+1)/2 \rfloor} = 20 g_{n-1,\lfloor (n+1)/2 \rfloor -1} + 24 g_{n-2,\lfloor (n+1)/2 \rfloor -1} \geq 24 g_{n-2,\lfloor (n+1)/2 \rfloor -1} > 0.$$

This confirms the minimum genus of Y_n is exactly $\lfloor (n+1)/2 \rfloor$.

Now consider i such that $\lfloor (n+1)/2 \rfloor + 1 \leq i \leq n$. By (3.11), and using (3.10) inductively, we deduce

$$\begin{aligned} g_{n,i} &= 11 g_{n-1,i-1} + 24 g_{n-2,i-1} + \left(9 g_{n-1,i-1} - 64 g_{n-2,i-2} - 384 g_{n-3,i-3}\right) \\ &> 11 g_{n-1,i-1} + 24 g_{n-2,i-1} + \left(35 g_{n-2,i-2} - 384 g_{n-3,i-3}\right) \\ &> 11 g_{n-1,i-1} + 24 g_{n-2,i-1} + g_{n-3,i-3} \\ &\geq 11 g_{n-1,i-1}. \end{aligned}$$

So Inequality (3.10) holds true. It follows that $g_{n,i} > 0$. Hence

$$\Gamma_{Y_n}(z) \in \mathcal{P}_{\lfloor (n+1)/2 \rfloor, n+1}.$$

This completes the proof. $\square$

4 Genus Polynomials for Iterated Claws are Real-Rooted

Our goal in this section is to establish in Theorem 4.3 the real-rootedness of the genus polynomials $\Gamma_{Y_n}(z)$ of the iterated claws, via an associated sequence $W_n(z)$ of normalized polynomials. It follows from this real-rootedness that the genus polynomials for iterated claws are log-concave, by the following theorem of Newton.

Theorem 4.1 (Newton's theorem)**.** *Let $a_0,\, a_1,\, \ldots,\, a_n$ be real numbers and let all the roots of the polynomial*

$$P(x) \;=\; \sum_{j=0}^{n} a_i x^i$$

be real. Then $a_j^2 \geq a_{j-1}a_{j+1}$ for $j = 1, \ldots, n-1$.
Proof. For instance, see Theorem 2 of [13]. □

To proceed, we "normalize" the polynomials $\Gamma_{Y_n}(z)$ by defining

$$W_n(z) \;=\; z^{-\lfloor (n+1)/2 \rfloor}\Gamma_{Y_n}(z), \tag{4.1}$$

so that $W_n(z)$ starts from a non-zero constant term, and has the same non-zero roots as $\Gamma_{Y_n}(z)$. We use the symbol d_n to denote the degree of $W_n(z)$, that is,

$$d_n \;=\; \deg W_n(z) \;=\; (n+1) - \left\lfloor \frac{n+1}{2} \right\rfloor \;=\; \left\lceil \frac{n+1}{2} \right\rceil. \tag{4.2}$$

By Theorem 3.3, we have $W_n(z) \in \mathcal{P}_{0,d_n}$. Substituting (4.1) into the recurrence relation (3.8), we derive

$$W_n(z) \;=\; \begin{cases} 20zW_{n-1}(z) + 8(3-8z)W_{n-2}(z) - 384z^2W_{n-3}(z), & \text{if } n \text{ is even,} \\ 20W_{n-1}(z) + 8(3-8z)W_{n-2}(z) - 384zW_{n-3}(z), & \text{if } n \text{ is odd,} \end{cases} \tag{4.3}$$

with the initial polynomials

$$\begin{aligned} W_0(z) &= 2(1+z), \\ W_1(z) &= 8(5+3z), \\ W_2(z) &= 16(3+45z+16z^2). \end{aligned} \tag{4.4}$$

Let $\mathcal{P}$ denote the union $\cup_{n\geq 0}\mathcal{P}_{0,n} \;=\; \cup_{n\geq 0}\{\sum_{k=0}^{n} a_k z^k \mid a_k \;\in\; \mathbb{Z}^+\}$. Lemma 4.2 is ultimately a consequence of the intermediate value theorem.

Lemma 4.2. *Let $P(x), Q(x) \in \mathcal{P}$. Suppose that $P(x)$ has roots $x_1 < x_2 < \cdots < x_{\deg P}$, and that $Q(x)$ has roots $y_1 < y_2 < \cdots < y_{\deg Q}$. If $\deg Q - \deg P \in \{0, 1\}$ and if the roots interlace so that*

$$x_1 \;<\; y_1 \;<\; x_2 \;<\; y_2 \;<\; \cdots\,,$$

then

$$(-1)^{i+\deg P}P(y_i) > 0 \qquad \textit{for all } 1 \leq i \leq \deg Q, \tag{4.5}$$

$$(-1)^{j+\deg Q}Q(x_j) < 0 \qquad \textit{for all } 1 \leq j \leq \deg P. \tag{4.6}$$

Proof. Since $P(x)$ is a polynomial with positive coefficients, we have

$$(-1)^{\deg P} P(-\infty) > 0. \tag{4.7}$$

We suppose first that $\deg P(x)$ is odd, and we consider the curve $P(x)$. We see that Inequality (4.7) reduces to $P(-\infty) < 0$. Thus, the curve $P(x)$ starts in the lower half plane and intersects the x-axis at its first root, x_1. From there, the curve $P(x)$ proceeds without going below the x-axis, until it meets the second root, x_2. Since $x_1 < y_1 < x_2$, we recognize that (4.5) holds for $i = 1$, i.e.,

$$P(y_1) > 0. \tag{4.8}$$

After passing through x_2, the curve $P(x)$ stays below the x-axis up to the third root, x_3. It is clear that the curve $P(x)$ continues going forward, intersecting the x-axis in this alternating way. It follows from this alternation that

$$P(y_k)P(y_{k+1}) < 0 \qquad \text{for all } 1 \le k \le \deg Q - 1. \tag{4.9}$$

From (4.8) and (4.9), we conclude that (4.5) holds for all $1 \le i \le \deg Q$, when $\deg P(x)$ is odd.

We next suppose that $\deg P(x)$ is even. In this case, we can draw the curve $P(x)$ so that it starts in the upper half plane, first intersects the x-axis at x_1, then goes below the axis up to x_2, and continues alternatingly. Therefore the sign-alternating relation (4.9) still holds. Since $P(y_1) < 0$ when $\deg P(x)$ is even, we have proved (4.5).

It is obvious that Inequality (4.6) can be shown along the same line. This completes the proof of Lemma 4.2. □

Now we proceed with our main theorem on the genus polynomial of iterated claws. Beyond proving real-rootedness of the genus polynomials, we derive two interlacing relationships on their roots.

Theorem 4.3. *For every $n \ge 0$, the polynomial $W_n(z)$ is real-rooted. Moreover, if the roots of $W_k(z)$ are denoted by $x_{k,1} < x_{k,2} < \cdots$, then we have the following interlacing properties:*

(i) for every $n \ge 2$, the polynomial $W_n(z)$ has one more root than $W_{n-2}(z)$, and the roots interlace so that

$$x_{n,1} < x_{n-2,1} < x_{n,2} < x_{n-2,2} < \cdots < x_{n,d_n-1} < x_{n-2,d_n-1} < x_{n,d_n};$$

(ii) for every $n \ge 1$, the polynomial $W_n(z)$ has either one more (when n is even) or the same number (when n is odd) of roots as $W_{n-1}(z)$, and the roots interlace so that

$$x_{n,1} < x_{n-1,1} < x_{n,2} < x_{n-1,2} < \cdots < x_{n-1,d_n-1} < x_{n,d_n} \text{ when } n \text{ even};$$

and

$$x_{n,1} < x_{n-1,1} < x_{n,2} < x_{n-1,2} < \cdots < x_{n,d_n} < x_{n-1,d_n} \text{ when } n \text{ odd}.$$

Proof. From the initial polynomials (4.4), it is easy to verify Theorem 4.3 for $n \leq 2$. We suppose that $n \geq 3$ and proceed inductively.

For every $k \leq n-1$, we denote the roots of $W_k(z)$ by $x_{k,1} < x_{k,2} < \cdots < x_{k,d_k}$. For convenience, we define $x_{k,0} = -\infty$ and $x_{k,d_k+1} = 0$, for all $k \leq n-1$. To clarify the interlacing properties, we now consider the signs of the function $W_m(z)$ at $-\infty$ and at the origin, for any $m \geq 0$. Since $W_m(z)$ is a polynomial of degree d_m, with all coefficients non-negative, we deduce that

$$(-1)^{d_m} W_m(-\infty) > 0. \tag{4.10}$$

Having the constant term positive implies that

$$W_m(0) = g_{n,0} > 0. \tag{4.11}$$

By the intermediate value theorem and Inequality (4.10), for the polynomial $W_n(z)$ to have $d_n = \deg W_n(z)$ distinct negative roots and for Part (i) of Theorem 4.3 to hold, it is necessary and sufficient that

$$(-1)^{d_n+j} W_n(x_{n-2,j}) > 0 \quad \text{for } 1 \leq j \leq d_{n-2}+1. \tag{4.12}$$

In fact, for $j = d_{n-2}+1$, Inequality (4.12) becomes

$$(-1)^{d_n+d_{n-2}+1} W_n(0) > 0. \tag{4.13}$$

By (4.11), Inequality (4.13) holds if and only if $d_n + d_{n-2}$ is odd, which is true since

$$d_n + d_{n-2} = \left\lceil \frac{n+1}{2} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil = 2\left\lceil \frac{n-1}{2} \right\rceil + 1.$$

Now consider any j such that $1 \leq j \leq d_{n-2}$. We are going to prove (4.12). We will use the particular indicator function I_{even}, which is defined by

$$\mathrm{I}_{\text{even}}(n) = \begin{cases} 1, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Note that $x_{n-2,j}$ is a root of $W_{n-2}(z)$. By Recurrence (4.3), we have

$$W_n(z_{n-2,j}) = x_{n-2,j}^{\mathrm{I}_{\text{even}}(n)} \Big(20 W_{n-1}(x_{n-2,j}) - 384 x_{n-2,j} W_{n-3}(x_{n-2,j}) \Big). \tag{4.14}$$

Since $x_{n-2,j} < 0$, the factor $x_{n-2,j}^{\mathrm{I}_{\text{even}}(n)}$ contributes $(-1)^{n+1}$ to the sign of the right hand side of (4.14). On the other hand, it is clear that the sign of the parenthesized factor can be determined if both the summands $20W_{n-1}(x_{n-2,j})$ and $-384x_{n-2,j}W_{n-3}(x_{n-2,j})$ have the same sign. Therefore, Inequality (4.12) holds if

$$(-1)^{d_n+j+n+1} W_{n-1}(x_{n-2,j}) > 0, \tag{4.15}$$

$$(-1)^{d_n+j+n+1} W_{n-3}(x_{n-2,j}) > 0. \tag{4.16}$$

By the induction hypothesis on part (ii) of this theorem, we can substitute $P = W_{n-1}$ and $Q = W_{n-2}$ into Lemma 4.2. Then Inequality (4.5) gives

$$(-1)^{d_{n-1}+j} W_{n-1}(x_{n-2,j}) > 0. \tag{4.17}$$

Thus, Inequality (4.15) holds if and only if the total power

$$d_n + j + n + 1 + d_{n-1} + j \ = \ \left\lceil \frac{n+1}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + n + 2j + 1$$

of (-1) in (4.15) and (4.17) is even, which is clear by a simple parity argument. Moreover, again using the induction hypothesis on part (ii), we can make substitutions $P(x) = W_{n-2}(x)$ and $Q(x) = W_{n-3}(x)$ into Lemma 4.2. Then Inequality (4.6) gives

$$(-1)^{d_{n-3}+j} W_{n-3}(x_{n-2,j}) < 0. \tag{4.18}$$

Thus, Inequality (4.16) holds if and only if the total power

$$d_n + j + n + 1 + d_{n-3} + j \ = \ \left\lceil \frac{n+1}{2} \right\rceil + \left\lceil \frac{n-2}{2} \right\rceil + n + 2j + 1 \tag{4.19}$$

of (-1) in (4.16) and (4.18) is odd, which is also clear by a simple parity argument. This completes the proof of (4.12), and the proof of Part (i).

The approach to proving Part (ii) is similar to that used to prove Part (i). By the intermediate value theorem and Inequality (4.10), Part (ii) holds if and only if

$$(-1)^{d_n+j} W_n(x_{n-1,j}) > 0 \quad \text{for } 1 \le j \le d_{n-1}, \tag{4.20}$$

and also for $j = d_{n-1} + 1$ when n is even. In fact, when n is even and $j = d_{n-1} + 1$, we have

$$(-1)^{d_n+d_{n-1}+1} W_n(0) > 0. \tag{4.21}$$

By (4.11), Inequality (4.21) holds if and only if $(-1)^{d_n+d_{n-1}+1} = 1$, which is clear since

$$d_n + d_{n-1} + 1 \ = \ \left\lceil \frac{n+1}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + 1 = n + 2.$$

For $1 \le j \le d_{n-1}$, we are now going to show (4.20). By setting $x = x_{n-1,j}$, Recurrence (4.3) turns into

$$W_n(x_{n-1,j}) \ = \ 8(3 - 8x_{n-1,j}) W_{n-2}(x_{n-1,j}) - 384 x_{n-1,j}^{1+\mathrm{I}_{\mathrm{even}}(n)} W_{n-3}(x_{n-1,j}). \tag{4.22}$$

Since $x_{n-1,j} < 0$, we see that $8(3-8x_{n-1,j}) > 0$, and that the factor $-384 x_{n-1,j}^{1+\mathrm{I}_{\mathrm{even}}(n)}$ contributes $(-1)^{n+1}$ to the sign of the right-hand side of (4.22). Therefore, Inequality (4.20) holds if

$$(-1)^{d_n+j} W_{n-2}(x_{n-1,j}) > 0, \tag{4.23}$$

$$(-1)^{d_n+j+n+1} W_{n-3}(x_{n-1,j}) > 0. \tag{4.24}$$

Substituting $P(x) = W_{n-1}(x)$ and $Q(x) = W_{n-2}(x)$ into Lemma 4.2, we find that Inequality (4.6) yields

$$(-1)^{d_{n-2}+j} W_{n-2}(x_{n-1,j}) < 0 \quad \text{when } 1 \le j \le d_{n-1}. \tag{4.25}$$

Thus, Inequality (4.23) holds if and only if the total power

$$d_n + j + d_{n-2} + j \ = \ \left\lceil \frac{n+1}{2} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil + 2j$$

of (-1) in (4.23) and (4.25) is odd, which holds true, obviously, by parity. On the other hand, by the induction hypothesis on Part (i) and substituting $P(x) = W_{n-1}(x)$ and $Q(x) = W_{n-3}(x)$ into Lemma 4.2, Inequality (4.6) becomes

$$(-1)^{d_{n-3}+j} W_{n-3}(x_{n-1,j}) < 0. \tag{4.26}$$

Therefore, Inequality (4.24) holds if and only if the total power $d_n + j + n + 1 + d_{n-3} + j$ of (-1) in (4.24) and (4.26) is odd, which coincides with (4.19). This completes the proof of (4.20), ergo the proof of Part (ii), and hence the entire theorem. □

Corollary 4.4. *The sequence of coefficients for every genus polynomial $\Gamma_{Y_n}(z)$ is log-concave.*

Proof. Recalling Equation (4.1), we have

$$\Gamma_{Y_n}(z) \;=\; z^{\lfloor (n+1)/2 \rfloor} W_n(z).$$

By Theorem 4.3, we know that the polynomial $W_n(z)$ is real-rooted. It follows that the polynomial $\Gamma_{Y_n}(z)$ is real-rooted. Applying Theorem 4.1 (Newton's theorem), we know that the polynomial $\Gamma_{Y_n}(z)$ is log-concave. □

5 On Real-Rootedness

In the study of genus polynomials, the role of real-rootedness may rise beyond being a sufficient condition for log-concavity. The introductory section presents two basic research problems specifically on real-rootedness. One may reasonably anticipate that continuing study of the roots of genus polynomials will lead to new insights into the imbeddings of graphs.

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2-Arc-Transitive regular covers of $K_{n,n} - nK_2$ with the covering transformation group $\mathbb{Z}_p^2$

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Abstract

In 2014, Xu and Du classified all regular covers of a complete bipartite graph $K_{n,n}$ minus a matching, denoted by $K_{n,n} - nK_2$, whose covering transformation group is cyclic and whose fibre-preserving automorphism group acts 2-arc-transitively. In this paper, a further classification is achieved for all the regular covers of $K_{n,n} - nK_2$, whose covering transformation group is isomorphic to $\mathbb{Z}_p^2$ with p a prime and whose fibre-preserving automorphism group acts 2-arc-transitively. Actually, there are only few covers with these properties and it is shown that all of them are covers of $K_{4,4} - 4K_2$.

Keywords: Arc-transitive graph, covering graph, 2-transitive group.

Math. Subj. Class.: 05C25, 20B25, 05E30

1 Introduction

Throughout this paper graphs are finite, simple and undirected. For the group- and graph-theoretic terminology we refer the reader to [15, 17]. For a graph X, let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}\, X$ denote the vertex set, edge set, arc set and the full automorphism group of X respectively. An edge and an arc of X are denoted by $\{u, v\}$ and (u, v), respectively. An *s-arc* of X is a sequence $(v_0, v_1, \ldots, v_s)$ of $s + 1$ vertices such that $(v_i, v_{i+1}) \in A(X)$ and $v_i \neq v_{i+2}$, and X is said to be 2-*arc-transitive* if $\mathrm{Aut}\, X$ acts transitively on the set of 2-arcs of X.

Let X be a graph, and let $\mathcal{P}$ be a partition of $V(X)$ into disjoint sets of equal size m. The quotient graph $Y := X/\mathcal{P}$ is the graph with the vertex set $\mathcal{P}$ and two vertices P_1 and P_2 of Y are adjacent if there is at least one edge between a vertex of P_1 and a vertex of

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P_2 in X. We say that X is an *m-fold cover* of Y if the edge set between P_1 and P_2 in X is a matching whenever $P_1P_2 \in E(Y)$. In this case Y is called the *base graph* of X and the sets P_i are called the *fibres* of X. An automorphism of X which maps a fibre to a fibre is said to be *fibre-preserving*. The subgroup K of all those automorphisms of X which fix each of the fibres setwise is called the *covering transformation group*. It is easy to see that if X is connected then the action of K on the fibres of X is necessarily semiregular, that is, $K_v = 1$ for each $v \in V(X)$. In particular, if this action is regular we say that X is a *regular cover* of Y.

The main motivation for the present paper is to contribute toward the classification of finite 2-arc-transitive graphs. In [23, Theorem 4.1], Professor Praeger divided all the finite 2-arc-transitive graphs X into the following three subclasses:

(1) *Quasiprimitive type*: every nontrivial normal subgroup of $\mathrm{Aut}\,X$ acts transitively on vertices;

(2) *Bipartite type*: every nontrivial normal subgroup of $\mathrm{Aut}\,X$ has at most two orbits on vertices and at least one of them has two orbits on vertices;

(3) *Covering type*: there exists a normal subgroup of $\mathrm{Aut}\,X$ having at least three orbits on vertices, and thus X is a regular cover of some graphs of types (1) or (2).

During the past twenty years, a lot of results regarding the primitive, quasiprimitive and bipartite 2-arc-transitive graphs have appeared [11, 18, 19, 20, 23, 24]. However, very few results concerning the 2-arc-transitive covers are known, except for some covers of graphs with small valency and small order. The first meaningful class of graphs to be studied might be complete graphs. In [7], a classification of covers of complete graphs is given, whose fibre-preserving automorphism groups act 2-arc-transitively and whose covering transformation group is either cyclic or $\mathbb{Z}_p^2$. This classification is generalized in [8] to covering transformation group $\mathbb{Z}_p^3$. In [26], the same problem as in [7] and [8] is considered, but the covering transformation group considered is metacyclic.

As for covers of bipartite type, in [25], all regular covers of complete bipartite graph minus a matching $K_{n,n} - nK_2$ were classified, whose covering transformation group is cyclic and whose fibre-preserving automorphism group acts 2-arc-transitively. In this paper, we consider the same base graphs while the covering transformation group is $\mathbb{Z}_p^2$ with p a prime. Remarkably, we shall show that all the regular covers with these properties are just covers of $K_{4,4} - 4K_2$.

Note that to classify regular covers of given graphs such as K_n and $K_{n,n}$, whose covering transformation group is an elementary group $\mathbb{Z}_p^k$ and whose fibre-preserving automorphism group acts 2-arc-transitively is a very difficult task. Essentially, it is related to the group extension theory, the group representation theory and other specific branches of group theory. We believe that the classification of all such covers for all the values k is almost not feasible. Therefore, the first step might be to study the problem for small values k and to construct some new interesting covers.

Except for the graph $K_{n,n} - nK_2$, another often considered graph is the complete bipartite graph $K_{n,n}$. In further research, we shall focus on the 2-arc-transitive regular elementary abelian covers of this graph. For further reading on the topic of covers, see [4, 5, 9, 13, 14, 22].

A cover of a given graph can be derived through a voltage assignment, see Gross and Tucker [15, 16]. Let Y be a graph and K a finite group. A *voltage assignment* (or, *K-voltage assignment*) on the graph Y is a function $f : A(Y) \to K$ with the property that

$f(u,v) = f(v,u)^{-1}$ for each $(u,v) \in A(Y)$. The values of f are called *voltages*, and K is called the *voltage group*. The *derived graph* $Y \times_f K$ *from a voltage assignment* f has for its vertex set $V(Y) \times K$, and its edge set

$$\{\{(u,g),(v,f(v,u)g)\} \mid \{u,v\} \in E(Y), g \in K\}.$$

By the definition, the derived graph $Y \times_f K$ is a covering of the graph Y with the first coordinate projection $p : Y \times_f K \to Y$, which is called the *natural projection* and with the covering transformation group isomorphic to K. Conversely, each connected regular cover X of Y with the covering transformation group K can be described by a derived graph $Y \times_f K$ from some voltage assignment f. Moreover, the voltage assignment f naturally extends to walks in Y. For any walk W of Y, let f_W denote the voltage of W. Finally, we say that an automorphism $\overline{\alpha}$ of Y *lifts* to an automorphism α of X if $\overline{\alpha}p = p\alpha$, where p is the covering projection from X to Y.

Before stating the main result, we first introduce a family of derived graphs. Let $Y = K_{4,4} - 4K_2$ with the bipartition $V(Y) = \{a,b,c,d\} \cup \{w,x,y,z\}$ as shown in Figure (a), and fix a spanning tree T of $K_{4,4} - 4K_2$ as shown in Figure (b). Identify the elementary group $\mathbb{Z}_p^2$ with the 2-dimensional linear vector space over $\mathbb{F}_p$. Then we define a family of derived graphs $X(p) := (K_{4,4} - 4K_2) \times_\phi \mathbb{Z}_p^2$ with voltage assignment ϕ such that

$$\phi(b,y) = (1,0),\ \phi(c,w) = \phi(d,w) = \phi(d,x) = (0,1),\ \phi(c,x) = (1,1)$$

$$\text{and } \phi(u,v) = 0 \text{ for any tree arc } (u,v).$$

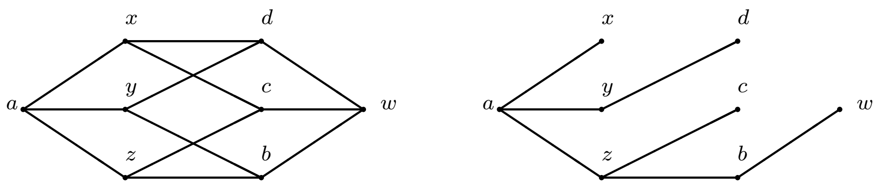

Figure (a): the graph $K_{4,4} - 4K_2$; (b): a spanning tree T of $K_{4,4} - 4K_2$.

The following theorem is the main result of this paper.

Theorem 1.1. *Let X be a connected regular cover of the complete bipartite graph minus a matching $K_{n,n} - nK_2$ $(n \geq 3)$, whose covering transformation group K is isomorphic to $\mathbb{Z}_p^2$ with p a prime and whose fibre-preserving automorphism group acts 2-arc-transitively. Then $n = 4$ and X is isomorphic to $X(p)$.*

2 Preliminaries

In this section we introduce some preliminary results needed in Section 3.

The first result may be deduced from the classification of doubly transitive groups (see [2] and [3, Corollary 8.3]).

Proposition 2.1. *Let G be a 3-transitive permutation group of degree at least 4. Then one of the following occurs.*

(i) $G \cong S_4$;

(ii) $\mathrm{soc}(G)$ *is* 4*-transitive;*

(iii) $\mathrm{soc}(G) \cong M_{22}$ *or* A_5*, which are* 3*-transitive but not* 4*-transitive;*

(iv) $\mathrm{PSL}(2,q) \le G \le \mathrm{P\Gamma L}(2,q)$*, where the projective special linear group* $\mathrm{PSL}(2,q)$ *is the socle of* G *which does not act* 3*-transitively, and* G *acts on the projective geometry* $\mathrm{PG}(1,q)$ *in a natural way, having degree* $q+1$*, with* $q \ge 5$ *an odd prime power;*

(v) $G \cong \mathrm{AGL}(m,2)$ *with* $m \ge 3$*;*

(vi) $G \cong \mathbb{Z}_2^4 \rtimes A_7 < \mathrm{AGL}(4,2)$.

Let G be a finite group and H be a proper subgroup of G, and let $D = D^{-1}$ be inverse-closed union of some double cosets of H in $G \setminus H$. Then the *coset graph* $X = X(G; H, D)$ is defined by taking $V(X) = \{Hg \mid g \in G\}$ as the vertex set and $E(X) = \{\{Hg_1, Hg_2\} \mid g_2g_1^{-1} \in D\}$ as the edge set. By the definition, the size of $V(X)$ is the number of right cosets of H in G and its valency is $|D|/|H|$. It follows that the group G in its coset action by right multiplication on $V(X)$ is transitive, and the kernel of this representation of G is the intersection of all the conjugates of H in G. If this kernel is trivial, then we say the subgroup H is *core-free*. In particular, if $H = 1$, then we get a *Cayley graph*. Conversely, each vertex-transitive graph is isomorphic to a coset graph (see [21]).

Let G be a group, let L and R be subgroups of G and let D be a union of double cosets of R and L in G, namely, $D = \bigcup_i Rd_iL$. By $[G : L]$ and $[G : R]$, we denote the set of right cosets of G relative to L and R, respectively. Define a bipartite graph $X = \mathbf{B}(G, L, R; D)$ with bipartition $V(X) = [G : L] \cup [G : R]$ and edge set $E(X) = \{\{Lg, Rdg\} \mid g \in G, d \in D\}$. This graph is called the *bicoset graph* of G with respect to L, R and D (see [10]).

Proposition 2.2. ([10, Lemmas 2.3, 2.4])

(i) *The bicoset graph* $X = \mathbf{B}(G, L, R; D)$ *is connected if and only if* G *is generated by elements of* $D^{-1}D$.

(ii) *Let* Y *be a bipartite graph with bipartition* $V(Y) = U(Y) \cup W(Y)$*, let* G *be a subgroup of* $\mathrm{Aut}\,(Y)$ *acting transitively on both* U *and* W*, let* $u \in U(Y)$ *and* $w \in W(Y)$*, and set* $D = \{g \in G \mid w^g \in Y_1(u)\}$*, where* $Y_1(u)$ *is the neighborhood of* u*. Then* D *is a union of double cosets of* G_w *and* G_u *in* G*, and* $Y \cong \mathbf{B}(G, G_u, G_w; D)$*. In particular, if* $\{u, w\} \in E(Y)$ *and* G_u *acts transitively on its neighborhood, then* $D = G_wG_u$.

Proposition 2.3. ([17, Satz 4.5]) *Let* H *be a subgroup of a group* G*. Then* $C_G(H)$ *is a normal subgroup of* $N_G(H)$ *and the quotient* $N_G(H)/C_G(H)$ *is isomorphic with a subgroup of* $\mathrm{Aut}\,H$.

Let G be a group and N a subgroup of G. If there exists a subgroup H of G such that $G = NH$ and $N \cap H = 1$, then the subgroup H is called a *complement* of N in G. The following proposition is due to Gaschütz.

Proposition 2.4. ([17, Satz 17.4]) *Let* G *be a finite group. Let* A *and* B *be two subgroups of* G *such that* A *is abelian normal in* G*,* $A \le B \le G$ *and* $(|A|, |G : B|) = 1$*. If* A *has a complement in* B*, then* A *has a complement in* G.

Proposition 2.5. ([7, Lemma 2.7]) *If p is a prime, then the general linear group* $\mathrm{GL}(2,p)$ *does not contain a nonabelian simple subgroup.*

A *central extension* of a group G is a pair (H,π) where H is a group and $\pi : H \to G$ is a surjective homomorphism with $\ker(\pi) \leq Z(H)$. A central extension $(\widetilde{G},\pi)$ of G is *universal* if for each central extension (H,σ) of G there exists the unique group homomorphism $\alpha : \widetilde{G} \to H$ with $\pi = \alpha\sigma$. If G is a perfect group, namely $G' = G$, then up to isomorphism, G has the unique universal central extension, say $(\widetilde{G},\pi)$, (see [1, pp.166-167]). In this case, $\widetilde{G}$ is called the *universal covering group* of G and $\ker(\pi)$ the *Schur multiplier* of G.

Proposition 2.6. ([6, page xv]) *The Schur multiplier of the simple group* $\mathrm{PSL}(2,q)$ *is* $\mathbb{Z}_2$ *for $q \neq 9$, and $\mathbb{Z}_6$ for $q = 9$.*

The following proposition is quoted from [9].

Proposition 2.7. ([9, Lemma 2.5]) *Let Y be a graph and let $\mathcal{B}$ be a set of cycles of Y spanning the cycle space $\mathcal{C}_Y$ of Y. If X is a cover of Y given by a voltage assignment f for which each $C \in \mathcal{B}$ is trivial, then X is disconnected.*

3 Proof of Theorem 1.1

Now we prove Theorem 1.1. Let $U = \{1,2,\cdots,n\}$ and $W = \{1',2',\cdots,n'\}$. Set $Y = K_{n,n} - nK_2$ ($n \geq 3$) with the vertex set $V(Y) = U \cup W$ and edge set $E(Y) = \{\{i,j'\} \mid i \neq j, i,j = 1,2\cdots,n\}$. Let X be a cover of Y with the covering projection $\phi : X \to Y$ and the covering transformation group $K \cong \mathbb{Z}_p^2$, where p is a prime.

Suppose that $n = 3$. Then Y is a 6-cycle and there is only one cotree arc. Since X is assumed to be connected, all the voltage assigned to the cotree arcs in Y should generate K. It means that K is a cyclic group, a contradiction.

Suppose that $n = 4$. In [12, Theorem 4.1], all regular covers of $K_{4,4} - 4K_2$ were classified, whose covering transformation group K is either cyclic or elementary abelian, and whose fibre-preserving automorphism group acts arc-transitively. Among them, $X(p)$ is the unique cover when $K \cong \mathbb{Z}_p^2$ and the fibre-preserving automorphism group acts 2-arc-transitively.

In what follows, we will assume $n \geq 5$. Since our aim is to find the covers of Y whose fibre-preserving automorphism group acts 2-arc-transitively, this group module the covering transformation group K should be isomorphic to a 2-arc-transitive subgroup of $\mathrm{Aut}\,Y$, in other word, there exists a 2-arc-transitive subgroup of $\mathrm{Aut}\,Y$ to be lifted. Now, let $A \leq \mathrm{Aut}\,Y$ be a 2-arc-transitive subgroup, and let $G \leq A$ be the corresponding index 2 subgroup of A fixing U and W setwise. Let $\widetilde{A}$ and $\widetilde{G}$ be the respective lifts of A and G. Clearly, $\mathrm{Aut}\,(Y) = S_n \times \langle\sigma\rangle$, where σ is the involution exchanging every pair i and i'.

Now, we show that G has a faithful 3-transitive representation on the two biparts of Y. Take arbitrary two different triples $\{u_1,v_1,w_1\}$ and $\{u_2,v_2,w_2\}$ with $u_i,v_i,w_i \in U$ and $i \in \{1,2\}$. Since (u_1,v_1',w_1) and (u_2,v_2',w_2) are both 2-arcs, and since A acts 2-arc-transitively on Y, there exists an element $g \in A$ such that $(u_1,v_1',w_1)^g = (u_2,v_2',w_2)$, noting that $v_1'^g = v_2'$ implying $v_1^g = v_2$. Moreover, it is obvious that g fixes two biparts setwise so that $g \in G$. So G acts 3-transitively on U. By the symmetry, G acts 3-transitively on another bipart. Therefore, G should be one of the 3-transitive groups listed in Proposition 2.1. Since $n \geq 5$, we conclude the following four cases from Proposition 2.1:

(1) either $\mathrm{soc}(G)$ is 4-transitive or $\mathrm{soc}(G) \cong M_{22}$;

(2) $n = 5$ and $\mathrm{soc}(G) = A_5$;

(3) $\mathrm{soc}(G) = \mathrm{PSL}(2, q)$ with $q \geq 5$;

(4) G is of affine type, that is the last two cases of Proposition 2.1.

To prove the theorem, we shall prove the non-existence for the above four cases separately in the following subsections.

3.1 Either $\mathrm{soc}(G)$ is 4-transitive or $\mathrm{soc}(G) \cong M_{22}$

Lemma 3.1. *There exist no regular covers X of $K_{n,n} - nK_2$, whose fibre-preserving automorphism group acts 2-arc-transitively and whose covering transformation group is isomorphic to $\mathbb{Z}_p^2$ with p a prime, such that either $\mathrm{soc}(G)$ acts 4-transitively on two biparts or $\mathrm{soc}(G) \cong M_{22}$.*

Proof. Suppose that G has a nonabelian simple socle $T := \mathrm{soc}(G)$ which is either 4-transitive or isomorphic to M_{22}. Let $\widetilde{T}$ be the lift of T so that $\widetilde{T}/K = T$. In view of Proposition 2.3, we have

$$(\widetilde{T}/K)/(C_{\widetilde{T}}(K)/K) \cong \widetilde{T}/C_{\widetilde{T}}(K) \leq \mathrm{Aut}\,(K) \cong \mathrm{GL}(2, p). \tag{3.1}$$

Since $C_{\widetilde{T}}(K)/K \rhd \widetilde{T}/K$ and $\widetilde{T}/K$ is simple, we get $C_{\widetilde{T}}(K)/K = 1$ or $\widetilde{T}/K$. If the first case happens, then Eq(3.1) implies that $\mathrm{GL}(2, p)$ contains a nonabelian simple subgroup, which contradicts Proposition 2.5. Thus, $C_{\widetilde{T}}(K) = \widetilde{T}$, that is, $K \leq Z(\widetilde{T})$. It was shown in [9, pp.1361-1364] that the voltages on all the 4-cycles and 6-cycles of the base graph Y are trivial, provided $K \leq Z(\widetilde{T})$ and either T is 4-transitive or $T \cong M_{22}$. Therefore, Proposition 2.7 implies that the covering graph X is disconnected. This completes the proof of the lemma. □

3.2 $n = 5$ and $\mathrm{soc}(G) = A_5$

Lemma 3.2. *Suppose that $n = 5$ and $\mathrm{soc}(G) = A_5$. Then, there are no connected graphs X arising as regular covers of Y whose covering transformation group K is isomorphic to $\mathbb{Z}_p^2$ with p a prime, and whose fibre-preserving automorphism group acts 2-arc-transitively.*

Proof. Since G is isomorphic to either A_5 or S_5, it suffices to consider the case $G \cong A_5$. Let $\widetilde{G}$ be a lift of G, that is, $\widetilde{G}/K = G$. As in Lemma 3.1, a similar argument shows that $K \leq Z(\widetilde{G})$. Set $\widetilde{T} := \widetilde{G}'$. In what follows, we divide our proof into four steps.

Step 1: Show $\widetilde{T} \cap K = 1$ or $\mathbb{Z}_2$.

Set $\widetilde{T} := \widetilde{G}'$. Since $G' = G$, we get

$$\widetilde{T}/\widetilde{T} \cap K \cong \widetilde{T}K/K = (\widetilde{G}/K)' = G' = G = \widetilde{G}/K \cong A_5, \tag{3.2}$$

which implies that $\widetilde{G} = \widetilde{T}K$. As $K \leq Z(\widetilde{G})$, we have

$$\widetilde{T} = [\widetilde{G}, \widetilde{G}] = [\widetilde{T}K, \widetilde{T}K] = [\widetilde{T}, \widetilde{T}] = \widetilde{T}'.$$

Thus, $\widetilde{T} \cap K \leq \widetilde{T}' \cap Z(\widetilde{T})$ and Eq(3.2) implies that $\widetilde{T}$ is a proper central extension of $\widetilde{T} \cap K$ by $G \cong A_5$. By Proposition 2.6, we know that the Schur Multiplier of A_5 is $\mathbb{Z}_2$. Thus, $\widetilde{T} \cap K$ is either 1 or $\mathbb{Z}_2$.

Let $u \in V(Y)$ be an arbitrary vertex, and take $\widetilde{u} \in \phi^{-1}(u)$, where ϕ is the covering projection from X to Y.

Step 2: Show $\mathbb{D}_4 \leq \widetilde{G}_{\widetilde{u}} \cap \widetilde{T}$.

Now, we have $\widetilde{G}_{\widetilde{u}} \cong G_u \cong A_4$ and so

$$\widetilde{G}_{\widetilde{u}}/\widetilde{G}_{\widetilde{u}} \cap \widetilde{T} \cong \widetilde{G}_{\widetilde{u}}\widetilde{T}/\widetilde{T} \leq \widetilde{G}/\widetilde{T} = \widetilde{T}K/\widetilde{T} \cong K/K \cap \widetilde{T}. \tag{3.3}$$

Since $\widetilde{G}_{\widetilde{u}} \cap \widetilde{T} \trianglerighteq \widetilde{G}_{\widetilde{u}} \cong A_4$, it follows that $\widetilde{G}_{\widetilde{u}} \cap \widetilde{T} \cong 1$, $\mathbb{D}_4$ or A_4. If $\widetilde{G}_{\widetilde{u}} \cap \widetilde{T} = 1$, then Eq(3) implies that $\widetilde{G}_{\widetilde{u}} \cong A_4$ is isomorphic to a quotient group of $K \cong \mathbb{Z}_p^2$, a contradiction. So, we get $\mathbb{D}_4 \leq \widetilde{G}_{\widetilde{u}} \cap \widetilde{T}$.

Step 3: Show $\widetilde{T} \cong A_5$ and $\widetilde{G} = \widetilde{T} \times K$.

By Step 1, we know that $\widetilde{T} \cap K = 1$ or $\mathbb{Z}_2$. If $\widetilde{T} \cap K \cong \mathbb{Z}_2$, then Eq(3.2) implies that $\widetilde{T} \cong \mathrm{SL}(2,5)$ which has the unique involution, contradicting the fact that $\mathbb{D}_4 \leq \widetilde{G}_{\widetilde{u}} \cap \widetilde{T}$. Hence, it follows that $\widetilde{T} \cap K = 1$, and so $\widetilde{T} \cong A_5$ and $\widetilde{G} = \widetilde{T} \times K$.

Step 4: Show the nonexistence of the covering graph X.

Suppose that

$$V(Y) = \{1,2,3,4,5\} \cup \{1',2',3',4',5'\} \quad \text{and} \quad E(Y) = \{\{i,j'\} \mid i \neq j, 1 \leq i,j \leq 5\}.$$

Since $\widetilde{T} \cong A_5$, we may identify $\widetilde{T}$ with A_5. In $\widetilde{T}$, set

$$x = (23)(45),\ y = (25)(34),\ z = (234),\ b = (15)(23).$$

Then, $\widetilde{G}_F = (\langle x,y\rangle \rtimes \langle z\rangle) \times K$, where $F = \phi^{-1}(1)$ is the fibre over the vertex $1 \in V(Y)$. Take $\widetilde{u} \in F$. Since $\mathbb{D}_4 \leq \widetilde{G}_{\widetilde{u}} \cap \widetilde{T}$, one may deduce that $\mathbb{D}_4 \cong \langle x,y\rangle \leq \widetilde{G}_{\widetilde{u}}$ so that $L := \widetilde{G}_{\widetilde{u}} = \langle x,y\rangle \rtimes \langle zk_1\rangle$ for some $k_1 \in K$. Note that $\widetilde{G}_F = \widetilde{G}_{F'}$, where $F' = \phi^{-1}(1')$ is the fibre over the vertex $1' \in V(Y)$. Then, one may assume that $R := \widetilde{G}_{\widetilde{w}} = \langle x,y\rangle \rtimes \langle zk_2\rangle$ for some $k_2 \in K$ and $\widetilde{w} \in F'$.

By Proposition 2.2, the covering graph X should be isomorphic to a bicoset graph $X' = \mathbf{B}(\widetilde{G}, L, R; D)$, where $D = Rbk_3L$ for some $k_3 \in K$ with two biparts:

$$\begin{aligned} \widetilde{U'} &= \{Lk \mid k \in K\} \cup \{Lbx^iy^jk \mid i,j = 0,1, k \in K\}, \\ \widetilde{W'} &= \{Rk \mid k \in K\} \cup \{Rbx^iy^jk \mid i,j = 0,1, k \in K\}. \end{aligned}$$

Moreover, X' should satisfy the following two conditions.

(i) $d(X') = 4$:

Since the length of the orbit of L containing the vertex Rbk_3L is 4, zk_1 must fix the vertex Rbk_3, that is,

$$Rbk_3 = Rbk_3zk_1 = Rbk_3zk_1(bk_3)^{-1}bk_3 = Rz^bk_1bk_3 = \\ Rz^{-1}k_2^{-1}k_2k_1bk_3 = Rbk_3k_2k_1,$$

which implies that

$$k_2 = k_1^{-1}. \tag{3.4}$$

(ii) *Connectedness property:*

By Eq(4), we have

$$\langle D^{-1}D\rangle \quad = \quad \langle LbRbL\rangle = \langle L,\ R^b\rangle = \langle x, y, zk_1, x^b, y^b, z^bk_2\rangle \leq \widetilde{T} \times \langle k_1\rangle \neq \widetilde{G}.$$

It follows from Proposition 2.2(i) that the bicoset graph X' is disconnected, which completes our proof. □

3.3 *G* is of affine type

Lemma 3.3. *Suppose that either $G \cong \mathrm{AGL}(m, 2)$, where $m \geq 3$ or $G \cong \mathbb{Z}_2^4 \rtimes A_7 < \mathrm{AGL}(4, 2)$. Then, there are no connected graphs X arising as regular covers of Y whose covering transformation group K is isomorphic to $\mathbb{Z}_p^2$ with p a prime, and whose fibre-preserving automorphism group acts 2-arc-transitively.*

Proof. The arguments in both cases are exactly the same, and so here we just discuss the first case in details. Suppose that $G \cong \mathrm{AGL}(m, 2) \cong \mathbb{Z}_2^m \rtimes \mathrm{GL}(m, 2)$, and let $\widetilde{G}$ be a lift of G, namely $\widetilde{G}/K = G$.

Since

$$C_{\widetilde{G}}(K)/K \trianglerighteq \widetilde{G}/K \cong \mathbb{Z}_2^m \rtimes \mathrm{GL}(m, 2),$$

it follows that $C_{\widetilde{G}}(K)/K$=1, $\mathbb{Z}_2^m$ or $\widetilde{G}/K$. By Proposition 2.3, we have

$$(\widetilde{G}/K)/(C_{\widetilde{G}}(K)/K) \cong \widetilde{G}/C_{\widetilde{G}}(K) \leq \mathrm{Aut}\,(K) \cong \mathrm{GL}(2, p). \tag{3.5}$$

If the first two cases happen, then Eq(3.5) implies that $\mathrm{GL}(2, p)$ contains a nonabelian simple subgroup, which contradicts Proposition 2.5. Thus, $C_{\widetilde{G}}(K) = \widetilde{G}$, that is $K \leq Z(\widetilde{G})$.

Let $\widetilde{A}$ be the group of fibre-preserving automorphism of X acting 2-arc-transitively. Let $\widetilde{U}$ and $\widetilde{W}$ be the two biparts of X. Take a fibre F in $\widetilde{U}$ and take a vertex $\widetilde{u}_1 \in F$. Set $\widetilde{M} := \widetilde{G}_{\widetilde{u}_1} \cong \mathrm{GL}(m, 2)$ and $\widetilde{T}/K = \mathrm{soc}(\widetilde{G}/K) \cong \mathbb{Z}_2^m$. Then $\widetilde{G} = \widetilde{T} \rtimes \widetilde{M}$. Let F' denote the unique corresponding fibre in $\widetilde{W}$ without edges leading to F and take a vertex $\widetilde{w}_1 \in F'$. Then $\widetilde{G}_F = \widetilde{G}_{F'}$. Since $\widetilde{M}$ is the unique subgroup isomorphic to $\mathrm{GL}(m, 2)$ in $K \times \widetilde{M}$, it follows that $\widetilde{G}_{\widetilde{w}_1} = \widetilde{M}$.

First, suppose that $p \neq 2$. Now, $\widetilde{G}_F = K \times \widetilde{M}$. Since $(|\widetilde{G} : \widetilde{G}_F|, |K|) = (2^m, p^2) = 1$, by Proposition 2.4, K has a complement in $\widetilde{G}$. So, we may suppose that $\widetilde{G} = K \times (\widetilde{L} \rtimes \widetilde{M})$, where $\widetilde{L} \cong \mathbb{Z}_2^m$. Since $\widetilde{G}$ is transitive on $\widetilde{W}$, there exists an element $x \in \widetilde{G}$

such that $(\widetilde{u}_1, \widetilde{w}_1^x) \in E(X)$. By Proposition 2.2(ii), X is isomorphic to a bicoset graph $\mathbf{B}(\widetilde{G}, \widetilde{M}, \widetilde{M}^x; D)$, where $D = \widetilde{M}\widetilde{M}^x$. Since $\widetilde{L} \rtimes \widetilde{M} \trianglerighteq \widetilde{G}$, we get $\langle D^{-1}D\rangle = \langle \widetilde{M}, \widetilde{M}^x\rangle \leq \widetilde{L} \rtimes \widetilde{M} \neq \widetilde{G}$. It follows from Proposition 2.2(i) that X is disconnected.

Next, suppose that $p = 2$, namely $K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $F = \{\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \widetilde{u}_4\}$ and $F' = \{\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3, \widetilde{w}_4\}$. Clearly, $\widetilde{M}$ has four orbits on $\widetilde{U} \setminus F$ and $\widetilde{W} \setminus F'$, respectively, say

$$\Delta_1,\ \Delta_2,\ \Delta_3,\ \Delta_4;\ \Delta_1',\ \Delta_2',\ \Delta_3',\ \Delta_4'.$$

For $i = 0, 1, 2, \cdots$, by $X_i(\widetilde{u}_1)$ we denote the set of vertices of distance i from $\widetilde{u}_1$. Without loss of generality, let $X_1(\widetilde{u}_1) = \Delta_1'$. Since $\widetilde{M}$ acts 2-arc-transitively on the arcs initialed from $\widetilde{u}_1$, it follows that $X_2(\widetilde{u}_1)$ is an orbit of $\widetilde{M}$, that is, $X_2(\widetilde{u}_1) = \Delta_i$ for some $i \in \{1, 2, 3, 4\}$. Then $X_3(\widetilde{u}_1) = \{\widetilde{w}_j\}$, for some $j \in \{1, 2, 3, 4\}$. Clearly, $X_4(\widetilde{u}_1) = \emptyset$ and therefore X is disconnected. □

3.4 $\mathrm{soc}(G) = \mathrm{PSL}(2, q)$ for $q \geq 5$

In this subsection, identify $V(Y)$ with two copies of the projective line $\mathrm{PG}(1, q)$.

Lemma 3.4. *Suppose that* $\mathrm{PSL}(2, q) \leq G \leq \mathrm{P\Gamma L}(2, q)$, *where* $q = r^l \geq 5$ *is an odd prime power. Then, there are no connected graphs X arising as regular covers of Y whose covering transformation group K is isomorphic to $\mathbb{Z}_p^2$ with p a prime, and whose fibre-preserving automorphism group acts* 2*-arc-transitively.*

Proof. Let $\widetilde{G}$ be the lift of G so that $\widetilde{G}/K = G$. Since $\mathrm{P\Gamma L}(2, q)' = \mathrm{PSL}(2, q)$ and $\mathrm{PSL}(2, q) \leq G \leq \mathrm{P\Gamma L}(2, q)$, we have $G' = \mathrm{PSL}(2, q)$. Hence, $\widetilde{G}$ is insolvable and there exists a positive integer m such that $\widetilde{G}^{(m)} = \widetilde{G}^{(m+1)}$. Suppose that $\widetilde{T} = \widetilde{G}^{(m)}$, it follows that

$$\widetilde{T}/\widetilde{T} \cap K \cong \widetilde{T}K/K = \widetilde{G}^{(m)}K/K = (\widetilde{G}/K)^{(m)} = G^{(m)} \cong \mathrm{PSL}(2, q). \tag{3.6}$$

Therefore, $\widetilde{T}K/K$ is simple and so $(\widetilde{T}K/K) \cap (C_{\widetilde{G}}(K)/K) = 1$ or $\widetilde{T}K/K$.

Again, by Proposition 2.3 and 2.5, we have $\widetilde{T}K/K \leq C_{\widetilde{G}}(K)/K$, implying that $\widetilde{T} \cap K \leq Z(\widetilde{T})$. Thus, by Eq(3.6), $\widetilde{T}$ is a proper central extension of $\widetilde{T} \cap K$ by $\mathrm{PSL}(2, q)$. In viewing of Proposition 2.6, the Schur Multiplier of $\mathrm{PSL}(2, q)$ is either $\mathbb{Z}_2$ for $q \neq 9$ or $\mathbb{Z}_6$ for $q = 9$.

It is obvious that $\widetilde{T} \cap K \cong 1$ or $\mathbb{Z}_2$ for $q \neq 9$. Next, we show it is also true for $q = 9$. Assume, the contrary, that $\widetilde{T} \cap K \cong \mathbb{Z}_3$ for $q = 9$. Since $\widetilde{T}K/K \cong \mathrm{PSL}(2, 9)$, we get $(\widetilde{T}K)_{\widetilde{u}} \cong \mathbb{Z}_3^2 \rtimes \mathbb{Z}_4$. Let $\mathbb{Z}_3^2 \cong \widetilde{H} \leq (\widetilde{T}K)_{\widetilde{u}}$. As $\widetilde{H} \cap K = 1$ and $(|\widetilde{T}K : \widetilde{H}K|, |K|) = 1$, it follows from Proposition 2.4 that K has a complement in $\widetilde{T}K$, say $\widetilde{N}$. Thus, $\widetilde{T}K = K \times \widetilde{N} \cong \mathbb{Z}_3^2 \times \mathrm{PSL}(2, 9)$. Since $[K, \widetilde{T}] = 1$, one may get

$$\widetilde{N} = \widetilde{N}' = (\widetilde{T}K)' = [\widetilde{T}K, \widetilde{T}K] = [\widetilde{T}, \widetilde{T}] = \widetilde{T}' = \widetilde{T},$$

contradicting $\widetilde{T} \cap K = \mathbb{Z}_3$. Therefore we have either $\widetilde{T} \cap K = 1$ or $\widetilde{T} \cap K = \mathbb{Z}_2$. In what follows, we discuss these two cases respectively. Set $\widetilde{M} := \widetilde{T}K$ so that $\widetilde{M}/K \cong \mathrm{PSL}(2, q)$.

Case 1: $\widetilde{T} \cap K = 1$

In this case, we have $\widetilde{M} = \widetilde{T} \times K$ and $\widetilde{T} \cong \mathrm{PSL}(2,q)$, and we shall identify $\widetilde{T}$ with $\mathrm{PSL}(2,q)$. In $\mathrm{PSL}(2,q)$, set

$$t_i = \overline{\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}}, \quad x = \overline{\begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}}, \quad y = \overline{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}.$$

where $\mathbb{F}_q^* = \langle \theta \rangle$ and $i \in \mathbb{F}_q$. Let $Q = \langle t_i \mid i \in \mathbb{F}_q \rangle \cong \mathbb{Z}_r^l$ and $\widetilde{Q} \leq \widetilde{T}$ be the lift of Q. Acting on $\mathrm{PG}(1,q)$, set $H := (\mathrm{PSL}(2,q))_\infty = Q \rtimes \langle x \rangle$ and the points $i \in \mathrm{PG}(1,q) \setminus \{\infty\}$ correspond to the cosets Hyt_i.

Take $\widetilde{u} \in \phi^{-1}(\infty)$ and set $\widetilde{H} := \widetilde{M}_{\widetilde{u}}$. Since $\widetilde{H}$ is a lift of H, we may assume that $\widetilde{H} = \widetilde{Q}_1 \rtimes \langle xk_1 \rangle$ for some $k_1 \in K$, and $\widetilde{Q}_1 \leq \widetilde{Q} \times K$. Actually, we are showing $\widetilde{Q}_1 = \widetilde{Q}$ below.

Suppose that $\widetilde{Q} \neq \widetilde{Q}_1$, it follows that $p = r$. Then, there exist two nontrivial elements $c_1 \in \widetilde{Q}$ and $k \in K$ such that $c_1 k \in \widetilde{Q}_1$. Moreover, we have $|\widetilde{Q}_1 \cap \widetilde{Q}| \geq r^{l-2}$.

If $l > 2$, then there exists a nontrivial element $c_2 \in \widetilde{Q}_1 \cap \widetilde{Q}$. Since $\langle x \rangle$ has two orbits both with length $\frac{q-1}{2}$ on $Q \setminus \{1\}$ by conjugacy action, $\langle xk_1 \rangle$ has the same property on $\widetilde{Q}_1 \setminus \{1\}$, whose two orbits should be $B_1 := \{(c_1k)^{\langle xk_1 \rangle}\} = \{c_1^{\langle xk_1 \rangle} k\}$ and $B_2 := \{c_2^{\langle xk_1 \rangle}\}$. Therefore, $\widetilde{Q}_1 = B_1 \cup B_2 \cup \{1\}$. Noting $r \geq 3$, the inverse $(c_1k)^{-1}$ of $c_1 k \in \widetilde{Q}_1$ is not contained in $B_1 \cup B_2 \cup \{1\}$, a contradiction.

If $l = 1$, then we get $\widetilde{Q}_1 \cap \widetilde{Q} = 1$. As $q = r^l = r \geq 5$, there exist two nontrivial elements $c_2 \in \widetilde{Q}$ and $k' \in K$ such that $c_2 k' \in \widetilde{Q}_1$. Again, $\widetilde{Q}_1 = \{c_1^{\langle xk_1 \rangle} k\} \cup \{c_2^{\langle xk_1 \rangle} k'\} \cup \{1\}$. Since $p = r \geq 5$, take $k^s \in K \setminus \{1, k, k'\}$ for some integer s. Then, $(c_1k)^s = {c_1}^s k^s \in \widetilde{Q}_1$ is neither contained in $\{c_1^{\langle xk_1 \rangle} k\}$ nor in $\{c_2^{\langle xk_1 \rangle} k'\}$, a contradiction.

If $l = 2$ and $r \geq 5$, we shall have the same discussion as in the case $l = 1$. Now, we only need to consider $l = 2$ and $r = 3$, that is, $q = r^l = 9$. Since $c_1 k \in \widetilde{Q}_1$, it is easy to check that

$$(xk_1)^{-1}(c_1k)(xk_1) = c_1^x k = c_1^{-1} k \in \{(c_1)^{\langle xk_1 \rangle} k\} \subset \widetilde{Q}_1.$$

Hence, $1 \neq (c_1k)(c_1^{-1}k) = k^2 \in \widetilde{Q}_1$, a contradiction again.

By the above discussion, we may assume that $L := \widetilde{M}_{\widetilde{u}} = \widetilde{Q} \rtimes \langle xk_1 \rangle$ and $R := \widetilde{M}_{\widetilde{u}'} = \widetilde{Q} \rtimes \langle xk_2 \rangle$ for some $k_1, k_2 \in K$ and $\widetilde{u}' \in \phi^{-1}(\infty')$. Then by Proposition 2.2, our graph X is isomorphic to a bicoset graph $X' = \mathbf{B}(\widetilde{M}, L, R; D)$ for some double coset D with two biparts:

$$\begin{array}{rcl} \widetilde{U}' & = & \{Lk \mid k \in K\} \cup \{Lyt_ik \mid i \in \mathbb{F}_q, k \in K\}, \\ \widetilde{W}' & = & \{Rk \mid k \in K\} \cup \{Ryt_ik \mid i \in \mathbb{F}_q, k \in K\}. \end{array}$$

Since there is only one edge from L to the block $\{Ryk \mid k \in K\}$, we may assume that the neighbor of L corresponds to the bicoset $D = Ryk_3L$ for some $k_3 \in K$. Then X' should satisfy the following two conditions.

(i) *$d(X') = q$:*

Since the length of the orbit of L containing the vertex Ryk_3L is q, we have xk_1 should fix the vertex Ryk_3, that is,

$$\begin{array}{rcl} Ryk_3 & = & Ryk_3xk_1 = Ryk_3xk_1(yk_3)^{-1}yk_3 = Ry^x yk_2^{-1}k_1yk_3 \\ & = & Rx^{-2}k_2^{-1}k_1yk_3 = Rk_2k_1yk_3, \end{array}$$

which implies that

$$k_2 = k_1^{-1}. \tag{3.7}$$

(ii) *Connectedness property:*

By Eq(3.7), we have

$$\begin{aligned}\langle D^{-1}D\rangle &= \langle L(yk_3)^{-1}R(yk_3)L\rangle = \langle L,\ R^y\rangle \\ &= \langle \widetilde{Q}, xk_1, \widetilde{Q}^y, x^y k_2\rangle = \langle \widetilde{Q}, xk_1, \widetilde{Q}^y, x^y k_1^{-1}\rangle \le \widetilde{T} \times \langle k_1\rangle \ne \widetilde{M}.\end{aligned}$$

Again, Proposition 2.2(i) implies that the graph X' is disconnected.

Case 2: $\widetilde{T} \cap K = \mathbb{Z}_2$ and $\widetilde{T} \cong$ SL$(2, q)$

In this case, we have $K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and identify $\widetilde{T}$ with SL$(2, q)$.
In SL$(2, q)$, set

$$e = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t_i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}, \quad y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $\mathbb{F}_q^* = \langle\theta\rangle$ and $i \in \mathbb{F}_q$. Let $\widetilde{Q} = \langle t_i \mid i \in \mathbb{F}_q\rangle \cong \mathbb{Z}_r^l$.

Take $\widetilde{u} \in \phi^{-1}(\infty)$, one may assume that $\widetilde{M_{\widetilde{u}}} = \widetilde{Q}_1 \rtimes \langle xk\rangle \cong \mathbb{Z}_r^l \rtimes \mathbb{Z}_{\frac{q-1}{2}}$, where $\widetilde{Q}_1 \le K \times \widetilde{Q}$ and $k \in K$. Since $\widetilde{Q} \cong \mathbb{Z}_r^l$ and r is an odd prime, we get $\widetilde{Q}_1 = \widetilde{Q}$. Moreover, as $(xk)^{\frac{q-1}{2}} = 1$ and $K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, it follows that $k^{\frac{q-1}{2}} = e$, that is, $k = e$ and $\frac{q-1}{2}$ is odd. Hence, we may assume that $L := \widetilde{M_{\widetilde{u}}} = \widetilde{Q} \rtimes \langle xe\rangle$ and $R := \widetilde{M_{\widetilde{u}'}} = \widetilde{Q} \rtimes \langle xe\rangle$, where $\widetilde{u}' \in \phi^{-1}(\infty')$.

Finally, with the same discussion as Case 1, one may get the nonexistence of X. □

Combining the lemmas in Subsections 3.1, 3.2, 3.3 and 3.4, we complete our proof of Theorem 1.1.

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Cube-contractions in 3-connected quadrangulations

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Abstract

A 3-connected quadrangulation of a closed surface is said to be $\mathcal{K}'_3$-irreducible if no face- or cube-contraction preserves simplicity and 3-connectedness. In this paper, we prove that a $\mathcal{K}'_3$-irreducible quadrangulation of a closed surface except the sphere and the projective plane is either (i) irreducible or (ii) obtained from an irreducible quadrangulation H by applying 4-cycle additions to $F_0 \subseteq F(H)$ where $F(H)$ stands for the set of faces of H. We also determine $\mathcal{K}'_3$-irreducible quadrangulations of the sphere and the projective plane. These results imply new generating theorems of 3-connected quadrangulations of closed surfaces.

Keywords: Quadrangulation, closed surface, generating theorem.

Math. Subj. Class.: 05C10

1 Introduction

In this paper, we only consider simple graphs which have no loops and no multiple edges. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. We say that $S \subset V(G)$ is a *cut* of G if $G - S$ is disconnected. In particular, S is called a k-*cut* if S is a cut with $|S| = k$. A cycle C of G is said to be *separating* if $V(C)$ is a cut. Similarly, a simple closed curve γ on a closed surface F^2 is said to be *separating* if $F^2 - \gamma$ is disconnected.

A *quadrangulation* G of a closed surface F^2 is a simple graph cellularly embedded on the surface so that each face is quadrilateral; thus, a 2-path on the sphere is not a quadrangulation. We denote the set of faces of G by $F(G)$ throughout the paper. For quadrangulations we consider applying three reductions, called a *face-contraction*, a *4-cycle removal*

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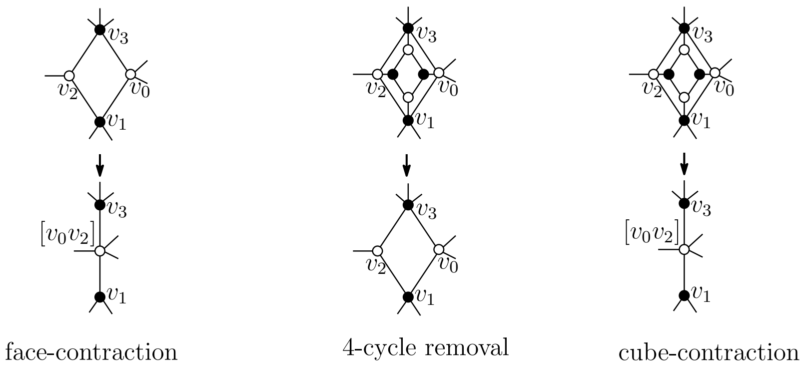


Figure 1: Reductions for quadrangulations.

and a *cube-contraction*, as shown in Figure 1. (Precise definitions of these reductions will be given in the next section.) The corresponding inverse operations are called a *vertex-splitting*, a *4-cycle addition* and a *cube-splitting*, respectively. In particular, the operations of a face-contraction and a 4-cycle removal were first introduced by Batagelj [1]

Irreducible quadrangulations, such that no face-contraction is applicable without making a loop or multiple edges, on a fixed closed surface with low genus were obtained in earlier papers. In [9], it was proven that a 4-cycle is the unique irreducible quadrangulation of the sphere, and that there exist precisely two irreducible quadrangulations of the projective plane shown in Figure 2, where Q_P^1 and Q_P^2 are the unique quadrangular embeddings of K_4 and $K_{3,4}$ on the projective plane, respectively. The irreducible quadrangulations of the torus and the Klein bottle have also been determined in [6, 5]. In [8], it was proven that for any closed surface F^2 there exist only finitely many irreducible quadrangulations of F^2, up to homeomorphism.

A 3-connected quadrangulation G of a closed surface F^2 is said to be $\mathcal{K}_3$*-irreducible* if any of a face-contraction and a 4-cycle removal breaks simplicity or 3-connectedness of G. The following theorem is the starting point of the study of 3-connected quadrangulations. (The definitions of a pseudo double wheel, a Möbius wheel and a double cube are given in the next section.)

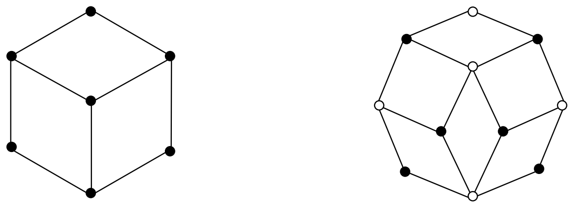
Theorem 1.1 (Brinkmann et al.[2]). *Any $\mathcal{K}_3$-irreducible quadrangulation of the sphere is isomorphic to a pseudo double wheel.*

Observe that a 3-connected quadrangulation of the sphere corresponds to a 4-regular 3-connected graph on the same surface by taking its dual. Broersma et al. [3] considered the same problem of the dual version with weaker conditions than those of Brinkmann. For the projective plane, Nakamoto proved the following.

Theorem 1.2 (Nakamoto[7]). *Any $\mathcal{K}_3$-irreducible quadrangulation of the projective plane is isomorphic to either a Möbius wheel or Q_P^2.*

Furthermore, the results in [4] imply the following.

Theorem 1.3 (Nagashima et al.[4]). *Let G be a quadrangulation of a closed surface other than the sphere and the projective plane. Then G is $\mathcal{K}_3$-irreducible if and only if G is irreducible.*



Q_P^1 Q_P^2

Figure 2: Irreducible quadrangulations on the projective plane.

In this paper, we determine other minimal subsets of 3-connected quadrangulations by replacing 4-cycle removals with cube-contractions. A 3-connected quadrangulation G is said to be $\mathcal{K}'_3$-irreducible if any of a face-contraction and a cube-contraction breaks the simplicity or the 3-connectedness of G. The followings are our main results in the paper. In these statements, $F(H)$ stands for the set of faces of a quadrangulation H.

Theorem 1.4. *Let G be a $\mathcal{K}'_3$-irreducible quadrangulation of a closed surface F^2 other than the sphere and the projective plane. Then, G is either (i) irreducible or (ii) obtained from an irreducible quadrangulation H by applying 4-cycle additions to $F_0 \subseteq F(H)$.*

Theorem 1.5. *Let G be a $\mathcal{K}'_3$-irreducible quadrangulation of the sphere. Then, G is either (i) a pseudo double wheel or (ii) a double cube.*

Theorem 1.6. *Let G be a $\mathcal{K}'_3$-irreducible quadrangulation of the projective plane. Then, G is (i) a Möbius wheel, (ii) Q_P^2 or (iii) obtained from Q_P^1 (resp. Q_P^2) by applying 4-cycle additions to $F_0 \subseteq F(Q_P^1)$ (resp. $F_0 \subseteq F(Q_P^2)$).*

Corollary 1.7. *For any closed surface F^2, there exist only finitely many quadrangulations which are $\mathcal{K}'_3$-irreducible but are not $\mathcal{K}_3$-irreducible, up to homeomorphism.*

This paper is organized as follows. In the next section, we define the reductions used in this paper and introduce typical 3-connected quadrangulations on the sphere and the projective plane called a pseudo double wheel and a Möbius wheel, respectively. In Section 3, we develop some theoretical tools and prove Theorem 1.4. The last section is devoted to prove the planar case and the projective-planar case individually, using some figures.

2 Reductions and typical quadrangulations

Let G be a quadrangulation of a closed surface F^2 and let f be a face of G bounded by a cycle $v_0v_1v_2v_3$. (We also use the notation like $f = v_0v_1v_2v_3$ in this paper.) The *face-contraction* of f at $\{v_0, v_2\}$ in G consists of identification of v_0 and v_2, and replacement of the resulting multiple edges $\{v_0v_1, v_2v_1\}$ and $\{v_0v_3, v_2v_3\}$ with two single edges, respectively. In the resulting graph, let $[v_0v_2]$ denote the vertex arisen by the identification of v_0

and v_2 (see the left-hand side of Figure 1). Similarly, we define the face-contraction of f at $\{v_1, v_3\}$. The inverse operation of a face-contraction is called a *vertex-splitting*. We say that f is *contractible* at $\{v_0, v_2\}$ in G, if the graph obtained from the face-contraction of f at $\{v_0, v_2\}$ is simple. Assume in addition that G is 3-connected. A face f of G is said to be 3-*contractible* at $\{v_0, v_2\}$ if f is contractible at $\{v_0, v_2\}$ and the graph obtained from the face-contraction is still 3-connected.

Let $f = v_0v_1v_2v_3$ be a face of a quadrangulation G of F^2. A 4-*cycle addition* to f consists of inserting a 4-cycle $C = u_0u_1u_2u_3$ inside f in G and joining v_i and u_i for $i = 0, 1, 2, 3$. The inverse operation of a 4-cycle addition is called a 4-*cycle removal* (of C), as shown in the center of Figure 1. We call the subgraph Q isomorphic to a cube with eight vertices u_i, v_i for $i = 0, 1, 2, 3$ an *attached cube*. For an attached cube Q, we call the above 4-cycle C an *inner* 4-*cycle* of Q. In addition, we denote $\partial Q = v_0v_1v_2v_3$. Let G be a 3-connected quadrangulation of a closed surface having an attached cube Q. We say that an inner 4-cycle C of Q (or easily an attached cube Q) is *removable* if the graph obtained from G by applying 4-cycle removal C preserves the 3-connectedness. (Observe that a 4-cycle removable never destroy simplicity of G.)

As mentioned in the introduction, there exist some results of 3-connected quadrangulations (or quadrangulations with minimum degree 3) on surfaces. In those results, the 4-cycle removal is necessary by the following reason: Let $\tilde{G}$ be the graph obtained from a 3-connected quadrangulation G of a closed surface by applying 4-cycle additions to all faces of G. Clearly $\tilde{G}$ is 3-connected, but we cannot apply any face-contraction to $\tilde{G}$ without creating a vertex of degree 2.

Our third reduction of quadrangulations of closed surfaces is defined as a sequence of the above two reductions. Assume that a quadrangulation G has an attached cube Q with an inner 4-cycle C and with $\partial Q = v_0v_1v_2v_3$. A *cube-contraction* of Q at $\{v_0, v_2\}$ in G consists of a 4-cycle removal of C followed by a face-contraction at $\{v_0, v_2\}$ (see the right-hand side of Figure 1). The inverse operation of a cube-contraction is called a *cube-splitting*. We say that an attached cube Q is *contractible* if the graph obtained from G by applying a cube-contraction of Q preserves the simplicity and the 3-connectedness. One might suspect that if an attached cube Q is contractible then Q is removable (and the face that appeared by the removal is contractible). However, this is not true in general since a 4-cycle removal might break the 3-connectedness of the graph.

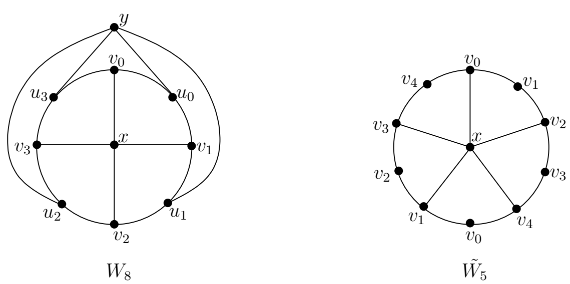


Figure 3: W_8 and $\tilde{W}_5$.

We need to describe two special types of embeddings. Firstly, embed a $2k$-cycle $C = v_0u_0v_1u_1 \dots v_{k-1}u_{k-1}$ ($k \geq 3$) into the sphere, put a vertex x on one side and a vertex y on the other side and add edges xv_i and yu_i for $i = 0, \ldots, k-1$. The resulting quadrangulation of the sphere with $2k+2$ vertices is said to be a *pseudo double wheel* and denoted by W_{2k} (see the left-hand side of Figure 3). The smallest pseudo double wheel is W_6, which is isomorphic to a cube, when the graphs are assumed to be 3-connected. The cycle C of length $2k$ is called the *rim* of W_{2k}. We call a quadrangulation of the sphere obtained from W_6 by a single 4-cycle addition a *double cube*, which is isomorphic to $C_4 \times P_2$.

Secondly, embed a $(2k-1)$-cycle $C = v_0v_1 \ldots v_{2k-2}$ $(k \geq 2)$ into the projective plane so that the tubular neighborhood of C forms a Möbius band. Next, put a vertex x on the center of the unique face of the embedding and join x to v_i for all i so that the resulting graph is a quadrangulation. The resulting quadrangulation of the projective plane with $2k$ vertices is said to be a *Möbius wheel* and denoted by $\tilde{W}_{2k-1}$ (see the right-hand side of Figure 3).

3 Lemmas to prove Theorem 1.4

The following lemma holds not only for quadrangulations but also for even embeddings of closed surfaces F^2, that is, for graphs embedded on F^2 with each face bounded by a cycle of even length. Taking a dual of an even embedding and using the odd point theorem, we can easily obtain this lemma.

Lemma 3.1. *An even embedding of a closed surface has no separating closed walk of odd length.*

Let G be a quadrangulation of a closed surface F^2 and let $f = v_0v_1v_2v_3$ be a face of G. Then a pair $\{v_i, v_{i+2}\}$ is called a *diagonal pair* of f in G, where the subscripts are taken modulo 4. A closed curve γ on F^2 is said to be a *diagonal k-curve* for G if γ passes only through distinct k faces $f_0, \ldots, f_{k-1}$ and distinct k vertices $x_0, \ldots, x_{k-1}$ of G such that for each i, f_i and f_{i+1} share x_i, and that for each i, $\{x_{i-1}, x_i\}$ forms a diagonal pair of f_i of G, where the subscripts are taken modulo k.

Lemma 3.2. *Let G be a quadrangulation of a closed surface F^2 with a 2-cut $\{x, y\}$. Then there exists a separating diagonal 2-curve for G only through x and y.*

Proof. Observe that every quadrangulation of any closed surface F^2 is 2-connected and admits no closed curve on F^2 crossing G at most once. Thus there exists a surface separating simple closed curve γ on F^2 crossing only x and y, since $\{x, y\}$ is a cut of G.

We shall show that γ is a diagonal 2-curve. Suppose that γ passes through two faces f_1 and f_2 meeting at two vertices x and y. If γ is not a diagonal 2-curve, then x and y are adjacent on ∂f_1 or ∂f_2. Since G has no multiple edges between x and y, and since $\{x, y\}$ is a 2-cut of G, we may suppose that x and y are adjacent in ∂f_1, but not in ∂f_2. Here we can take a separating 3-cycle of G along γ. This contradicts Lemma 3.1. □

Lemma 3.3. *Let G be a 3-connected quadrangulation of a closed surface F^2, and let $f = v_0v_1v_2v_3$ be a face of G. If the face-contraction of f at $\{v_0, v_2\}$ breaks 3-connectedness of the graph but preserves simplicity, then G has a separating diagonal 3-curve passing through v_0, v_2 and another vertex $x \in V(G) - \{v_0, v_1, v_2, v_3\}$.*

Proof. Let G' be the quadrangulation of F^2 obtained from G by the face-contraction of f at $\{v_0, v_2\}$. Since G' has connectivity 2, G' has a 2-cut. By Lemma 3.2, G' has a separating diagonal 2-curve γ' passing through two vertices of the 2-cut. Clearly, one of the two vertices must be $[v_0v_2]$ of G', which is the image of v_0 and v_2 by the face-contraction of f. (Otherwise, G would not be 3-connected, a contradiction.) Let x be a vertex of G' on γ' other than $[v_0v_2]$. Note that x is not a neighbor of $[v_0v_2]$ in G'. Now apply the vertex-splitting of $[v_0v_2]$ to G' to recover G. Then a diagonal 3-curve for G passing through only v_0, v_2 and x arises from γ' for G'. □

The next lemma plays an important role in a later argument.

Lemma 3.4. *Let G be a 3-connected quadrangulation on a closed surface F^2. If G has a separating 4-cycle $C = x_0x_1x_2x_3$ and a face f of G such that*

(i) one of the diagonal pairs of f is $\{x_i, x_{i+2}\}$ for some i, and

(ii) f has a separating diagonal 3-curve γ intersecting C only at x_i and x_{i+2} transversely,

then there exists a 3-contractible face in G.

Proof. Suppose that G has a separating 4-cycle $C = x_0x_1x_2x_3$ and a face f bounded by ax_1cx_3. Since C is separating, G has two subgraphs G_R and G_L such that $G_R \cup G_L = G$ and $G_R \cap G_L = C$. Suppose that f is contained in G_R. Furthermore, we assume that G_R contains as few vertices of G as possible.

Since C is separating, we have $\partial f \neq C$. By (ii), f has a separating diagonal 3-curve γ through x_1, x_3 and some vertex x. Note that $x \in V(G_L) - V(C)$ by the condition (ii) in the lemma. Now assume that f is not 3-contractible at $\{a, c\}$. Observe that γ (or the 3-cut $\{x_1, x, x_3\}$) separates a from c. Further, G does not have both of edges ax and cx since $\partial f \neq C$. Therefore, there is no path of G of length at most 2 joining a and c other than ax_1c and ax_3c. Moreover, if $\{a, c\} \cap \{x_0, x_2\} = \emptyset$, then f has no separating diagonal 3-curve joining a and c. This contradicts our assumption by Lemma 3.3 and so we may suppose that $a = x_0$ and $c \neq x_2$, and f has a separating diagonal 3-curve, say γ', through a $(= x_0)$ and c.

Since γ' separates x_1 and x_3 and since x_2 is a common neighbors of x_1 and x_3, γ' must pass through x_2, and hence we can find a face f' of G_R one of whose diagonal pair is $\{c, x_2\}$. Let C' be the 4-cycle $x_1x_2x_3c$ of G. Since $\deg(c) \geq 3$, we have $\partial f' \neq C'$, and hence C' is a separating 4-cycle in G_R such that $C' \neq C$. Moreover, γ' and C' cross transversely at x_2 and c. Therefore, C' and f' are a 4-cycle and a face which satisfy the assumption of the lemma, and moreover, C' can cut a strictly smaller graph than G_R from G. Therefore, this contradicts the choice of C. □

Lemma 3.5. *Let G be a 3-connected quadrangulation of a closed surface F^2. If G is $\mathcal{K}_3$-irreducible then G is $\mathcal{K}_3'$-irreducible.*

Proof. Let G be a 3-connected quadrangulation of a closed surface. Assume that G is not $\mathcal{K}_3'$-irreducible. Then, G has either a 3-contractible face or a contractible cube. If G has a 3-contractible face, then G is not $\mathcal{K}_3$-irreducible. Therefore, we suppose that G has no 3-contractible face but has a contractible cube Q with an inner 4-cycle C in the following argument.

Now, we apply a 4-cycle removal of C to G and let G' be the resulting quadrangulation. Let $f' = \partial Q$ be the new face of G' into which C was inserted. If G' is 3-connected, G is not $\mathcal{K}_3$-irreducible by the definition, and we are done. Therefore, we assume that G' is not 3-connected. By Lemma 3.2, there is a diagonal 2-curve γ passing through f' and another face f''; otherwise, G would have a 2-cut, contrary to our assumption. Note that f'' is also a face in G. Now ∂Q and f'' satisfy the conditions of Lemma 3.4, and hence there exists a 3-contractible face in G. However, this contradicts the above assumption. Thus, the lemma follows. $\square$

In the following argument, we denote the set of $\mathcal{K}_3$-irreducible (resp. $\mathcal{K}_3'$-irreducible) quadrangulations of a closed surface F^2 by $\mathcal{K}_3\mathcal{I}(F^2)$ (resp. $\mathcal{K}_3'\mathcal{I}(F^2)$).

Lemma 3.6. *Let G be a 3-connected quadrangulation of F^2. If $G \in \mathcal{K}_3'\mathcal{I}(F^2) \setminus \mathcal{K}_3\mathcal{I}(F^2)$, then G has an attached cube Q such that the graph obtained from G by applying a 4-cycle removal of Q is in $\mathcal{K}_3'\mathcal{I}(F^2)$.*

Proof. Let G be in $\mathcal{K}_3'\mathcal{I}(F^2) \setminus \mathcal{K}_3\mathcal{I}(F^2)$. By the definition, G has an attached cube Q with an inner 4-cycle C which is removable, but is not contractible. We apply a 4-cycle removal of C and let G^- be the resulting quadrangulation. We denote the new face of G^- by f^-, where $f^- = \partial Q$.

First, we confirm that G^- is 3-connected. Otherwise, G^- has a 2-cut and has a separating diagonal 2-curve γ on F^2 by Lemma 3.2. If γ does not pass through f^- then γ would also be a diagonal 2-curve in G, a contradiction. Let f_0 be the other face passed by γ. Here, f_0 and ∂Q in G satisfy the conditions in Lemma 3.4 and there exists a 3-contractible face, contrary to G being $\mathcal{K}_3'$-irreducible.

By way of contradiction, assume that G^- is not in $\mathcal{K}_3'\mathcal{I}(F^2)$. That is, G^- has either (a) a 3-contractible face or (b) a contractible cube. First, we assume (a) and let f be a 3-contractible face in G^-. If $f^- = f$, the attached cube Q in G would be contractible, contrary to G being $\mathcal{K}_3'$-irreducible. Thus, suppose $f^- \neq f$. In this case, let G' be the resulting 3-connected quadrangulation after applying a face-contraction of f in G^-. Since any 4-cycle addition doesn't break the 3-connectedness of a quadrangulation, the graph obtained from G' by a 4-cycle addition to f^- is clearly 3-connected. This means that f is also 3-contractible in G, a contradiction.

Next, suppose (b) and let Q' be such a contractible cube with $\partial Q' = v_0v_1v_2v_3$. If Q' does not contain f^- as one of its five faces, Q' is also contractible in G and G would not be $\mathcal{K}_3'$-irreducible by the similar argument as above. Thus, we assume that Q' contains f^-. Let $C = u_0u_1u_2u_3$ denotes the inner 4-cycle of Q' where $u_iv_i \in E(Q')$ for $i = 0, 1, 2, 3$. We consider the following two cases up to symmetry; (b-1) $f^- = C$ and (b-2) $f^- = v_0u_0u_1v_1$. At first, suppose (b-1). Here, we apply a face-contraction of $f_1 = v_0u_0u_1v_1$ at $\{u_0, v_1\}$ to G. If the above face-contraction breaks the 3-connectedness of G, there exists a face $f_2 = v_1xv_3y$ in the outside of Q' by Lemma 3.3; note that it clearly preserves the simplicity of the graph since $v_1 \neq v_3$. Now, a separating diagonal 3-curve passing through $\{v_1, u_0, v_3\}$ satisfies the conditions of Lemma 3.4 and hence G is not $\mathcal{K}_3'$-irreducible, contrary to our assumption. In fact, an analogous proof is valid for (b-2) if we try to apply a face contraction at $\{v_1, u_2\}$ to G. Therefore the lemma follows. $\square$

Lemma 3.7. *Let G be a 3-connected quadrangulation of a closed surface F^2. If $G \in \mathcal{K}_3'\mathcal{I}(F^2) \setminus \mathcal{K}_3\mathcal{I}(F^2)$, then G can be obtained from $H \in \mathcal{K}_3\mathcal{I}(F^2)$ by applying 4-cycle additions to $F_0 \subseteq F(H)$.*

Proof. Assume that $G \in \mathcal{K}_3'\mathcal{I}(F^2) \setminus \mathcal{K}_3\mathcal{I}(F^2)$. By the previous lemma, there exists a sequence of $\mathcal{K}_3'$-irreducible quadrangulations $G = G_0, G_1, \ldots, G_k$ such that G_{i+1} is obtained from G_i by a single 4-cycle removal of C_i, where $G_k \in \mathcal{K}_3\mathcal{I}(F^2)$. (Since the number of vertices of G is finite, $G_k \in \mathcal{K}_3\mathcal{I}(F^2)$.) Let Q_i denote an attached cube in G_i with an inner 4-cycle C_i.

For a contradiction, we assume that there exists $l \in \{0, \ldots, k-2\}$ such that G_l is obtained from G_{l+1} by a 4-cycle addition which is put on a face not of $F(G_k)$; this l should be maximal. This implies that C_l is put on a face of Q_{l+1} as one of its five faces. Then the same argument as the proof of Lemma 3.6 holds and hence G_l would not be $\mathcal{K}_3'$-irreducible, contrary to our assumption. Thus for each $i \in \{0, \ldots, k-1\}$, G_i is obtained from G_{i+1} by a 4-cycle addition which is put on a face of $F(G_k)$. □

Proof of Theorem 1.4. By Lemma 3.5, we have $\mathcal{K}_3\mathcal{I}(F^2) \subseteq \mathcal{K}_3'\mathcal{I}(F^2)$. Furthermore, by Theorem 1.3 and Lemma 3.7, we obtain (i) and (ii) in the statement. Thus, we have got a conclusion. □

4 Spherical and projective-planar cases

In this section, we discuss the spherical case and the projective-planar case.

Proof of Theorem 1.5. Let G be a $\mathcal{K}_3'$-irreducible quadrangulation of the sphere. We have $\mathcal{K}_3\mathcal{I}(S^2) \subseteq \mathcal{K}_3'\mathcal{I}(S^2)$ by Lemma 3.5, where S^2 stands for the sphere.

If G is $\mathcal{K}_3$-irreducible, then G is isomorphic to a pseudo double wheel by Theorem 1.1. If G is in $\mathcal{K}_3'\mathcal{I}(S^2) \setminus \mathcal{K}_3\mathcal{I}(S^2)$, G can be obtained from a pseudo double wheel W_{2k} ($k \geq 3$) by some 4-cycle additions to faces of W_{2k} by Lemma 3.7. However if $k \geq 4$, G has a 3-contractible face (or a contractible cube), as shown in the first operation in Figure 4. (For example, the entire Figure 4 presents a sequence of a face-contraction and a cube-contraction which deforms W_8 with an attached cube Q into W_6, preserving the 3-connectedness.)

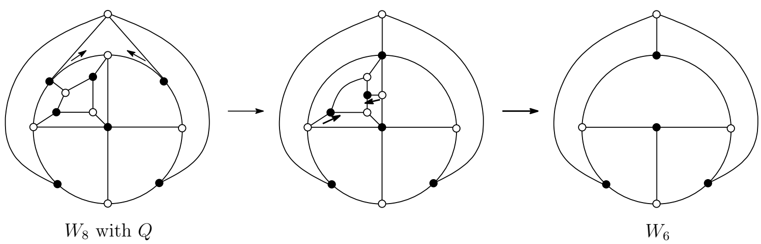


Figure 4: W_8 with an attached cube Q deformed into W_6.

Therefore, we only consider the case of $k = 3$ in the following argument. Assume that G is obtained from W_6 by at least two 4-cycle additions to faces of W_6. Similarly to the above argument, G would have a 3-contractible face (or a contractible cube) , as in Figure 5, contrary to G being $\mathcal{K}_3'$-irreducible; note that it suffices to discuss these two cases, up to symmetry. Therefore, we conclude that G is obtained from W_6 by exactly one 4-cycle addition. This is nothing but a double cube; observe that a double cube has no 3-contractible face and no contractible cube. □

To conclude with, we prove the projective-planar case.

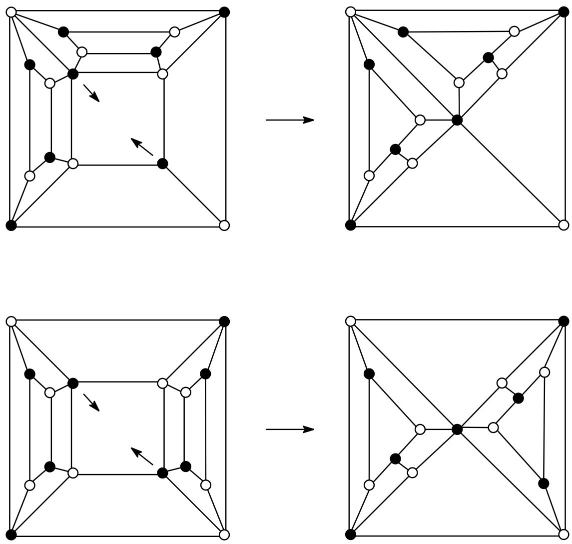


Figure 5: W_6 with two attached cubes can be reduced.

Proof of Theorem 1.6. In this case, we use Möbius wheels $\tilde{W}_k (k \geq 3)$ and Q_P^2 as base graphs by Theorem 1.2.

First we consider the former case. Similarly to the previous proof (and see Figure 6), we consider only a Möbius wheel $\tilde{W}_3$ as a base to which we apply some 4-cycle additions. However, $\tilde{W}_3$ $(= Q_P^1)$ is isomorphic to the complete graph with four vertices, and hence it is irreducible. This fact implies that every G obtained from $\tilde{W}_3$ by applying at most three 4-cycle additions is $\mathcal{K}_3'$-irreducible since any face-contraction and any cube-contraction to G destroys the simplicity of the graph, or results in a vertex of degree 2. From this case, we obtain exactly three quadrangulations in $\mathcal{K}_3'\mathcal{I}(P^2) \setminus \mathcal{K}_3\mathcal{I}(P^2)$, up to homeomorphism, where P^2 stands for the projective plane.

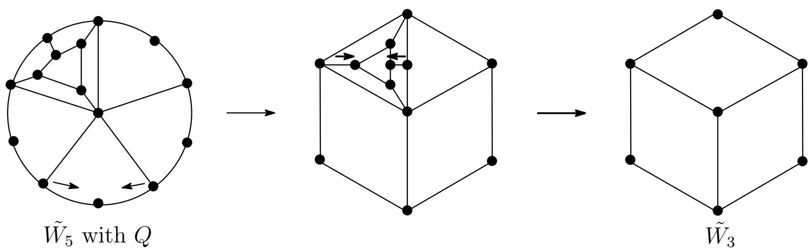


Figure 6: $\tilde{W}_5$ with an attached cube Q deformed into $\tilde{W}_3$.

Similarly, as the latter case, we obtain the other ten quadrangulations in $\mathcal{K}_3'\mathcal{I}(P^2) \setminus \mathcal{K}_3\mathcal{I}(P^2)$ from Q_P^2; consider all the way to put attached cubes into faces of Q_P^2, up to symmetry. As a result, we have $|\mathcal{K}_3'\mathcal{I}(P^2) \setminus \mathcal{K}_3\mathcal{I}(P^2)| = 13$ in total. $\square$

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One-point extensions in n_3 configurations

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Abstract

Given an n_3 configuration, a 1-point extension is a technique that constructs an $(n+1)_3$ configuration from it. It is proved that all $(n+1)_3$ configurations can be constructed from an n_3 configuration using a 1-point extension, except for the Fano, Pappus, and Desargues configurations, and a family of Fano-type configurations. A 3-point extension is also described. A 3-point extension of the Fano configuration produces the Desargues and anti-Pappian configurations.

The significance of the 1-point extension is that it can frequently be used to construct real and/or rational coordinatizations in the plane of an $(n+1)_3$ configuration, whenever it is geometric, and the corresponding n_3 configuration is also geometric.

Keywords: Fano configuration, Pappus, Desargues, $(n, 3)$*-configuration.*

Math. Subj. Class.: 51E20, 51E30

1 Projective Configurations

A *projective configuration* consists of a set Σ of points and lines, and an incidence relation Π, such that two lines intersect in at most one point. We denote this by (Σ, Π). For example, a triangle with points A, B, C and lines a, b, c can be represented by the pair $(\{A, B, C, a, b, c\}, \{Ab, Ac, Ba, Bc, Ca, Cb\})$. A configuration (Σ, Π) can also be viewed as a bipartite incidence graph of points versus lines. We will always assume that the incidence graph of a configuration is connected. Excellent references on configurations are the recent books by Grünbaum [7], and by Pisanski and Servatius [11].

An n_3-configuration is a projective configuration with n points and n lines such that every line is incident on 3 points, and every point is incident on 3 lines. There is a unique 7_3-configuration, the Fano configuration, and a unique 8_3-configuration, the Möbius-Kantor

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configuration. In 1887, Martinetti [10] presented a method to construct the $(n+1)_3$ configurations from the n_3 configurations. This is described in [7, 6]. Boben [1, 2] has analysed and extended Martinetti's construction significantly. Important related work has also been done by Carstens, Dinski and Steffen [4]. See also [12]. A recent paper [13] by Stokes studies extensions of configurations in a very general setting. The 1-point extension presented here can be related to Stokes's construction, but does not follow directly from it.

An n_3 configuration which can be represented by a collection of points and straight lines in the real or rational plane, such that all incidences are respected, and no two points or two lines coincide, and no unwanted incidences occur, is termed a *geometric* n_3 configuration. In order to show that an n_3 configuration is geometric, the usual method is to assign suitable homogeneous coordinates to its points and lines. We call this a *coordinatization* of the configuration. Some n_3 configurations are not geometric configurations, although it is currently an unsolved problem to determine which n_3 configurations are geometric.

The purpose of this paper is to present a theorem, the 1-point extension theorem, which describes another method to construct an $(n+1)_3$-configuration from an n_3-configuration; and to characterize which configurations can be obtained in this way. The significance of this construction is that if the n_3 configuration is geometric, with a given coordinatization, then there is usually a simple method to extend the coordinatization to the $(n+1)_3$ configuration, that is, the $(n+1)_3$ configuration will also be geometeric. This is too long to include here, it will be the subject of another paper, currently in preparation [8].

In particular the following theorem is proved.

Theorem 1.1. *Let* (Σ, Π) *be an* $(n+1)_3$*-configuration. Then* (Σ, Π) *can be constructed by a 1-point extension from an* n_3*-configuration if and only if* (Σ, Π) *is not one of the following configurations:*

a) the Fano configuration,

b) the Pappus configuration,

c) the Desargues configuration,

d) a Fano-type configuration (to be described).

We begin with the idea of a 1-point extension in an n_3-configuration.

Theorem 1.2. (1-Point Extension) *Let* (Σ, Π) *be an* n_3*-configuration. Let* a_1, a_2, a_3 *be 3 distinct points in* Σ*, and let* ℓ_1, ℓ_2, ℓ_3 *be 3 distinct lines in* Σ *such that* $a_1 = \ell_1 \cap \ell_2$*,* $a_2 = \ell_2 \cap \ell_3$ *and* $a_3 \in \ell_3$*, where* $a_3 \notin \ell_1$*. We can represent this in tabular form as*

$$\begin{array}{lcccc} (\Sigma, \Pi) & \ell_1 & \ell_2 & \ell_3 & \cdots \\ & a_1 & a_1 & a_2 & \cdots \\ & \cdot & a_2 & a_3 & \cdots \\ & \cdot & \cdot & \cdot & \cdots \end{array}$$

where the dots indicate other points of the configuration. Let ℓ' *be the third line containing* a_1*. Suppose further that if* $\ell' \cap \ell_3 \neq \emptyset$*, then* $\ell' \cap \ell_3 = a_3$*. Construct a new configuration* (Σ', Π') *as follows.* $\Sigma' = \Sigma \cup \{a_0, \ell_0\}$ *where* a_0 *is a new point and* ℓ_0 *is a new line.* $\Pi' = \Pi - \{a_1\ell_1, a_2\ell_2, a_3\ell_3\} \cup \{a_1\ell_3, a_2\ell_0, a_3\ell_0, a_0\ell_0, a_0\ell_1, a_0\ell_2\}$*. We can represent this in tabular form as*

(Σ',Π')	ℓ_0	ℓ_1	ℓ_2	ℓ_3	$\cdots$
	a_2	a_0	a_1	a_1	$\cdots$
	a_3	$\cdot$	a_0	a_2	$\cdots$
	a_0	$\cdot$	$\cdot$	$\cdot$	$\cdots$

Here the dots represent exactly the same points as in the previous table. Then (Σ',Π') is an $(n+1)_3$-configuration.

Proof. The only incidences in which (Σ',Π') and (Σ,Π) differ are those involving $\ell_0, \ell_1, \ell_2, \ell_3$. It is easy to verify from the tables that each of a_1, a_2 and a_3 occurs in exactly 3 lines in both (Σ',Π') and (Σ,Π), and that a_0 also occurs in exactly 3 lines. We must still verify that any two lines of (Σ',Π') intersect in at most one point. Notice that ℓ_0 intersects ℓ_1 and ℓ_2 in exactly one point, since $a_3 \notin \ell_1, \ell_2$. Also, ℓ_0 intersects ℓ_3 in exactly one point. If $\ell \neq \ell_1, \ell_2, \ell_3$ is any line of (Σ,Π) intersecting ℓ_1, then in (Σ',Π'), it intersects ℓ_1 in either 0 or 1 point. If ℓ intersects ℓ_2 in (Σ,Π), then in (Σ',Π'), it intersects ℓ_2 in either 0 or 1 point. If $\ell = \ell'$, the third line of (Σ,Π) containing a_1, then in (Σ',Π'), ℓ intersects ℓ_3 in only a_1, because of the condition concerning ℓ'. If $\ell \neq \ell'$ and ℓ intersects ℓ_3 in (Σ,Π), then then since $a_1 \notin \ell_3$ in (Σ,Π), it follows that ℓ intersects ℓ_3 in 0 or 1 point in (Σ',Π'). Finally, if ℓ is any line of (Σ,Π) not intersecting ℓ_1, ℓ_2, then it does not intersect ℓ_1, ℓ_2 in (Σ',Π'). If ℓ does not intersect ℓ_3 in (Σ,Π), it may intersect ℓ_3 in a_1 in (Σ',Π'). This completes the proof of the theorem.

□

Example 1.3. The Fano configuration can be represented by the following table.

Fano	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7
	1	2	3	4	5	6	7
	2	3	4	5	6	7	1
	4	5	6	7	1	2	3

Choose ℓ_1, ℓ_2, ℓ_3 as indicated, and choose $a_1 = 2$, $a_2 = 3$, $a_3 = 6$, and let $a_0 = 8$. Notice that the third line containing a_1 is $\ell' = \ell_6$, which intersects ℓ_3 in $a_3 = 6$. Then by Theorem 1.2, the following table represents an 8_3-configuration, which is known to be unique.

8_3-config	ℓ_0	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7
	3	1	2	2	4	5	6	7
	6	4	5	3	5	6	7	1
	8	8	8	4	7	1	2	3

The 8_3-configuration can be viewed as a double cover of the cube [9]. It is possible to apply a 1-point extension to this configuration in two possible ways, resulting in two distinct 9_3-configurations. The third 9_3-configuration, known as the Pappus configuration, cannot be obtained in this way.

The 1-point extension theorem can be illustrated by the diagram of Figure 1. In (Σ,Π), we have a substructure consisting of 3 points a_1, a_2, a_3, and 3 lines, ℓ_1, ℓ_2, ℓ_3, sequentially incident, forming a self-dual substructure contained in the n_3-configuration. After the extension, we find that (Σ',Π') contains a triangle with vertices a_1, a_2, a_0 and sides ℓ_2, ℓ_3, ℓ_0, where the third point on ℓ_0 is a_3, and the third line through a_0 is ℓ_1. This is again a self-dual substructure in the configuration.

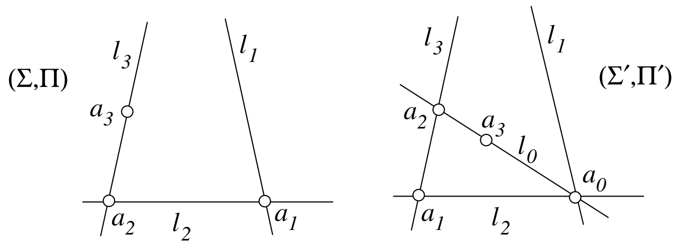

Figure 1: A 1-point extension with 3 points

Corollary 1.4. *In (Σ', Π'), the third line through a_1 does not intersect ℓ_1; the third point on ℓ_3 is not collinear with a_3; and the third line through a_2 does not intersect ℓ_2.*

Proof. If there were a line ℓ in (Σ', Π') through a_1 which intersected ℓ_1 in a point u, then in (Σ, Π), ℓ would intersect ℓ_1 in u and a_1, which is impossible. If there were a point x in (Σ', Π') on ℓ_3 collinear with a_3, then the line ℓ containing a_3 and x would also be a line in (Σ, Π), where it would intersect ℓ_3 in two points. Finally, if there were a line ℓ in (Σ', Π') through a_2 which intersected ℓ_2 in a point u, then in (Σ, Π), ℓ would intersect ℓ_2 in a_2 and u, which is impossible. □

The purpose of this paper is to characterize the configurations that can be obtained using 1-point extensions. In practice, the 1-point extensions are very easy to find and apply, and can easily be done by computer. However, the characterization of which configurations can be obtained by them is very long and tedious. We shall refer to the Fano, Pappus, and Desargues configurations, illustrated in Figure 1.1.

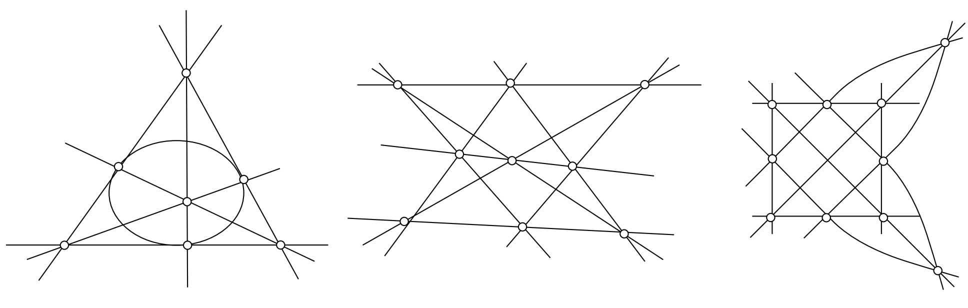

Figure 2: The Fano, Pappus, and Desargues configurations

The conditions of Corollary 1.4 will be used frequently in the characterization. We state them here. We are concerned with an ordered triangle, denoted $\Delta(i, j, k)$, where i, j and k are the first, second, and third vertices, respectively, of the triangle. The line containing i and j is denoted ℓ_{ij}, etc.

Definition 1.5. Let (Σ, Π) be a configuration containing an ordered triangle $\Delta(i, j, k)$. We define the following 3 conditions:

A) The third line through k intersects ℓ_{ij};

B) The third line through i intersects the third line through j;

C) The third point on ℓ_{ik} is collinear with the third point on ℓ_{jk}.

The definition is illustrated in Figure 3.

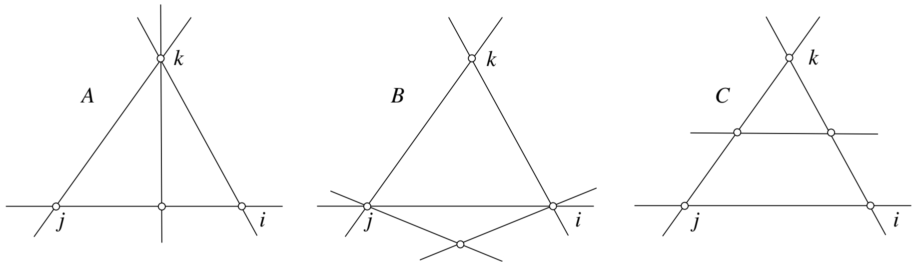

Figure 3: Conditions A, B and C for triangle $\Delta(i, j, k)$

Theorem 1.6. *Let (Σ', Π') be an $(n + 1)_3$-configuration containing a triangle Δ. If conditions A, B and C do not apply to some ordering of the triangle, then (Σ', Π') can be derived from an n_3-configuration by a 1-point extension.*

Proof. Let the ordered triangle to which conditions A, B and C do not apply be $\Delta(a_0, a_1, a_2)$, and let the sides of the triangle be ℓ_0, ℓ_2, ℓ_3, where $a_0 = \ell_0 \cap \ell_2$, $a_1 = \ell_2 \cap \ell_3$, $a_2 = \ell_3 \cap \ell_0$. Let a_3 be the third point on ℓ_0, and let ℓ_1 be the third line through a_0. Observe that $a_3 \notin \ell_1$. These incidences are characterized by the following table.

(Σ, Π)	ℓ_0	ℓ_1	ℓ_2	ℓ_3
	a_2	a_0	a_1	a_1
	a_3	·	a_0	a_2
	a_0	·	·	·

We can then delete a_0 and ℓ_0, and change the incidences to the following.

(Σ', Π')	ℓ_1	ℓ_2	ℓ_3
	a_1	a_1	a_2
	·	a_2	a_3
	·	·	·

Call the result (Σ', Π'). If ℓ is the third line through a_2 in (Σ, Π), then since condition A does not apply, we know that in (Σ', Π'), ℓ and ℓ_2 intersect in just one point. If ℓ is the third line through a_1 in (Σ, Π), then since condition B does not apply, we know that in (Σ', Π'), ℓ and ℓ_1 intersect in just one point, a_1. Since $\ell \cap \ell_3 = a_1$ in (Σ, Π), it follows that in (Σ', Π'), if ℓ and ℓ_3 intersect, they intersect in a_3.

If ℓ is any line other than ℓ_0 through a_3 in (Σ, Π), then since condition C does not apply, we know that in (Σ', Π'), ℓ and ℓ_3 intersect in just one point. The result is an n_3-configuration to which Theorem 1.2 applies.

□

Given an ordered triangle $\Delta(i,j,k)$, the dual is an ordered triangle whose sides are lines which can be denoted i', j', k'. The dual of condition A is that the third point on k' is collinear with $i' \cap j'$. But this is just condition A again applied to the triangle $\Delta(i' \cap k', j' \cap k', i' \cap j')$. So condition A is self-dual. The dual of condition B is that the third point on i' is collinear with the third point on j'. This is just condition C applied to the triangle $\Delta(i' \cap k', j' \cap k', i' \cap j')$. So B and C are dual conditions.

Theorem 1.6 is the main tool which we will use to characterize the extensions. We will find all configurations such that at least one of conditions A, B, and C apply to every ordering of every triangle. We will also need longer cycles than triangles.

2 The General Extension Theorem

Before beginning the characterization of the n_3-configurations that can be obtained by 1-point extensions, we generalize Theorem 1.2 to m points and m lines, sequentially incident.

Theorem 2.1. (General 1-Point Extension) *Let (Σ, Π) be an n_3-configuration. Let $a_1, a_2, \ldots, a_m$ be m distinct points in Σ, where $3 \leq m \leq n$, and let $\ell_1, \ell_2, \ldots, \ell_m$ be m distinct lines in Σ such that $a_1 = \ell_1 \cap \ell_2$, $a_2 = \ell_2 \cap \ell_3$, …, $a_{m-1} = \ell_{m-1} \cap \ell_m$, and $a_m \in \ell_m$. Suppose that $a_{m-1}, a_m \notin \ell_1, \ell_2$, and that $a_i \notin \ell_{i+3}$, where $i = 1, 2, \ldots, m-3$. We can represent this in tabular form as*

$$\begin{array}{lcccccc} (\Sigma,\Pi) & \ell_1 & \ell_2 & \ell_3 & \ldots & \ell_{m-1} & \ell_m \\ & a_1 & a_1 & a_2 & \ldots & a_{m-2} & a_{m-1} \\ & \cdot & a_2 & a_3 & \ldots & a_{m-1} & a_m \\ & \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \end{array}$$

where the dots indicate other points of the configuration. Let ℓ_i' be the third line containing a_i, where $1 \leq i \leq m-2$. Suppose further that if $\ell_i' \cap \ell_{i+2} \neq \emptyset$, then $\ell_i' \cap \ell_{i+2} = a_{i+2}$. Construct a new configuration (Σ', Π') as follows. $\Sigma' = \Sigma \cup \{a_0, \ell_0\}$ where a_0 is a new point and ℓ_0 is a new line. $\Pi' = \Pi - \{a_1\ell_1, a_2\ell_2, \ldots, a_m\ell_m\} \cup \{a_1\ell_3, a_2\ell_4, \ldots, a_{m-2}\ell_m, a_{m-1}\ell_0, a_m\ell_0, a_0\ell_0, a_0\ell_1, a_0\ell_2\}$. We can represent this in tabular form as

$$\begin{array}{lccccccc} (\Sigma',\Pi') & \ell_0 & \ell_1 & \ell_2 & \ell_3 & \ldots & \ell_{m-1} & \ell_m \\ & a_{m-1} & a_0 & a_0 & a_1 & \ldots & a_{m-3} & a_{m-2} \\ & a_m & \cdot & a_1 & a_2 & \ldots & a_{m-2} & a_{m-1} \\ & a_0 & \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \end{array}$$

Here the dots represent exactly the same points as in the previous table. Then (Σ', Π') is an $(n+1)_3$-configuration.

Proof. The only incidences in which (Σ', Π') and (Σ, Π) differ are those involving $\ell_0, \ell_1, \ell_2, \ldots, \ell_m$. It is easy to verify from the tables that each of $a_1, a_2, \ldots, a_m$ occurs in exactly 3 lines in both (Σ', Π') and (Σ, Π), and that a_0 also occurs in exactly 3 lines. We must still verify that any two lines of (Σ', Π') intersect in at most one point. Notice that ℓ_0 intersects ℓ_1 and ℓ_2 in exactly one point, since $a_{m-1}, a_m \notin \ell_1, \ell_2$. It does not intersect $\ell_3, \ldots, \ell_{m-1}$, and it intersects ℓ_m in exactly one point.

Let $\ell \neq \ell_1, \ell_2, \ldots, \ell_m$ be a line of (Σ, Π). If ℓ intersects ℓ_1 in (Σ, Π), then in (Σ', Π'), it intersects ℓ_1 in either 0 or 1 point. If ℓ intersects ℓ_2 in (Σ, Π), then in (Σ', Π'), it intersects

ℓ_2 in either 0 or 1 point. Suppose that ℓ intersects ℓ_3 in (Σ, Π). If $\ell = \ell_1'$, then $\ell \cap \ell_3 = a_3$ in (Σ, Π) according to the condition of the theorem concerning ℓ_i'. It follows that $\ell \cap \ell_3 = a_1$ in (Σ', Π'). If $\ell \neq \ell_1'$, then ℓ intersects ℓ_3 in either 0 or 1 point in (Σ', Π'). An identical argument holds if ℓ intersects one of $\ell_4, \ldots, \ell_m$ in (Σ, Π).

Suppose that ℓ does not intersect ℓ_1 in (Σ, Π). Then it also does not intersect ℓ_1 in (Σ', Π'). Similarly, if ℓ does not intersect ℓ_2 in (Σ, Π), then it also does not intersect ℓ_2 in (Σ', Π'). Suppose that ℓ does not intersect ℓ_3 in (Σ, Π). Then in (Σ', Π'), it may intersect ℓ_3 only in a_1. A similar argument holds if ℓ does not intersect $\ell_4, \ldots, \ell_m$.

Finally, let ℓ_i and ℓ_j, where $1 \leq i < j \leq m$, be two lines of (Σ, Π). If $j = i + 1$, then ℓ_i and ℓ_j intersect in one point in both (Σ, Π) and (Σ', Π'). Suppose that $j = i + 2$. If $\ell_i \cap \ell_j = \emptyset$ in (Σ, Π), then it is also $\emptyset$ in (Σ', Π'). Now $\ell_i \cap \ell_j \neq a_{i-1}$ in (Σ, Π) (when $i > 1$), because of the hypothesis that $a_k \notin \ell_{k+3}$. Also, $\ell_i \cap \ell_j \neq a_i$, because ℓ_{i+1} contains a_i and a_{i+1}. It follows that $|\ell_i \cap \ell_j|$ is the same in (Σ, Π) and (Σ', Π') when $j = i + 2$. Suppose now that $j \geq i + 3$. It is easy to see that $|\ell_i \cap \ell_j| \leq 1$ in (Σ', Π'). This completes the proof of the theorem. □

Theorem 2.1 is illustrated in Figure 4, with $m = 4$. This general form of Theorem 2.1 is stated separately from Theorem 1.2, because the form with $m = 3$ is simpler, and because we shall mostly only require Theorems 1.2 and 1.6 when characterizing extensions.

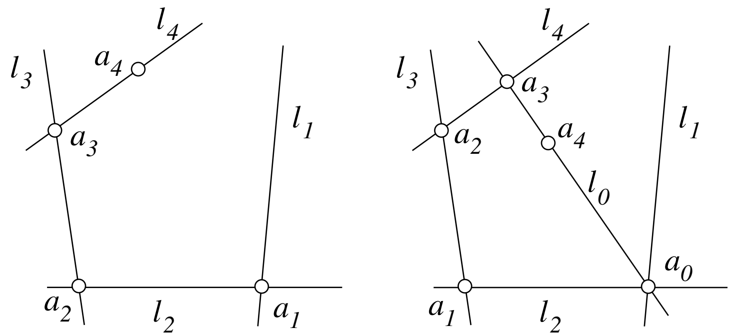

Figure 4: A 1-point extension with 4 points

An ordered *cycle* in a configuration is a sequence of distinct points and lines which are cyclicly incident, for example $C = (a_1, \ell_1, a_2, \ell_2, \ldots, a_m, \ell_m)$, where $a_i = \ell_{i-1} \cap \ell_i$ for $i = 2, 3, \ldots, m$, and $a_1 = \ell_m \cap \ell_1$. Here $m \geq 3$. Each point of C is incident on two lines of C, and vice versa.

Corollary 2.2. *Let* (Σ, Π) *and* (Σ', Π') *be as in Theorem 2.1, so that* $C = (a_0, \ell_2, a_1, \ell_3, \ldots, a_{m-2}, \ell_m, a_{m-1}, \ell_0)$ *is an ordered cycle in* (Σ', Π')*. Then in* (Σ', Π')*:*

i) the third points of ℓ_m *and* ℓ_0 *are not collinear;*

ii) the third point on ℓ_i *is not contained in the third line through* a_i*, for* $i = 2, \ldots, m-1$*;*

iii) the third lines through a_0 *and* a_1 *do not intersect.*

Proof. The third point of ℓ_0 is a_m. If there were a line ℓ in (Σ', Π') containing a_m and the third point of ℓ_m, then in (Σ, Π), ℓ and ℓ_m would intersect in two points, which is impossible.

Let ℓ be the third line through a_i in (Σ', Π'), for some $i = 2, \ldots, m-1$, and let u be the third point on ℓ_i. Suppose that $u \in \ell$. In (Σ', Π'), a_i is contained in ℓ_{i+1} and ℓ_{i+2}, but in (Σ, Π), a_i is contained in ℓ_i and ℓ_{i+1}. We then find that in (Σ, Π), $\ell \cap \ell_i = \{u, a_i\}$, which is impossible.

The third line through a_0 is ℓ_1. Let ℓ be the third line through a_1. If $\ell \cap \ell_1 = u$ in (Σ', Π'), then in (Σ, Π), $\ell \cap \ell_1 = \{u, a_1\}$, which is impossible. □

Observe that a triangle is a set of three distinct points and lines that are cyclically incident. Similarly, we define a *quadrangle* to be a set of four distinct points and lines that are cyclically incident. We will also need conditions similar to A, B, C for quadrangles. An ordered quadrangle with vertices i, j, k, m is denoted $\square(i, j, k, m)$. In analogy with Definition 1.5 and Corollary 2.2, we make the following definition for a quadrangle.

Definition 2.3. Let (Σ, Π) be a configuration containing an ordered quadrangle $\square(i, j, k, m)$. We define the following 4 conditions:

D) The third point on ℓ_{im} is collinear with the third point on ℓ_{km};

E) The third line through m intersects ℓ_{jk};

F) The third line through k intersects ℓ_{ij};

G) The third line through j intersects the third line through i.

These conditions are illustrated in Figure 5.

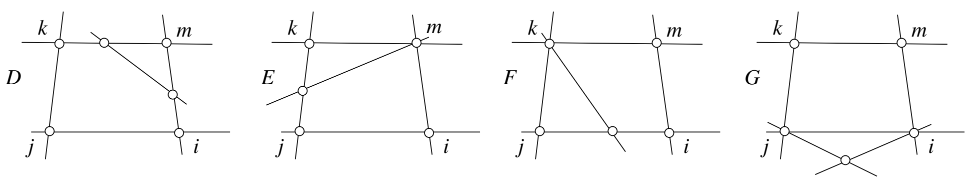

Figure 5: Conditions D, E, F, G for quadrangle $\square(i, j, k, m)$

The analog of Theorem 1.6 for general 1-point extensions is the following.

Theorem 2.4. *Let (Σ', Π') be an $(n+1)_3$-configuration containing an ordered cycle $C = (a_0, \ell_2, a_1, \ell_3, a_2, \ell_4, \ldots, a_{m-2}, \ell_m, a_{m-1}, \ell_0)$, where $m \geq 4$; $a_0, a_1, \ldots, a_{m-1}$ are distinct points; and $\ell_0, \ell_2, \ell_3, \ldots, \ell_{m-1}$ are distinct lines. Let ℓ_1 denote the third line containing a_0 and let a_m denote the third point on ℓ_0. Suppose that ℓ_1 is distinct from $\ell_0, \ell_2, \ell_3, \ldots, \ell_{m-1}$ and that $a_2 \notin \ell_1$. Let ℓ_i' denote the third line containing a_i, for $i = 1, 2, \ldots, m-1$. Suppose that ℓ_i' does not not contain the third point of ℓ_i, for $i = 2, \ldots, m-1$; that $\ell_1' \cap \ell_1 = \emptyset$; and that a_m is not collinear with the third point of ℓ_m. Then (Σ', Π') can be derived from an n_3-configuration by a 1-point extension.*

Proof. The incidences of the ordered cycle can be represented by the following table.

$$\begin{array}{lllllll}
(\Sigma,\Pi) & \ell_0 & \ell_1 & \ell_2 & \ell_3 & \ldots & \ell_{m-1} & \ell_m \\
 & a_{m-1} & a_0 & a_0 & a_1 & \ldots & a_{m-3} & a_{m-2} \\
 & a_m & \cdot & a_1 & a_2 & \ldots & a_{m-2} & a_{m-1} \\
 & a_0 & \cdot & \cdot & \cdot & \ldots & \cdot & \cdot
\end{array}$$

We can then delete a_0 and ℓ_0, and change the incidences to the following.

$$\begin{array}{lllllll}
(\Sigma',\Pi') & \ell_1 & \ell_2 & \ell_3 & \ldots & \ell_{m-1} & \ell_m \\
 & a_1 & a_1 & a_2 & \ldots & a_{m-2} & a_{m-1} \\
 & \cdot & a_2 & a_3 & \ldots & a_{m-1} & a_m \\
 & \cdot & \cdot & \cdot & \ldots & \cdot & \cdot
\end{array}$$

Call the result (Σ,Π). It is clear that each point of (Σ,Π) is contained in exactly three lines. We have to show that any two lines intersect in at most one point in (Σ,Π), and that $\ell_1,\ell_2,\ell_3,\ldots,\ell_m$ are distinct lines in (Σ,Π). Any two of $\ell_1,\ell_2,\ldots,\ell_m$ intersect in at most one point because we began with an ordered cycle of distinct points, and because $a_2 \notin \ell_1$. Let ℓ be any line not in this set. Suppose that ℓ intersects ℓ_i in two points, for some $i = 2,\ldots,m-1$. Now ℓ_i contains a_{i-1}, a_i and a third point z. If ℓ contained a_i, then $\ell = \ell'_i$, which does not intersect ℓ_i in (Σ',Π'), by assumption. Therefore $a_i \notin \ell$. Otherwise ℓ must contain a_{i-1} and z. But these points are in ℓ_i in (Σ',Π'), and ℓ is unchanged. It follows that ℓ intersects $\ell_2,\ldots,\ell_{m-1}$ in at most one point each.

Suppose that ℓ intersects ℓ_1 in two points in (Σ,Π). Now ℓ_1 contains a_1 and two other points u, v. As u and v are both on ℓ_1 in (Σ',Π'), it follows that ℓ does not contain both u and v. Therefore $\ell = \ell'_1$. But by assumption, $\ell'_1 \cap \ell_1 = \emptyset$ in (Σ',Π').

Suppose that ℓ intersects ℓ_m in two points in (Σ,Π). The two points cannot be a_{m-1}, a_m, because these points occur on ℓ_0 in (Σ',Π'). They cannot be a_{m-1} and a third point w, because these points occur on ℓ_m in (Σ',Π'). And they cannot be a_m and the third point w, because by assumption, a_m is not collinear with the third point of ℓ_m in (Σ',Π'). We conclude that (Σ,Π) is an n_3-configuration to which the conditions of Theorem 2.1 apply. □

Corollary 2.5. *Let (Σ',Π') be an $(n+1)_3$-configuration containing a quadrangle $\square(i,j,k,m)$. If conditions D, E, F and G do not apply to some ordering of the quadrangle, and if the third line through i does not contain k, then (Σ',Π') can be derived from an n_3-configuration by a 1-point extension.*

Proof. The conditions D, E, F, G, and $a_2 = k \notin \ell_1$ are the conditions of Theorem 2.4 applied to an ordered quadrangle. □

Theorem 2.6. *Let (Σ',Π') be an $(n+1)_3$-configuration. If (Σ',Π') does not contain a triangle, then it can be derived by a 1-point extension from an n_3-configuration.*

Proof. Choose a cycle of smallest possible length in (Σ',Π'). Denote the cycle by

$$(a_0,\ell_2,a_1,\ell_3,a_2,\ell_4,\ldots,a_{m-2},\ell_m,a_{m-1},\ell_0),$$

where $m \geq 4$. Let ℓ_1 be the third line containing a_0, and let a_m be the third point on ℓ_0. This can be denoted in tabular from by

(Σ, Π)	ℓ_0	ℓ_1	ℓ_2	ℓ_3	$\dots$	ℓ_{m-1}	ℓ_m
	a_{m-1}	a_0	a_0	a_1	$\dots$	a_{m-3}	a_{m-2}
	a_m	$\cdot$	a_1	a_2	$\dots$	a_{m-2}	a_{m-1}
	a_0	$\cdot$	$\cdot$	$\cdot$	$\dots$	$\cdot$	$\cdot$

Let ℓ_i' denote the third line containing a_i, where $i = 1, 2, \dots, m-1$. If ℓ_i' were to intersect ℓ_i in a point z, where $i = 2, \dots, m-1$, this would create a triangle $\Delta(a_{i-1}, a_i, z)$. If ℓ_1' were to intersect ℓ_1 in a point u, this would create a triangle $\Delta(a_0, a_1, u)$. If a_m were collinear with the third point w of ℓ_m, this would create a triangle $\Delta(a_{m-1}, a_m, w)$. If ℓ_1 contained a_2, this would create a triangle $\Delta(a_0, a_1, a_2)$. It follows that the conditions of Theorem 2.4 apply, so that (Σ', Π') can be derived by a 1-point extension from an n_3-configuration. $\square$

3 Fano-Type Configurations

Let F denote the Fano configuration, the unique 7_3 configuration. We will use three sub-configurations to build a family of n_3 configurations which cannot be obtained by 1-point extensions.

Definition 3.1. Denote by F' the unique configuration obtained from F by removing a single incidence. Denote by F_ℓ the unique configuration obtained from F by removing a line. Denote by F_p the unique configuration obtained from F by removing a point. Note that F_ℓ and F_p are dual configurations.

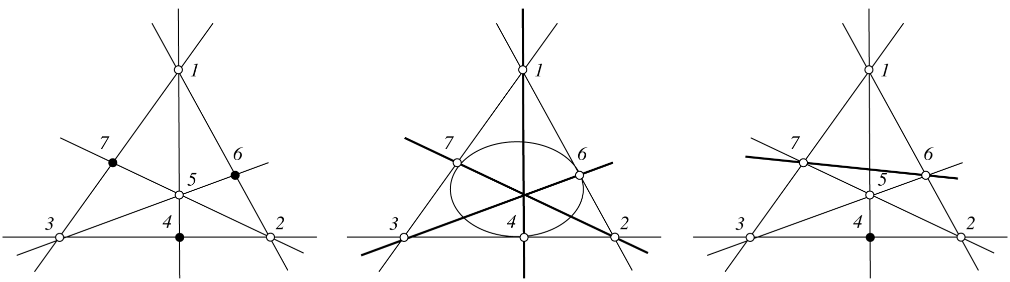

Figure 6: The configurations F_ℓ, F_p and F'

The configurations F_ℓ, F_p and F' are not n_3-configurations. They can be used as building blocks of n_3 configurations, which we call *Fano-type* configurations. F' has one point on only two lines, and one line containing only two points. F_p has three lines containing only two points. Every point is in three lines. F_ℓ has three points in only two lines. Every line contains three points. These are illustrated schematically in Figure 7, where the points missing a line are indicated as black circles, and the lines missing a point are indicated as lines.

These sub-configurations can be used as modules, which can be connected together like vertices of a graph, to create graphs representing n_3 configurations. For example, two or more copies of F' can be connected into a cycle or path of arbitrary length. If only F_ℓ and F_p are used, the resulting structure is a bipartite graph.

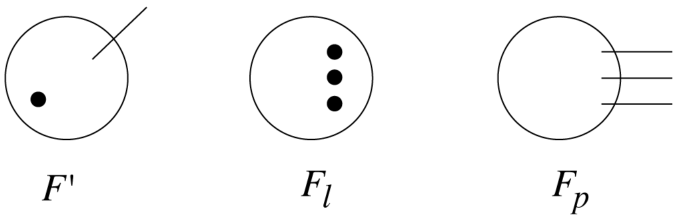

Figure 7: F', F_ℓ and F_p schematically

Theorem 3.2. *Let G be a multigraph which is isomorphic to either a cycle of length ≥ 2, or a subdivision of a 3-regular bipartite multigraph, with bipartition (X, Y). Replace each vertex of X by a configuration F_p, replace each vertex of Y by a configuration F_ℓ, and replace each vertex of degree two by a configuration F'. The result is an n_3 configuration which can not be obtained by a 1-point extension.*

Proof. Refer to Figure 8, showing a cycle of length four, and a configuration constructed from the unique 3-regular bipartite multigraph on four vertices.

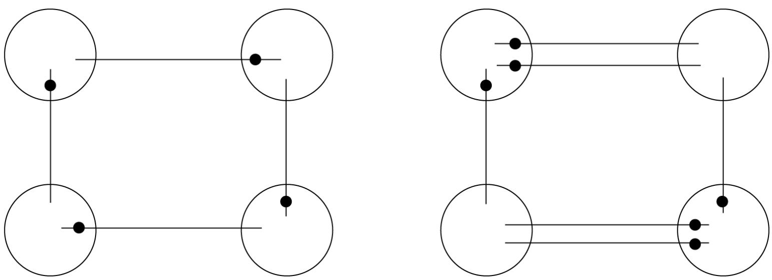

Figure 8: Configurations constructed from F', F_ℓ and F_p

We must show that the n_3 configurations constructed like this cannot be obtained by a 1-point extension. Observe first that the Fano configuration F is a projective plane, so that every two points are contained in a line, and every two lines intersect in a point. Consequently, every triangle contained in F', F_ℓ or F_p has an ordering which satisfies one of conditions A, B or C. By Corollary 1.4, a Fano-type configuration cannot be obtained by a triangular 1-point extension (Theorem 1.2). Suppose that it can be obtained by a general 1-point extension (Theorem 2.1). By Corollary 2.2, there must be an ordered cycle C of length ≥ 4 satisfying certain conditions. Let $C = (a_0, \ell_2, a_1, \ell_3, \ldots, a_{m-2}, \ell_m, a_{m-1}, \ell_0)$ be as in Corollary 2.2, and let ℓ_i' denote the third line containing a_i, where $i = 1, 2, \ldots, m-1$. Let ℓ_1 denote the third line containing a_0, and let a_m denote the third point on ℓ_0. If C were contained within an F', F_ℓ or F_p, then C would have length 4, because any 5 points of F necessarily contain three collinear points. But in F', F_ℓ or F_p, every ordered quadrangle satisfies at least one of conditions D, E, F, G, since the Fano configuration is a projective plane.

It follows that C is not contained within an F', F_ℓ or F_p. Consider the portion of C contained within some F', F_ℓ or F_p. It is a sequence of sequentially incident points and lines. Suppose first that it is contained within an F'. Referring to Figure 6 we see that the shortest possible portion of C contained within an F' is $(a_i, \ell_{i+2}, a_{i+1}, \ell_{i+3}, a_{i+2}, \ell_{i+4})$, for some $i = 0, 1, \ldots, m-1$ where subscripts are reduced modulo m. If $a_{i+2} \neq a_0, a_1$, then ℓ'_{i+2} contains the third point of ℓ_{i+2}, which is in F'. If $a_{i+2} = a_0$, then $a_{i+1} = a_{m-1}$ and $\ell_{i+2} = \ell_m$, so that a_m is collinear in F' with the third point of ℓ_m. If $a_{i+2} = a_1$, then $a_{i+1} = a_0$, so that ℓ_1 and ℓ'_1 are in F' and $\ell'_1 \cap \ell_1 \neq \emptyset$. Thus, the conditions of Corollary 2.2 are never satisfied if a portion of C is contained within an F'.

Suppose next that a portion of C is contained within an F_ℓ. Referring to Figure 6 we see that the shortest possible portion of C contained within an F_ℓ is $(a_i, \ell_{i+2}, a_{i+1}, \ell_{i+3}, a_{i+2})$, for some $i = 0, 1, \ldots, m-1$ where subscripts are reduced modulo m. If $a_{i+2} \neq a_0, a_1$, then ℓ'_{i+2} contains the third point of ℓ_{i+2}, which is in F_ℓ. If $a_{i+2} = a_0$, then $a_{i+1} = a_{m-1}$ and $\ell_{i+2} = \ell_m$, so that a_m is collinear in F_ℓ with the third point of ℓ_m. If $a_{i+2} = a_1$, then $a_{i+1} = a_0$, so that ℓ_1 and ℓ'_1 are in F_ℓ and $\ell'_1 \cap \ell_1 \neq \emptyset$. Thus, the conditions of Corollary 2.2 are never satisfied if a portion of C is contained within an F_ℓ. A similar result holds for F_p, which is the dual of F_ℓ. We conclude that the Fano-type configurations can not be obtained by a 1-point extension. □

4 The Characterization Theorem

In this section we will assume that (Σ, Π) is an n_3-configuration which cannot be derived by a 1-point extension. It follows from Theorem 2.6 that we can assume that (Σ, Π) has a triangle. Let the points of (Σ, Π) be numbered $1, 2, \ldots, n$. Without loss of generality, we can assume that $\Delta(2, 3, 1)$ is a triangle in (Σ, Π). This is illustrated in Figure 9. It will be convenient to omit the commas and brackets from expressions like $\Delta(2, 3, 1)$, and write simply $\Delta 231$.

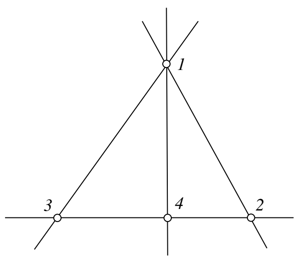

Figure 9: Triangle $\Delta 231$ with condition A

We divide the analysis into two cases according to whether or not (Σ, Π) has a triangle satisfying condition A. The theorem obtained will be the following.

Theorem 4.1. *If* (Σ, Π) *is an* n_3*-configuration which cannot be obtained from a 1-point extension, then either:*

i) (Σ, Π) *is one of the Fano, Pappus, or Desargues configurations; or*

ii) (Σ, Π) *is a Fano-type configuration.*

Proof. The proof of this theorem is very long, involving an analysis of many possible cases.

Case A. (Σ, Π) has a triangle satisfying condition A.

Let the ordered triangle be $\Delta 231$, as above. Condition A tells us that the third line through 1 intersects ℓ_{23}. Call the point of intersection 4. This is shown in Figure 9. We will show that any n_3 configuration that cannot be obtained by a 1-point extension, and which satisfies Condition A, is either a Fano-type configuration, or the Fano configuration. Now consider $\Delta 142$. It currently does not satisfy conditions A, B, or C. Since every triangle must satisfy at least one of these conditions, there are three possibilities, which we indicate by $\Delta 142A$, $\Delta 142B$, and $\Delta 142C$. These are shown in Figure 10. In $\Delta 142A$, the third line through 4 intersects ℓ_{12} (in point 5). In $\Delta 142B$, the third lines through 1 and 4 intersect (in point 5). In $\Delta 142C$, the third points on ℓ_{12} (point 5) and ℓ_{24} (point 3) are collinear.

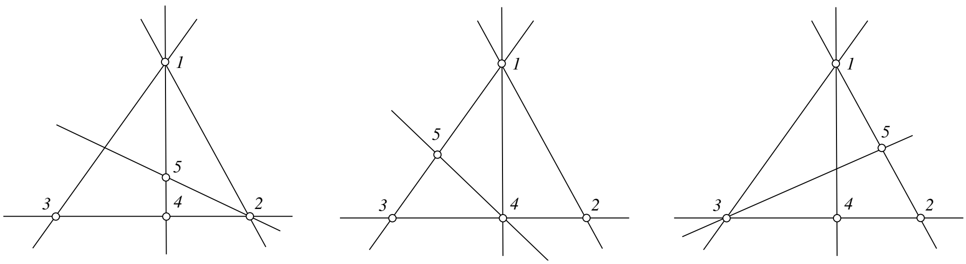

Figure 10: $\Delta 142A$, $\Delta 142B$, and $\Delta 142C$

These three structures are easily seen to be isomorphic, by relabelling the points. Each structure is self-dual, having two points incident on 3 lines each, and two lines each containing 3 points. Thus, without loss of generality, we can assume that the subconfiguration $\Delta 142A$ exists in (Σ, Π) in Case A. Consider triangle $\Delta 124$. It currently does not satisfy condition A, B, or C. Since it must satisfy at least one of these conditions, there are three possibilities, which we indicate by $\Delta 142A\Delta 124A$, $\Delta 142A\Delta 124B$, and $\Delta 142A\Delta 124C$. These are shown in Figure 11.

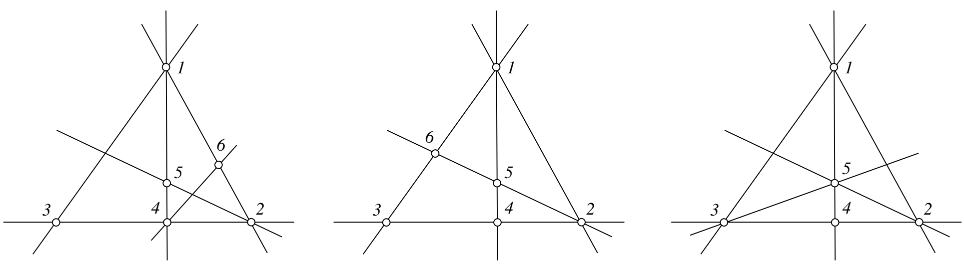

Figure 11: $\Delta 142A\Delta 124A$, $\Delta 142A\Delta 124B$, and $\Delta 142A\Delta 124C$

The structures $\Delta 142A\Delta 124B$ and $\Delta 142A\Delta 124C$ are duals of each other. The first has 6 points and 5 lines, while the other has 5 points and 6 lines. It can be verified by exhaustion that every ordered triangle in these structures satisfies at least one of conditions A, B, or C.

Case $\mathbf{\Delta 142A\Delta 124A}$.

Consider the quadrangle $\square 6431$ in $\Delta 142A\Delta 124A$. It must satisfy at least one of conditions D, E, F, G (see Figure 5). Condition D is possible only if ℓ_{25} intersects ℓ_{13}. Condition E is not possible. Condition F is possible only if the third line through 3 intersects ℓ_{46}. Condition G is possible only if there is a line ℓ_{56}. These cases are illustrated in Figure 12.

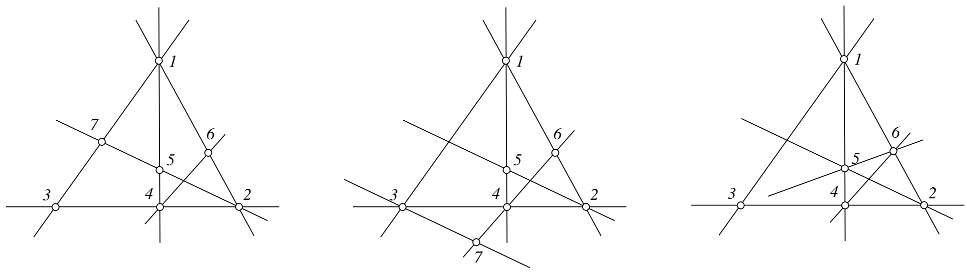

Figure 12: $\Delta 142A\Delta 124A\square 6431D$, $\Delta 142A\Delta 124A\square 6431F$, $\Delta 142A\Delta 124A\square 6431G$

Now the diagrams $\Delta 142A\Delta 124A\square 6431D$ and $\Delta 142A\Delta 124A\square 6431G$ are duals of each other, for the mapping which sends points $1, 2, 3, 4, 5, 6, 7$ of D to $\ell_{15}, \ell_{16}, \ell_{25}, \ell_{24}, \ell_{46}, \ell_{13}, \ell_{56}$ of G is an isomorphism. Therefore we need only consider cases D and F.

Case $\mathbf{\Delta 142A\Delta 124A\square 6431D}$.

It can be verified that all triangles of the diagram satisfy one of conditions A, B, C. Consider the quadrangle $\square 3164$. Condition D is only possible if point 7 lies on line ℓ_{46}. Condition E is not possible. Condition F is only possible if there is a line ℓ_{67}. Condition G is only possible if there is a line ℓ_{35}. These cases are illustrated in Figure 13.

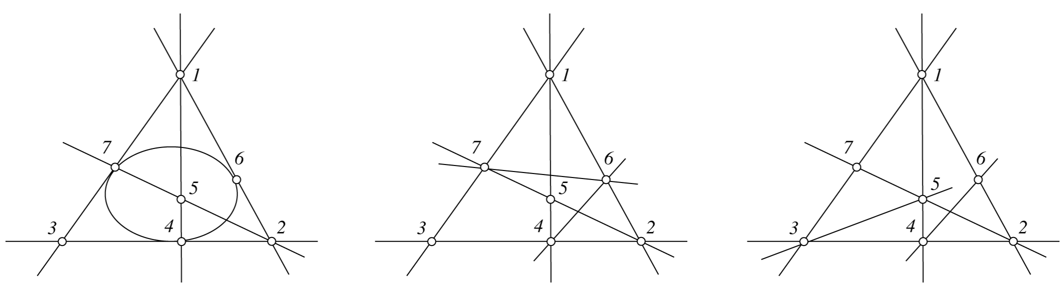

Figure 13: $\Delta 142A\Delta 124A\square 6431D\square 3164D$, F, and G

Case $\mathbf{\Delta 142A\Delta 124A\square 6431D\square 3164D}$.

It can be verified that every triangle satisfies at least one of conditions A, B, C, and every quadrangle satisfies at least one of conditions D, E, F, G. This configuration is isomorphic to the **Fano configuration, with one line removed** (ℓ_{356}), which we denote as F_ℓ. The dual configuration is the **Fano configuration, with one point removed**, which we denote as F_p.

Case $\mathbf{\Delta 142A\Delta 124A\square 6431D\square 3164F}$.

Consider the quadrangle $\square 2376$. Condition D requires that ℓ_{15} intersects ℓ_{67}, which is impossible. Condition E requires that ℓ_{46} contains point 1, which is impossible. Condition F requires that ℓ_{75} contains point 4, which is impossible. Condition G requires a line ℓ_{35}. The result is illustrated in Figure 14.

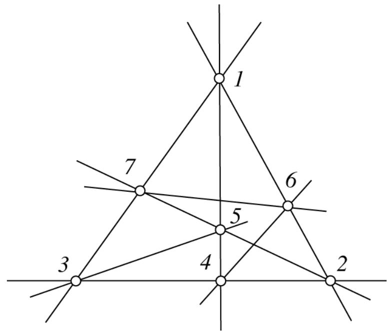

Figure 14: Case **$\mathbf{\Delta 142A\Delta 124A\square 6431D\square 3164F\square 2376G}$**

We then consider quadrangle $\square 6237$. Condition D requires that ℓ_{15} intersects ℓ_{67}, which is impossible. Condition E requires that ℓ_{75} contains point 4, which is impossible. Condition F requires that ℓ_{35} contains point 1, which is impossible. Condition G requires that ℓ_{46} and ℓ_{25} intersect in point 5, which is impossible. We conclude that case $\Delta 142A\Delta 124A\square 6431D\square 3164F$ is not possible.

Case $\mathbf{\Delta 142A\Delta 124A\square 6431D\square 3164G}$.
Consider the quadrangle $\square 4316$. Condition D requires that ℓ_{25} intersects ℓ_{46}. The point of intersection can only be 7. Condition E requires that ℓ_{75} contains point 6, which is impossible. Condition F requires that ℓ_{15} contains point 2, which is impossible. Condition G requires a line ℓ_{356}. These cases are illustrated in Figure 15.

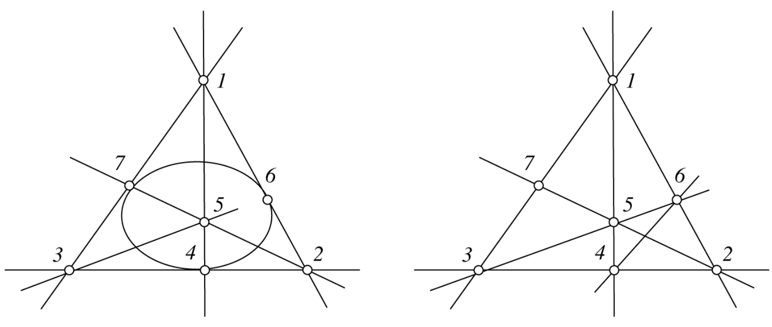

Figure 15: Cases $\Delta 142A\Delta 124A\square 6431D\square 3164G\square 4316D$ and G

These two configurations are easily seen to be isomorphic, by the permutation of the points given by $(2,3,4)(5,6,7)$, mapping D onto G. They are both isomorphic to the **Fano configuration, with one incidence removed**, denoted by F'. Every triangle satisfies at least one of conditions A, B, C, and every quadrangle satisfies at least one of conditions D, E, F, G.

Note that we can complete F' to the Fano configuration, which can not be constructed by a 1-point extension.

We summarise Case A as follows:

Consider an n_3 configuration (Σ, Π), where $n > 7$, which cannot be constructed by a 1-point extension. Every triangle satisfying condition A is contained in a unique sub-configuration isomorphic to one of F_ℓ, F_p or F'.

Case B. (Σ, Π) has no triangle satisfying condition A.

We begin with triangle $\Delta 231$. It must satisfy condition B or C. These two possibilities are shown in Figure 16.

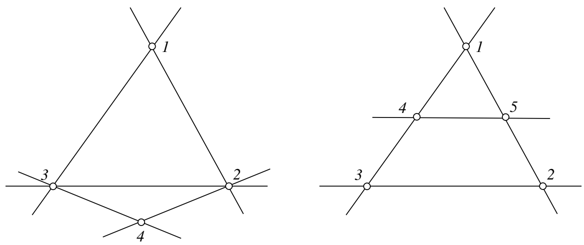

Figure 16: $\Delta 231B$ and $\Delta 231C$

These two structures are duals of each other. Hence we can assume without loss of generality that (Σ, Π) contains the structure $\Delta 231B$.

Consider the triangle $\Delta 123$. It must satisfy condition B or C. We must take these as two separate cases, Case $B\Delta 123B$ and Case $B\Delta 123C$. They are shown in Figure 17. It will be necessary to examine a great many subcases.

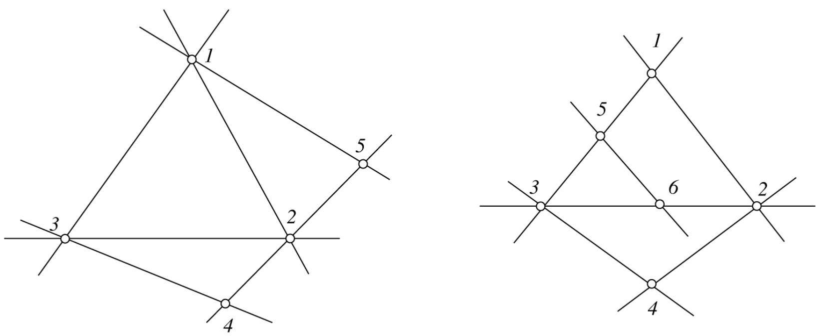

Figure 17: Cases $B\Delta 123B$ and $B\Delta 123C$

Case B$\boldsymbol{\Delta}$123B.
Consider triangle $\Delta 132$. There are two possibilities, cases $B\Delta 123B\Delta 132B$ and $B\Delta 123B\Delta 132C$, which must both be considered. They are shown in Figure 18.

Case B$\boldsymbol{\Delta}$123B$\boldsymbol{\Delta}$132B.
Consider triangle $\Delta 243$. There are two choices $B\Delta 123B\Delta 132B\Delta 243B$ and

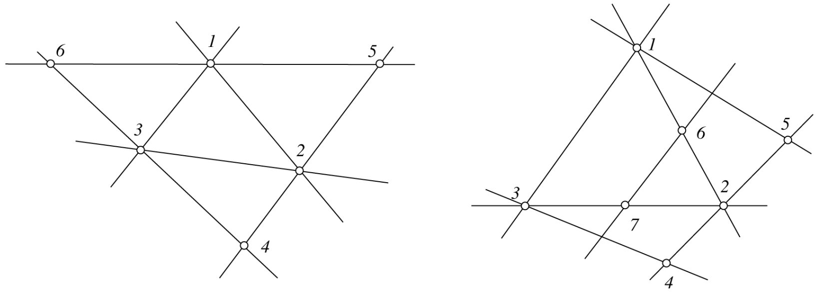

Figure 18: Cases $B\Delta 123B\Delta 132B$ and $B\Delta 123B\Delta 132C$

$B\Delta 123B\Delta 132B\Delta 243C$. They are shown in Figure 19. These structures both have 7 points $\{1, 2, \ldots, 7\}$, so that a mapping from the first to the second can be denoted by a permutation. It is easy to see that the permutation $(1, 2, 3)(4, 6, 5)(7)$ maps the first to the second. Thus, without loss of generality, we can suppose that (Σ, Π) contains the structure $B\Delta 123B\Delta 132B\Delta 243B$.

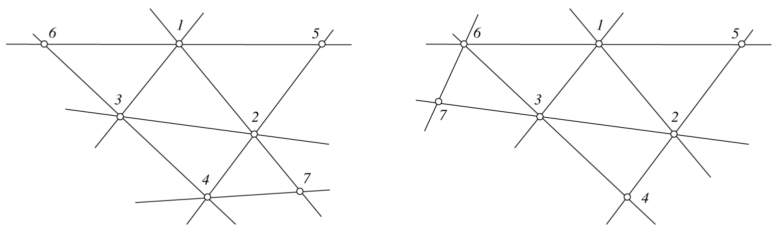

Figure 19: Isomorphic cases $B\Delta 123B\Delta 132B\Delta 243$ B and C

Consider triangle $\Delta 342$. There are two possibilities, $B\Delta 123B\Delta 132B\Delta 243B$ $\Delta 342B$ and $B\Delta 123B\Delta 132B\Delta 243B\Delta 342C$. They are shown in Figure 20. We must consider both possibilities.

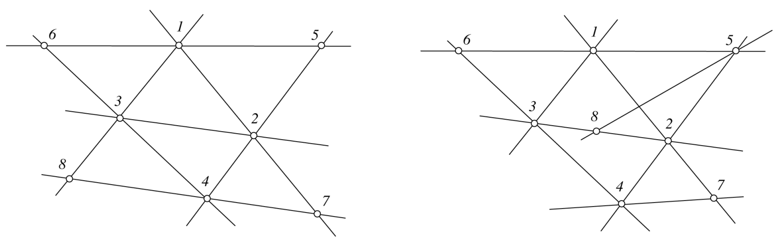

Figure 20: Cases $B\Delta 123B\Delta 132B\Delta 243B\Delta 342B$ and $B\Delta 123B\Delta 132B\Delta 243B\Delta 342C$

This is beginning to look remarkably like the Pappus configuration.

Case B$\boldsymbol{\Delta}$123B$\boldsymbol{\Delta}$132B$\boldsymbol{\Delta}$243B$\boldsymbol{\Delta}$342B.

Consider the quadrangle $\square 1248$. At least one of conditions D, E, F, G must be satisfied. Of these, it is only possible to satisfy condition E, namely the third line through 8 must intersect ℓ_{24}. The point of intersection can only be 5. Therefore the left diagram of Figure 21 must exist in (Σ, Π).

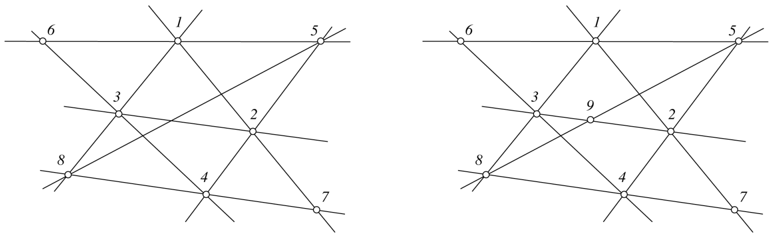

Figure 21: Cases $B\square 1248E$ and $B\square 1248E\square 7238E$

Consider the quadrangle $\square 7238$. At least one of conditions D, E, F, G must be satisfied. Of these, it is only possible to satisfy condition E, namely the third line through 8 must intersect ℓ_{23}. Therefore the right diagram of Figure 21 must exist in (Σ, Π).

Consider the quadrangle $\square 3159$. It is only possible to satisfy condition E, namely the third line through 9 must intersect ℓ_{15} in point 6. Therefore the following structure (Figure 22) must exist in (Σ, Π).

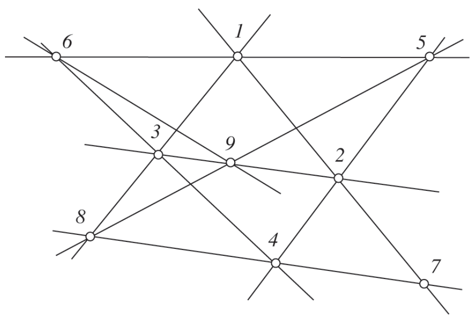

Figure 22: Case $B\square 1248E\square 7238E\square 3159E$

Consider the quadrangle $\square 1347$. It is only possible to satisfy condition E, namely the third line through 7 must intersect ℓ_{34}. The point of intersection must be 6, so that point 7 is incident with ℓ_{69}. Therefore the diagram is completed to a 9_3-configuration, so that (Σ, Π) can only be the **Pappus configuration.**

Case B$\boldsymbol{\Delta}$123B$\boldsymbol{\Delta}$132B$\boldsymbol{\Delta}$243B$\boldsymbol{\Delta}$342C.

This case is illustrated in Figure 20. Consider the triangle $\Delta 274$. There are two possibilities, $\Delta 274B$ and $\Delta 274C$, shown in Figure 23. These are duals of each other. The mapping which sends the points $1, 2, \ldots, 8$ of $\Delta 274B$ to the lines $\ell_{15}, \ell_{25}, \ell_{34}, \ell_{32}, \ell_{12}, \ell_{13}, \ell_{58}, \ell_{47}$ of $\Delta 274C$ is an isomorphism. Hence we only need to consider one of them, the first, say.

Consider the quadrangle $\square 1783$. It is only possible to satisfy condition E, namely the third line through 3 must intersect ℓ_{78}. The point of intersection must be 6, so that ℓ_{78} must be extended to include point 6. Consider next quadrangle $\square 1745$. It is

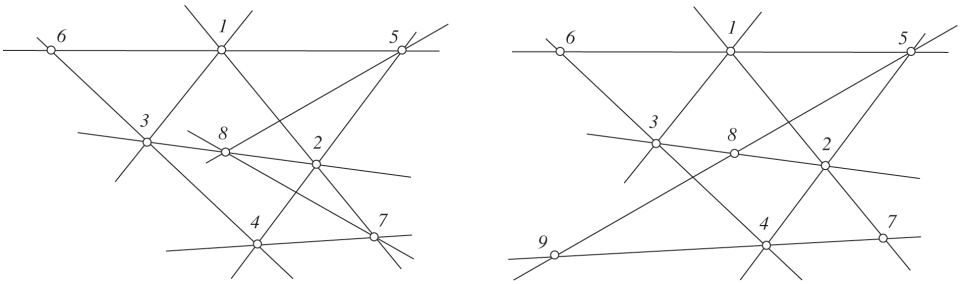

Figure 23: Case $B\Delta123B\Delta132B\Delta243B\Delta342C\Delta274$, B and C

only possible to satisfy condition E, namely the third line through 5 must intersect ℓ_{47}. The result is illustrated in Figure 24.

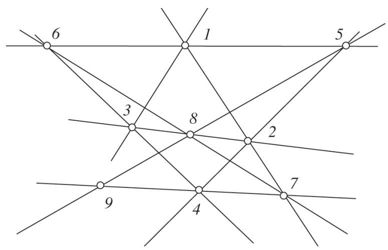

Figure 24: Case $B\Delta123B\Delta132B\Delta243B\Delta342C\Delta274B\Box1783\Box1745$

Finally, consider quadrangle $\Box7138$. It is only possible to satisfy condition E, namely the third line through 8 must intersect ℓ_{13}. The point of intersection must be 9, so that ℓ_{13} must be extended to include point 9. Once again we have the **Pappus configuration.**

Case BΔ123BΔ132C.

This case is illustrated in Figure 18. Consider the triangle $\Delta267$. There are two possible ways to satisfy condition B, namely the third line through 6 could contain either 4 or 5. The first of these choices is illustrated in Figure 25. The second is not allowed, as it would create a triangle $\Delta125$ satisfying condition A. There are two possible ways to satisfy condition C, namely ℓ_{67} could intersect ℓ_{13} or ℓ_{34}. Call these two results C_1 and C_2, respectively, also shown in Figure 25.

Case BΔ123BΔ132CΔ267B.

Consider the quadrangle $\Box1673$. It is not possible to satisfy conditions D or F. Condition E can only be satisfied if ℓ_{34} intersects ℓ_{67}. Condition G can only be satisfied if ℓ_{15} intersects ℓ_{46}. These cases are shown in Figure 26.

Now case G (the right diagram) leads to a contradiction, for consider the quadrangle $\Box3167$. Conditions E, F, G are not possible. Condition D is only possible if $5 \in \ell_{67}$. But this creates a triangle $\Delta156$ satisfying condition A, a contradiction. Therefore we consider case E (the left diagram). Consider the quadrangle $\Box3761$. Conditions D, F, G cannot be satisfied. Condition E can only be satisfied if ℓ_{15} intersects ℓ_{67} in point 8, as shown in Figure 27. Consider next the quadrangle $\Box6137$. Conditions

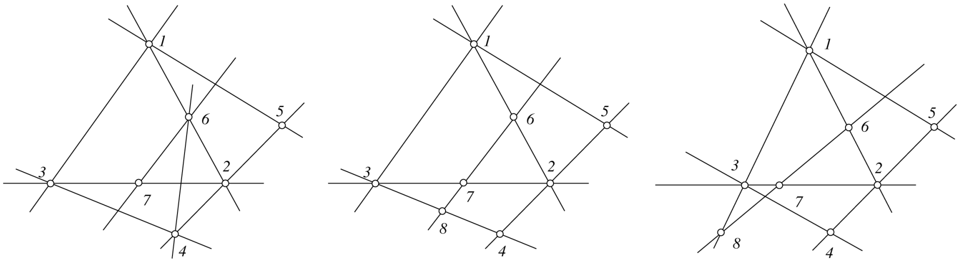

Figure 25: Cases $B\Delta123B\Delta132C\Delta267$ B, C_1, and C_2

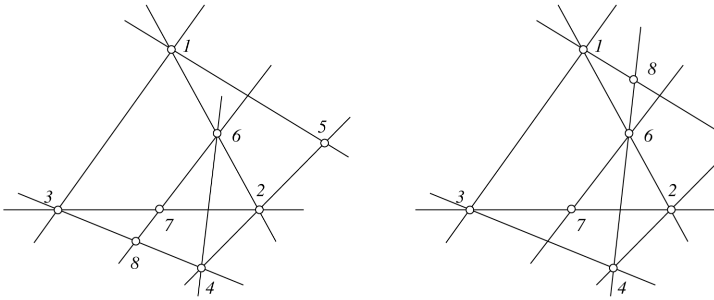

Figure 26: Cases $B\Delta123B\Delta132C\Delta267B\square1673$ E and G

D, F, G cannot be satisfied. Condition E can only be satisfied if the third line through 7 intersects ℓ_{13} in a point 9, also illustrated in Figure 27.

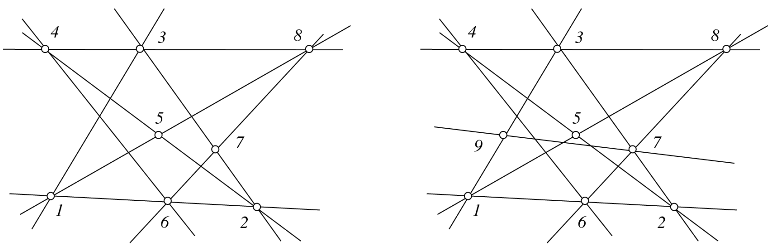

Figure 27: Cases $E\square1673E$ and $E\square1673E\square6137E$

Consider now the quadrangle $\square2685$ in the right diagram of Figure 27. Conditions D, F, G cannot be satisfied. Condition E can only be satisfied if the third line through 5 contains point 7, which is only possible if $5 \in \ell_{79}$. The result is isomorphic to the diagram of Figure 24. Once again, we obtain the **Pappus configuration**.

Case $\mathbf{B\Delta123B\Delta132C\Delta267C_1}$.

Refer to Figure 25. Consider the quadrangle $\square2784$. Conditions D and F cannot be satisfied. Condition E can only be satisfied if there is a line ℓ_{46}, which gives a result identical to the left diagram of Figure 26. Condition G can only be satisfied if the third line through 7 intersects ℓ_{26} in point 1, but this creates a triangle $\Delta127$

satisfying condition A, which is not allowed. This completes this case.

Case BΔ123BΔ132CΔ267C$_2$.

Refer to Figure 25. Consider the quadrangle $\square 1376$. Conditions D, E, F are not possible. Condition G is only possible if ℓ_{15} and ℓ_{34} intersect, shown in Figure 28. Consider now the quadrangle $\square 1872$. Conditions D, E, F are not possible. Condition G is possible if ℓ_{15} intersects the third line through 8. The point of intersection can be either 5 or 9, resulting in G_1 and G_2, also shown in Figure 28.

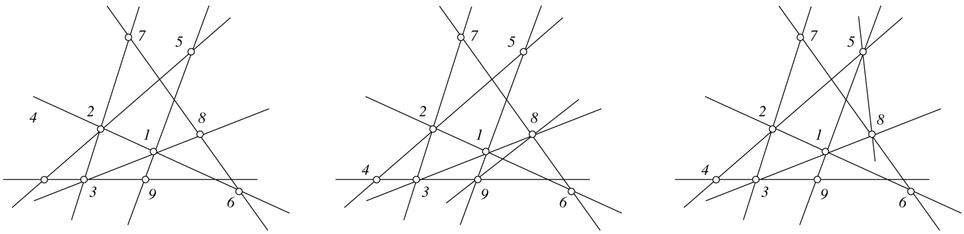

Figure 28: Cases $C_2\square 1376G$, $G\square 1872G_1$ and $G\square 1872G_2$

Consider the quadrangle $\square 7218$ in diagram $G\square 1872G_1$. Conditions D, E, F cannot be satisfied. Condition G can only be satisfied if the third line through 7 intersects ℓ_{24}. The point of intersection can be 4 or 5. But 4 creates a triangle $\Delta 734$ satisfying condition A, a contradiction. Therefore the intersection must be point 5, as shown in Figure 29. Then consider quadrangle $\square 7812$. Conditions D, E, F cannot be satisfied. Condition G can only be satisfied if ℓ_{15} and ℓ_{89} intersect, also shown in Figure 29. Next, consider quadrangle $\square 1572$. Conditions D, E, F, G cannot be satisfied, a contradiction. This completes this case.

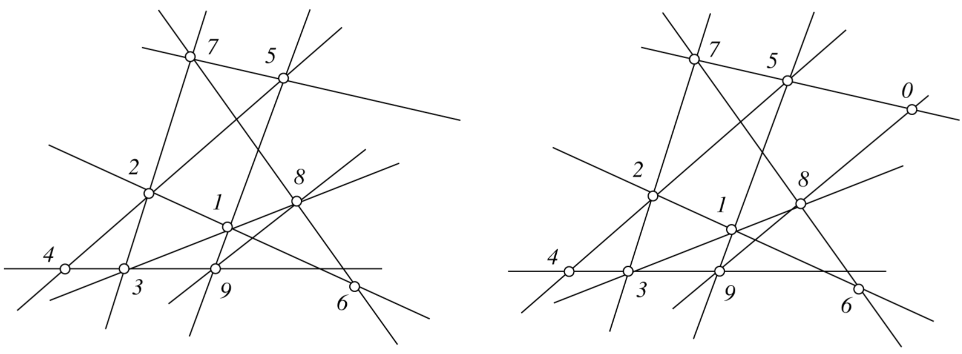

Figure 29: Cases $G_1 : \quad \square 7218G$ and $\square 7218G\square 7812G$

Consider next $G\square 1872G_2$, and quadrangle $\square 7218$. Conditions D, E, F cannot be satisfied. Condition G can only be satisfied if the third line through 7 intersects ℓ_{24}. The point of intersection must be 4. But this creates a triangle $\Delta 734$ satisfying condition A, a contradiction. This completes this case, and also case $B\Delta 123B\Delta 132C\Delta 267C_2$, and case $B\Delta 123B\Delta 132C$ and case $B\Delta 123B$.

Case BΔ123C.

Refer to Figure 17. Consider the triangle $\Delta 132$. Condition B can be satisfied if the third line through 1 intersects ℓ_{34}. There are two ways this can occur – the intersection can be point 4, or a new point. This gives B_1 and B_2, shown in Figure 30. Condition C can be satisfied if point 6 is collinear with the third point on ℓ_{12}. There are two ways this can occur. The line through 6 intersecting ℓ_{12} can be ℓ_{56} or a new line. This gives C_1 and C_2, shown in Figure 31.

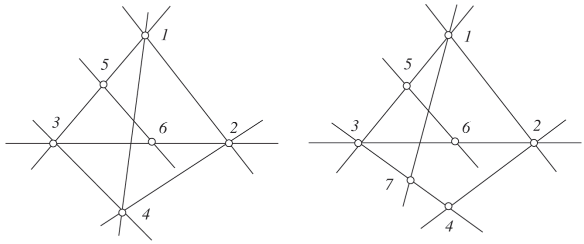

Figure 30: Case $B\Delta 123C\Delta 132$ B_1 and B_2

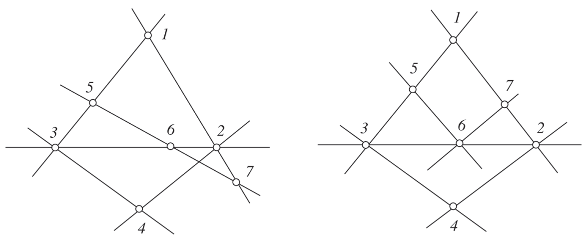

Figure 31: Case $B\Delta 123C\Delta 132$ C_1 and C_2

It can be observed that C_1 is isomorphic to the dual of B_1. If we map points $1, 2, 3, 4, 5, 6, 7$ of C_1 to lines $\ell_{12}, \ell_{23}, \ell_{13}, \ell_{56}, \ell_{14}, \ell_{34}, \ell_{24}$, respectively, of B_1, we have an isomorphism. Similarly, C_2 is isomorphic to the dual of B_2. An isomorphism maps points $1, 2, 3, 4, 5, 6, 7$ of C_2 to lines $\ell_{12}, \ell_{13}, \ell_{23}, \ell_{56}, \ell_{24}, \ell_{34}, \ell_{17}$, respectively, of B_2. Consequently, we have only cases B_1 and B_2 to deal with.

Case $\mathbf{B\Delta 123C\Delta 132B_1}$.

Consider the quadrangle $\square 1562$. Condition D can only be satisfied if the third point on ℓ_{12} is collinear with point 3. But then triangle $\Delta 123$ would satisfy condition A, which is not allowed. Condition E can be satisfied if ℓ_{24} intersected ℓ_{56}. This is shown in Figure 32. Condition F can only be satisfied if the third line through 6 intersected ℓ_{15} in point 3. However, 6 and 3 are already collinear. Condition G can be satisfied if the third line through 5 intersected ℓ_{14}. The third line through 5 cannot be ℓ_{24}, for $\Delta 124$ would then satisfy condition A. Thus, the third line through 5 must be a new line, as shown also in Figure 32.

Case $\mathbf{B\Delta 123C\Delta 132B_1\square 1562E}$.

Consider the triangle $\Delta 267$. Condition B can be satisfied if the third line through 6 intersected ℓ_{12}. The third line through 6 cannot be ℓ_{14}, as the triangle $\Delta 123$ would then satisfy condition A. Hence, the third line through 6 must be a new line, as shown in Figure 33. Condition C can only be satisfied if points 4 and 5 are collinear.

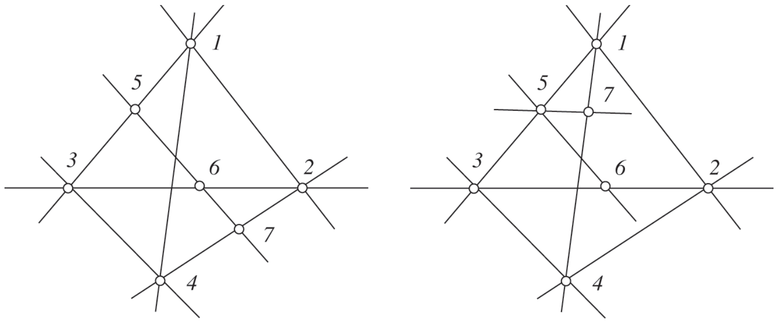

Figure 32: Case $B\Delta 123C\Delta 132B_1\square 1562$ E and G

The line containing 4 and 5 cannot be ℓ_{14} and it cannot be ℓ_{34}. Therefore Condition C is impossible, and we must have $B\Delta 123C\Delta 132B1\square 1562E\Delta 267B$, shown in Figure 33.

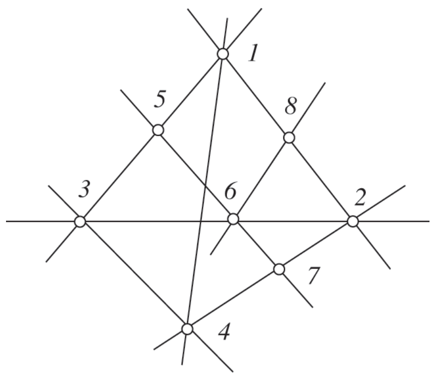

Figure 33: Case $B\Delta 123C\Delta 132B_1\square 1562E\Delta 267B$

This structure is found to be isomorphic to the dual of $B\Delta 123B\Delta 132C\Delta 267B$ $\square 1673G$, shown in Figure 26. The isomorphism maps points $1, 2, 3, 4, 5, 6, 7, 8$ to lines $\ell_{24}, \ell_{26}, \ell_{56}, \ell_{15}, \ell_{34}, \ell_{18}, \ell_{68}, \ell_{14}$. This completes case $B\Delta 123C\Delta 132B1$ $\square 1562E$.

Case B$\boldsymbol{\Delta}$123C$\boldsymbol{\Delta}$132B$_1\square$1562G.

Consider the quadrangle $\square 2651$. Condition D can only be satisfied if the third point on ℓ_{23} is collinear with point 3. However triangle $\Delta 132$ would then satisfy condition A. Condition E can only be satisfied if ℓ_{14} intersected ℓ_{56}. The point of intersection cannot be 7. If it were point 4, then $\Delta 563$ would then satisfy condition A. Hence condition E is not possible. Condition F can only be satisfied if ℓ_{57} intersected ℓ_{26} in point 3. However 5 and 3 are already collinear. Condition G can be satisfied if the third line through 6 intersected ℓ_{24}. The point of intersection cannot be 4. The only possibility is a new line through 6, as shown in Figure 34.

Consider the quadrangle $\square 4863$. Condition D can only be satisfied if the third point on ℓ_{34} is collinear with point 2. The triangle $\Delta 342$ would then satisfy condition A,

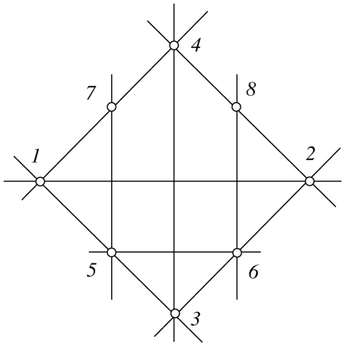

Figure 34: Cases $B\Delta123C\Delta132B_1\square1562G : \square2651G$ and $\square2651G\square4863G$

which is not allowed. Condition E can only be satisfied if ℓ_{13} intersected ℓ_{68} in either 1 or 5. However, 1 and 5 are already each on 3 lines. Condition F can only be satisfied if ℓ_{56} intersected ℓ_{48} in 2. However 6 and 2 are already collinear. Condition G can be satisfied if the third line through 8 intersected ℓ_{14}. The point of intersection can only be 7, shown in the right diagram of Figure 34.

Consider the quadrangle $\square6512$. Condition D can only be satisfied if the third point on ℓ_{12} were collinear with point 3. But triangle $\Delta123$ would then satisfy condition A. Condition E can only be satisfied if ℓ_{24} intersected ℓ_{15} in 3. This is not possible. Condition F can only be satisfied if ℓ_{14} intersected ℓ_{56}. This is not possible. Condition G can only be satisfied if ℓ_{57} intersected ℓ_{68}. This is shown in Figure 35.

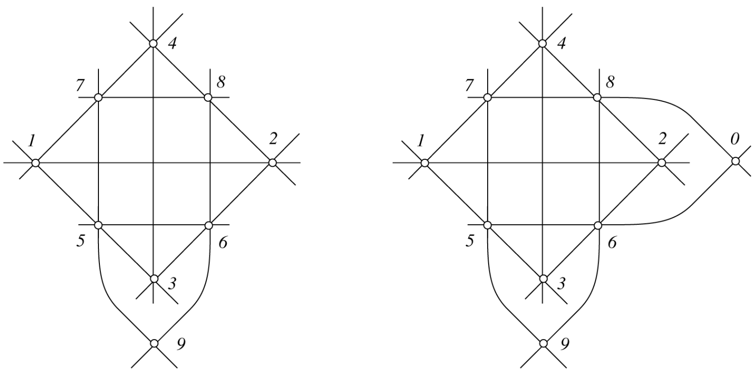

Figure 35: Cases $\square6512G$ and $\square6512G\square5743G$

Consider the quadrangle $\square5743$. Condition D can only be satisfied if the third point on ℓ_{34} were collinear with point 1. But then triangle $\Delta341$ would satisfy condition A. Condition E can only be satisfied if ℓ_{23} intersected ℓ_{47} in point 1. This is not possible. Condition F can only be satisfied if ℓ_{24} intersected ℓ_{57} in 9. This is not possible. Condition G can only be satisfied if ℓ_{78} intersected ℓ_{56} in a new point, also shown in Figure 35.

Consider the triangle $\Delta157$. Condition B can only be satisfied if ℓ_{12} intersected

ℓ_{56}. The point of intersection must be point 0. Condition C can only be satisfied if points 4 and 9 are collinear. The line of collinearity must be ℓ_{34}. The resulting two structures are both isomorphic to the **Desargues configuration**, with one incidence missing, as can be seen from Figure 1.1. If we then consider $\Delta 268$, the remaining incidence is forced. This completes case $B\Delta 123C\Delta 132B_1\Box 1562G$ and also case $B\Delta 123C\Delta 132B_1$.

Case $\mathbf{B\Delta 123C\Delta 132B_2}$.

Refer to Figure 30. Consider the triangle $\Delta 173$. Condition B can be satisfied if the third line through 7 intersected ℓ_{12}. The point of intersection cannot be point 2. Therefore it is a new point, as shown in Figure 36. Condition C can be satisfied if points 4 and 5 are collinear. The line of collinearity cannot be ℓ_{56}, for triangle $\Delta 453$ would then satisfy condition A. Hence ℓ_{45} is a new line, also shown in Figure 36. be satisfied if ℓ_{57} intersected ℓ_{68}. This is shown in Figure 35.

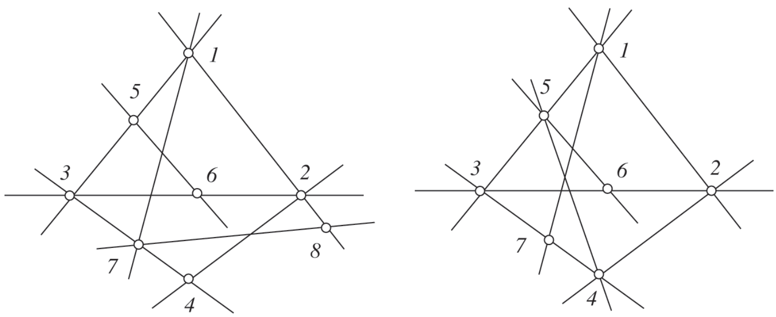

Figure 36: Cases $B\Delta 123C\Delta 132B_2\Delta 173$ B and C

Now case $B\Delta 123C\Delta 132B_2\Delta 173C$ is isomorphic to case $B\Delta 123B\Delta 132C\Delta 267B$, shown in Figure 25. As both diagrams have 7 points, the isomorphism can be given by a permutation, $(1,5,6)(2,3,4)$, which maps diagram $B\Delta 123B\Delta 132C\Delta 267B$ to $B\Delta 123C\Delta 132B_2\Delta 173C$. Thus we need only consider case $B\Delta 123C\Delta 132B_2$ $\Delta 173B$.

Consider the triangle $\Delta 781$ in the left diagram of Figure 36. Condition B can be satisfied if the third line through 8 intersected ℓ_{37}. The point of intersection cannot be 3. Therefore there must be a line ℓ_{48}, as shown in Figure 37. Condition C can be satisfied if the third point on ℓ_{17} is collinear with point 2. The line of collinearity cannot be ℓ_{26}, for if point 6 were on ℓ_{17}, triangle $\Delta 173$ would satisfy condition A. Hence ℓ_{24} must intersect ℓ_{17} in a new point. This is also shown in Figure 37.

Case $\mathbf{B\Delta 123C\Delta 132B_2\Delta 173B\Delta 781B}$.

Consider the triangle $\Delta 365$. Condition B can be satisfied if the third line through 6 intersected ℓ_{37}. The point of intersection cannot be 4, because ℓ_{48} would then contain 6, causing a triangle $\Delta 682$ satisfying condition A. Line ℓ_{17} cannot contain 6, for then triangle $\Delta 136$ would satisfy condition A. Therefore condition B requires that ℓ_{78} contain 6, shown in Figure 38. Condition C can be satisfied if the third point on ℓ_{56} were collinear with point 1. The line of collinearity must be ℓ_{17}, also shown in Figure 38.

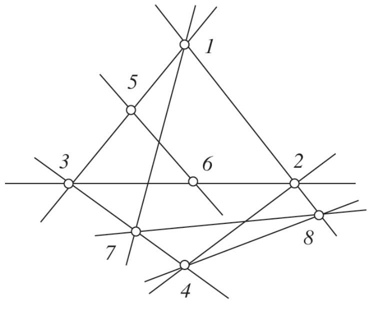

Figure 37: Cases $B\Delta 123 C\Delta 132 B_2\Delta 173 B\Delta 781$ B and C

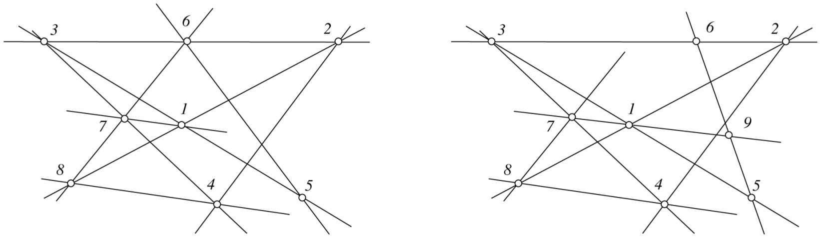

Figure 38: Case $B\Delta 123 C\Delta 132 B_2\Delta 173 B\Delta 781 B\Delta 365$ B and C

Case B$\boldsymbol{\Delta}$123C$\boldsymbol{\Delta}$132B$_2\boldsymbol{\Delta}$173B$\boldsymbol{\Delta}$781B$\boldsymbol{\Delta}$365B.

Refer to the left diagram of Figure 38. Consider the quadrangle $\square 2176$. Condition D can only be satisfied if points 3 and 8 were collinear. This is not possible as 3 and 8 are already incident on 3 lines each. Condition E can only be satisfied if ℓ_{56} intersected ℓ_{17}, shown in Figure 39. Condition F can only be satisfied if ℓ_{37} intersected ℓ_{12} in 8. However, 7 and 8 are already collinear. Condition G can only be satisfied if ℓ_{15} and ℓ_{24} intersected. The point of intersection must be 5, making triangle $\Delta 132$ satisfy condition A. We conclude that only E is possible.

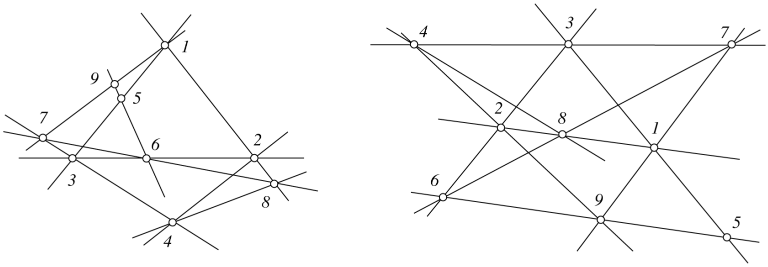

Figure 39: Case $B\Delta 123 C\Delta 132 B_2\Delta 173 B\Delta 781 B\Delta 365 B\square 2176 E$

Consider the quadrangle $\square 2156$. Condition D can only be satisfied if points 3 and 9 were collinear, which is impossible. Condition E can only be satisfied if ℓ_{67} intersected ℓ_{15} in point 3, which is impossible. Condition F can only be satisfied if the third line through 5 intersected ℓ_{12} in point 8, which is impossible. Condition G can only be satisfied if ℓ_{17} and ℓ_{24} intersected. The point of intersection must be point 9, also shown in Figure 39. As can be seen from the diagram, this is the Pappus configuration with one incidence missing. We conclude that this case results in the **Pappus configuration**.

Case $\mathbf{B\Delta 123C\Delta 132B_2\Delta 173B\Delta 781B\Delta 365C}$.

Refer to the right diagram of Figure 38. Consider the quadrangle $\square 7123$. Condition D can only be satisfied if points 4 and 6 are collinear, which is impossible. Condition E can only be satisfied if ℓ_{13} contains 8, which is impossible. Condition F can only be satisfied if ℓ_{24} contains point 9. Condition G can only be satisfied if ℓ_{78} intersected ℓ_{13}. The point of intersection must be 5, creating a triangle $\Delta 195$ satisfying condition A, a contradiction. We conclude that only condition F is possible, shown in Figure 40.

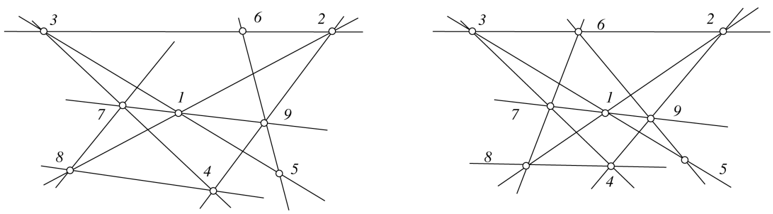

Figure 40: Cases $B\Delta 123C\Delta 132B_2\Delta 173B\Delta 781B\Delta 365C\square 7123F$ and $\square 2371F$

Consider the quadrangle $\square 2371$. Condition D can only be satisfied if points 8 and 9 are collinear, which is impossible. Condition E is only possible if ℓ_{13} contains 4, which is impossible. Condition F is possible only if ℓ_{78} contains 6. Condition G is only possible if ℓ_{29} and ℓ_{35} intersected, which is impossible. We conclude that condition F is necessary.

We next consider quadrangle $\square 4862$. Condition D can only be satisfied if points 9 and 3 are collinear, which is impossible. Condition E can only be satisfied if ℓ_{21} contains point 7, which is impossible. Condition F is possible only if ℓ_{69} and ℓ_{48} intersect in point 5. Condition G is only possible if ℓ_{47} and ℓ_{81} intersected, which is impossible. We conclude that condition F is necessary, giving the **Pappus configuration**. This completes case $B\Delta 123C\Delta 132B_2\Delta 173B\Delta 781B$.

Case $\mathbf{B\Delta 123C\Delta 132B_2\Delta 173B\Delta 781C}$.

Refer to the right diagram of Figure 37. Consider triangle $\Delta 243$. Condition B can only be satisfied if the third line through 4 intersected ℓ_{28}. The point of intersection can only be 8, as shown in Figure 41. Condition C can only be satisfied if points 6 and 7 are collinear. The line of collinearity cannot be ℓ_{17}, as triangle $\Delta 231$ would then satisfy condition A. Hence, the line can only be ℓ_{78}, which must contain 6, as shown in Figure 41.

Case C is isomorphic to the dual of $B\Delta 123C\Delta 132B2\Delta 173B\Delta 178B\Delta 365B$, shown in Figure 38. An isomorphism maps points $1, 2, \ldots, 9$ of C to lines ℓ_{67}, ℓ_{34}, $\ell_{23}, \ell_{24}, \ell_{56}, \ell_{13}, \ell_{12}, \ell_{17}, \ell_{48}$, respectively, of B. Thus we only need consider case B.

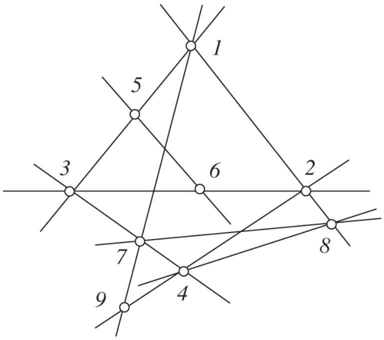

Figure 41: Case $B\Delta123C\Delta132B2\Delta173B\Delta178C\Delta243B$ and C

Consider the quadrangle $\square8731$. Condition D can only be satisfied if points 2 and 6 are collinear, which is impossible, as the line of collinearity could only be ℓ_{24}. Condition E cannot be satisfied. Condition F can only be satisfied if ℓ_{36} intersects ℓ_{87}. The point of intersection must be 6, as shown in Figure 42. Condition G can only be satisfied if ℓ_{84} and ℓ_{79} intersect, which is impossible. Thus, only condition F is possible. But this diagram is isomorphic to case $B\Delta123B\Delta132C\Delta267B\square1673E\square6137E$, shown in Figure 27. An isomorphism is given by $(5,9)(6,7,8)$.

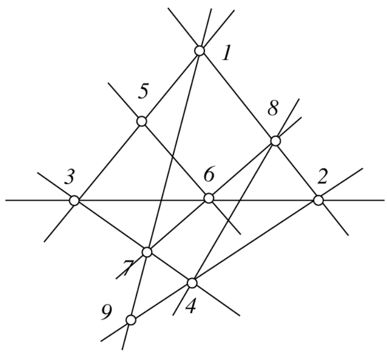

Figure 42: Case $B\Delta123C\Delta132B2\Delta173B\Delta178C\Delta243B\square8731F$

We summarise Case B as follows:

An n_3 configuration (Σ, Π), which cannot be constructed by a 1-point extension, and having no triangle satisfying condition A, is one of the Pappus or Desargues configurations.

We still must show that the Fano, Pappus, and Desargues configurations cannot be obtained by 1-point extensions. This is clearly so for the Fano configuration, as there are no 6_3 configurations. Consider the Pappus configuration. One way to show that it cannot be obtained by a 1-point extension is to start with the unique 8_3 configuration and to show that the possible 1-point extensions do not produce the Pappus configuration. Another way is to show that every ordering of every triangle and quadrilateral in the Pappus configuration satisfies one of conditions A, B, C, D, E, F, G, so that the Pappus configuration does not arise by a 1-point extension. The collineation group of the Pappus configuration has order 108. It is transitive on points, lines, triangles, and quadrangles, so that only one triangle and one quadrilateral need be tested. We omit the proof.

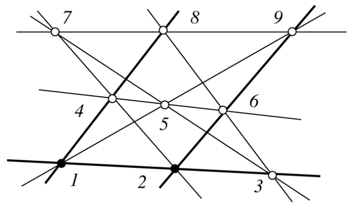

Figure 43: The Pappus configuration

Consider next the Desargues configuration. Its collineation group has order 120. It is transitive on points, lines, triangles, quadrangles, and also on quadruples $(a_0, \ell_2, a_1, \ell_3)$, where $a_0, a_1 \in \ell_2$, $a_0 \neq a_1$, $a_1 \in \ell_3$, and $\ell_2 \neq \ell_3$. It is not transitive on pentagons, hexagons, etc. Refer to Figure 44. We look for a cycle beginning $(a_0, \ell_2, a_1, \ell_3, \ldots, \ell_0) = (1, \ell_{13}, 3, \ell_{34}, \ldots)$, satisfying the conditions of Theorem 2.4. Since $\ell'_1 \cap \ell_1 = \emptyset$, where $\ell'_1 = \ell_{37}$, and ℓ_1 is the third line through $a_0 = 1$, we must have $\ell_1 = \ell_{15}$, so that $\ell_0 = \ell_{17}$. Since $a_2 \notin \ell_1$, by Theorem 2.4, we cannot have $a_2 = 5$. Hence, $a_2 = 4$.

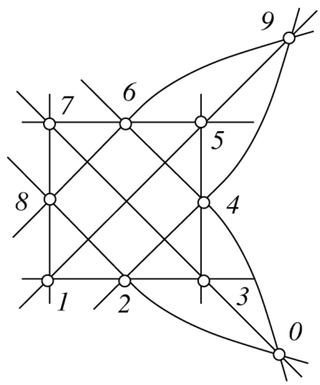

Figure 44: The Desargues configuration

Then since $\ell'_2 \cap \ell_2 = \emptyset$, we cannot have $\ell'_2 = \ell_{42}$, as ℓ_{42} intersects $\ell_2 = \ell_{13}$ in 2. Therefore $\ell_4 = \ell_{49}$, from which we have $a_3 = 9$, and the cycle is $(1, \ell_{13}, 3, \ell_{34}, 4, \ell_{49}, 9, \ldots, \ell_{17})$. Since $\ell'_3 \cap \ell_3 = \emptyset$, we cannot have $\ell'_3 = \ell_{59}$, as ℓ_{59} intersects $\ell_3 = \ell_{34}$ in 5. It follows that $\ell_5 = \ell_{59}$. But then a_4 must be either 1 or 5, both of which are impossible. We conclude that the Desargues configuration cannot be obtained by a 1-point extension. This completes the proof of Theorem 4.1. □

Observe that we have only used 1-point extensions based on triangles and quadrangles in the proof of Theorem 4.1. Hence we have proved that if an $(n+1)_3$ configuration cannot be obtained using a 1-point extensions based on triangles or quadrangles, then it is the Fano, Pappus, Desargues, or a Fano-type configuration. Therefore we have the following corollary.

Corollary 4.2. *Every $(n+1)_3$ configuration that can be obtained from an n_3 configuration by a 1-point extension, can be obtained using a 1-point extension based on triangles or quadrangles.*

A consequence of this corollary is that the $(n+1)_3$ configurations can be constructed from the n_3 configurations by constructing all sequences of sequentially incident points and lines of length at most 4, and testing whether they satisfy the conditions required for a 1-point extension. Isomorphism testing of the resulting $(n+1)_3$ configurations then gives all configurations that can be constructed by 1-point extensions. Those which cannot be constructed in this way are the Fano-type configurations, which can be constructed from cycles and subdivisions of bipartite 3-regular multigraphs, using Theorem 3.2.

One of the central problems in the theory of n_3 configurations is to determine whether they are geometric, that is, whether they can be *coordinatized* over the reals and/or rationals. See [3, 14, 15, 16]. This means to assign homogeneous coordinates in the real and/or rational projective plane, so that the lines are straight lines, and all incidences and non-incidences are respected. The application of 1-point extensions to geometric configurations will be described in another article (in preparation).

5 The 3-Point Extension

Let (Σ, Π) be an n_3-configuration. Choose a line ℓ, and let its points be a_1, a_2, a_3. Construct a new configuration (Σ', Π') as follows. $\Sigma' = \Sigma \cup \{b_1, b_2, b_3, \ell_1, \ell_2, \ell_3\}$, where b_1, b_2, b_3 are new points and ℓ_1, ℓ_2, ℓ_3 are new lines. The incidences Π' are constructed as follows. ℓ_1 contains the points a_1, b_2, b_3. ℓ_2 contains the points b_1, a_2, b_3, and ℓ_3 contains the points b_1, b_2, a_3. Choose 3 lines $\ell_1', \ell_2', \ell_3' \neq \ell$ such that ℓ_i' contains a_i. Remove a_i from ℓ_i' and place b_i on ℓ_i'. This is illustrated in the following table. Then Π' contains all remaining incidences of Π, except for the incidences $a_1\ell_1', a_2\ell_2', a_3\ell_3'$.

ℓ	ℓ_1	ℓ_2	ℓ_3	ℓ_1'	ℓ_2'	ℓ_3'
a_1	a_1	b_1	b_1	b_1	b_2	b_3
a_2	b_2	a_2	b_2	$\cdot$	$\cdot$	$\cdot$
a_3	b_3	b_3	a_3	$\cdot$	$\cdot$	$\cdot$

Theorem 5.1. *(Σ', Π') is an $(n+3)_3$-configuration.*

Proof. Note that each b_i is incident on exactly 3 lines, and that each of $\ell_1', \ell_2', \ell_3'$ is incident on exactly 3 points. We must verify that any 2 lines of (Σ', Π') intersect in at most one point. Clearly $\ell, \ell_1, \ell_2, \ell_3$ intersect each other in at most one point. Similarly for $\ell, \ell_1', \ell_2', \ell_3'$. The same is true for all other lines of Σ', because it is true for (Σ, Π). □

Example 5.2. The Fano configuration has 7 points and 7 lines, all of which are equivalent under automorphisms. There is one way to choose 3 points a_1, a_2, a_3. The incidences of $\ell, \ell_1, \ell_2, \ell_3$ are uniquely determined. The choice of $\ell_1', \ell_2', \ell_3'$ is not unique, as each a_i is incident on two lines other than ℓ. There results two possible 3-point extensions of the Fano configuration. One of these is the Desargues configuration. The other is known as the "anti-Pappian" configuration [5].

A *complete quadrilateral* in an n_3 configuration is a set of four distinct lines intersecting in six distinct points. Notice that the extended configuration (Σ', Π') always contains a complete quadrilateral $\ell, \ell_1, \ell_2, \ell_3$, intersecting in the six points $a_1, a_2, a_3, b_1, b_2, b_3$. The

3-point extension can also be constructed from the dual point of view – rather than beginning with 3 collinear points a_1, a_2, a_3, we begin with 3 concurrent lines, and so forth. This is equivalent to using the 3-point extension in the dual of (Σ, Π), and then dualizing (Σ', Π'). In this case, the 3-point extension will always contain a complete quadrangle, that is, the dual of a complete quadrilateral.

Theorem 5.3. *The Fano-type configurations cannot be obtained by a 3-point extension.*

Proof. Suppose that a Fano-type configuration (Σ, Π) were obtained by a 3-point extension. It would then contain a complete quadrilateral $\ell, \ell_1, \ell_2, \ell_3$, intersecting in the six points $a_1, a_2, a_3, b_1, b_2, b_3$. These four lines and six points must all be part of a single F', F_p, or F_ℓ. Refer to Figure 6. Now the points a_1, a_2, a_3 must be collinear. Furthermore, there must be a line containing a_1, b_2, b_3, and so forth. This determines the labelling of an F', F_p, or F_ℓ. But we then find there is a line containing at least one of the pairs a_1, b_1; a_2, b_2; a_3, b_3, which is not possible in a 3-point extension. □

6 Acknowledgement

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The number of edges of the edge polytope of a finite simple graph

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Abstract

Let $d \geq 3$ be an integer. It is known that the number of edges of the edge polytope of the complete graph with d vertices is $d(d-1)(d-2)/2$. In this paper, we study the maximum possible number μ_d of edges of the edge polytope arising from finite simple graphs with d vertices. We show that $\mu_d = d(d-1)(d-2)/2$ if and only if $3 \leq d \leq 14$. In addition, we study the asymptotic behavior of μ_d. Tran–Ziegler gave a lower bound for μ_d by constructing a random graph. We succeeded in improving this bound by constructing both a non-random graph and a random graph whose complement is bipartite.

Keywords: Finite simple graph, edge polytope.

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1 Introduction

The number of i-dimensional faces of a convex polytope has been studied by many researchers for a long time. One of the most famous classical results is "Euler's formula." The extremal problem concerning the number of faces is an important topic in the study of convex polytopes. On the other hand, the study of edge polytopes of finite graphs has been conducted by many authors from viewpoints of commutative algebra on toric ideals and combinatorics of convex polytopes. We refer the reader to [2, 3] for foundations of edge polytopes. Faces of edge polytopes are studied in, e.g., [2, 4, 5]. Recently, Tran and Ziegler [6] studied this extremal problem on edge polytopes. In particular, using [5, Lemma 1.4], they gave bounds for the maximum possible number μ_d of edges of the edge polytope arising from finite simple graphs with d vertices. Following [1, Question 1.3], we wish to find a finite simple graph G with d vertices such that the edge polytope of G has μ_d edges and to compute μ_d.

Recall that a finite *simple* graph is a finite graph with no loops and no multiple edges. Let $[d] = \{1, \ldots, d\}$ be the vertex set and Ω_d the set of finite simple graphs on $[d]$, where $d \geq 3$. Let $\mathbf{e}_i$ denote the ith unit coordinate vector of the Euclidean space $\mathbb{R}^d$. Let $G \in \Omega_d$ and $E(G)$ the set of edges of G. If $e = \{i, j\} \in E(G)$, then we set $\rho(e) = \mathbf{e}_i + \mathbf{e}_j \in \mathbb{R}^d$. The *edge polytope* $\mathcal{P}_G$ of $G \in \Omega_d$ is the convex hull of the finite set $\{\rho(e) : e \in E(G)\}$ in $\mathbb{R}^d$. Let $\varepsilon(G)$ denote the number of edges, namely 1-dimensional faces, of $\mathcal{P}_G$. For example, consider the case of the complete graph K_d on $[d]$. By [5, Lemma 1.4], for edges e and f $(e \neq f)$ of K_d, the convex hull of $\{\rho(e), \rho(f)\}$ is an edge of the edge polytope $\mathcal{P}_{K_d}$ if and only if e and f have a common vertex. Hence, $\varepsilon(K_d) = d\binom{d-1}{2} = d(d-1)(d-2)/2$. On the other hand, $\varepsilon(K_{m,n}) = mn(m+n-2)/2$, where $K_{m,n}$ is the complete bipartite graph on the vertex set $[\,m\,] \cup \{m+1, \ldots, m+n\}$ for which $m, n \geq 1$ (see [4, Theorem 2.5]). In this paper, we are interested in $\mu_d = \max\{\, \varepsilon(G) : G \in \Omega_d \,\}$ for $d \geq 3$.

Theorem 1.1. *For an integer $d \geq 3$, let Ω_d be the set of finite simple graphs on $[d]$. Given a graph $G \in \Omega_d$, let $\varepsilon(G)$ denote the number of edges of the edge polytope $\mathcal{P}_G$ of G. Then, the following holds:*

(a) *If $3 \leq d \leq 13$ and $G \in \Omega_d$ with $G \neq K_d$, then $\varepsilon(G) < \varepsilon(K_d)$.*

(b) *Let $G \in \Omega_{14}$ with $G \neq K_{14}$. Then $\varepsilon(G) \leq \varepsilon(K_{14})$. Moreover, $\varepsilon(G) = \varepsilon(K_{14})$ if and only if either $G = K_{14} - K_{4,5}$ or $G = K_{14} - K_{5,5}$.*

(c) *If $d \geq 15$, then there exists $G \in \Omega_d$ such that $\varepsilon(G) > \varepsilon(K_d)$.*

We devote Section 2 to giving a proof of Theorem 1.1. At present, for $d \geq 15$, it remains unsolved to find $G \in \Omega_d$ with $\mu_d = \varepsilon(G)$ and to compute μ_d. (Later, we will see that $\mu_{15} \geq \varepsilon(K_{15}) + 50 = 1415$.) In Section 3, we study the asymptotic behavior of μ_d. Recently, Tran–Ziegler [6] gave a lower bound for μ_d by a random graph:

$$\varepsilon(G(d, 1/\sqrt{3})) = \frac{1}{54}d^4 + \frac{1}{18}d^3 - \frac{8}{27}d^2 + \frac{1}{3}d.$$

They also gave an upper bound for μ_d: $\mu_d \leq (\frac{1}{32} + o(1))d^4$. (However, this upper bound is not sharp. See [6, Remark].) In this paper, we succeeded in improving their lower bound by constructing a non-random graph (see Example 3.1) and a random graph whose complement is bipartite (see Theorem 3.2):

$$\varepsilon(\mathbb{G}) = \frac{5\sqrt{5} - 11}{8}\, d^4 - \frac{12\sqrt{5} - 27}{2} d^3 + \frac{19\sqrt{5} - 44}{2} d^2 + d,$$

where $\mathbb{G} = K_d - G(K_{d/2,d/2}, p)$ with $p = 3 - \sqrt{5}$. These results suggest the following:

Conjecture 1.2. *Let $G \in \Omega_d$ with $\mu_d = \varepsilon(G)$. Then, the complement of G is a bipartite graph.*

Note that, by Theorem 1.1, this conjecture is true for $3 \leq d \leq 14$.

2 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. The following lemma is studied in [5, Lemma 1.4].

Lemma 2.1. *Let e and f $(e \neq f)$ be edges of a graph $G \in \Omega_d$. Then, the convex hull of $\{\rho(e), \rho(f)\}$ is an edge of the edge polytope $\mathcal{P}_G$ if and only if one of the following conditions is satisfied.*

(i) *e and f have a common vertex in $[d]$.*

(ii) *$e = \{i, j\}$ and $f = \{k, l\}$ have no common vertices, and the induced subgraph of G on the vertex set $\{i, j, k, l\}$ has no cycles of length 4.*

The *complement* graph $\overline{G}$ of a graph $G \in \Omega_d$ is the graph whose vertex set is $[d]$ and whose edges are the non-edges of G. For a vertex i of a graph G, let $\deg_G(i)$ denote the degree of i in G. We translate Lemma 2.1 in terms of the complement $\overline{G}$ of G.

Lemma 2.2. *Let H be the complement of a graph $G \in \Omega_d$. Then, we have*

$$\begin{aligned}\varepsilon(G) &= \sum_{i=1}^{d} \binom{d-1-\deg_H(i)}{2} + a(H) + b(H) + c(H) \\ &= \varepsilon(K_d) + \frac{1}{2}\sum_{i=1}^{d} \deg_H^2(i) - (2d-3)|E(H)| + a(H) + b(H) + c(H),\end{aligned}$$

where $a(H)$, $b(H)$ and $c(H)$ are the number of induced subgraphs of H on 4 vertices of the form (a) *a path of length* 3; (b) *a cycle of length* 4; (c) *a path of length* 2 *and one isolated vertex, respectively.*

Proof. First, the number of pairs of edges satisfying Lemma 2.1 (i) is equal to

$$\begin{aligned}\sum_{i=1}^{d} \binom{d-1-\deg_H(i)}{2} &= \sum_{i=1}^{d} \frac{(d-1)(d-2) - (2d-3)\deg_H(i) + \deg_H^2(i)}{2} \\ &= \varepsilon(K_d) + \frac{1}{2}\sum_{i=1}^{d} \deg_H^2(i) - (2d-3)|E(H)|.\end{aligned}$$

Second, the number of pairs of edges satisfying Lemma 2.1 (ii) is equal to the number of the induced subgraphs W of G where W is one of the following: (a') W is a path of length 3; (b') W consists of two disjoint edges; (c') W is a graph on $\{i, j, k, \ell\}$ with $E(W) = \{\{i, j\}, \{j, k\}, \{i, k\}, \{k, \ell\}\}$. Note that each induced subgraph has exactly one such pair of edges. The complement of each (a'), (b'), and (c') is (a), (b) and (c), respectively. □

For a graph $H \in \Omega_r$ with $r \leq d$, let $K_d - H$ denote the graph $G \in \Omega_d$ such that $E(G) = E(K_d) \setminus E(H)$. Using Lemma 2.2, we have the following:

Proposition 2.3. *Let $H \in \Omega_r$ and let $\psi(H)$ denote the number of induced paths in H of length 2. Then, the function $\varphi(d) = \varepsilon(K_d - H) - \varepsilon(K_d)$ for $d = r, r+1, r+2, \ldots$ is a linear polynomial of d whose leading coefficient is $\psi(H) - 2|E(H)|$.*

Proof. Since d is a natural number it is sufficient to show that $\varphi(d+1) - \varphi(d) = \psi(H) - 2|E(H)|$ for any d. Let $H_1 = \overline{K_d - H}$ and $H_2 = \overline{K_{d+1} - H}$. Then, H_2 is obtained by adding one isolated vertex $d+1$ to H_1. Hence, it follows that $a(H_1) = a(H_2)$, $b(H_1) = b(H_2)$, $c(H_1) + \psi(H) = c(H_2)$ and $\deg_{H_1}(i) = \deg_{H_2}(i)$ for all $1 \leq i \leq d$. Thus, by Lemma 2.2, we have

$$\begin{aligned}
& \varphi(d+1) - \varphi(d) \\
= \; & \varepsilon(K_{d+1} - H) - \varepsilon(K_{d+1}) - \varepsilon(K_d - H) + \varepsilon(K_d) \\
= \; & \sum_{i=1}^{d+1} \binom{d - \deg_{H_2}(i)}{2} - \sum_{i=1}^{d} \binom{d - 1 - \deg_{H_1}(i)}{2} + \psi(H) \\
& + \frac{d(d-1)(d-2)}{2} - \frac{(d+1)d(d-1)}{2} \\
= \; & \binom{d}{2} + \sum_{i=1}^{d} \left(\binom{d - \deg_{H_1}(i)}{2} - \binom{d - 1 - \deg_{H_1}(i)}{2} \right) + \psi(H) - \frac{3d(d-1)}{2} \\
= \; & \binom{d}{2} + \sum_{i=1}^{d} (d - 1 - \deg_{H_1}(i)) + \psi(H) - \frac{3d(d-1)}{2} \\
= \; & \psi(H) - \sum_{i=1}^{d} \deg_{H_1}(i) \\
= \; & \psi(H) - 2|E(H)|,
\end{aligned}$$

as desired. □

Proposition 2.4. *Let $G \in \Omega_d$ and let $H_1, H_2, \ldots, H_m$ be all the nonempty connected components of $\overline{G}$. Then, $\varepsilon(K_d) - \varepsilon(G) = \sum_{j=1}^{m} (\varepsilon(K_d) - \varepsilon(K_d - H_j))$.*

Proof. Let $H = \overline{G}$ and let $H'_j = \overline{K_d - H_j}$ for $1 \leq j \leq m$. Then, it is easy to see that $|E(H)| = \sum_{j=1}^{m} |E(H'_j)|$, $\sum_{i=1}^{d} \deg_H^2(i) = \sum_{j=1}^{m} \sum_{i=1}^{d} \deg_{H'_j}^2(i)$, $a(H) = \sum_{j=1}^{m} a(H'_j)$, $b(H) = \sum_{j=1}^{m} b(H'_j)$, and $c(H) = \sum_{j=1}^{m} c(H'_j)$. Thus, by Lemma 2.2, we are done. □

A graph $G \in \Omega_d$ is called *bipartite* if $[d]$ admits a partition into two sets of vertices V_1 and V_2 such that, for every edge $\{i, j\}$ of G, either $i \in V_1, j \in V_2$ or $j \in V_1, i \in V_2$ is satisfied. A *complete bipartite* graph is a bipartite graph such that every pair of vertices i, j with $i \in V_1$ and $j \in V_2$ is adjacent. Let $K_{m,n}$ denote the complete bipartite graph with $|V_1| = m$ and $|V_2| = n$.

Proposition 2.5. *Let $G = K_d - K_{m,n}$ such that $m + n \leq d$ and $m, n \geq 1$. Then,*

$$\varepsilon(G) - \varepsilon(K_d) = \frac{1}{2} mn(m+n-6)d - \frac{1}{4} mn(3mn + 2m^2 + 2n^2 - 5m - 5n - 13).$$

Proof. Let $H = K_{m,n}$. Then,

$$\psi(H) - 2|E(H)| = m\binom{n}{2} + n\binom{m}{2} - 2mn = \frac{1}{2}mn(m+n-6).$$

Moreover, since $K_{m+n} - K_{m,n}$ is the disjoint union of K_m and K_n, we have

$$\begin{aligned}
\varphi(m+n) &= \frac{m(m-1)(m-2)}{2} + \frac{n(n-1)(n-2)}{2} + \binom{m}{2}\binom{n}{2} \\
&\quad - \frac{(m+n)(m+n-1)(m+n-2)}{2} \\
&= \frac{1}{4}mn(mn - 7m - 7n + 13)
\end{aligned}$$

by Lemma 2.1. Hence, by Proposition 2.3,

$$\begin{aligned}
\varepsilon(G) - \varepsilon(K_d) &= \frac{1}{2}mn(m+n-6)(d-(m+n)) + \frac{1}{4}mn(mn-7m-7n+13) \\
&= \frac{1}{2}mn(m+n-6)d - \frac{1}{4}mn(3mn + 2m^2 + 2n^2 - 5m - 5n - 13),
\end{aligned}$$

as desired. □

Let $k_3(H)$ denote the number of triangles (i.e., cycles of length 3) of H. The following lemma is important.

Lemma 2.6. *Let H be the complement graph of $G \in \Omega_d$. Then, we have*

$$\varepsilon(G) \leq \varepsilon(K_d) + \frac{d^2 - 16d + 29}{7}|E(H)| - \frac{3}{7}(d-8)k_3(H).$$

Proof. The number of pairs of edges satisfying Lemma 2.1 (i) is, by Lemma 2.2, $\varepsilon(K_d) - (2d-3)|E(H)| + \frac{1}{2}\sum_{i=1}^{d}\deg_H^2(i)$. For an edge $\{i,j\}$ of H, let $k_3(i,j)$ be the number of triangles in H containing $\{i,j\}$. We define three subsets of $[d] \setminus \{i,j\}$:

$$\begin{aligned}
X_{i,j} &= \{\ell \in [d] \setminus \{i,j\} : \{i,\ell\} \in E(H), \{j,\ell\} \notin E(H)\}, \\
Y_{i,j} &= \{\ell \in [d] \setminus \{i,j\} : \{j,\ell\} \in E(H), \{i,\ell\} \notin E(H)\}, \\
Z_{i,j} &= \{\ell \in [d] \setminus \{i,j\} : \{i,\ell\} \notin E(H), \{j,\ell\} \notin E(H)\}.
\end{aligned}$$

It then follows that, $|X_{i,j}| + |Y_{i,j}| + |Z_{i,j}| + k_3(i,j) = d-2$, and

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{d}\deg_H^2(i) &= \frac{1}{2}\sum_{\{i,j\}\in E(H)}(\deg_H(i) + \deg_H(j)) \\
&= \frac{1}{2}\sum_{\{i,j\}\in E(H)}(|X_{i,j}| + |Y_{i,j}| + 2k_3(i,j) + 2) \\
&= |E(H)| + 3k_3(H) + \frac{1}{2}\sum_{\{i,j\}\in E(H)}(|X_{i,j}| + |Y_{i,j}|).
\end{aligned}$$

Second, we count the number of pairs satisfying Lemma 2.1 (ii). By Lemma 2.2, this number is equal to $a(H) + b(H) + c(H)$. Here, we count the number of the induced subgraphs H' of type (a), (b) and (c) containing an edge $e = \{i, j\}$ of H. If e is an edge of H', then the other two vertices ℓ and m of H' satisfy exactly one of the following conditions:

(i) $\ell \in X_{i,j}, m \in Y_{i,j}$;

(ii) $\ell \in Y_{i,j}, m \in Z_{i,j}$;

(iii) $\ell \in Z_{i,j}, m \in X_{i,j}$.

If i, j, ℓ, m satisfy condition (i), then one of the following holds:

- H' is a path (e_1, e_2, e_3) and $e = e_2$ (type (a)) ;
- H' is a cycle of length 4 and e is one of four edges (type (b)).

It then follows that

$$a(H) + 4b(H) = \sum_{\{i,j\}\in E(H)} |X_{i,j}||Y_{i,j}|.$$

If i, j, ℓ, m satisfy either condition (ii) or (iii), then one of the following holds:

- H' is a path (e_1, e_2, e_3) and $e \in \{e_1, e_3\}$ (type (a)) ;
- H' is a path (e_1, e_2) with one isolated vertex and $e \in \{e_1, e_2\}$ (type (c)).

It then follows that

$$2a(H) + 2c(H) = \sum_{\{i,j\}\in E(H)} \left(|Y_{i,j}||Z_{i,j}| + |Z_{i,j}||X_{i,j}|\right).$$

Thus, we have

$$a(H)+b(H)+c(H) = -\frac{a(H)}{4} + \sum_{\{i,j\}\in E(H)} \left(\tfrac{1}{4}|X_{i,j}||Y_{i,j}| + \tfrac{1}{2}|Y_{i,j}||Z_{i,j}| + \tfrac{1}{2}|Z_{i,j}||X_{i,j}|\right).$$

Subject to $|X_{i,j}| + |Y_{i,j}| + |Z_{i,j}| = d - 2 - k_3(i, j)$, we study an upper bound of

$$\alpha = \sum_{\{i,j\}\in E(H)} \left(\frac{|X_{i,j}| + |Y_{i,j}|}{2} + \frac{1}{4}|X_{i,j}||Y_{i,j}| + \frac{1}{2}|Y_{i,j}||Z_{i,j}| + \frac{1}{2}|Z_{i,j}||X_{i,j}|\right).$$

Each summand of α satisfies

$$\begin{aligned}
& \frac{|X_{i,j}| + |Y_{i,j}|}{2} + \frac{1}{4}|X_{i,j}||Y_{i,j}| + \frac{1}{2}|Y_{i,j}||Z_{i,j}| + \frac{1}{2}|Z_{i,j}||X_{i,j}| \\
= \; & \frac{1}{4}|X_{i,j}||Y_{i,j}| + \frac{1}{2}(|X_{i,j}| + |Y_{i,j}|)(d - 1 - k_3(i, j) - (|X_{i,j}| + |Y_{i,j}|)) \\
\leq \; & \frac{1}{4}\left(\frac{|X_{i,j}| + |Y_{i,j}|}{2}\right)^2 + \frac{1}{2}(|X_{i,j}| + |Y_{i,j}|)(d - 1 - k_3(i, j) - (|X_{i,j}| + |Y_{i,j}|)) \\
= \; & -\frac{7}{16}(|X_{i,j}| + |Y_{i,j}|)^2 + \frac{d - 1 - k_3(i, j)}{2}(|X_{i,j}| + |Y_{i,j}|).
\end{aligned}$$

The last function has the maximum number $\frac{1}{7}(d-1-k_3(i,j))^2$ when $|X_{i,j}|+|Y_{i,j}| = \frac{4}{7}(d-1-k_3(i,j))$. Hence,

$$\begin{aligned}\sum_{\{i,j\}\in E(H)} \frac{1}{7}(d-1-k_3(i,j))^2 &\leq \sum_{\{i,j\}\in E(H)} \frac{1}{7}(d-1)(d-1-k_3(i,j)) \\ &= \frac{1}{7}\sum_{\{i,j\}\in E(H)}(d-1)^2 - \frac{1}{7}\sum_{\{i,j\}\in E(H)}(d-1)k_3(i,j) \\ &= \frac{1}{7}(d-1)^2|E(H)| - \frac{3}{7}(d-1)k_3(H)\end{aligned}$$

is an upper bound of α. Thus,

$$\varepsilon(K_d)-(2d-3)|E(H)|+|E(H)|+3k_3(H)+\frac{1}{7}(d-1)^2|E(H)|-\frac{3}{7}(d-1)k_3(H)$$

is an upper bound of $\varepsilon(G)$ as desired. $\square$

Using Proposition 2.5 and Lemma 2.6, we prove Theorem 1.1.

Proof of Theorem 1.1. (a) Let $3 \leq d \leq 13$ and $G \in \Omega_d$ with $G \neq K_d$. If $d = 3$, then $\varepsilon(G) < \varepsilon(K_d)$ is trivial. If $d = 4$, then $\varepsilon(K_4) = 12$. Since $|E(G)| < 6$, we have $\varepsilon(G) \leq \binom{5}{2} = 10 < \varepsilon(K_4)$. Let $d \geq 5$ and let H be the complement graph of G. By Lemma 2.6,

$$\varepsilon(G)-\varepsilon(K_d) \leq \frac{d^2-16d+29}{7}|E(H)| - \frac{3}{7}(d-8)k_3(H).$$

If $8 \leq d \leq 13$, then $\varepsilon(G)-\varepsilon(K_d) < 0$ since $\frac{d^2-16d+29}{7} < 0$, $|E(H)| > 0$ and $k_3(H) \geq 0$. Let $5 \leq d \leq 7$. Then,

$$\varepsilon(G)-\varepsilon(K_d) \leq \begin{cases} -\frac{26}{7}|E(H)|+\frac{9}{7}k_3(H) & \text{if } d=5, \\ -\frac{31}{7}|E(H)|+\frac{6}{7}k_3(H) & \text{if } d=6, \\ -\frac{34}{7}|E(H)|+\frac{3}{7}k_3(H) & \text{if } d=7. \end{cases}$$

Hence, if $k_3(H) \leq 2$, then $\varepsilon(G)-\varepsilon(K_d)$ is negative. On the other hand, if $k_3(H) \geq 3$, then $|E(H)| \geq 5$. Since $k_3(H) \leq \binom{d}{3}$, it follows that $\varepsilon(G)-\varepsilon(K_d)$ is negative.

(b) Let $G \in \Omega_{14}$ with $G \neq K_{14}$ and let $H = \overline{G}$. We need to evaluate the function which appears in the proof of Lemma 2.6 more accurately by focusing on $d = 14$. Let $|Z_{i,j}| = 12 - k_3(i,j) - |X_{i,j}| - |Y_{i,j}|$ and

$$\begin{aligned} f &= \frac{|X_{i,j}|+|Y_{i,j}|}{2} + \frac{1}{4}|X_{i,j}||Y_{i,j}| + \frac{1}{2}|Y_{i,j}||Z_{i,j}| + \frac{1}{2}|Z_{i,j}||X_{i,j}| \\ g &= -\frac{7}{16}(|X_{i,j}|+|Y_{i,j}|)^2 + \frac{13-k_3(i,j)}{2}(|X_{i,j}|+|Y_{i,j}|) \end{aligned}$$

be functions of $|X_{i,j}|$ and $|Y_{i,j}|$. Recall that $f \leq g \leq \frac{1}{7}(13-k_3(i,j))^2$ and $g = \frac{1}{7}(13-k_3(i,j))^2$ when $|X_{i,j}|+|Y_{i,j}| = \frac{4}{7}(13-k_3(i,j))$. If $1 \leq k_3(i,j) \leq 12$, then

$$\frac{1}{7}(13-k_3(i,j))^2 = 24-\frac{13}{7}k_3(i,j)-\frac{11}{7}+\frac{1}{7}(k_3(i,j)-1)(k_3(i,j)-12) < 24-\frac{13}{7}k_3(i,j).$$

If $k_3(i,j)=0$, then $\frac{1}{7}(13-k_3(i,j))^2 = 24+1/7$. However, since

$$4\left(\frac{|X_{i,j}|+|Y_{i,j}|}{2}+\frac{1}{4}|X_{i,j}||Y_{i,j}|+\frac{1}{2}|Y_{i,j}||Z_{i,j}|+\frac{1}{2}|Z_{i,j}||X_{i,j}|\right)$$

is an integer, the value of f is at most 24 if $|X_{i,j}|$ and $|Y_{i,j}|$ are non-negative integers. Thus, for $k_3(i,j)=0,1,\dots,12$, the value of f is at most $24-\frac{13}{7}k_3(i,j)$ if $|X_{i,j}|$ and $|Y_{i,j}|$ are non-negative integers. Thus, by the same argument in the proof of Lemma 2.6, $\varepsilon(G)-\varepsilon(K_{14})$ is at most

$$-24|E(H)|+3k_3(H)+24|E(H)|-\frac{3\cdot 13}{7}k_3(H)-\frac{a(H)}{4}=-\frac{18}{7}k_3(H)-\frac{a(H)}{4}\le 0.$$

Therefore, $\varepsilon(G)\le\varepsilon(K_{14})$.

Suppose that $\varepsilon(G)=\varepsilon(K_{14})$. Then, $-\frac{18}{7}k_3(H)-\frac{a(H)}{4}\ge 0$. Since $k_3(H),a(H)\ge 0$, we have $k_3(H)=a(H)=0$. Moreover,

$$\frac{|X_{i,j}|+|Y_{i,j}|}{2}+\frac{1}{4}|X_{i,j}||Y_{i,j}|+\frac{1}{2}|Y_{i,j}||Z_{i,j}|+\frac{1}{2}|Z_{i,j}||X_{i,j}|=24$$

and $|X_{i,j}|+|Y_{i,j}|+|Z_{i,j}|=12$ for an arbitrary edge $\{i,j\}$ of H. It is easy to see that $|X_{i,j}|+|Y_{i,j}|\in\{7,8\}$. It then follows that, for an arbitrary $\{i,j\}\in E(H)$, $(|X_{i,j}|,|Y_{i,j}|,|Z_{i,j}|)\in\{(3,4,5),(4,3,5),(4,4,4)\}$. In particular, the degree of each vertex is either 0, 4 or 5. Moreover, since $k_3(H)=0$, $\{j\}\cup X_{i,j}$ and $\{i\}\cup Y_{i,j}$ are independent sets. Hence, by $a(H)=0$, the induced subgraph of H on $\{i,j\}\cup X_{i,j}\cup Y_{i,j}$ is the complete bipartite graph $K_{|X_{i,j}|+1,|Y_{i,j}|+1}$.

Suppose that an edge $\{i,j\}$ of H satisfies $(|X_{i,j}|,|Y_{i,j}|,|Z_{i,j}|)=(4,4,4)$. Then, the induced subgraph of H on $\{i,j\}\cup X_{i,j}\cup Y_{i,j}$ is $K_{5,5}$. Since the degree of any vertex of H is either, 0, 4 or 5, other four vertices are isolated. Therefore, $G=K_{14}-K_{5,5}$.

It is enough to consider the case that $(|X_{s,t}|,|Y_{s,t}|,|Z_{s,t}|)\ne(4,4,4)$ holds for every edge $\{s,t\}$. Suppose that $(|X_{i,j}|,|Y_{i,j}|)=(3,4)$. Then, the induced subgraph of H on $\{i,j\}\cup X_{i,j}\cup Y_{i,j}$ is $K_{4,5}$. Since $(|X_{s,t}|,|Y_{s,t}|,|Z_{s,t}|)\ne(4,4,4)$ for each edge $\{s,t\}$, the degree of every vertex in $\{i\}\cup Y_{i,j}$ is 4. In this case, $K_{4,5}$ is a connected component of H. Since the degree of other five vertices is at most 4, it follows that they are isolated vertices. Therefore, $G=K_{14}-K_{4,5}$.

(c) Let $d\ge 15$ and let $G=K_d-K_{m,n}\in\Omega_d$. By Proposition 2.5, we have

$$\varepsilon(G)-\varepsilon(K_d)=\frac{1}{2}mn(m+n-6)d-\frac{1}{4}mn(3mn+2m^2+2n^2-5m-5n-13).$$

When $m=n=5$, we obtain $\varepsilon(G)-\varepsilon(K_d)=50(d-14)>0$ as desired. □

3 Asymptotic behavior of μ_d

For $0<p<1$ and an integer $d>0$, let $G(d,p)$ denote the random graph on the vertex set $[d]$ in which the edges are chosen independently with probability p. For a graph H on the vertex set $[d]$ and $0<p<1$, let $G(H,p)$ denote the random graph on the vertex set $[d]$ in which the edges of H are chosen independently with probability p and the edges not belonging to H are not chosen. Tran–Ziegler [6] showed that, for the random graph $G(d,1/\sqrt{3})$,

$$\varepsilon(G(d,1/\sqrt{3}))=\frac{1}{54}d^4+\frac{1}{18}d^3-\frac{8}{27}d^2+\frac{1}{3}d,$$

and hence this is a lower bound for μ_d.

First, for $d \gg 0$, we give an example of a (non-random) graph G on the vertex set $[d]$ such that $\varepsilon(G) > \varepsilon(G(d, 1/\sqrt{3}))$.

Example 3.1. Let $G = K_d - K_{ad,ad} - K_{(1/2-a)d,(1/2-a)d}$ where $a = \frac{1}{28}(7+\sqrt{21})$ and $d \gg 0$. By Propositions 2.4 and 2.5, it follows that

$$\varepsilon(G) = \frac{9}{448}d^4 + \frac{1}{7}d^3 - \frac{103}{112}d^2 + d.$$

Since $1/54 \fallingdotseq 0.0185$ and $9/448 \fallingdotseq 0.0201$, we have $\varepsilon(G) > \varepsilon(G(d, 1/\sqrt{3}))$ for $d \gg 0$.

Second, we give a random graph $\mathbb{G}$ on the vertex set $[d]$ such that $\varepsilon(\mathbb{G}) > \varepsilon(G(d, 1/\sqrt{3}))$ for $d \gg 0$.

Theorem 3.2. *For an integer d, let $\mathbb{G}$ be a random graph $K_d - G(K_{d/2,d/2}, p)$ with $p = 3 - \sqrt{5}$. Then,*

$$\varepsilon(\mathbb{G}) = \frac{5\sqrt{5}-11}{8}\, d^4 - \frac{12\sqrt{5}-27}{2}d^3 + \frac{19\sqrt{5}-44}{2}d^2 + d.$$

In particular, we have $\varepsilon(\mathbb{G}) > \varepsilon(G(d, 1/\sqrt{3}))$ for all $d \gg 0$.

Proof. Let $m = d/2$ and let $[d] = V_1 \cup V_2$ be a partition of the vertex set of $K_{m,m}$. The number of pairs of edges $\{i,j\}, \{i,k\}$ satisfying Lemma 2.1 (i) is

$$\eta_1 = m(m-1)(m-2) + 2m^2(m-1)(1-p) + m^2(m-1)(1-p)^2$$

where each term corresponds to the case when (i) $i, j, k \in V_s$, (ii) $i, j \in V_s$, $k \notin V_s$ and (iii) $i \in V_s$, $j, k \notin V_s$, respectively.

Next, we study the number of pairs of edges $\{i,j\}, \{k,\ell\}$ satisfying Lemma 2.1 (ii). Let $\mathbb{G}_{ijk\ell}$ denote the induced subgraph of $\mathbb{G}$ on the vertex set $\{i,j,k,\ell\} \subset [d]$. If either "$i,j,k,\ell \in V_s$" or "$i,\ell \in V_s$ and $j,k \notin V_s$" holds, then $\{i,j,k,\ell\}$ is a cycle of $\mathbb{G}_{ijk\ell}$ whenever $\{i,j\}, \{k,\ell\} \in E(\mathbb{G})$. Hence, we consider the following two cases:

Case 1. Suppose $i, j \in V_s$ and $k, \ell \notin V_s$. Then, $\mathbb{G}_{ijk\ell}$ has a cycle of length 4 if and only if either $\{i,k\}, \{j,\ell\} \in E(\mathbb{G})$ or $\{i,\ell\}, \{j,k\} \in E(\mathbb{G})$ holds. Thus, the expected number of pairs of edges is $\eta_2 = \binom{m}{2}^2(1-(1-p)^2)^2$.

Case 2. Suppose that $i \in V_s$ and $j, k, \ell \notin V_s$ hold. Then, all of $\{k,\ell\}$, $\{j,k\}$ and $\{j,\ell\}$ are edges of $\mathbb{G}$. On the other hand, $\{i,j\}$ is an edge of $\mathbb{G}$ with probability $1-p$. If $\{i,j\}$ is an edge of $\mathbb{G}$, then $\mathbb{G}_{ijk\ell}$ has a cycle of length 4 if and only if either $\{i,k\} \in E(\mathbb{G})$ or $\{i,\ell\} \in E(\mathbb{G})$ holds. Thus, the expected number of pairs of edges is $\eta_3 = m^2(m-1)(m-2)(1-p)p^2$.

Therefore, $\varepsilon(\mathbb{G}) = \eta_1 + \eta_2 + \eta_3$. If $m = d/2$ and $p = 3 - \sqrt{5}$, then

$$\varepsilon(\mathbb{G}) = \frac{5\sqrt{5}-11}{8}\, d^4 - \frac{12\sqrt{5}-27}{2}d^3 + \frac{19\sqrt{5}-44}{2}d^2 + d,$$

whose leading coefficient is $\frac{5\sqrt{5}-11}{8} \fallingdotseq 0.0225425$. $\square$

Remark 3.3. By Theorem 3.2, the graph G in Example 3.1 does not satisfy $\mu_d = \varepsilon(G)$ for $d \gg 0$. In fact, for $d = 20$, by Propositions 2.4 and 2.5, it follows that

$$\max\left\{\varepsilon(G) : \begin{array}{c} G \in \Omega_{20} \text{ and each non-empty connected} \\ \text{component of } \overline{G} \text{ is a complete bipartite graph} \end{array}\right\} = 4176.$$

Let $G' \in \Omega_{20}$ be the graph such that $\overline{G'}$ is the bipartite graph with $E(\overline{G'}) =$

$$\begin{gathered}\{\{1,12\},\{1,14\},\{1,15\},\{1,16\},\{1,18\},\{1,19\},\{1,20\},\{2,11\},\{2,12\},\{2,13\},\{2,15\},\\ \{2,17\},\{2,19\},\{2,20\},\{3,11\},\{3,12\},\{3,13\},\{3,14\},\{3,15\},\{3,16\},\{3,18\},\{4,14\},\\ \{4,15\},\{4,16\},\{4,17\},\{4,18\},\{4,19\},\{4,20\},\{5,11\},\{5,12\},\{5,13\},\{5,15\},\{5,17\},\\ \{5,18\},\{5,20\},\{6,12\},\{6,16\},\{6,17\},\{6,18\},\{6,19\},\{6,20\},\{7,11\},\{7,12\},\{7,13\},\\ \{7,14\},\{7,16\},\{7,17\},\{7,19\},\{8,11\},\{8,12\},\{8,13\},\{8,14\},\{8,15\},\{8,18\},\{8,19\},\\ \{8,20\},\{9,11\},\{9,14\},\{9,15\},\{9,16\},\{9,17\},\{9,18\},\{9,19\},\{10,11\},\{10,13\},\{10,15\},\\ \{10,16\},\{10,18\},\{10,19\},\{10,20\}\}.\end{gathered}$$

Then, $\varepsilon(G') = 4203 > 4176$.

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Equitable coloring of corona products of cubic graphs is harder than ordinary coloring*

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Abstract

A graph is equitably k-colorable if its vertices can be partitioned into k independent sets in such a way that the number of vertices in any two sets differ by at most one. The smallest k for which such a coloring exists is known as the *equitable chromatic number* of G and it is denoted by $\chi_=(G)$. In this paper the problem of determinig $\chi_=$ for coronas of cubic graphs is studied. Although the problem of ordinary coloring of coronas of cubic graphs is solvable in polynomial time, the problem of equitable coloring becomes NP-hard for these graphs. We provide polynomially solvable cases of coronas of cubic graphs and prove the NP-hardness in a general case. As a by-product we obtain a simple linear time algorithm for equitable coloring of such graphs which uses $\chi_=(G)$ or $\chi_=(G)+1$ colors. Our algorithm is best possible, unless P=NP. Consequently, cubical coronas seem to be the only known class of graphs for which equitable coloring is harder than ordinary coloring.

Keywords: Corona graph, cubic graph, equitable chromatic number, equitable graph coloring, NP-hardness, polynomial algorithm.

Math. Subj. Class.: 05C15, 05C10

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1 Introduction

All graphs considered in this paper are connected, finite and simple, i.e. undirected, loopless and without multiple edges, unless otherwise is stated.

If the set of vertices of a graph G can be partitioned into k (possibly empty) classes $V_1, V_2, \ldots, V_k$ such that each V_i is an independent set and the condition $\big||V_i| - |V_j|\big| \leq 1$ holds for every pair (i, j), then G is said to be *equitably k-colorable*. If $|V_i| = l$ for every $i = 1, 2, \ldots, k$, then G on $n = kl$ vertices is said to be *strong equitably k-colorable*. The smallest integer k for which G is equitably k-colorable is known as the *equitable chromatic number* of G and it is denoted by $\chi_=(G)$ [14]. Since equitable coloring is a proper coloring with an additional constraint, we have $\chi(G) \leq \chi_=(G)$ for any graph G.

The notion of equitable colorability was introduced by Meyer [14]. However, an earlier work of Hajnal and Szemerédi [9] showed that a graph G with maximal degree Δ is equitably k-colorable if $k \geq \Delta + 1$. Recently, Kierstead et al. [11] have given an $O(\Delta n^2)$-time algorithm for obtaining a $(\Delta + 1)$-coloring of a graph G on n vertices.

This model of graph coloring has many practical applications. Every time when we have to divide a system with binary conflict relations into equal or almost equal conflict-free subsystems we can model this situation by means of equitable graph coloring. In particular, one motivation for equitable coloring suggested by Meyer [14] concerns scheduling problems. In this application, the vertices of a graph represent a collection of tasks to be performed and an edge connects two tasks that should not be performed at the same time. A coloring of this graph represents a partition of tasks into subsets that may be performed simultaneously. Due to load balancing considerations, it is desirable to perform equal or nearly-equal numbers of tasks in each time slot, and this balancing is exactly what equitable colorings achieve. Furmańczyk [5] mentions a specific application of this type of scheduling problem, namely, assigning university courses to time slots in a way that avoids scheduling incompatible courses at the same time and spreads the courses evenly among the available time slots.

The topic of equitable coloring was widely discussed in literature. It was considered for some particular graph classes and also for several graph products: cartesian, weak or strong tensor products [13, 5] as well as for coronas [6, 10]. Graph products are interesting and useful in many situations. The complexity of many problems, also equitable coloring, that deal with very large and complicated graphs is reduced greatly if one is able to fully characterize the properties of less complicated prime factors. Moreover, corona graphs lie often close to the boundary between easy and hard problems.

The *corona* of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of G, called the *center graph*, $|V(G)|$ copies of H, named the *outer graph*, and making the i-th vertex of G adjacent to every vertex in the i-th copy of H. Such type of graph products was introduced by Frucht and Harary in 1970 [4] (for an example see Fig. 1). After that many works have been devoted to study its structure and to obtain some relationships between the corona graph and its factors [1, 4, 12, 15].

In general, the problem of optimal equitable coloring, in the sense of the number of colors used, is NP-hard and remains so for corona products of graphs. In fact, Furmańczyk et al. [6] proved that the problem of deciding whether $\chi_=(G \circ K_2) \leq 3$ is NP-complete even if G is restricted to the line graph of a cubic graph.

Let us recall some basic facts concerning cubic graphs. It is well known from Brook's theorem [2] that for any cubic graph $G \neq K_4$, we have $\chi(G) \leq 3$. On the other hand, Chen et al. [3] proved that for any cubic graph with $\chi(G) = 3$, its equitable chromatic number equals 3 as well. Moreover, since a connected cubic graph G with $\chi(G) = 2$ is a bipartite graph with partition sets of equal size, we have the equivalence of the classical and equitable chromatic numbers for 2-chromatic cubic graphs. Since the only cubic graph for which the chromatic number is equal to 4 is the complete graph K_4, we have

$$2 \leq \chi_=(G) = \chi(G) \leq 4, \tag{1.1}$$

for any cubic graph G.

In the paper we will consider the equitable coloring of coronas. We assume that in corona $G \circ H$, $|V(G)| = n$ and $|V(H)| = m$. A vertex with color i is called an *i-vertex*. We use color 4 instead of 0, in all colorings in the paper, including cases when color label is implied by an expresion $(\mathrm{mod} 4)$.

Let

- Q_2 denote the class of equitably 2-chromatic cubic graphs,
- Q_3 denote the class of equitably 3-chromatic cubic graphs,
- Q_4 denote the class of equitably 4-chromatic cubic graphs.

  Clearly, $Q_4 = \{K_4\}$.

Next, let $Q_2(t) \subset Q_2$ ($Q_3(t) \subset Q_3$) denote the class of bipartite (tripartite) cubic graphs with partition sets of cardinality t, and let $Q_3(u, v, w) \subset Q_3$ denote the class of 3-partite graphs with color classes of cardinalities u, v and w, respectively, where $u \geq v \geq w \geq u - 1$. Observe that

$$\chi(K_4 \circ H) = \begin{cases} 4 & \text{if } H \in Q_2, \\ \chi(H) + 1 & \text{otherwise.} \end{cases} \tag{1.2}$$

In the next section we show a way to color $G \circ H$ with 3 colors provided that the corona admits such a coloring. Next, in Section 3 we give a linear-time procedure for coloring corona products of cubic graphs with 5 colors. It turns out that this number of colors is sufficent for equitable coloring of any corona of cubic graphs, but in some cases less than 5 colors suffice. In Section 4 we give our main result that deciding whether $G \circ H$ is equitably 4-colorable is NP-complete when $H \in Q_3(t)$ and 10 divides t, in symbols $10|t$. Hence, our 5-coloring algorithm of Section 3 is 1-absolute approximate and the problem of equitable coloring of cubical coronas belongs to very few NP-hard problems that have approximation algorithms of this kind. Most of our results are summarized in Table 1.

G \ H	Q_2	Q_3	Q_4
Q_2	3 or 4 [Thm. 2.3]	4 or 5* [Thms. 3.3, 4.3]	5 [Thm. 3.2]
Q_3	3 or 4 [Thm. 2.3]	4 or 5* [Thm. 3.4, Col. 4.4]	5 [Thm. 3.2]
Q_4	4 [Thm. 2.3]	4	5 [Thm. 3.2]

Table 1: Possible values of $\chi_=(G \circ H)$, where G and H are cubic graphs. Asterix (*) means that deciding this case is NP-complete.

To the best of our knowledge, cubical coronas are so far the only class of graphs for which equitable coloring is harder than ordinary coloring. And, since $\chi_=(G \circ H) \leq 5$ and

$\Delta(G \circ H) \geq 7$, our results confirm Meyer's Equitable Coloring Conjecture [14], which claims that for any connected graph G, other than a complete graph or an odd cycle, we have $\chi_=(G) \leq \Delta$.

2 Equitable 3-coloring of corona of cubic graphs

First, let us recall a result concerning coronas $G \circ H$, where H is a 2- or 3-partite graph.

Theorem 2.1 ([6]). *Let G be an equitably k-colorable graph on $n \geq k$ vertices and let H be a $(k-1)$-partite graph. If $k|n$, then*

$$\chi_=(G \circ H) \leq k.$$

Proposition 2.2. *If G and H are cubic graphs, then $\chi_=(G \circ H) = 3$ if and only if $G \in Q_2 \cup Q_3$, $H \in Q_2$, and G has a strong equitable 3-coloring.*

Proof. ($\Leftarrow$) Since G is strong equitably 3-colorable, the cardinality of its vertex set must be divisible by 3. The thesis follows now from Theorem 2.1.
($\Rightarrow$) Assume that $\chi_=(G \circ H) = 3$. This implies:

- H must be 2-chromatic, and due to (1.1) it must be also equitably 2-chromatic,
- G must be 3-colorable (not necessarily equitably), $\chi(G) \leq \chi_=(G) \leq 3$, which implies $G \in Q_2 \cup Q_3$.

Otherwise, we would have $\chi(G \circ H) \geq 4$ which is a contradiction.

Since $H \in Q_2$ is connected, its bipartition is determined. Let $H \in Q_2(t)$, $t \geq 3$. Observe that every 3-coloring of G determines a 3-partition of $G \circ H$. Let us consider any 3-coloring of G with color classes of cardinality n_1, n_2 and n_3, respecively, where $n = n_1 + n_2 + n_3$. Then the cardinalities of color classes in the implied 3-coloring of $G \circ H$ form a sequence $((n_2 + n_3)t, (n_1 + n_3)t, (n_1 + n_2)t)$. Such a 3-coloring of $G \circ H$ is equitable if and only if $n_1 = n_2 = n_3$. This means that G must have a strong equitable 3-coloring, which, keeping in mind that $\chi_=(G \circ H) \geq 3$ for all cubic graphs G and H, completes the proof. □

In the remaining cases of coronas $G \circ H$, where $H \in Q_2$, we have to use more than three colors. However, it turns out that in all such cases four colors suffice.

Theorem 2.3. *If G is a cubic graph, $H \in Q_2$, then*

$$\chi_=(G \circ H) = \begin{cases} 3 & \textit{if } G \in Q_2(s) \cup Q_3, 3|s \textit{ and } G \textit{ is equitably 3-colorable}, \\ 4 & \textit{otherwise.} \end{cases}$$

Proof. Due to Proposition 2.2, we only have to define an equitable 4-coloring of $G \circ H$. The cases of $G \in Q_2 \cup Q_4$ are easy. We start from an equitable 4-coloring of the center graph and extend it to the corona.

Let us assume that $G \in Q_3$. First, we color equitably G with 3 colors and then extend this coloring to equitable 4-coloring of $G \circ H$, $H = H(U, V) \in Q_2(t)$. Since the number of vertices of cubic graph G is even, we have to consider two cases.

Case 1: $n = 4k$, for some $k \geq 2$.

Since G is equitably 3-colorable, the color classes of equitable 3-coloring of G are of cardinalities $\lceil 4k/3 \rceil$, $\lceil (4k-1)/3 \rceil$ and $\lceil (4k-2)/3 \rceil$, respectively. And, since $|V(G \circ H)| = 4k(2t+1)$, in every equitable 4-coloring of $G \circ H$ each color class must be of cardinality $2kt + k$.

We extend our 3-coloring of G to $G \circ H$ as follows (see Fig. 1a)). We color:

- the vertices in one copy of H linked to a 1-vertex in G using t times color 3 (vertices in partition U), $t - (\lceil (4k-1)/3 \rceil - k)$ times color 2 and $\lceil (4k-1)/3 \rceil - k$ times color 4 (vertices in partition V),
- the vertices in one copy of H linked to a 2-vertex in G using t times color 1 (vertices in partition U), $t - (\lceil (4k-2)/3 \rceil - k)$ times color 3 and $\lceil (4k-2)/3 \rceil - k$ times color 4 (vertices in partition V),
- the vertices in one copy of H linked to a 3-vertex in G using t times color 2 (vertices in partition U), $t - (\lceil 4k/3 \rceil - k)$ times color 1 and $\lceil 4k/3 \rceil - k$ times color 4 (vertices in partition V).

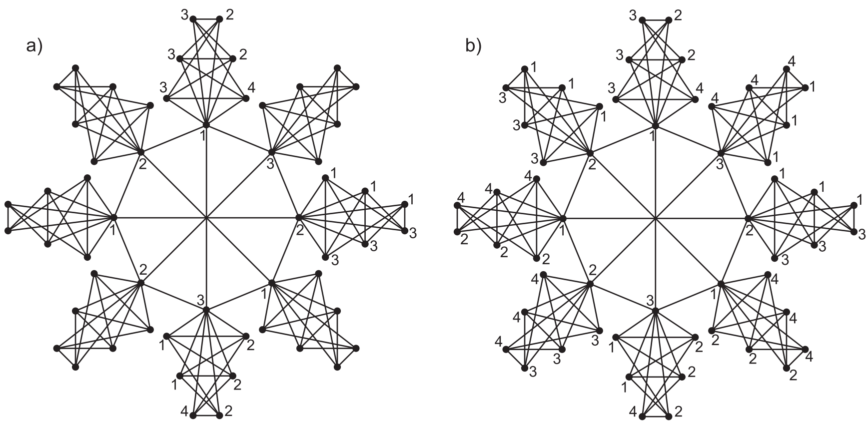

Figure 1: An example of coloring of $W \circ K_{3,3}$, where W is the Wagner graph (C_8 with 4 diagonals): a) partial 4-coloring; b) equitable 4-coloring.

So far, colors 1, 2 and 3 have been used $2t + k$ times, while color 4 has been used k times.

Now, we color each of uncolored copy of H with two out of three allowed colors in such a way that in this step colors 1, 2 and 3 are used $(2k-2)t$ times and color 4 is used $2kt$ times, which results in an equitable 4-coloring of the whole corona $G \circ H$ (see Fig. 1b)).

Case 2: $n = 4k + 2$, for some $k \geq 1$.

Since G is equitably 3-colorable, its color classes are of cardinalities $\lceil (4k+2)/3 \rceil$, $\lceil (4k+1)/3 \rceil$ and $\lceil 4k/3 \rceil$, respectively, in any equitable coloring of G. Since $|V(G \circ H)| = (4k+2)(2t+1) = 8kt + 4t + 4k + 2$, in every equitable 4-coloring the color classes must be of cardinality $2kt + t + k$ or $2kt + t + k + 1$.

We color:

- the vertices in one copy of H linked to a 1-vertex of G using t times color 3 (vertices in partition U), $t - (\lceil (4k+1)/3 \rceil - k - 1)$ times color 2 and $\lceil (4k+1)/3 \rceil - k - 1$ times color 4 (vertices in partition V),
- the vertices in one copy of H linked to a 2-vertex of G using t times color 1 (vertices in partition U), $t - (\lceil 4k/3 \rceil - k)$ times color 3 and $\lceil 4k/3 \rceil - k$ times color 4 (vertices in partition V),
- the vertices in one copy of H linked to a 3-vertex of G using t times color 2 (vertices in partition U), $t - (\lceil (4k+2)/3 \rceil - k - 1)$ times color 1 and $\lceil (4k+2)/3 \rceil - k - 1$ times color 4 (vertices in partition V).

So far, colors 1 and 2 have been used $2t + k + 1$ times, while color 3 has been used $2t + k$ times and color 4 has been used k times.

Finally, we color still uncolored copies of H with two (out of three) allowed colors so that colors 1, 2 and 3 are used $(2k-1)t$ times and color 4 is used $2kt$ times, which results in an equitable 4-colorings of the whole corona $G \circ H$.

□

3 Equitable 5-coloring of coronas of cubic graphs

We start by considering cases when 5 colors are necessary for such graphs to be colored equitably.

Proposition 3.1 ([6]). *If G is a graph with $\chi(G) \leq m+1$, then $\chi_=(G \circ K_m) = m+1$.*

This proposition immediately implies

Corollary 3.2. *If G is a cubic graph, then*

$$\chi_=(G \circ K_4) = 5.$$

It turns out that 5 colors may be required also in some coronas $G \circ H$, where $G \in Q_2 \cup Q_3$ and $H \in Q_3$.

Theorem 3.3. *If $G \in Q_2(s)$ and $H \in Q_3$, then*

$$4 \leq \chi_=(G \circ H) \leq 5.$$

Proof. Since $H \in Q_3$, we obviously have $\chi_=(G \circ H) \geq 4$.

To prove the upper bound, we consider two cases. Let $H = H(U, V, W)$ with tripartition of H satisfying $|U| \geq |V| \geq |W|$.

Case 1: $s = 2k+1$, $k \geq 1$.

We start with the following 4-coloring of $G \circ H$.

1. Color graph G with 4 colors, using each of colors 1 and 2 k times and colors 3 and 4 $(k+1)$ times, respectively.
2. Color the vertices of each copy of $H(U,V,W)$ linked to an i-vertex of G using color $(i+1) \bmod 4$ for vertices in U, color $(i+2) \bmod 4$ for vertices in V, and color $(i+3) \bmod 4$ for vertices in W (we use color 4 instead of 0).

Now, we have to consider three subcases, where we bound the number of vertices that have to be recolored to 5.

Subcase 1.1: $H \in Q_3(t+1,t,t)$, where $t=v=w$.

The color sequence of the 4-coloring of this corona is $\mathcal{C}_4 = (c_1,c_2,c_3,c_4) = (3kt+2k+2t+1, 3kt+2k+2t, 3kt+2k+t+1, 3kt+2k+t+2)$.

In every equitable 5-coloring of the corona $G \circ H$, where $G \in Q_2(2k+1)$ and $H \in Q_3(t+1,t,t)$, every color must be used $\gamma_5^1 = \lceil(12kt+8k+6t+4)/5\rceil = (2kt+t+k+\lceil(2kt+t+3k+4)/5\rceil)$ or $\gamma_5^2 = (2kt+t+k+\lfloor(2kt+t+3k+4)/5\rfloor)$ times. The number d_i of vertices colored with i, $1 \le i \le 4$, that have to be recolored is equal to $c_i - \gamma_5^1$ or $c_i - \gamma_5^2$. We have

$$d_1 \le c_1 - \gamma_5^1 \le c_1 - \gamma_5^2 = kt+t+k+1 - \lfloor(2kt+t+3k+4)/5\rfloor =$$

$$= (k+1)(t+1) - \lfloor(2kt+t+3k+4)/5\rfloor \le (k+1)(t+1).$$

Similarly, we have

$$\begin{aligned} d_2 &\le k(t+1)+t, \\ d_3 &\le k(t+1), \text{ and} \\ d_4 &\le k(t+1). \end{aligned}$$

Subcase 1.2: $H \in Q_3(t+1,t+1,t)$, where $t=w$.

The color sequence of the 4-coloring of this corona is $\mathcal{C}_4 = (c_1,c_2,c_3,c_4) = (3kt+3k+2t+2, 3kt+3k+2t+1, 3kt+3k+t+1, 3kt+3k+t+2)$.

In every equitable 5-coloring of the corona $G \circ H$, where $G \in Q_2(2k+1)$ and $H \in Q_3(t+1,t+1,t)$, every color must be used $\gamma_5^1 = \lceil(12kt+12k+6t+6)/5\rceil = (2kt+t+2k+1+\lceil(2kt+t+2k+1)/5\rceil)$ or $\gamma_5^2 = (2kt+t+2k+1+\lfloor(2kt+t+2k+1)/5\rfloor)$ times.

Similarly, as in Subcase 1.1, we have

$$\begin{aligned} d_1 &\le c_1 - \gamma_5^1 \le c_1 - \gamma_5^2 \le (k+1)(t+1), \\ d_2 &\le k(t+1)+y, \\ d_3 &\le k(t+1), \text{ and} \\ d_4 &\le k(t+1). \end{aligned}$$

Subcase 1.3: $H \in Q_3(t)$, where $t=u=v=w$.

The color sequence of the 4-coloring of this corona is $\mathcal{C}_4 = (c_1,c_2,c_3,c_4) = (3kt+k+2t, 3kt+k+2t, 3kt+k+t+1, 3kt+k+t+1)$.

In every equitable 5-coloring of the corona $G \circ H$, where $G \in Q_2(2k+1)$ and $H \in Q_3(t,t,t)$, every color must be used $\lceil(12kt+4k+6t+2)/5\rceil =$

$(2kt + t + \lceil (2kt + t + 4k + 2)/5 \rceil)$ or $(2kt + t + \lfloor (2kt + t + 4k + 2)/5 \rfloor)$ times.

Similarly, as in previous subcases, we have

$$\begin{aligned} d_1 &\leq (k+1)t, \\ d_2 &\leq kt + t, \\ d_3 &\leq kt, \text{ and} \\ d_4 &\leq kt. \end{aligned}$$

Consequently, in all subcases, the number of i-vertices that have to be recolored is bounded by:

- $(k+1)u$ for $i = 1$,
- $ku + w$ for $i = 2$,
- ku for $i = 3, 4$.

To obtain an equitable 5-coloring from the 4-coloring of $G \circ H(U, V, W)$, $|U| \geq |V| \geq |W|$, we recolor the appropriate number of i-vertices in partitions U linked to $(i-1)$-vertices of G for the vertices which were colored with color i. Due to the above, this is possible in the cases of colors 1, 3 and 4. In the case of 2-vertices, the number of vertices recolored in partition U in copies of H can be insufficient. In this case, we can recolor the vertices in partition W (of cardinality w) in one copy of H linked to 3-vertex of G.

Case 2: $s = 2k$, $k \geq 2$.

Again, we start with 4-coloring of $G \circ H$, as follows.

1. Color graph G with 4 colors, using each of colors 1,2, 3 and 4 k times.
2. Color the vertices of each copy of $H(U, V, W)$ linked to an i-vertex of G using color $(i+1) \bmod 4$ for vertices in U, color $(i+2) \bmod 4$ for vertices in V, and color $(i+3) \bmod 4$ for vertices in W (we use color 4 instead of 0).

Notice that the resulting 4-coloring does not require recoloring: it is equitable and establishes that the lower bound is tight.

□

Similar technique for obtaining an equitable coloring is used in the proof of the following theorem, by introducing the fifth color.

Theorem 3.4. *If $G, H \in Q_3$, then*

$$4 \leq \chi_=(G \circ H) \leq 5.$$

Proof. Let $G = G(A, B, C)$, where $|A| \geq |B| \geq |C| \geq |A| - 1$, and let $H = H(U, V, W)$, where $|U| \geq |V| \geq |W| \geq |U| - 1$. We start with a 4-coloring of $G \circ H$.

1. Color the vertices of graph G with 3 colors: the vertices in A with color 1, in B with 2, and in C with color 3.

2. Color the vertices of each copy of H linked to an i-vertex using color $(i+1) \bmod 4$ for vertices in U, color $(i+2) \bmod 4$ for vertices in V, and color $(i+3) \bmod 4$ for vertices in W, $i = 1, 2, 3$ (see Fig. 2a)).

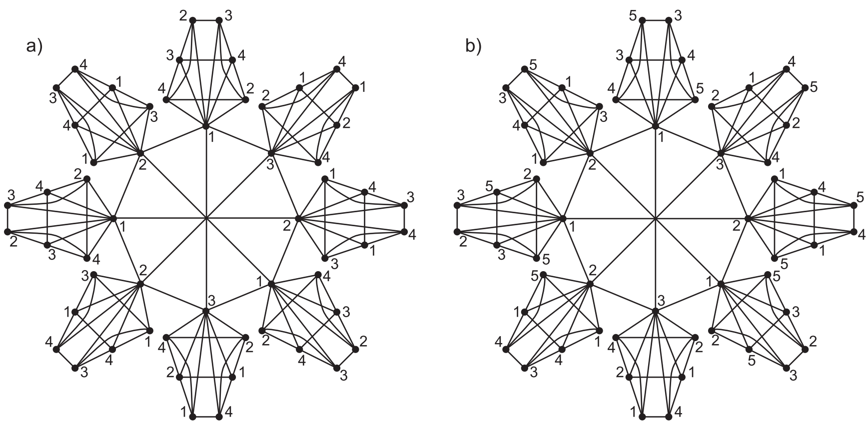

Figure 2: An example of coloring of $W \circ P$, where W is the Wagner graph and P is the prism graph: a) ordinary 4-coloring; b) equitable 5-coloring.

Since $|V(G \circ H)| = (m+1)n$, the color cardinality sequence $\mathcal{C} = (c_1, c_2, c_3, c_4)$ of the above 4-coloring of $G \circ H$ is as follows:

$$\begin{aligned}\Big(\quad & \lceil n/3 \rceil + \lceil (n-1)/3 \rceil \lceil (m-2)/3 \rceil + \lceil (n-2)/3 \rceil \lceil (m-1)/3 \rceil, \\ & \lceil n/3 \rceil \lceil m/3 \rceil + \lceil (n-1)/3 \rceil + \lceil (n-2)/3 \rceil \lceil (m-1)/3 \rceil, \\ & \lceil n/3 \rceil \lceil (m-1)/3 \rceil + \lceil (n-1)/3 \rceil \lceil m/3 \rceil + \lceil (n-2)/3 \rceil, \\ & \lceil n/3 \rceil \lceil (m-2)/3 \rceil + \lceil (n-1)/3 \rceil \lceil (m-1)/3 \rceil + \lceil (n-2)/3 \rceil \lceil m/3 \rceil \Big),\end{aligned}$$

respectively. This 4-coloring is not equitable. We have to recolor some vertices colored with 1, 2, 3 and 4 into 5. The number of vertices colored with i, $1 \le i \le 4$, that have to be recolored is equal to $c_i - \lceil ((m+1)n - i + 1)/5 \rceil$.

We have the following claims:

$$c_1 - \left\lceil \frac{(m+1)n}{5} \right\rceil \leq \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{3} \right\rceil \right\rfloor \left\lceil \frac{m-1}{3} \right\rceil = \left\lfloor \frac{1}{2}|C| \right\rfloor |V|, \tag{3.1}$$

$$c_2 - \left\lceil \frac{(m+1)n-1}{5} \right\rceil \leq \left\lfloor \frac{1}{2} \left\lceil \frac{n}{3} \right\rceil \right\rfloor \left\lceil \frac{m}{3} \right\rceil = \left\lfloor \frac{1}{2}|A| \right\rfloor |U|, \tag{3.2}$$

$$c_3 - \left\lceil \frac{(m+1)n-2}{5} \right\rceil \leq \left\lfloor \frac{3}{4} \left\lceil \frac{n-1}{3} \right\rceil \right\rfloor \left\lceil \frac{m}{3} \right\rceil = \left\lfloor \frac{3}{4}|B| \right\rfloor |U|, \text{ and} \tag{3.3}$$

$$\begin{aligned} c_4 - \left\lceil \frac{(m+1)n-3}{5} \right\rceil &\leq \left\lceil \frac{1}{2} \left\lceil \frac{n-2}{3} \right\rceil \right\rceil \left\lceil \frac{m-2}{3} \right\rceil + \left\lceil \frac{1}{4} \left\lceil \frac{n-1}{3} \right\rceil \right\rceil \left\lceil \frac{m-1}{3} \right\rceil + \\ &+ \left\lceil \frac{1}{2} \left\lceil \frac{n-2}{3} \right\rceil \right\rceil \left\lceil \frac{m}{3} \right\rceil = \\ &= \left\lceil \frac{1}{2}|A| \right\rceil |W| + \left\lceil \frac{1}{4}|B| \right\rceil |V| + \left\lceil \frac{1}{2}|C| \right\rceil |U|. \end{aligned} \tag{3.4}$$

Proof of inequalities (3.1)-(3.4). Let us consider three cases, $G \in Q_3(s), Q_3(s+1,s,s)$, and $Q_3(s+1,s+1,s)$, and in each case three subcases, $H \in Q_3(t), Q_3(t+1,t,t), Q_3(t+1,t+1,t)$, respectively. The estimation technique for the number of vertices that have to be recolored to color 5 is similar to that used in the proof of Theorem 3.3.

Case 1: $G \in Q_3(s)$, *where* $s = 2k$ *for some* $k \geq 1$.

Subcase 1.1: $H \in Q_3(t)$, *where* $t = 2l$ *for some* $l \geq 1$.

We have $|V(G \circ H)| = (3t+1)3s = 5(7kl+k)+kl+k$, while the color cardinality sequence $\mathcal{C}$ of the 4-coloring of $G \circ H$ is $\mathcal{C} = (s+2st, s+2st, s+2st, 3st) = (8kl+2k, 8kl+2k, 8kl+2k, 12kl)$.

Since in every equitable 5-coloring of $G \circ H$ each of 5 colors has to be used $(7kl + k + \lceil (kl+k)/5 \rceil)$ or $(7kl+k+\lfloor (kl+k)/5 \rfloor)$ times, we have to recolor some vertices colored with 1, 2, 3 and 4 into 5. The number of vertices that have to be recolored is as follows:

- the vertices colored with 1:
  $8kl + 2k - 7kl - k - \lceil (kl+k)/5 \rceil\rceil \leq 2kl = \lfloor \frac{1}{2}|C| \rfloor |V|$,
- the vertices colored with 2:
  $8kl + 2k - 7kl - k - \lceil (kl+k-1)/5 \rceil\rceil \leq 2kl = \lfloor \frac{1}{2}|A| \rfloor |U|$,
- the vertices colored with 3:
  $8kl + 2k - 7kl - k - \lceil (kl+k-2)/5 \rceil \leq 2kl \leq \lfloor \frac{3}{4}|B| \rfloor |U|$,
- the vertices colored with 4:
  $12kl - 7kl - k - \lceil (kl+k-3)/5 \rceil \leq 4kl + \lceil \frac{k}{2} \cdot 2l \rceil =$
  $= \lceil \frac{1}{2}|A| \rceil |W| + \lceil \frac{1}{4}|B| \rceil |V| + \lceil \frac{1}{2}|C| \rceil |U|$.

Subcase 1.2: $H \in Q_3(t+1,t,t)$, *where* $t = 2l+1$ *for some* $l \geq 1$.

We have $|V(G \circ H)| = (3t+2)3s = 5(7kl+6k)+kl$, while the color cardinality sequence $\mathcal{C}$ of the 4-coloring of $G \circ H$ is $\mathcal{C} = (s+2st, 2s+2st, 2s+2st, 3st+s) = (8kl+6k, 8kl+8k, 8kl+8k, 12kl+8k)$.

Since in every equitable 5-coloring of $G \circ H$ each of 5 colors has to be used $(7kl + 6k + \lceil kl/5 \rceil)$ or $(7kl + 6k + \lfloor kl/5 \rfloor)$ times, we have to recolor some vertices colored with 1, 2, 3 and 4 into 5. The number of vertices that have to be recolored is as follows:

- the vertices colored with 1:
  $kl - \lceil kl/5 \rceil \le 2kl + k = \lfloor \frac{1}{2}|C| \rfloor \, |V|$,
- the vertices colored with 2:
  $k(l+1) + k - \lceil (kl-1)/5 \rceil \le 2k(l+1) = \lfloor \frac{1}{2}|A| \rfloor \, |U|$,
- the vertices colored with 3:
  $k(l+1) + k - \lceil (kl-2)/5 \rceil \le \lfloor \frac{3}{4}k \rfloor (2l+2) = \lfloor \frac{3}{4}|B| \rfloor \, |U|$,
- the vertices colored with 4:
  $5kl + 2k - \lceil (kl-3)/5 \rceil \le 4kl + 2k + \lceil \frac{k}{2} \rceil (2l+1) =$
  $= \lceil \frac{1}{2}|A| \rceil \, |W| + \lceil \frac{1}{4}|B| \rceil \, |V| + \lceil \frac{1}{2}|C| \rceil \, |U|$.

Subcase 1.3: *$H \in Q_3(t+1, t+1, t)$, where $t = 2l$ for some $l \ge 1$.*

We have $|V(G \circ H)| = (3t+3)3s = 5(7kl+3k) + kl + 3k$, while the color cardinality sequence $\mathcal{C}$ of the 4-coloring of $G \circ H$ is $\mathcal{C} = (2s+2st, 2s+2st, 3s+2st, 3st+2s) = (8kl+4k, 8kl+4k, 8kl+6k, 12kl+4k)$.

Since in every equitable 5-coloring of $G \circ H$ each of 5 colors has to be used $(7kl + 3k + \lceil (kl+3k)/5 \rceil)$ or $(7kl + 3k + \lfloor (kl+3k)/5 \rfloor)$ times, we have to recolor some vertices colored with 1, 2, 3 and 4 into 5. The number of vertices that have to be recolored is as follows:

- the vertices colored with 1:
  $kl + k - \lceil (kl+3k)/5 \rceil \le 2kl + k = \lfloor \frac{1}{2}|C| \rfloor \, |V|$,
- the vertices colored with 2:
  $kl + k - \lceil (kl+3k-1)/5 \rceil \le 2kl + k = \lfloor \frac{1}{2}|A| \rfloor \, |U|$,
- the vertices colored with 3:
  $kl + 3k - \lceil (kl+3k-2)/5 \rceil \le \lfloor \frac{3}{2}k \rfloor (2l+1) = \lfloor \frac{3}{4}|B| \rfloor \, |U|$,
- the vertices colored with 4:
  $5kl + k - \lceil (kl+3k-3)/5 \rceil \le 4kl + k + \lceil \frac{k}{2} \rceil (2l+1) =$
  $= \lceil \frac{1}{2}|A| \rceil \, |W| + \lceil \frac{1}{4}|B| \rceil \, |V| + \lceil \frac{1}{2}|C| \rceil \, |U|$.

Case 2: *$G \in Q_3(s+1, s, s)$, where $s = 2k+1$ for some $k \ge 1$.* The proof follows by a similar argument to that in Case 1, we omit the details.

Case 3: *$G \in Q_3(s+1, s+1, s)$, where $s = 2k$ for some $k \ge 1$.* The proof follows by a similar argument to that in Case 1, we omit the details.

End of the proof of inequalities (3.1)-(3.4).

Now, to obtain an equitable 5-coloring of $G \circ H$, we choose the vertices that have to be recolored.

- Since the number of 1-vertices that have to be recolored to 5 is not greater than $\lfloor \frac{1}{2}|C| \rfloor |V|$, then the vertices colored with 1 are chosen from the partitions V of $\lfloor \frac{1}{2}|C| \rfloor$ copies of H linked to the vertices from partition C of G.

- Similarly, 2-vertices that have to be recolored are chosen from the partitions U of $\lfloor \frac{1}{2}|A| \rfloor$ copies of H linked to the vertices from partition A of G.
- 3-vertices to be recolored are chosen from the partitions U of $\lfloor \frac{3}{4}|B| \rfloor$ copies of H linked to the vertices from partition B of G.
- 4-vertices are chosen from:
  - partitions W of $\lceil \frac{1}{2}|A| \rceil$ copies of H linked to the vertices from the partition A of G (different copies than in recoloring of 2-vertices),
  - partitions V of $\lceil \frac{1}{4}|B| \rceil$ copies of H linked to the vertices from the partition B of G (different copies than in recoloring of 3-vertices),
  - partitions U of $\lceil \frac{1}{2}|C| \rceil$ copies of H linked to the vertices from the partition C of G (different copies than in recoloring of 1-vertices) (see Fig. 2b)).

Taking into account our claim, such recoloring is possible. □

As we have already observed, the lower bound in Theorem 3.3 is tight. Also upper bounds in Theorems 3.3 and 3.4 are tight. There are infinitely many coronas $G \circ H$, where $G \in Q_2 \cup Q_3$ and $H \in Q_3$, that require five colors to be equitably colored. For example, in such coronas graph $H \in Q_3$ may be built of $3t$ (t must be even) vertices and it must contain t disjoint triangles (cycles C_3) (cf. Fig. 3). Let us consider for example $G = K_{3,3}$. In the corona $K_{3,3} \circ H$, where H is defined as above, the number of vertices is equal to $36k + 6$, for some positive integer k. In any equitable 4-coloring of the corona, the color sequence must be $(9k+2, 9k+2, 9k+1, 9k+1)$. Since modifying the tripartite structure of H is impossible (it contains $t = 2k$ disjoint triangles), such a coloring does not exist for $k \geq 2$.

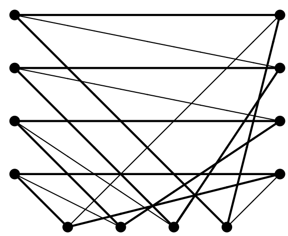

Figure 3: An example of graph $H \in Q_3$ for which $\chi_=(G \circ H) = 5$, for $G \in Q_3$.

4 Complexity results

Although we have only two possible values, 4 and 5, for $\chi_=(G \circ H)$, where $G \in Q_2 \cup Q_3$ and $H \in Q_3$, it is hard to decide which is correct[1]. All G, H are still cubic.
We consider the following combinatorial decision problems:

Note that the $\mathrm{IS}_3(H, k)$ problem is NP-complete and remains so even if $10|m$ [8]. This is so because we can enlarge H by adding j $(0 \leq j \leq 4)$ isolated copies of $K_{3,3}$ to it so that the number of vertices in the new graph is divisible by 10. Graph H has an independent set of size at least k if and only if the new graph has an independent set of size at least $k + 3j$.

[1] graphs considered in this section need not be connected

$\mathrm{IS}_3(H,k)$: Given a cubic graph H on m vertices and an integer k, the question is: does H have an independent set I of size at least k?

and its subproblem for $m = 10q$, $k = 4m/10 = 4q$, i.e. $\mathrm{IS}_3(H, 4q)$.

Lemma 4.1. *Problem* $\mathrm{IS}_3(H, 4m/10)$ *is* NP-*complete.*

Proof. Our polynomial reduction is from $\mathrm{IS}_3(H,k)$. For an m-vertex cubic graph H, $10|m$, and an integer k, let $r = |4m/10 - k|$. If $k \geq 4m/10$ then we construct a cubic graph $G = H + rK_4 + rP$ else we construct $G = H + rK_4 + 2rP + 4rK_{3,3}$, where $P \in Q_3(2)$ is the prism graph. It is easy to see that the answer to problem $\mathrm{IS}_3(H,k)$ is 'yes' if and only if the answer to problem $\mathrm{IS}_3(G, 4m/10)$ is 'yes'. □

Lemma 4.2. *Let H be a cubic graph and let $k = 4/10m$, where m is the number of vertices of H. The problem of deciding whether H has a coloring of type* $(4m/10, 3m/10, 3m/10)$ *is* NP-*complete.*

Proof. We prove that H has a coloring of type $(4m/10, 3m/10, 3m/10)$ if and only if there is an affirmative answer to $\mathrm{IS}_3(H, 4m/10)$.

Suppose first that H has the above 3-coloring. Then the color class of size $4m/10$ is an independent set that forms a solution to $\mathrm{IS}_3(H, 4m/10)$.

Now suppose that there is a solution I to $\mathrm{IS}_3(H, 4m/10)$. Thus $|I| \geq 4m/10$. We know from [7] that in this case there exists an independent set I' of size exactly $4m/10$ such that the subgraph $H - I'$ is equitably 2-colorable bipartite graph. This means that H can be 3-colored so that the color sequence is $(4m/10, 3m/10, 3m/10)$. □

In the following we show that, given such an unequal coloring of H, we can color $K_{3,3} \circ H$ equitably with 4 colors.

(i) Color the vertices of $K_{3,3}$ with 4 colors - the color sequence is $(2, 2, 1, 1)$.

(ii) Color the vertices in copies of $H = H(U, V, W)$, $|U| = 4m/10$, $|V| = |W| = 3m/10$, in the following way:

- vertices in partitions U of H adjacent to a 1-vertex of $K_{3,3}$ are colored with color 2, in partitions V - with 3, and in partitions W - with 4,
- vertices in partitions U of H adjacent to a 2-vertex of $K_{3,3}$ are colored with color 1, in partitions V - with 3, and in partitions W - with 4,
- vertices in partition U of H adjacent to the 3-vertex of $K_{3,3}$ are colored with color 1, in partition V - with 2, and in partition W - with 4,
- vertices in partition U of H adjacent to the 4-vertex of $K_{3,3}$ are colored with color 2, in partition V - with 1, and in partition W - with 3.

Color sequence of the corona is $(15m/10+2, 15m/10+2, 15m/10+1, 15m/10+1)$.

On the other hand, let us assume that the corona $K_{3,3} \circ H$, where $H \in Q_3(t)$ and $t = 10k$, is equitably 4-colorable, where the color sequence for $K_{3,3}$ is $(2, 2, 1, 1)$. Since $|V(K_{3,3} \circ H)| = 6(3t + 1) = 18t + 6$ and $t = 10k$ for some k, then each of the four colors in every equitable coloring is used $45k+1$ or $45k+2$ times. Since color 1 (similarly color 2) can be used only in four copies of H, then in at least one copy we have to use it $12k = 12t/10$ times. It follows that there must exist an independent set of cardinality $12t/10$ in H. Since H has $3t$ vertices, the size of this set is $4m/10$.

The above considerations lead us to the following

Theorem 4.3. *The problem of deciding whether $\chi_=(K_{3,3} \circ H) = 4$ is* NP-*complete even if $H \in Q_3(t)$ and $10|t$.* □

A similar argument implies the following

Corollary 4.4. *The problem of deciding whether $\chi_=(P \circ H) = 4$, where P is the prism graph, is* NP-*complete even if $H \in Q_3(t)$ and $10|t$.* □

In this way we have obtained the full classification of complexity for equitable coloring of cubical coronas.

5 Conclusion

In this paper, we presented all the cases of corona of cubic graphs for which 3 colors suffice for equitable coloring. In the remaining cases we have proved constructively that 5 colors are enough for equitable coloring. Since there are only two possible values for $\chi_=(G \circ H)$, namely 4 or 5, our algorithm is 1-absolute approximate. Due to Theorem 4.3 and Corollary 4.4 the algorithm cannot be improved unless P=NP. Since time spend to assign a final color to each vertex is constant, the complexity of our algorithm is linear. Finally, the algorithm confirms the Equitable Coloring Conjecture [14].

Our results are summarized in Table 2. This table contains also the values of classical chromatic numbers of appropriate coronas and the complexity classification. Let us notice that all cases are polynomially solvable for ordinary coloring.

G \ H	Q_2		Q_3		Q_4	
Q_2, Q_3	*3*	**3** or **4**	*4*	**4** or **5***	*5*	**5**
Q_4	*4*	**4**	*4*	**4**	*5*	**5**

Table 2: The exact values of classical chromatic number (in *italics*) and possible values of the equitable chromatic number (in **bold**) of coronas $G \circ H$. Asterix $(^*)$ means that this case is NP-complete. The other cases are solvable in linear time.

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Petrie polygons, Fibonacci sequences and Farey maps

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Abstract

We consider the regular triangular maps corresponding to the principal congruence subgroups $\Gamma(n)$ of the classical modular group. We relate the sizes of the Petrie polygons on these maps to the periods of reduced Fibonacci sequences.

Keywords: Regular map, Petrie polygon, Fibonacci sequence.

Math. Subj. Class.: 05C10, 11B39, 20H05

1 Introduction

An interesting number theoretic problem is to determine the period of the Fibonacci sequence mod n. Here we look at the period $\sigma(n)$ of the Fibonacci sequence mod n up to sign. A Petrie polygon on a regular map is a zig-zag path through the map and an important invariant of a regular map is the length of a Petrie polygon. The maps we consider here are those that arise out of principal congruence subgroups $\Gamma(n)$ of the classical modular group Γ. In this case It is shown that these lengths are equal to $\sigma(n)$. A particularly nice example is when $n = 7$. Here the regular map is the famous map on the Klein quartic and we find $\sigma(7) = 8$ giving the title "The Eightfold Way" to the sculpture by Helaman Ferguson that represents Klein's Riemann surface of genus 3 derived from the Klein quartic. This is described in the book "The eightfold way: the beauty of Klein's quartic curve", a collection of papers related to the Klein quartic edited by Silvio Levy [5].

Let X be a compact orientable surface. By a map (or clean dessin d'enfant) on X we mean an embedding of a graph $\mathcal{G}$ into X such that $X \setminus \mathcal{G}$ is a union of simply-connected polygonal regions, called faces. A map thus has vertices, edges and faces. A directed edge is called a *dart* and a map is called *regular* if its automorphism group acts transitively on its darts. The platonic solids are the most well-known examples of regular maps. These are the regular maps on the Riemann sphere. We recall how we study maps using triangle groups. The *universal map of type* $(m, n)$is the tessellation of one of the three simply connected

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Riemann surfaces, that is the Riemann sphere Σ, the Euclidean plane $\mathbb{C}$, or the hyperbolic plane, $\mathbb{H}$ (depending on whether the genus of X is 0,1, or > 1) by regular m-gons with n meeting at each vertex. This map is denoted by $\hat{\mathcal{M}}(m, n)$. The automorphism group, and also the conformal automorphism group, of $\hat{\mathcal{M}}(m, n)$ is the triangle group $\Gamma[2, m, n]$. In general, a map is of type (m, n) if m is the least common multiple of the face sizes and n is the least common multiple of the vertex valencies. As shown in [3] every map of type (m, n) is a quotient of $\hat{\mathcal{M}}(m, n)$ by a subgroup M of the triangle group $\Gamma[2, m, n]$. Then M is called a map subgroup of $\hat{\mathcal{M}}(m, n)$ or sometimes a fundamental group of $\hat{\mathcal{M}}(m, n)$, inside $\Gamma[2, m, n]$. A *platonic surface* is one that underlies a regular map. The map is regular if and only if M is a normal subgroup of $\Gamma[2, m, n]$. Thus a platonic surface is one of the form $\mathbb{U}/M$ where M is a normal subgroup of a triangle group and $\mathbb{U}$ is a simply connected Riemann surface.

It is permissible to let m or n, or both to be ∞. In this paper we are particularly interested in the case where $m = 3$, $n = \infty$. This means that the corresponding maps are triangular though in general we are not concerned with the vertex valencies. However, if the map is regular then we must have all vertex valencies equal. For example, the icosahedron is a triangular map with all vertices of valency 5.

To study triangular maps we use the triangle group $[2, 3, \infty]$ which is known to be the modular group Γ =PSL$(2, \mathbb{Z})$ one of the most significant groups in mathematics. The regular maps correspond to normal subgroups of Γ. The most well-known normal subgroups of Γ are the *principal congruence subgroups* $\Gamma(n)$ defined in section 5. We let $\mathcal{M}_3(n) = \hat{\mathcal{M}}_3(3, \infty)/\Gamma(n)$. We call these maps *principal congruence maps* or *PC maps*.

For low values of n these maps are well-known. For $n = 2, 3, 4, 5$ we get the triangle, tetrahedron, octahedron and icosahedron respectively. These are the only PC maps of genus 0. For $n = 6$ we get the regular map $\{3, 6\}_{2.2}$ on the torus and for $n = 7$ we get the Klein map on Klein's Riemann surface of genus 3. (See [2, 1]).

2 Petrie polygons

A *Petrie polygon* in a map $\mathcal{M}$ is defined as a zig-zag path in the map. More precisely, we start at a vertex, then go along an edge to an adjacent vertex, the turn left and go to the next vertex and then turn right, etc., (or interchange left and right.) We have a path in which two consecutive edges belong to the same face but no three consecutive edges belong to the same face, [1, p. 54]. Eventually, in a finite regular map, we will come back to the original vertex.This path is called a *Petrie path* or *Petrie polygon.* The number of edges of this Petrie polygon is called the *Petrie length* of the map.

We now relate the Petrie polygons to triangle groups. From the triangle group $\Gamma[2, m, n]$, we can form the extended triangle group $\Gamma(2, m, n)$ which is the group generated by the reflections R_1, R_2, R_3 in the edges of a triangle with angles $\pi/2, \pi/m, \pi/n$ where we choose our ordering so that $\Gamma(2, m, n)$ has a presentation

$$\langle R_1, R_2, R_3 | R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^2 = (R_2R_3)^m = (R_3R_1)^n = 1\rangle.$$

If we let $X = R_1R_2, Y = R_2R_3, Z = R_3R_1$, then we find that $\Gamma[2, m, n]$ has a presentation

$$\langle X, Y, Z | X^2 = Y^m = Z^n = XYZ = 1\rangle.$$

In section 5.2 of [1, p. 54], it is shown that $R_1R_2R_3$ is a transformation that goes one step around a Petrie polygon. Now

$$(R_1R_2R_3)^2 = R_1R_2R_3R_1R_2R_3 = R_1R_2R_3R_2R_2R_1R_2R_3 = XY^{-1}X^{-1}Y$$

showing that Petrie length is twice the order of this commutator which implies that the Petrie length is independent of the Petrie polygon chosen; it is just a property of the map.

3 The Farey map

This is basically the map $\hat{\mathcal{M}}(3,\infty)$, which we abbreviate to $\mathcal{M}_3$. We construct it as follows. The vertices are the extended rationals $\mathbb{Q}\cup\{\infty\}$ and two rationals $\frac{a}{b}$ and $\frac{c}{d}$ are joined by an edge if and only if $ad-bc=\pm 1$.

This map has the following properties.

(a) There is a triangle with vertices $\frac{1}{0}, \frac{1}{1}, \frac{0}{1}$ called the principal triangle.

(b) The modular group Γ acts as a group of automorphisms of $\mathcal{M}_3$.

(c) The general triangle has vertices $\frac{a}{c}, \frac{a+b}{c+d}, \frac{b}{d}$.

Thus the Farey map (Figure 1) is a triangular map with triangular faces given by (c). In [7] it is shown that this is the universal triangular map in the sense that any other triangular map on an orientable surface is a quotient of $\mathcal{M}_3$ by a subgroup Λ of the modular group Γ. As $\mathcal{M}_3$ has vertices the extended rationals this means that every triangular map the vertices can be given coordinates which are Λ orbits of points in $\mathbb{Q}\cup\{\infty\}$. We shall denote the orbit of $\frac{a}{b}$ by $[\frac{a}{b}]$. This is illustrated in [2] where there are many examples, in particular coordinates for the triangular platonic solids are given. Also see Figure 2.

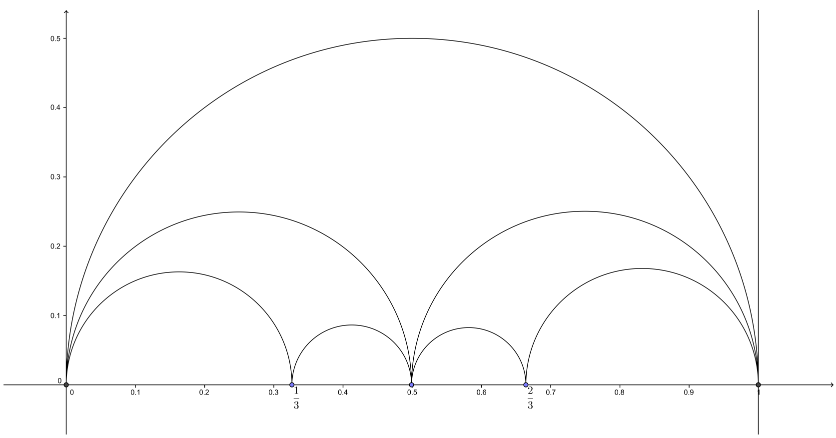

Figure 1: Farey map

4 The Petrie polygons of the Farey map

We consider a Petrie path in $\mathcal{M}_3$. By transitivity we may assume it's first edge goes from $W_1 = \frac{1}{0}$ to $W_2 = \frac{0}{1}$. A left turn then takes us to $W_3 = \frac{1}{1}$ Now a right turn takes us to $W_4 = \frac{1}{2}$. By applying a modular transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to the vertices ∞, 0 and 1 to the principal triangle we find that three consecutive vertices of the Petrie polygon are $\frac{a}{c}, \frac{b}{d}, \frac{a+b}{c+d}$, that is the third vertex is the *Farey median* of the previous two. As the first two vertices of the Petrie polygon are $\frac{1}{0}$ and $\frac{1}{1}$ the kth vertex of the Petrie polygon is equal to $\frac{f_{k-1}}{f_k}$ where f_k is the kth element of the Fibonacci sequence defined by $f_0 = 0, f_1 = 1, f_{k+1} = f_k + f_{k-1}$. for $k \geq 1$. Thus the Petrie polygon is

$$\frac{1}{0}, \frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5} \dots$$

Lemma 4.1. *The matrix* $P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ *maps each vertex of the Petrie polygon of* $\mathcal{M}_3$ *to the next one and also* $P^k = \begin{pmatrix} f_{k-1} & f_k \\ f_k & f_{k+1} \end{pmatrix}$

The proof follows immediately from the definition of the Fibonacci sequence, and induction.

Note that P having determinant -1 is not an element of Γ but $T = P^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ is an element of Γ.

In the following sections we will consider the Petrie polygon modulo n. As $\frac{a}{b} = \frac{-a}{-b}$, we introduce the following concept.

Definition 4.2. We call the least positive integer m with the property that $f_{m-1} \equiv \pm 1$, $\bmod\, n$, $f_m \equiv 0 \bmod n$ the *semi-period* $\sigma(n)$ of the Fibonacci sequence $\bmod\, n$. The *period* $\pi(n)$ is the least positive integer m such that $f_{m-1} \equiv 1 \bmod n$, $f_m \equiv 0 \bmod n$.

For example if $m = 7$, the Fibonacci sequence mod 7 is 0,1,1,2,3,5,1,6,0, so that $\sigma(7) = 8$ and $\pi(7) = 16$. The function π has been quite well-studied in the literature and is often called the *Pisano period*. See [8].

5 The principal congruence subgroups

The most well-known normal subgroup of the modular group are the *principal congruence subgroups*. Let $n \in \mathbb{Z}$, Then the principal congruence subgroup of level n in Γ is the subgroup

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod n \right\}$$

Now $\Gamma(n)$ is a normal subgroup of Γ and so corresponds to a regular map $\mathcal{M}_3(n)$ which lies on the surface $\mathbb{H}^*/\Gamma(n)$ where $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$.

Another important group for us is $\Gamma_1(n)$. This is defined as

$$\Gamma_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \bmod n \right\}$$

where $0 \leq b < n$.

We will not make use of this subgroup but in [2] it was shown that the left cosets of $\Gamma_1(n)$ in Γ are in one-to-one correspondence with the vertices of $\mathcal{M}_3(n)$.

$\Gamma(n)$ is a normal subgroup of Γ of index

$$\frac{n^3}{2} \Pi_{p|n} (1 - \frac{1}{p^2}). \tag{1}$$

6 The Petrie polygons of $\mathcal{M}_3(n)$

Our principle object of study are the Petrie polygons of the PC-maps $\mathcal{M}_3(n)$. We can regard $\mathcal{M}_3(n)$ as $\hat{\mathcal{M}}_3(3,\infty)/\Gamma(n)$, that is as a quotient of the Farey map. We illustrate our study with the classical regular map $\mathcal{M}_3(7)$. This is known as the Klein map and is a map of type $\{3,7\}$. This lies on Klein's Riemann surface of genus 3, known as the Klein quartic. Petrie polygons for this map appear on page 320 in the classic paper [4], although they were not called Petrie polygons there. In fact, Petrie polygons are named after John Flinders Petrie (1907-1972), and Klein's paper [4] was written in 1878. Three of the Petrie polygons are drawn on page 320 of "The Eightfold Way" [5]. The eight in the title comes from the fact that the size of the Petrie polygons is 8. This will be a special case of results in this paper where we determine the sizes of of the Petrie polygons in PC maps.

In general we observe that the group $\Gamma/\Gamma(n)$ has a transitive action on the Petrie polygons of $\mathcal{M}_3(n)$. For Γ clearly has a transitive action on the darts of $\mathcal{M}_3(\infty)$, and so $\Gamma/\Gamma(n)$ has an induced action on the darts of $\mathcal{M}_3(n)$. Clearly, this action will give a transitive action on the set of Petrie polygons of $\mathcal{M}_3(n)$.

The vertices of $\mathcal{M}_3(n)$ are equivalence classes of vertices of $\hat{\mathcal{M}}_3(3,\infty)$. We let $[\frac{a}{b}]$ denote the equivalence class of $\frac{a}{b}$ in $\mathcal{M}_3(n)$ and $[\frac{a}{b}]$ is joined by an edge to $[\frac{c}{d}]$ in $\mathcal{M}_3(n)$ if and only if $ad - bc \equiv 1 \bmod n$.

The points $[\frac{1}{0}], [\frac{0}{1}], \cdots [\frac{f_{r-1}}{f_r}] \cdots$ form the vertices of a Petrie polygon which we call $Pe(n)$.

Recall the definition of the semiperiod $\sigma(n)$ in section 4.

Theorem 6.1. *The Petrie length of $Pe(n)$ is equal to $\sigma(n)$.*

Proof. $f_{\sigma(n)-1} = f_{\sigma(n)+1} = \pm 1$, $f_{\sigma(n)} = 0$, so the result follows. □

Note that $Pe(n)$ is a Petrie polygon on $\mathcal{M}_3(n)$.

The Fibonacci sequence mod 7 is 0,1,1,2,3,5,1,6,0 and the Petrie polygon $Pe(7)$ has vertices $[\frac{1}{0}], [\frac{0}{1}], [\frac{1}{1}], [\frac{1}{2}][\frac{2}{3}], [\frac{3}{5}], [\frac{5}{1}], [\frac{1}{6}]$. The next vertex is $[\frac{-1}{0}]$ which is equal to $[\frac{1}{0}]$ so we have closed up our polygon, which has 8 vertices. This polygon is illustrated in Figure 2, where we denote $[\frac{a}{b}]$ by (a,b). This picture of the Klein surface comes from [2]. The same picture also appears in at the paper [6]. We can apply the same idea for other values of n. For example $\mathcal{M}_5$ is the icosahedron and $\sigma(5) = 10$ which is the known Petrie length for the icosahedron.

7 The universal Petrie polygon

To determine the stabiliser of a Petrie polygon it is useful to introduce a new idea. We first extend the standard Fibonacci sequence to include negative numbers. We still want the basic recurrence relation $f_{t-1} + f_t = f_{t+1}$ to hold, so this extended Fibonacci sequence is

$$\cdots -3, 2, -1, 1, 0, 1, 1, 2, 3, \cdots$$

so that $f_{-k} = (-1)^{k-1} f_k$. The universal Petrie polygon $\mathrm{Pe}(\infty)$ is the infinite polygon with vertices $\frac{f_k}{f_{k+1}}$ where $k \in \mathbb{Z}$ and edges the closed intervals $[\frac{f_{k-1}}{f_k}, \frac{f_k}{f_{k+1}}]$. Note that $\frac{f_{-k}}{f_{-k+1}} = \frac{(-1)^{k-1} f_k}{(-1)^k f_{k-1}} = -\frac{f_k}{f_{k-1}}$ and hence the transformation $R : t \mapsto -\frac{1}{t}$ represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is an automorphism of $\mathrm{Pe}(\infty)$. Also $T = P^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ is an

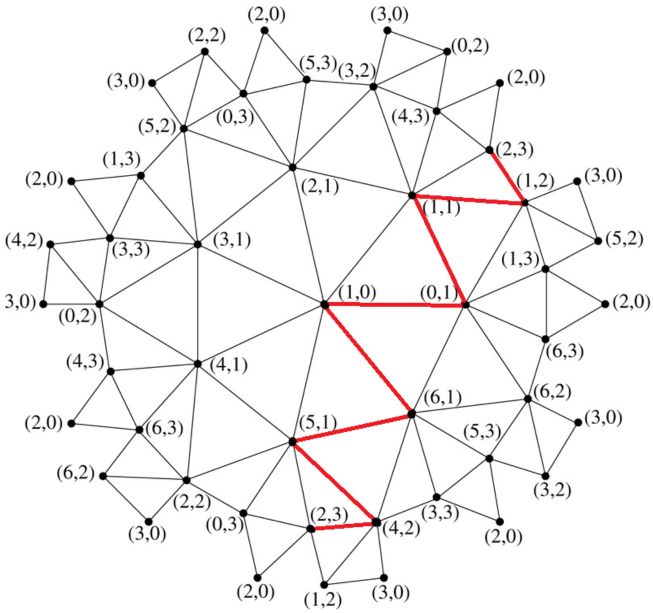

Figure 2: Petrie Polygon

automorphism of Pe(∞) that belongs to Γ. Note that $R^2 = (TR)^2 = I$ so that T and R generate an infinite dihedral group of automorphisms of Pe(∞).

Theorem 7.1. *The automorphism group of Pe(∞) in Γ is equal to $\langle T, R\rangle \cong D_\infty$.*

Proof. The group $\langle T, R\rangle$ is a group of automorphisms of Pe(∞). We show that it acts transitively on Pe(∞). First of all, P maps each Farey fraction two steps along the Farey sequence as

$$T(\frac{f_{k-1}}{f_k}) = \frac{f_{k-1} + f_k}{f_{k-1} + 2f_k} = \frac{f_{k+1}}{f_{k+1} + f_k} = \frac{f_{k+1}}{f_{k+2}}$$

Thus the union of the orbits of $\frac{0}{1}$ and $\frac{1}{0}$ under $\langle T\rangle$ is the whole of Pe(∞), and as $R(\frac{1}{0}) = \frac{0}{1}$, $\langle T, R\rangle$ acts transitively on Pe(∞). We note that the stabiliser of $\frac{1}{0} = \infty$ in Aut Pe(∞) is trivial. For the stabilizer of ∞ in Γ consists of the transformations $z \mapsto z + m$, where $m \in \mathbb{Z}$, and the only translation that preserves Pe(∞) is the identity, Now suppose that $A \in AutPe(\infty)$. Then $A(\infty) = \frac{f_{k-1}}{f_k}$. By transitivity, there exists $B \in \langle T, R\rangle$such that $B(\infty) = \frac{f_{k-1}}{f_k}$. Thus $A^{-1}B$ fixes ∞ and thus $A = B$. $\square$

We now search for the automorphism group of $Pe(n)$.

Theorem 7.2. *The automorphism group of Pe(n) is isomorphic to $D_{\sigma(n)/2}$.*

Proof. We have an epimorphism $\theta : \Gamma \longrightarrow PSL(2, \mathbb{Z}_n)$ and $\theta(T) = P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, where we think of this matrix as lying in PSL$(2, \mathbb{Z}_n)$.

Now $P^{\sigma(n)} = \begin{pmatrix} f_{\sigma(n)-1} & f_{\sigma(n)} \\ f_{\sigma(n)} & f_{\sigma(n)+1} \end{pmatrix}$.

Now $f_{\sigma(n)} \equiv 0 \bmod n$ and $f_{\sigma(n)-1} = f_{\sigma(n)+1} = \pm 1$, by the definition of $\sigma(n)$. Thus $P^{\sigma(n)} = \pm I$ and so $T^{\sigma(n)/2} = \pm I$. which is the identity in PSL (2,$\mathbb{Z}_n$) Thus the automorphism group of Pe(n) is generated by R and T with $R^2 = (RT)^2 = T^{\sigma(n)/2} = 1$ and hence $\langle R, T \rangle \cong D_{\sigma(n)/2}$ of order $\sigma(n)$. □

It is interesting to see how this theorem works in practice, so let us go back to our example of $n = 7$ as illustrated in Figure 2. As $\sigma(7) = 8$ we have an action of D_4 on Pe(7) an 8-sided polygon . The element T has two cycles of length 4, namely

$(1, 0) \longrightarrow (1, 1) \longrightarrow (2, 3) \longrightarrow (5, 1) \longrightarrow (1, 0)$

$(0, 1) \longrightarrow (1, 2) \longrightarrow (3, 5) \longrightarrow (1, 6) \longrightarrow (0, 1)$

and for the involution R we have $(1, 0) \leftrightarrow (0, 1), (1, 1) \leftrightarrow (6, 1), (1, 2) \leftrightarrow (5, 1), (2, 3) \leftrightarrow (4, 2)$.

(Note that $[\frac{3}{5}] = [\frac{-4}{-2}] = [\frac{4}{2}]$ so that $(3, 5) = (4, 2)$, etc.)

As $\Gamma/\Gamma(n)$ acts transitively on the darts of $\mathcal{M}_3(n)$ we use equation (1) in section 5 to obtain

Corollary 7.3. *The number of Petrie polygons on $\mathcal{M}_3(n)$ is equal to*

$$\frac{n^3}{2\sigma(n)}\Pi_{p|n}(1 - \frac{1}{p^2}).$$

Example. Let $n = 7$. Then $\sigma(7) = 8$. The number of Petrie polygons of $\mathcal{M}_3(7)$ is equal to 21. Klein drew three of them in [5]. The others can be found by rotating these through $2\pi k/7$, for $k = 1, \cdots 6$.

8 More about $\sigma(n)$

Theorem 8.1. *For all positive integers $m > 2$, $\sigma(m)$ is even.*

Proof. $P^{\sigma(m)} = \begin{pmatrix} f_{\sigma(m)-1} & f_{\sigma(m)} \\ f_{\sigma(m)} & f_{\sigma(m)+1} \end{pmatrix} \equiv \pm 1 \bmod m$

Thus $(detP)^{\sigma(m)} \equiv 1 \bmod m$, so $(-1)^{\sigma(m)} \equiv 1 \bmod m$ and thus $\sigma(m)$ is even. □

Exactly the same proof shows that $\pi(m)$ is even for $m > 2$. A much easier proof than that given in [8].

Let $\rho = \frac{1+\sqrt{5}}{2}$ (the golden ratio) and $\rho^* = \frac{1-\sqrt{5}}{2}$. Note that $\rho\rho^* = -1$ and $\rho + \rho^* = 1$.

Let $\mathbb{Z}_n[\rho] = \{a + b\rho | a, b \in \mathbb{Z}/(n)\}$ and if $\alpha = a + b\rho$, define $\alpha^* = a + b\rho^*$. Then $(\alpha\beta)^* = \alpha^*\beta^*$.

We define the norm N on $\mathbb{Z}_n[\rho]$ by $N(\alpha) = \alpha\alpha^*$. Then $N(\alpha\beta) = N(\alpha)N(\beta)$. We call α a unit if $N(\alpha) = \pm 1$, so that ρ is a unit. The units of $\mathbb{Z}_n[\rho]$ form a group $\mathbb{Z}_n^*[\rho]$ under multiplication.

Theorem 8.2. *$\sigma(n)$ is the order of ρ in $\mathbb{Z}_n^*[\rho]$, if $f_{\sigma(n)-1} = 1$ and is equal to half the order of ρ if $f_{\sigma(n)-1} = -1$. In all cases $\pi(n)$ is equal to the order of ρ in $\mathbb{Z}_n^*[\rho]$.*

Proof. From $\rho^2 = \rho + 1$ we can use induction to prove that $\rho^m = f_m\rho + f_{m-1}$ Thus if $m = \sigma(n)$, $f_m = 0$ and $f_{m-1} = \pm 1$. Thus $\rho^{\sigma(n)} = 1$ if $f_{\sigma(n)-1} = 1$ and is equal to -1 if $f_{\sigma(n)-1} = -1$ and in the latter case the order of ρ is $2n$. □

The proof for π is similar.

9 The Pisano period and the semiperiod

There is a lot about the Pisano period $\pi(m)$ in the literature . For example, see [8], where the Pisano period is calculated for all primes less than 2000. Very little is known about the semiperiod $\sigma(m)$. We end with a few results comparing the two.

Lemma 9.1. $\pi(m) = \sigma(m)$ *if and only if* $f_{\sigma(m)-1} = 1$. $\pi(m) = 2\sigma(m)$ *if and only if* $f_{\sigma(m)-1} = -1$.

Proof. Let k be the least integer such that, modulo m, $f_{k-1} = -1$, $f_k = 0$. Then $k = \sigma(m)$ and $f_{\sigma(m)+r} = -f_{\sigma(r)}$, so that $f_{2\sigma(m)-1} = -f_{\sigma(m)-1} = 1, f_{2\sigma(m)} = 0$ and $\pi(m) = 2\sigma(m)$.

Alternatively, $f_{k-1} = +1$ and then $\pi(m) = \sigma(m)$. □

We want to determine which of these occur. We give some partial answers. From Theorem 9.2 we see that $\sigma(m) = \pi(m)$ if and only if $\rho^{m-1} = 1$ and $\sigma(m) = 2\pi(m)$ if and only if $\rho^{m-1} = +1$.

Theorem 9.2. *Let* $p \equiv \pm 2 \,mod\, 5$ *be an odd prime. Then* $\pi(p) = 2\sigma(p)$.

Proof. The point is that 5 is a quadratic residue mod p if and only if $p \equiv \pm 1 \bmod 5$. Otherwise 5 is a non-residue and by adjoining $\sqrt{5}$ to F_p, the finite field of p elements which we can take to be $\mathbb{Z}/p\mathbb{Z}$, we get a finite field$K$ of characteristic p with p^2 elements. This field can be considered to be F_p/I, Where I is the ideal generated by $x^2 - x - 1$, which we can identify with all elements of the form $a + b\rho$, where $a, b \in F_p$ the field with p elements. The polynomial $x^2 - x - 1$ has no roots in F_p but two roots in K interchanged by the Frobenius automorphism $\phi : a \longrightarrow a^p$. If α is a root of this polynomial then the other root is $a^p = 1 - \alpha$ and hence so that $a^{p+1} = \alpha - \alpha^2 = -1$. Thus, by Theorem 8.2 and Lemma 9.1, $\pi(p) = 2\sigma(p)$. □

Theorem 9.3. *Let* $p \equiv 11, 19 \,mod\, 20$. *Then* $\pi(p) = \sigma(p)$.

Proof. We have $p \equiv \pm 1 \bmod 5$ and so 5 has a square root in F_p the finite field with p elements and hence $\rho \in F_p$. Its multiplicative group has order $p - 1$. Now

$$\rho^{\pi(p)} = f_{\pi(p)}\rho + f_{\pi(p)-1} \equiv 1 \text{mod}\, p$$

Therefore $\pi(p)$ is a divisor of $p - 1$. Now $p \equiv 3 \bmod 4$ so that $p = 4k + 3$ for some integer k. This $p - 1 = 4k + 2$. If $\pi(p) = 2\sigma(p)$, then $\sigma(p)$ is a divisor of $2k + 1$ and thus $\sigma(p)$ is odd contradicting Theorem 8.1. Therefore $\sigma(p) = \pi(p)$. □

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Odd edge-colorability of subcubic graphs*

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Abstract

An edge-coloring of a graph G is said to be odd if for each vertex v of G and each color c, the vertex v either uses the color c an odd number of times or does not use it at all. The minimum number of colors needed for an odd edge-coloring of G is the odd chromatic index $\chi'_o(G)$. These notions were introduced by Pyber in [7], who showed that 4 colors suffice for an odd edge-coloring of any simple graph. In this paper, we consider loopless subcubic graphs, and give a complete characterization in terms of the value of their odd chromatic index.

Keywords: Subcubic graph, odd edge-coloring, odd chromatic index, odd edge-covering, T-join.

Math. Subj. Class.: 05C15

1 Introduction

1.1 Terminology and notation

Throughout the article we mainly follow the terminology and notation used in [1, 11]. A graph $G = (V(G), E(G))$ is always regarded as being finite, i.e. having a finite nonempty

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set of vertices $V(G)$ and a finite (possibly empty) set of edges $E(G)$. An edge with identical ends is called a *loop*, and an edge with distinct ends a *link*. Two or more links with the same pair of ends are said to be *parallel edges*. The parameters $n(G) = |V(G)|$ and $m(G) = |E(G)|$ are called *order* and *size* of G, respectively. A graph of order 1 is said to be *trivial*, whereas a graph of size 0 *empty*. For every $v \in V(G)$, $E_G(v)$ denotes the set of edges incident to v, and the size of $E_G(v)$ (every loop being counted twice) is the *degree* of v in G, with notation $d_G(v)$. The maximum (resp. minimum) vertex degree in G is denoted by $\Delta(G)$ (resp. $\delta(G)$). We speak of G as a *subcubic graph* whenever $\Delta(G) \leq 3$. Each vertex v having an even (resp. odd) degree $d_G(v)$ is an *even* (resp. *odd*) *vertex*. In particular, if $d_G(v)$ equals 0 (resp. 1), we say that v is an *isolated* (resp. *pendant*) *vertex* of G. Any vertex of degree d is also called a d*-vertex*. A graph is *even* (resp. *odd*) whenever all its vertices are even (resp. odd). The set of neighboring vertices of $v \in V(G)$ is denoted by $N_G(v)$. For every $u \in N_G(v)$, the edge set $E_G(u) \cap E_G(v)$ is called the uv*-bouquet* in G, with notation $\mathcal{B}_{uv}$. The maximum size of a bouquet in G is its *multiplicity*. We say that G is a *simple graph* whenever it is loopless and of multiplicity at most 1. Whenever the underlying graph G is clear from the context, the edge-complement of a subgraph H is denoted by $\hat{H}$, i.e. $\hat{H} = G - E(H)$. A *co-forest* in G is a subgraph whose edge-complement is a forest. Every maximal path whose interior consists entirely of 2-vertices (of G) is called an *open thread*; similarly, every cycle all of whose vertices except one are 2-vertices of G is a *closed thread*. For every connected graph G that is not a cycle, each of its 2-vertices belongs to a unique thread, either open or closed.

1.2 Odd edge-colorings and odd chromatic index

Any mapping $\varphi : E(G) \to S$ is referred to as an *edge-coloring of* G, and then S is called the *color set* of φ. We say that φ is a k-edge-coloring when $|S| \leq k$. Since the nature of the colors is irrelevant, it is conventional to use $S = [k] := \{1, 2, \ldots, k\}$ whenever the color set is of size k. For each color $c \in S$, $E_c(G, \varphi)$ denotes the *color class of* c, being the set of edges colored by c (i.e. $E_c(G, \varphi) = \varphi^{-1}(c)$); whenever G and φ are clear from the context, we denote the color class of c simply by E_c. Given an edge-coloring φ and a vertex v of G, we say the color c *appears at* v if $E_c \cap E_G(v) \neq \emptyset$. Any decomposition $\{H_1, \ldots, H_k\}$ of G can be interpreted as its k-edge-coloring for which the color classes are $E(H_1), \ldots, E(H_k)$.

An *odd edge-coloring* of a given graph G is an edge-coloring such that each nonempty color class induces an odd subgraph. In other words, at each vertex v, for any appearing color c the degree $d_{G[E_c]}(v)$ is odd. Equivalently, an odd edge-coloring can be seen as a decomposition of G into (edge disjoint) odd subgraphs. Such decompositions represent a counterpart to decompositions into even subgraphs, which were mainly used while proving various flow problems (see e.g. [6, 9]). Historically speaking, as a topic in graph theory, decomposing into subgraphs of a particular kind started with the paper of Erdös et al. [2]. An odd edge-coloring of G using at most k colors is referred to as an *odd* k*-edge-coloring*, and then we say that G is *odd* k*-edge-colorable*. Whenever G is odd edge-colorable, the *odd chromatic index* $\chi'_o(G)$ is defined to be the minimum integer k for which G is odd k-edge-colorable.

It is obvious that a necessary and sufficient condition for odd edge-colorability of G is the absence of vertices incident only to loops. Apart from this, the presence of loops does not influence the existence nor changes the value of the index $\chi'_o(G)$. Therefore, while studying these matters it is enough to confine to loopless graphs.

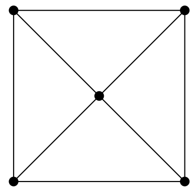

Figure 1: A simple graph with odd chromatic index equal to 4.

As a notion, odd edge-coloring was introduced by Pyber in his survey on graph coverings [7]. The mentioned work considers simple graphs and (among others) contains a proof of the following result.

Theorem 1.1 (Pyber, 1991)**.** *For every simple graph G, it holds that* $\chi'_o(G) \leq 4$.

Pyber remarked that the upper bound is realized by the wheel on four spokes W_4 (see Fig. 1). This upper bound of four colors does not apply to the class of all looplees graphs G. For instance, Fig. 2 depicts four graphs with the following characteristic property: each of their odd subgraphs is of order 2 and size 1, i.e. a copy of K_2. Consequently, for each of them the odd chromatic index equals the size of the graph.

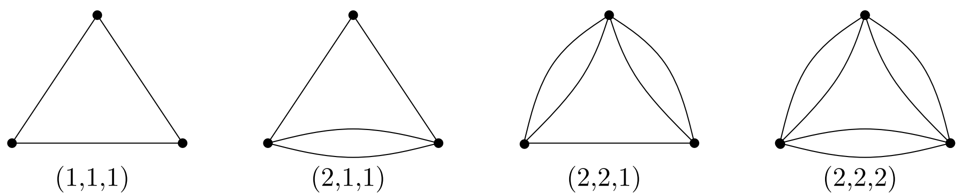

Figure 2: Four Shannon triangles (the smallest one of each type).

As defined in [4], a *Shannon triangle* is a loopless graph on three pairwise adjacent vertices. Observe that for any Shannon triangle, as a direct consequence of the handshake lemma, the edge set of every odd subgraph is fully contained in a single bouquet. Let p, q, r be the parities of the sizes of the bouquets of a Shannon triangle G in non-increasing order, with 2 (resp. 1) denoting that a bouquet consists of an even (resp. odd) number of parallel edges. Then G is a Shannon triangle of *type* (p, q, r), and it holds that $\chi'_o(G) = p + q + r$. The following result was proven in [4].

Theorem 1.2. *For every connected loopless graph G, it holds that* $\chi'_o(G) \leq 6$. *Equality is achieved if and only if G is a Shannon triangle of type* $(2, 2, 2)$.

In this paper we study the odd chromatic index for the class of loopless subcubic graphs G. We shall prove that over that class of graphs holds $\max_G \chi'_o(G) = 4$. Moreover, we will give a complete characterization of the loopless subcubic graphs in terms of the value of their odd chromatic index. In doing so, we will use methods such as eliminating characteristic subtrees and unicyclic subgraphs, or odd co-forests, developed in [3, 10, 12].

The rest of the article is divided into three sections. In the next one, as a preliminary, are collected several 'easy' results (most of them previously known). Section 3 is devoted to a derivation of our main result - a characterization of the loopless subcubic graphs G in terms of the value of $\chi'_o(G)$. The final section briefly conveys some ideas on odd edge-coverability of loopless subcubic graphs.

2 Preliminary results

We begin by recalling the definition of a T-join. For a graph G, let T be an even-sized subset of $V(G)$. Following [1], a spanning subgraph H of G is said to be a *T-join* if $d_H(v)$ is odd for all $v \in T$ and even for all $v \in V(G) \setminus T$. For example, if P is an x-y path in G, the spanning subgraph of G with edge set $E(P)$ is an $\{x, y\}$-join. Observe that the symmetric difference of a T-join and an S-join is a $T \triangle S$-join. With the use of this simple fact and the mentioned example, it can be readily deduced (see [8]) that for any connected graph G and any even-sized subset T of $V(G)$, there exists a T-join of G. Note also that by taking $S = \emptyset$ we infer that the symmetric difference of a T-join and a spanning even subgraph is again a T-join. In particular, removal (resp. addition) of the edges of an edge-disjoint cycle from (resp. to) a T-join, furnishes a T-join. Thus, whenever a T-join of G exists, there also exists such a forest (resp. co-forest). The above discussion yields the following conclusion.

Lemma 2.1. *Given a connected graph G of even order, there exists an odd co-forest in G.*

The next lemma originally appears in [7]. For a proof we refer the reader to [4].

Lemma 2.2. *If F is a forest, then $\chi'_o(F) \leq 2$.*

With the use of Lemmas 2.1 and 2.2, it can be easily shown that every connected graph of even order is odd 3-edge-colorable.

Proposition 2.3. *For every connected graph G of even order, it holds that $\chi'_o(G) \leq 3$.*

Proof. There exists an odd co-forest H in G. Take an odd edge-coloring of $\hat{H}$ with the color set $\{1, 2\}$ and extend it to $E(G)$ by coloring $E(H)$ with 3. Note that we have thus constructed an odd 3-edge-coloring of G. □

Corollary 2.4. *Let v be a 2-vertex in a connected graph G of odd order. Then G admits a 3-edge-coloring that is nearly odd with the only exception being that $E_G(v)$ is monochromatic.*

Proof. Suppress the vertex v, i.e. remove it and then add an edge e with ends in $N_G(v)$ (the edge e is either a link or a loop depending on whether $N_G(v)$ is of size 1 or 2). Denote the obtained graph by H. Since H is connected and of even order, the previous proposition assures its odd 3-edge-colorability. Apply such an edge-coloring to H, and then 'reinstate' the vertex v on the edge e. We thus regain the graph G with a required edge-coloring. □

3 Odd edge-colorability

As already mentioned, throughout this section we consider loopless subcubic graphs. We begin by showing that four colors suffice for an odd edge-coloring of any such graph.

Proposition 3.1. *If G is a loopless subcubic graph, then $\chi'_o(G) \leq 4$.*

Proof. We may assume that G is connected and non-trivial. Moreover, by Proposition 2.3 we may assume that $n(G)$ is odd. In case $\delta(G) = 1$ it is easily shown that $\chi'_o(G) \leq 3$. Indeed, say v is one of its pendant vertices. Since the graph $G - v$ is connected and of even order, by Lemma 2.1 there exists an odd co-forest K in $G - v$. Let us denote its edge-complement in G by F, i.e. $F = G - E(K)$. Then $\{K, F\}$ is a decomposition of G into an odd subgraph K and a forest F. By coloring $E(K)$ with 1, and applying to F an odd 2-edge-coloring with the color set $\{2, 3\}$, we furnish an odd 3-edge-coloring of G.

Henceforth we assume that $\delta(G) = 2$. Let v be one of its non-cut vertices. Either $d_G(v) = 2$ or $d_G(v) = 3$. We study first the case when $d_G(v) = 2$ (see Fig. 3).

Figure 3: The two possibilities when $d_G(v) = 2$.

Let $E_G(v) = \{e, f\}$. By Lemma 2.1, consider a decomposition $\{K, F\}$ of $G - v$ consisting of an odd subgraph K and a forest F. Then the graph $F + e$ is also a forest. Color $E(K)$ with 1, the edge f by 2, and combine with an odd edge-coloring of $F + e$ with the color set $\{3, 4\}$. This confirms that G is odd 4-edge-colorable.

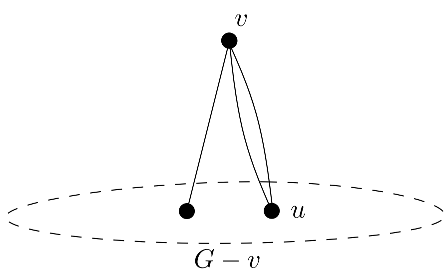

Figure 4: The only possibility when $d_G(v) = 3$.

Now we study the case when $d_G(v) = 3$ (see Fig. 4). Denote by u the neighbor of v for which the uv-bouquet is of size 2. Clearly, u is a pendant vertex of $G - v$. Select an odd co-forest K in $G - v$. Observe that in its edge-complement $\hat{K}$ (taken in $G - v$), the vertex u is isolated. Color $E(K)$ with 1; apply to the forest $\hat{K}$ an odd edge-coloring with the color set $\{2, 3\}$; color the bouquet $\mathcal{B}_{uv}$ with 2 and 3; finally, color the remaining non-colored edge (incident to v) by 4. This gives an odd 4-edge-coloring of G. $\square$

The established upper bound (of four colors) for the odd chromatic index of any loopless subcubic graph is sharp. For example, consider the smallest Shannon triangle G of type $(2, 1, 1)$ (the second of the graphs depicted in Fig. 2). As already observed in the introduction, $\chi'_o(G) = 4$. Note that this particular G can be obtained from a cubic bipartite graph (of order 2) by a single edge subdivision. As it turns out, every subcubic graph obtainable in this manner requires four colors for an odd edge-coloring. On the other hand, for any other connected loopless graph three colors suffice. In order to prove this assertion we will use the following lemma.

Lemma 3.2. *Let G be a connected graph having at least two 2-vertices. Then there exists a tree T in G that satisfies the following two conditions:*

(i) *every 2-vertex of G belongs to $V(T)$,*

(ii) *every pendant vertex of T is a 2-vertex of G.*

Proof. We argue by induction on the number k of 2-vertices in G. In case $k = 2$, we merely take T to be a path in G connecting the only two 2-vertices. Assume that $k > 2$ and let the statement be true whenever the number of 2-vertices is less than k. Suppress a 2-vertex v of G, i.e. remove v and add a new edge e between its neighbors; denote the obtained graph by G'. The inductive hypothesis provides us with a tree T' satisfying the conditions (i) and (ii) for G'. If $e \in E(T')$, then by reversing the suppression, i.e. by subdividing e, we arrive at the desired tree. Otherwise, T' is a subtree of $G - v$. If that is the case, then let P be a v-$V(T')$ path in G and set $T = P \cup T'$. Note that T is a tree in G for which both (i) and (ii) hold. □

Proposition 3.3. *For any connected loopless subcubic graph G, the following two statements are equivalent:*

(i) $\chi'_o(G) = 4$*;*

(ii) *G is obtainable from a cubic bipartite graph by a single edge subdivision.*

Proof. $(i) \Rightarrow (ii)$: Let G be a connected loopless subcubic graph that cannot be obtained from a cubic bipartite graph by a single edge subdivision. We shall prove that $\chi'_o(G) \leq 3$. As in the proof of Proposition 3.1, we may assume that $n(G)$ is odd and $\delta(G) = 2$. There are two cases to be considered.

Case 1: *G has at least two 2-vertices.* Let T be a tree in G as in Lemma 3.2. Note that for each non-isolated vertex u of its edge-complement $\hat{T}$ the degree $d_{\hat{T}}(u)$ is odd. Therefore, the combination of an odd 2-edge-coloring of T with the color set $\{1, 2\}$ and a monochromatic edge-coloring of $\hat{T}$ with the color 3 constitutes an odd 3-edge-coloring of G.

Case 2: *G has a unique 2-vertex.* Denote this particular vertex by v. Assume first the existence of an odd cycle (i.e. a cycle of odd length) C_o in G that does not pass through v. Since G is connected, there exists a nontrivial v-$V(C_o)$ path P. Let w be the other endpoint of P (besides v) and consider the subgraph $G' = P \cup C_o$. Note that w is the only isolated vertex of $\hat{G}'$; moreover, every other vertex of $\hat{G}'$ has an odd degree. Color the set $E(\hat{G}')$ with 3; use the color 1 for $E_G(w)$; color the remaining non-colored edges of P and C_o alternately by 1 and 2 such that the obtained edge-coloring of G' fails to be proper only at w (such a 2-edge-coloring of G' is possible because C_o is an odd cycle). This completes an odd 3-edge-coloring of G.

Assume now that such an odd cycle does not exist in G, meaning that every cycle avoiding v is even. We claim that there exists an even cycle C_e passing through v. To prove this, we argue as follows. Suppress the vertex v, and let e be the new edge. The obtained graph G^* is cubic (since v is the only 2-vertex of G and $\delta(G) = 2$), which further implies that G^* is not bipartite (otherwise, G would be obtainable from the cubic bipartite graph G^* by a single edge subdivision). Consider an odd cycle C^* of G^*. By our current assumption, C^* is not a cycle in G, which implies that $e \in E(C^*)$. Therefore, $v \cup V(C^*)$ constitutes the vertex set of an even cycle C_e passing through v. Once the existence of C_e is established, we can construct an odd 3-edge-coloring of G as follows: take a proper 2-edge-coloring of C_e with the color set $\{1, 2\}$; then color the edge set of $\hat{C}_e$ with 3.

$(ii) \Rightarrow (i)$: Let G be obtainable from a cubic bipartite graph by a single edge subdivision. We shall show that $\chi'_o(G) = 4$. Denote by v the unique 2-vertex of G, and let G' be the graph obtained from G by suppressing v. Since G' is bipartite, there exists a partition X, Y of $V(G')$ such that $E(G') = E(X, Y)$. By Proposition 3.1, $\chi'_o(G) \leq 4$. Suppose this inequality is strict, i.e. suppose there exists an odd 3-edge-coloring of G with the color set $\{1, 2, 3\}$. Without loss of generality, we may assume that the v-X edge is colored by 1, whereas the v-Y edge is colored by 2. Let x_1, x_2, x_3, x_{123} be respectively the number of vertices u from X such that $E_G(u)$ is colored entirely with 1, entirely with 2, entirely with 3, or by all the three colors $1, 2, 3$. Analogously, we employ notation y_1, y_2, y_3, y_{123} for the sizes of the respective subsets of Y.

By double counting the color class E_1, we derive the equality

$$3x_1 + x_{123} = 1 + 3y_1 + y_{123}\,. \tag{3.1}$$

Reasoning similarly for the class E_2, we deduce

$$1 + 3x_2 + x_{123} = 3y_2 + y_{123}\,. \tag{3.2}$$

Let us now consider the difference $x_{123} - y_{123}$. From (3.1) it follows that $x_{123} - y_{123} \equiv 1 (\bmod 3)$. On the other hand, (3.2) yields $x_{123} - y_{123} \equiv -1 (\bmod 3)$. This is the desired contradiction. □

It is a trivial task to characterize the connected loopless subcubic graphs G that are odd 1-edge-colorable. Namely, $\chi'_o(G) = 0$ if and only if G is K_1, whereas $\chi'_o(G) = 1$ precisely when G is odd. We proceed to characterize odd 2-edge-colorability.

Proposition 3.4. *If G is a connected loopless subcubic graph, then the following two statements are equivalent:*

- (i) $\chi'_o(G) \leq 2$;
- (ii) *for every cycle C of G, the set $\{v \in V(C) : d_G(v) = 2\}$ is even-sized.*

Proof. $(i) \Rightarrow (ii)$: Assume (i) and apply to G an odd 2-edge-coloring. Consider an arbitrary cycle C of G. Note that for every $v \in V(C)$ the edge set $E_C(v)$ is either monochromatic (when $d_G(v) = 3$) or dichromatic (when $d_G(v) = 2$). This clearly implies that the set $\{v \in V(C) : d_G(v) = 2\}$ is even-sized.

$(ii) \Rightarrow (i)$: Assume that (ii) holds. In case G is a cycle, it is readily seen that (i) follows. Henceforth, we prove that (i) holds when G is not a cycle. For each pair x, y of non-even vertices of G consider an arbitrary x-y walk W, and count the number of traversed 2-vertices, i.e. count the 2-vertices of G appearing (possibly with repetition) in the interior of W. We claim that the parity of this number is an invariant of the unordered pair x, y, i.e. does not dependent on the choice of W. Indeed, if we suppose the existence of an x-y walk W' which presents a counterexample combined with W, then the symmetric difference $E(W) \oplus E(W')$ must contain the edge set of a cycle C of G for which the set $\{v \in V(C) : d_G(v) = 2\}$ is odd-sized.

Let us employ notation $x \sim y$ (resp. $x \approx y$) whenever the parity of the considered number is odd (resp. even). Seen as binary relations on the set on non-even vertices, both $\sim$ and $\approx$ are symmetric. Moreover, by concatenating suitable walks, one readily deduces that $\approx$ is an equivalence relation, whereas $\sim$ is non-transitive (i.e. $x \sim y$ & $y \sim z \Rightarrow x \approx z$).

This means that there are at most two equivalence classes of $\approx$. In other words, the set of non-even vertices of G can be written as a disjoint union of two (possibly empty) subsets A, B such that $x \sim y$ holds if and only if x and y belong to distinct subsets. Note that there is no A-B edge in G. For each $u \in A$ color $E_G(u)$ with 1; similarly, for each $u \in B$ color $E_G(u)$ with 2. This gives a partial edge-coloring of G such that any non-colored edge is incident to a 2-vertex. Apply the following procedure: as long as there exists a 2-vertex, say v, with $E_G(v)$ not fully colored, consider the unique thread H that contains v. Two edges of H are already colored, and this pre-coloring extends to an edge-coloring of H with the color set $\{1, 2\}$ that is proper at each 2-vertex belonging to $V(H)$. (In case the two pre-colored edges received the same color then the length of H is odd; on the other hand, if they are of different colors, then the length is even.) This eventually completes an odd 2-edge-coloring of G. □

Since all the threads of a given connected loopless subcubic graph G can be detected in linear time, it is linearly decidable whether the set on non-even vertices of G admits a partition into two (possibly empty) subsets A and B as in the proof of the implication $(ii) \Rightarrow (i)$. Thus, it can be checked in linear time whether $\chi'_o(G) \leq 2$. Moreover, the proof of $(ii) \Rightarrow (i)$ suggests the following constructive characterization of odd 2-edge-colorability.

Corollary 3.5. *Every connected loopless subcubic graph G satisfying $\chi'_o(G) \leq 2$ is either an even cycle or can be obtained from a connected odd subcubic graph G_o (loops allowed) in the following manner:*

1. *split $V(G_o)$ arbitrarily into two (possibly empty) subsets A and B;*
2. *subdivide an odd number of times each edge from $E(A, B)$;*
3. *subdivide an even non-zero number of times each loop from $E(G_o)$;*
4. *subdivide an even (possibly zero) number of times each link whose endvertices belong to the same set from the pair A, B.*

To summarize this section, we state the promised characterization of all connected loopless subcubic graphs in terms of their odd chromatic index.

Theorem 3.6. *Let G be a connected loopless subcubic graph. Then*

$$\chi'_o(G) = \begin{cases} 0 & \textit{if } G \textit{ is empty}\,; \\ 1 & \textit{if } G \textit{ is odd}\,; \\ 2 & \textit{if } G \textit{ has 2-vertices, with an even number of them on each cycle}\,; \\ 4 & \textit{if } G \textit{ is obtained from a cubic bipartite graph by a single edge} \\ & \quad \textit{subdivision}\,; \\ 3 & \textit{otherwise}\,. \end{cases}$$

The above comments on the algorithmic aspects of odd 2-edge-colorability, combined with the well-known fact that the decision problem whether a given graph is bipartite can be solved in polynomial time (by using Breadth-First Search), assure that our characterization is good.

Corollary 3.7. *For any loopless subcubic graph G, the odd chromatic index $\chi'_o(G)$ can be determined in polynomial time of $n(G)$.*

4 Odd edge-coverability

In this section we present an application of Theorem 3.6 while briefly studying the odd edge-coverability of subcubic graphs, a related concept to odd edge-colorability. An *edge-covering* of a graph G is a family $\{H_1, \ldots, H_k\}$ of subgraphs such that $\bigcup_{i=1}^{k} E(H_i) = E(G)$. Any edge-covering of G can be interpreted as a 'generalized edge-coloring', i.e. a mapping $\varphi^* : E(G) \to \mathcal{P}^*([k])$ assigning to each edge of G a nonempty subset of the set of colors $\{1, \ldots, k\}$. In other words, we pass from edge-colorings to edge-coverings by allowing more than one color per edge. In the context of an edge-covering φ^*, the color class E_c of any color $c \in [k]$ consists of the edges $e \in E(G)$ for which $c \in \varphi^*(e)$. If each non-empty color class induces an odd subgraph, then we speak of an *odd edge-covering* of G. More verbosely, we say that G is *odd k-edge-coverable* whenever it admits an odd edge-covering with at most k colors. The minimum size (i.e. minimum number of colors) of an odd edge-covering of G is denoted by $\mathrm{cov}_o(G)$. Similar to odd edge-colorability, a given graph G is odd edge-coverable if and only if there are no vertices incident only to loops, and apart from this, the presence of loops does not influence the existence nor changes the value of $\mathrm{cov}_o(G)$. Therefore, any study of odd edge-coverability should be restricted to loopless graphs. Since every odd edge-coloring of G is also an odd edge-covering, it holds that

$$\mathrm{cov}_o(G) \leq \chi'_o(G)\,. \tag{4.1}$$

As a notion, odd edge-covering was introduced in [5]. The scope of the mentioned work are all simple graphs, and the following result is proven.

Theorem 4.1 (Mátrai, 2006). *For every simple graph G, it holds that $\mathrm{cov}_o(G) \leq 3$.*

In this section we consider the possible values of the index $\mathrm{cov}_o(G)$ taken over all connected loopless subcubic graphs G. When G is the smallest Shannon triangle of type $(2, 1, 1)$ (the second graph in Fig. 2), the handshake lemma readily implies that $\mathrm{cov}_o(G) = 4$; indeed, for every graph G of order $n(G) = 3$ the equality $\mathrm{cov}_o(G) = \chi'_o(G)$ holds. We shall prove that this is the only exception to odd 3-edge-coverability of connected loopless subcubic graphs. For this we should note that, according to (4.1) and Theorem 3.6, any exception must be obtainable from a cubic bipartite graph by a single edge subdivision. Thus, it is enough to consider the odd 3-edge-coverability of that particular class of graphs.

Proposition 4.2. *Apart from the smallest Shannon triangle of type* $(2, 1, 1)$*, every other connected loopless subcubic graph is odd* 3*-edge-coverable.*

Proof. Suppose the opposite, i.e. let G present a counterexample. Hence, G can be obtained from a cubic bipartite graph H by a single edge subdivision. Say the subdivided edge $e \in E(H)$ has endpoints x and y, and let v be the introduced 2-vertex. Denote by e_x and e_y the respective 'parts' of e in G (see Fig. 5).

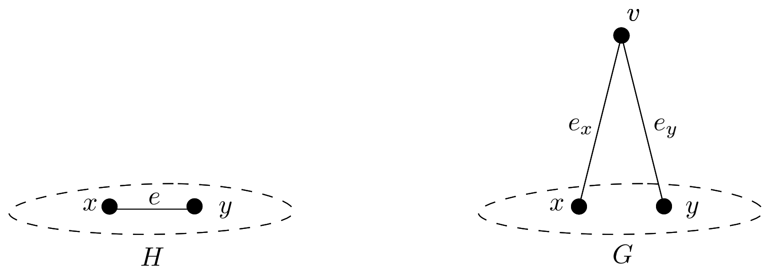

Figure 5: The graphs H and G. (The possibility of another xy-edge in H, i.e. an xy-edge in G, is not excluded.)

Let $\mathcal{B}_{xy}$ be the xy-bouquet of H. Since H is a cubic graph and G is not the smallest Shannon triangle of type $(2, 1, 1)$, the size of $\mathcal{B}_{xy}$ is either 1 or 2. We claim the latter. To confirm this, we argue by contradiction. Suppose the opposite, i.e. let x and y be non-adjacent in G. First we show that the graph $G - v$ is connected. Otherwise, it must consist of two components H_x and H_y, containing x and y, respectively. Moreover, since the only even vertex of the graph H_x (resp. H_y) is the 2-vertex x (resp. y), the handshake lemma implies that both $n(H_x)$ and $n(H_y)$ are odd. By Corollary 2.4, there exists an edge-coloring φ_x (resp. φ_y) of H_x (resp. H_y) with the color set $\{1, 2, 3\}$ that is nearly odd, the only exception being that the edge set $E_{H_x}(x)$ is colored with 1 (resp. the edge set $E_{H_y}(y)$ is colored with 2). Apply $\varphi_x \cup \varphi_y$, and then color e_x by 1 and e_y by 2. We thus obtain an odd 3-edge-coloring of G, a contradiction. This confirms that $G - v$ is indeed connected.

Denote by P a shortest x-y path in $G-v$, and say $x, u_1, \ldots, u_{k-1}, y$ are the consecutive vertices met on a traversal of P. Since H is bipartite with x and y belonging to different partite sets, the length k is an odd integer greater than 2. Suppose that $N_{G-v}(x) = \{u_1\}$, i.e. let the bouquet $\mathcal{B}_{xu_1}$ be of size 2. We can then apply to G the following edge-covering with the color set $\{1, 2, 3\}$: color e_x with 1; for e_y use both 2 and 3; color $\mathcal{B}_{xu_1}$ with 1; for the u_1u_2-edge of P use both 1 and 3; color the rest of $E(G)$ with 3. This clearly implies $\mathrm{cov}_o(G) \leq 3$, a contradiction. Therefore, it must be that, besides u_1, there exists another neighbor of x in $G - v$; let us denote this particular vertex by u. The choice of P assures $u \notin V(P)$. We construct an odd 3-edge-covering of G as follows: color e_x by 1; color e_y by 2; for the unique xu-edge use both 1 and 3; apply to P a proper edge-coloring with the color set $\{1, 2\}$ such that the xu_1-edge receives the color 1; for the rest of $E(G)$ use 3. But the obtained odd 3-edge-covering presents a contradiction, thus establishing that $|\mathcal{B}_{xy}| = 2$, as claimed.

Now, let u be the third neighbor of x in G (besides v and y). Apply to G the following edge-covering φ^* with the color set $\{1, 2, 3\}$: color e_x by 1; color e_y by 2; color the unique xy-edge of G by 1; for the unique xu-edge use both 1 and 3; color the rest of $E(G)$ with 3. It is readily checked that φ^* is an odd 3-edge-covering of G. But this contradicts the choice of G as a counterexample, and thus settles the proposition.

□

There are plentyful of connected loopless subcubic graphs G satisfying $\mathrm{cov}_o(G) = 3$. For instance, every nontrivial odd cycle is such. As another example we may take any G that possesses an even cycle C passing through only one 3-vertex (i.e. the rest of $V(C)$ consists entirely of 2-vertices of G). Yet another example is the graph obtained from $K_{3,3}$

by a single edge-subdivision. In order to derive a result for odd edge-coverability analogous to Theorem 3.6, we need to characterize the odd 2-edge-coverability of connected loopless subcubic graphs. The final proposition of this section can be seen as a step towards such a characterization.

Proposition 4.3. *For every connected loopless subcubic graph G the following two statements are equivalent:*

(i) $\mathrm{cov}_o(G) \leq 2$*;*

(ii) *There exists an $S \subseteq E(G)$ such that*

- *S is not incident to any 2-vertex of G,*
- *every pendant vertex of $G - S$ is a pendant vertex of G,*
- *for every cycle C of $G - S$, the set $\{v \in V(C) : d_G(v) = 2\}$ is even-sized.*

Proof. $(i) \Rightarrow (ii)$: Assume the existence of an odd 2-edge-covering of G with the color set $\{1, 2\}$. Define $S = E_1 \cap E_2$, i.e. let S be the collection of edges that are colored by both colors. It is easily seen that this particular choice for S meets all the requirements of (ii).

$(ii) \Rightarrow (i)$: Let $S \subseteq E(G)$ be as stated in (ii). We claim that the third requirement for S assures $G - S$ admits an edge-coloring with the color set $\{1, 2\}$ which is dichromatic precisely at each 2-vertex of G. To construct such a 2-edge-coloring of $G - S$ we follow a similar pattern to the one in the proof of the second implication from Proposition 3.4: namely, in the graph $G - S$, for each pair of vertices x, y neither of which is a 2-vertex of G, we consider an arbitrary x-y walk W and count the 2-vertices of G appearing (possibly with repetition) in the interior of W; the third requirement for S assures that the parity of this number is an invariant of the unordered pair x, y; let the notation $x \approx y$ mean that this parity is even; as before, it is easily shown that $\approx$ is an equivalence relation with at most two equivalence classes; and so on.

Once such an edge-coloring of $G - S$ is constructed, we can extend it to a 2-edge-covering φ^* of G simply by coloring each edge in S by both 1 and 2. The first two requirements for S clearly imply that φ^* is odd. □

We conclude the paper with the following remark regarding potential further work. In [4] it was conjectured that the problem of determining whether an arbitrary loopless graph G is odd 2-edge-colorable is NP-hard. Perhaps for some values of $\Delta(G)$ beyond 3 this is still decidable in polynomial time. The authors believe that an analogous result to Theorem 3.6 is possible for $\Delta(G) = 4$.

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The spectrum of α-resolvable λ-fold $(K_4 - e)$-designs

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Abstract

A λ-fold G-design is said to be α-resolvable if its blocks can be partitioned into classes such that every class contains each vertex exactly α times. In this paper we study the α-resolvability for λ-fold $(K_4 - e)$-designs and prove that the necessary conditions for their existence are also sufficient, without any exception.

Keywords: α-resolvable G-design, α-parallel class, $(K_4 - e)$-design.

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1 Introduction

For any graph Γ, let $V(\Gamma)$ and $\mathcal{E}(\Gamma)$ be the vertex-set and the edge-set of Γ, respectively, and $\lambda\Gamma$ be the graph Γ with each of its edges replicated λ times. Throughout the paper K_v will denote the complete graph on v vertices, while $K_n \setminus K_h$ will denote the graph with $V(K_n)$ as vertex-set and $\mathcal{E}(K_n) \setminus \mathcal{E}(K_h)$ as edge-set (this graph is sometimes referred to as a complete graph of order n with a *hole* of size h); finally, $K_{n_1,n_2,\ldots,n_t}$ will denote the complete multipartite graph with t-parts of sizes $n_1, n_2, \ldots, n_t$.

Let G and H be simple finite graphs. A *λ-fold G-design* of H ($(\lambda H, G)$-design in short) is a pair $(X, \mathcal{B})$ where X is the vertex-set of H and $\mathcal{B}$ is a collection of isomorphic copies (called *blocks*) of the graph G, whose edges partition the edges of λH. If $\lambda = 1$, we drop the term "1-fold". If $H = K_v$, we refer to such a λ-fold G-design as one of order v. A $(\lambda H, G)$-design is *balanced* if for every vertex x of H the number of blocks containing x is a costant r.

A $(\lambda H, G)$-design is said to be *α-resolvable* if it is possible to partition the blocks into classes (often referred to as *α-parallel classes*) such that every vertex of H appears in exactly α blocks of each class. When $\alpha = 1$, we simply speak of resolvable design and parallel classes. The existence problem of resolvable G-decompositions has been the subject of an extensive research (see [1, 4, 5, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 19, 21, 24]). The α-resolvability, with $\alpha > 1$, has been studied for: $G = K_3$ by D. Jungnickel, R. C. Mullin, S. A. Vanstone [13], Y. Zhang and B. Du [25]; $G = K_4$ by M. J. Vasiga, S. Furino and A.C.H. Ling [22]; $G = C_4$ by M.X. Wen and T.Z. Hong [17].

In this paper we investigate the existence of an α-resolvable λ-fold $(K_4 - e)$-design (where $K_4 - e$ is the complete graph K_4 with one edge removed). In what follows, by $(a, b, c; d)$ we will denote the graph $K_4 - e$ having $\{a, b, c, d\}$ as vertex-set and $\{\{a, b\}, \{a, c\}, \{b, c\}, \{a, d\}, \{b, d\}\}$ as edge-set. Basing on the definitions given above, we can derive the following necessary conditions:

(1) $\lambda v(v - 1) \equiv 0 \pmod{10}$;

(2) $\alpha v \equiv 0 \pmod 4$;

(3) $2\lambda(v - 1) \equiv 0 \pmod{5\alpha}$.

Note that, since the number of α-parallel classes of an α-resolvable λ-fold $(K_4 - e)$-design of order v is $\frac{2\lambda(v-1)}{5\alpha}$ and every vertex appears exactly α times in each of them, we have the following theorem.

Theorem 1.1. *Any α-resolvable λ-fold $(K_4 - e)$-design is balanced.*

From Conditions $(1) - (3)$ we can desume minimum values for α and λ, say α_0 and λ_0, respectively. Similarly to Lemmas 2.1, 2.2 in [22], we have the following lemmas.

Lemma 1.2. *If an α-resolvable λ-fold $(K_4 - e)$-design of order v exists, then $\alpha_0 | \alpha$ and $\lambda_0 | \lambda$.*

Lemma 1.3. *If an α-resolvable λ-fold $(K_4 - e)$-design of order v exists, then a $t\alpha$-resolvable $n\lambda$-fold $(K_4 - e)$-design of order v exists for any positive integers n and t with $t \mid \frac{2\lambda(v-1)}{5\alpha}$.*

The above two lemmas imply the following theorem (for the proof see Theorem 2.3 in [22]).

Theorem 1.4. *If an α_0-resolvable λ_0-fold $(K_4 - e)$-design of order v exists and α and λ satisfy Conditions $(1) - (3)$, then an α-resolvable λ-fold $(K_4 - e)$-design of order v exists.*

Therefore, in order to show that the necessary conditions for α-resolvable designs are also sufficient, we simply need to prove the existence of an α_0-resolvable λ_0-fold (K_4-e)-design of order v, for any given v.

2 Auxiliary definitions

A $(\lambda K_{n_1,n_2,\ldots,n_t}, G)$-design is known as a λ-fold *group divisible design*, G-GDD in short, of type $\{n_1, n_2, \ldots, n_t\}$ (the parts are called the *groups* of the design). We usually use an "exponential" notation to describe group-types: the group-type $1^i 2^j 3^k$... denotes i occurrences of 1, j occurrences of 2, etc. When $G = K_n$ we will call it an n-GDD.

If the blocks of a λ-fold G-GDD can be partitioned into *partial* α-parallel classes, each of which contains all vertices except those of one group, we refer to the decomposition as a λ-fold (α, G)*-frame*; when $\alpha = 1$, we simply speak of λ-fold G-frame (n-frame if additionally $G = K_n$). In a λ-fold (α, G)-frame the number of partial α-parallel classes missing a specified group of size g is $\frac{\lambda g|V(G)|}{2\alpha|\mathcal{E}(G)|}$.

An *incomplete* α-resolvable λ-fold G-design of order $v+h$, $h \geq 1$, with a hole of size h is a $(\lambda(K_{v+h} \setminus K_h), G)$-design in which there are two types of classes, $\frac{\lambda(h-1)|V(G)|}{2\alpha|\mathcal{E}(G)|}$ *partial* classes which cover every vertex α times except those in the hole and $\frac{\lambda v|V(G)|}{2\alpha|\mathcal{E}(G)|}$ *full* classes which cover every vertex of K_{v+h} α times.

3 $v \equiv 0 \pmod 4$

In [4, 5, 23] it was showed that there exists a resolvable $(K_4 - e)$-design of order $v \equiv 16 \pmod{20}$; while, for every $v \equiv 0, 4, 8, 12 \pmod{20}$ Gionfriddo et al. ([7]) proved that there exists a resolvable 5-fold $(K_4 - e)$-design of order v. Hence the necessary conditions are also sufficient.

4 $v \equiv 1 \pmod 2$

4.1 $v \equiv 1 \pmod{10}$

If $v \equiv 1 \pmod{10}$, then $\lambda_0 = 1$ and $\alpha_0 = 4$ and so a solution is given by a cyclic $(K_4 - e)$-design ([2]), where every base block generates a 4-parallel class. If $v = 10k + 1$, $k \geq 4$, the desired design can be obtained by developing in Z_{10k+1} the base blocks listed below:

$(1 + 2i, 4k + 1 + i, 1; 2k + 2)$, $i = 3, 4, \ldots, \lfloor \frac{k}{2} \rfloor$;
$(2k + 3 - 2i, 5k + 2 - i, 1; 2k + 2)$, $i = 1, 2, \ldots, \lceil \frac{k}{2} \rceil$;
$(1, 4k + 1, 3; 6k)$;
$(1, 2k + 2, 5; 6k + 1)$;

where $\lfloor x \rfloor$ (or $\lceil x \rceil$) denote the greatest (or lower) integer that does not exceed (or that exceed) x. If $v = 11, 21, 31$, the base blocks are:

$v = 11$: $(1, 10, 2; 5)$ developed in Z_{11};
$v = 21$: $(1, 11, 3; 15)$, $(1, 7, 2; 10)$ developed in Z_{21};
$v = 31$: $(2, 13, 1; 5)$, $(1, 27, 10; 11)$, $(1, 7, 3; 14)$ developed in Z_{31}.

4.2 $v \equiv 3,5,7,9 \pmod{10}$

If $v \equiv 3,5,7,9 \pmod{10}$, then $\lambda_0 = 5$ and $\alpha_0 = 4$ and so a solution is given by a cyclic 5-fold $(K_4 - e)$-design, where every base block generates a 4-parallel class. The required design is obtained by developing in Z_v the following blocks:

$(1+i, v-1-i, 0; 1), \ \ i = 1, 2, \ldots, \frac{v-3}{2}$;

$(0, 1, 2; v-1)$.

5 $v \equiv 2 \pmod 4$

5.1 $v \equiv 6 \pmod{20}$

If $v \equiv 6 \pmod{20}$, then $\lambda_0 = 1$ and $\alpha_0 = 2$. In order to prove the existence of a 2-resolvable $(K_4 - e)$-design of order v for every $v \equiv 6 \pmod{20}$, preliminarly we need to construct one of order 6.

Lemma 5.1. *There exists a* 2*-resolvable* $(K_4 - e)$*-design of order* 6.

Proof. Let $V = \{0,1,2,3,4,5\}$ be the vertex-set and $\{(0,1,2;3), (2,3,4;5), (4,5,0;1)\}$ be the class. $\square$

For constructing a 2-resolvable $(K_4 - e)$-design of any order $v \equiv 6 \pmod{20}$ and for later use, note that starting from a $(K_4 - e)$-frame of type h^n also a λ-fold $(2, K_4 - e)$-frame of type h^n can be obtained for any $\lambda > 0$, since necessarily $h \equiv 0 \pmod 5$ and so the number of partial parallel classes missing any group is even.

Lemma 5.2. *For every* $v \equiv 6 \pmod{20}$*, there exists a* 2*-resolvable* $(K_4 - e)$*-design of order* v.

Proof. Let $v = 20k + 6$. The case $k = 0$ follows by Lemma 5.1. For $k > 0$, consider a $(2, K_4 - e)$-frame of type 5^{4k+1} ([5]) with groups G_i, $i = 1, 2, \ldots, 4k+1$ and a new vertex ∞. For each $i = 1, 2, \ldots, 4k+1$, let P_i the unique partial 2-parallel class which misses the group G_i. Place on $G_i \cup \{\infty\}$ a copy of a 2-resolvable $(K_4 - e)$-design of order 6, which exists by Lemma 5.1, and combine its full class with the partial class P_i so to obtain the desired design. $\square$

5.2 $v \equiv 2, 10, 14, 18 \pmod{20}$

To prove the existence of an α-resolvable λ-fold $(K_4 - e)$-design of order $v \equiv 2, 10, 14, 18 \pmod{20}$, with minimum values $\lambda_0 = 5$ and $\alpha_0 = 2$, we will construct some small examples most of which will be used as ingredients in the constructions given by the following theorems.

Theorem 5.3. *Let v, g, u, and h be positive integers such that $v = gu + h$. If there exists*

$i)$ a 5*-fold* $(2, K_4 - e)$*-frame of type* g^u*;*

$ii)$ a 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* g*;*

$iii)$ an incomplete 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* $g + h$ *with a hole of size* h*;*

then there exists a 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* $v = gu + h$.

Proof. Take a 5-fold $(2, K_4-e)$-frame of type g^u with groups G_i, $i=1,2,\ldots,u$ and a set H of size h such taht $H\cap(\cup_{i=1}^{u}G_i)=\emptyset$. For $j=1,2,\ldots,g$, let $P_{i,j}$ be the j-th 2-partial class which misses the group G_i. Place on $H\cup G_1$ a copy $\mathcal{D}_1$ of a 2-resolvable 5-fold (K_4-e)-design of order $g+h$ having $g+h-1$ classes $R_{1,1}, R_{1,2},\ldots,, R_{1,g}, H_{1,1}, H_{1,2},\ldots, H_{1,h-1}$. For $i=2,3,\ldots,u$, place on $H\cup G_i$ a copy $\mathcal{D}_i$ of an incomplete 2-resolvable 5-fold (K_4-e)-design of order $g+h$ with H as hole and having $h-1$ partial classes $H_{i,1}, H_{i,2},\ldots, H_{i,h-1}$ and g full classes $R_{i,1}, R_{i,2},\ldots,, R_{i,g}$. Combine the g partial classes $P_{1,j}$ with the full classes $R_{1,1}, R_{1,2},\ldots,, R_{1,g}$ of $\mathcal{D}_1$ and for $i=2,3,\ldots,u$ the g partial classes $P_{i,j}$ of $\mathcal{D}_i$ with the full classes $R_{i,1}, R_{i,2},\ldots, R_{i,g}$ so to obtain gu 2-parallel classes on $H\cup(\cup_{i=1}^{u}G_i)$. Combine the classes $H_{1,1}, H_{1,2},\ldots, H_{1,h-1}$ with the partial classes $H_{i,1}, H_{i,2},\ldots, H_{i,h-1}$ so to obtain $h-1$ 2-parallel classes. The result is a 2-resolvable 5-fold (K_4-e)-design of order $gu+h$ with $gu+h-1$ 2-parallel classes. □

The following lemma gives an input design in the construction of Theorem5.5.

Lemma 5.4. *There exists a* 2*-resolvable* 5*-fold* (K_4-e)*-GDD of type* 2^3.

Proof. Let $\{0,3\}$, $\{1,4\}$ and $\{2,5\}$ be the groups and consider the following classes: $P_1=\{(0,2,1;4),(1,5,0;3),(3,4,2;5)\}$, $P_2=\{(3,5,1;4),(1,2,0;3),(0,4,2;5)\}$, $P_3=\{(0,5,1;4),(2,4,0;3),(1,3,2;5)\}$, $P_4=\{(2,3,1;4),(4,5,0;3),(0,1,2;5)\}$. □

Theorem 5.5. *Let v, g, m, h and u be positive integers such that $v=2gu+2m+h$. If there exists*

- $i)$ *a 3-frame of type m^1g^u;*
- $ii)$ *a* 2*-resolvable* 5*-fold* (K_4-e)*-design of order* $2m+h$*;*
- $iii)$ *an incomplete* 2*-resolvable* 5*-fold* (K_4-e)*-design of order* $2g+h$ *with a hole of size h;*

then there exists a 2-resolvable 5*-fold* (K_4-e)*-design of order* $2gu+2m+h$.

Proof. Let $\mathcal{F}$ be a 3-frame with one group G of cardinality m and u groups G_i, $i=1,2,\ldots,u$ of cardinality g; such a frame has $\frac{m}{2}$ partial classes which miss G, each containing $\frac{gu}{3}$ triples, and, for $i=1,2,\ldots,u$, $\frac{g}{2}$ partial classes which miss G_i, each containing $\frac{g(u-1)+m}{3}$ triples. Expand each vertex 2 times and add a set H of h new vertices. Place on $H\cup(G\times\{1,2\})$ a copy $\mathcal{D}$ of a 2-resolvable 5-fold (K_4-e)-design of order $2m+h$ having $2m+h-1$ classes $R_1, R_2,\ldots, R_{2m}, H_1, H_2,\ldots, H_{h-1}$. For each $i=1,2,\ldots,u$ place on $H\cup(G_i\times\{1,2\})$ a copy $\mathcal{D}_i$ of an incomplete 2-resolvable 5-fold (K_4-e)-design of order $2g+h$ with H as hole and having $h-1$ partial classes $H_{i,j}$ with $j=1,2,\ldots,h-1$ and $2g$ full classes $R_{i,t}$, $t=1,2,\ldots,2g$. For each block $b=\{x,y,z\}$ of a given class of $\mathcal{F}$ place on $b\times\{1,2\}$ a copy of a 2-resolvable 5-fold (K_4-e)-GDD of type 2^3 from Lemma 5.4, having $\{x_1,x_2\}$, $\{y_1,y_2\}$ and $\{z_1,z_2\}$ as groups. This gives $2m$ partial classes (whose blocks are copies of K_4-e) which miss $G\times\{1,2\}$ and $2g$ partial classes which miss $G_i\times\{1,2\}$, $i=1,2,\ldots,u$. Combine the $2m$ partial classes which miss the group $G\times\{1,2\}$ with the classes $R_1, R_2,\ldots, R_{2m}$ so to obtain $2m$ classes. For $i=1,2,\ldots,u$ combine the $2g$ partial classes which miss the group $G_i\times\{1,2\}$ with the full classes of $\mathcal{D}_i$ so to obtain $2gu$ classes. Finally, combine the $h-1$ classes $H_1, H_2,\ldots, H_{h-1}$ of $\mathcal{D}$ with the partial classes of $\mathcal{D}_i$ so to obtain $h-1$ classes. This gives a 2-resolvable 5-fold (K_4-e)-design of order v and $v-1$ 2-parallel classes. □

Theorem 5.6. *Let v, k and h be non-negative integers. If there exists*

$i)$ an incomplete α-resolvable λ-fold $(K_4 - e)$-design of order $v + k + h$ with a hole of size $k + h$;

$ii)$ an incomplete α-resolvable λ-fold $(K_4 - e)$-design of order $k + h$ with a hole of size h;

then there exists an incomplete α-resolvable λ-fold $(K_4 - e)$-design of order $v + k + h$ with a hole of size h.

Lemma 5.7. *There exists a resolvable $(K_4 - e)$-GDD of type $5^2 10^1$.*

Proof. Let $Z_{10} \cup \{\infty_0, \infty_1, \ldots, \infty_9\}$ be the vertex-set and $2Z_{10}$, $2Z_{10} + 1$, $\{\infty_0, \infty_1, \ldots, \infty_9\}$ be the groups. The desired design is obtained by adding $2 \pmod{10}$ to the following base blocks, including the subscripts of ∞: $(0, 1, \infty_0; \infty_1)$, $(2, 5, \infty_0; \infty_1)$, $(4, 9, \infty_0; \infty_1)$, $(6, 3, \infty_0; \infty_1)$, $(8, 7, \infty_0; \infty_1)$. The parallel classes are generate by every base block. □

Lemma 5.8. *There exists a 2-resolvable 5-fold $(K_4 - e)$-GDD of type 10^3.*

Proof. Start with the 2-resolvable 5-fold $(K_4 - e)$-GDD $\mathcal{G}$ of type 2^3 of Lemma 5.4 with groups G_i, $i = 1, 2, 3$. For each block $b = (x, y, z; t)$ of a given 2-parallel class of $\mathcal{G}$ consider a copy of a resolvable $(K_4 - e)$-GDD of type $5^2 10^1$ where $\{x\} \times Z_5$, $\{y\} \times Z_5$, $\{z, t\} \times Z_5$ are the groups. □

Lemma 5.9. *There exists an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 6 with a hole of size 2.*

Proof. On $V = Z_4 \cup H$, where $H = \{\infty_1, \infty_2\}$ is the hole, consider the partial class $\{(1, 3, 0; 2), (0, 2, 1; 3)\}$ and the four full classes obtained by developing $\{(0, 2, \infty_1; \infty_2), (\infty_1, 1, 0; 3), (\infty_2, 2, 3; 1)\}$ in Z_4, where $\infty_i + 1 = \infty_i$ for $i = 1, 2$. □

Lemma 5.10. *There exists an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 10 with a hole of size 2.*

Proof. On $V = Z_8 \cup H$, where $H = \{\infty_1, \infty_2\}$ is the hole, consider the partial class $\{(0, 4, 2; 6), (1, 5, 3; 7), (2, 6, 4; 0), (3, 7, 5; 1)\}$ and the eight full classes obtained by developing $\{(0, 1, \infty_1; 3), (2, 3, \infty_2; 7), (\infty_1, 5, 6; 2), (\infty_2, 6, 4; 5), (4, 7, 1; 0)\}$ in Z_8, where $\infty_i + 1 = \infty_i$ for $i = 1, 2$. □

Lemma 5.11. *There exists an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 14 with a hole of size 4.*

Proof. Let $V = Z_{10} \cup H$ be the vertex-set, where $H = \{\infty_1, \infty_2, \infty_3, \infty_4\}$ is the hole. The partial classes are obtained by adding $2 \pmod{10}$ to the base blocks $(2, 6, 9; 5)$, $(5, 9, 2; 8)$, $(8, 7, 6; 9)$, each block generating a partial class; while, the full classes are obtained by adding $2 \pmod{10}$ to the following base blocks partitioned into two full classes, each class generating five full classes: $\{(0, 8, \infty_1; \infty_2)$, $(1, 5, \infty_3; \infty_4)$, $(\infty_1, 4, 0; 9)$, $(\infty_2, 6, 2; 3)$, $(\infty_3, 3, 7; 8)$, $(\infty_4, 9, 1; 4)$, $(2, 7, 6; 5)\}$, $\{(1, 5, \infty_1; \infty_2)$, $(0, 8, \infty_3; \infty_4)$, $(\infty_1, 3, 9; 4)$, $(\infty_2, 9, 7; 0)$, $(\infty_3, 2, 6; 1)$, $(\infty_4, 6, 8; 3)$, $(4, 7, 2; 5)\}$, where $\infty_i + 1 = \infty_i$ for $i = 1, 2, 3, 4$. □

Lemma 5.12. *There exists an incomplete* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* 14 *with a hole of size* 2.

Proof. On $V = Z_{12} \cup H$, where $H = \{\infty_1, \infty_2\}$ is the hole, consider the partial class $\{(0,6,3;9),(1,7,4;10),(2,8,5;11),(3,9,6;0),(4,10,7;1),(5,11,8;2)\}$ and the twelve full classes obtained by developing $\{(0,1,\infty_1;11),(2,4,\infty_2;10),(\infty_1,10,6;5),(\infty_2,9,2;0),(3,7,8;1),(5,8,7;9),(6,11,3;4)\}$ in Z_{12}, where $\infty_i + 1 = \infty_i$ for $i = 1,2$. $\square$

Lemma 5.13. *There exists an incomplete* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* 22 *with a hole of size 6.*

Proof. Let $V = Z_{16} \cup H$ be the vertex-set, where $H = \{\infty_1, \infty_2, \ldots, \infty_6\}$ is the hole. In Z_{16} develop the full 2-parallel base class $\{(0,3,\infty_1;12),(1,5,\infty_2;2),(8,13,\infty_3;4),(14,15,\infty_4;11),(6,11,\infty_5;\infty_6),(\infty_1,2,1;3),(\infty_2,4,13;8),(\infty_3,7,0;14),(\infty_4,9,6;10),(\infty_5,10,5;15),(\infty_6,12,7;9)\}$. Additionally, include the partial 2-parallel class $\{(0,8,2;10),(1,9,3;11),(2,10,4;12),(3,11,5;13),(4,12,6;14),(5,13,7;15),(6,14,8;0),(7,15,9;1)\}$ repeated five times. $\square$

As consequence of Lemmas 5.9 and 5.13, by Theorem 5.6 the following lemma follows.

Lemma 5.14. *There exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* 22 *with a hole of size* 2.

Lemma 5.15. *There exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* 10.

Proof. Let $V = Z_9 \cup \{\infty\}$ be the vertex-set. The required design is obtained by developing the base class $\{(\infty,0,6;5),(1,5,4;3),(7,8,1;\infty),(2,6,7;8),(3,4,2;0)\}$ in Z_9. $\square$

Lemma 5.16. *There exists an incomplete* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* 30 *with a hole of size* 10.

Proof. Start from a 2-resolvable 5-fold $(K_4 - e)$-GDD of type 10^3 (which exists by Lemma 5.8) having G_i, $i = 1,2,3$, as groups. Fill in the groups G_2 and G_3 with a copy of a 2-resolvable 5-fold $(K_4 - e)$-design of order 10, which exists by Lemma 5.15. This gives an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 30 with G_1 as hole. $\square$

Lemma 5.17. *There exists an incomplete* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* 38 *with a hole of size* 12.

Proof. Let $V = Z_{26} \cup H$ be the vertex-set, where $H = \{\infty_1, \infty_2, \ldots, \infty_{12}\}$ is the hole. The partial classes are: $\{(i, 13+i, 2+i; 15+i) : i = 0,1,\ldots,12\}$, repeaeated five times; $\{(2i, 10+2i, 3+2i; 7+2i) : i = 0,1,\ldots,12\}$ and $\{(1+2i, 11+2i, 4+2i; 8+2i) : i = 0,1,\ldots,12\}$, repeated twice; $\{(2i, 10+2i, 1+2i; 9+2i) : i = 0,1,\ldots,12\}$; $\{(1+2i, 11+2i, 2+2i; 10+2i) : i = 0,1,\ldots,12\}$. The full classes are obtained by developing in $V = Z_{26}$ the full base class $\{(\infty_1,2,1;7),(\infty_2,12,3;24),(\infty_3,16,4;11),(\infty_4,13,5;25),(\infty_5,15,9;22),(\infty_6,17,11;23),(\infty_7,19,18;20),(\infty_8,14,10;18),(\infty_9,4,0;8),(\infty_{10},9,17;19),(\infty_{11},7,2;12),(\infty_{12},15,3;24),(1,5,\infty_1;\infty_2),(10,20,\infty_3;\infty_4),(6,23,\infty_5;\infty_6),(16,21,\infty_7;\infty_8),(22,25,\infty_9;\infty_{10}),(13,21,\infty_{11};\infty_{12}),(0,14,6;8)\}$. $\square$

As consequence of the existence of a 2-resolvable 5-fold $(K_4 - e)$-design of order $v = 4, 12$ (see Section 3 and Theorem 1.4) and Lemmas 5.1, 5.11, 5.13, 5.16, 5.17, 5.15, by Theorem 5.6 the following lemma follows.

Lemma 5.18. *There exists a* 2*-resolvable* 5*-fold* $(K_4{-}e)$*-design of order* $v = 14, 22, 30, 38$.

Lemma 5.19. *There exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* $v = 42, 234$.

Proof. Start with a resolvable 3-GDD of type $3^{\frac{v}{6}}$ ([20]). Expand each vertex 2 times and for each triple b of a given parallel class place on $b \times \{1, 2\}$ a copy of a 2-resolvable 5-fold $(K_4 - e)$-GDD of type 2^3, which exists by Lemma 5.4. Finally, fill each group of size 6 with a copy of a 2-resolvable 5-fold $(K_4 - e)$-design of order 6, which exists by Lemma 5.1. □

Lemma 5.20. *There exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* $v = 50, 62$.

Proof. Start from a 3-frame of type $6^{\frac{v-2}{12}}$ ([3]) and apply Contruction 5.5 with $m = g = 6$, $h = 2$ and $u = \frac{v-14}{12}$ to obtain a 2-resolvable 5-fold $(K_4 - e)$-design of order $v = 50, 62$ (the input designs are: a 2-resolvable 5-fold $(K_4 - e)$-design of order 14, which exists by Lemma 5.18; a 2-resolvable 5-fold $(K_4 - e)$-GDD of type 2^3, which exists by Lemma 5.4; an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 14 with a hole of size 2, which exists by Lemma 5.12). □

Lemma 5.21. *There exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* $v = 34, 274$.

Proof. Start from a 3-frame of type $4^{\frac{v-2}{8}}$ ([3]) and apply Theorem 5.5 with $m = g = 4$, $h = 2$ and $u = \frac{v-10}{8}$ to obtain a 2-resolvable 5-fold $(K_4 - e)$-design of order $v = 34, 274$ (the input designs are: a 2-resolvable 5-fold $(K_4 - e)$-design of order 10, which exists by Lemma 5.15; a 2-resolvable 5-fold $(K_4 - e)$-GDD of type 2^3, which exists by Lemma 5.4; an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 10 with a hole of size 2, which exists by Lemma 5.10). □

Lemma 5.22. *There exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* 70.

Proof. Start from a 3-frame of type 8^4 ([3]) and apply Theorem 5.5 with $m = g = 8$, $h = 6$ and $u = 3$ to obtain a 2-resolvable 5-fold $(K_4 - e)$-design of order 70 (the input designs are; a 2-resolvable 5-fold $(K_4 - e)$-design of order 22, which exists by Lemma 5.18; a 2-resolvable 5-fold $(K_4 - e)$-RGDD of type 2^3, which exists by Lemma 5.4; an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 22 with a hole of size 6, which exists by Lemma 5.13). □

Lemma 5.23. *For every* $v \equiv 2 \pmod{20}$*, there exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* v.

Proof. Let v=$20k + 2$. The case $v = 22, 42, 62$ are covered by Lemmas 5.18, 5.19 and 5.20. For $k \geq 4$, start from a 5-fold $(2, K_4 - e)$-frame of type 20^k ([5]) and apply Theorem 5.3 with $h = 2$ to obtain a 2-resolvable 5-fold $(K_4 - e)$-design of order v (the input designs are a 2-resolvable 5-fold $(K_4 - e)$-design of order 22, which exists by Lemma 5.18, and an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 22 with a hole of size 2, which exists by Lemma 5.14). □

Lemma 5.24. *For every* $v \equiv 10 \pmod{20}$*, there exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* v.

Proof. Let v=$20k + 10$. The case $v = 10, 30, 50, 70$ are covered by Lemmas 5.15, 5.18, 5.20 and 5.22. For $k \geq 4$, start from a 5-fold $(2, K_4 - e)$-frame of type 20^k ([5]) and apply Theorem 5.3 with $g = 20$ and $h = 10$ to obtain a 2-resolvable 5-fold $(K_4 - e)$-design of order v (the input designs are a 2-resolvable 5-fold $(K_4 - e)$-design of order 10, which exists by Lemma 5.15, and an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 30 with a hole of size 10, which exists by Lemma 5.16). □

Lemma 5.25. *For every* $v \equiv 14 \pmod{20}$*, there exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* v*.*

Proof. Let v=$20k + 14$. The case $v = 14, 34, 234, 274$ are covered by Lemmas 5.18, 5.19 and 5.21. For $k \geq 2$, $k \notin \{11, 13\}$, start from a 5-fold $(2, K_4 - e)$-frame of type 10^{2k+1} ([5]), apply Theorem 5.3 with $h = 4$ and proceed as in Lemma 5.24. □

Lemma 5.26. *For every* $v \equiv 18 \pmod{60}$*, there exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* v*.*

Proof. Let v=$60k+18$. Take a resolvable 3-GDD of type 3^{10k+3} ([6]). Expand each vertex 2 times and for each block b of a parallel class place on $b \times \{1, 2\}$ a copy of a 2-resolvable 5-fold $(K_4 - e)$-GDD of type 2^3 which exists by Lemma 5.4, so to obtain a 2-resolvable 5-fold $(K_4 - e)$-GDD of type 6^{10k+3}. Finally, fill in each group of size 6 with a copy of a 2-resolvable 5-fold $(K_4 - e)$-design, which exists by Lemma 5.1. □

Lemma 5.27. *For every* $v \equiv 38 \pmod{60}$*, there exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* v*.*

Proof. Let $v = 60k + 38$. The case $v = 38$ follows by Lemmas 5.18. For $k \geq 1$, start from a 3-frame of type 6^{5k+3} ([6]) and apply Theorem 5.5 with $m = g = 6$, $h = 2$ and $u = 5k + 2$ to obtain a 2-resolvable 5-fold $(K_4 - e)$-design of order v (the input designs are: a 2-resolvable 5-fold $(K_4 - e)$-design of order 14, which exists by Lemma 5.18; a 2-resolvable 5-fold $(K_4 - e)$-GDD of type 2^3, which exists by Lemma 5.4; an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 14 with a hole of size 2, which exists by Lemma 5.11) □

Lemma 5.28. *For every* $v \equiv 58 \pmod{120}$*, there exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* v*.*

Proof. Let $v = 120k + 58$. Start from a 3-frame of type 4^{15k+7} ([6]) and apply Theorem 5.5 with $m = g = 4$, $h = 2$ and $u = 15k + 6$ to obtain a 2-resolvable 5-fold $(K_4 - e)$-design of order v (the input designs are: a 2-resolvable $(K_4 - e)$-design of order 10, which exists by Lemma 5.15; a 2-resolvable 5-fold $(K_4 - e)$-RGDD of type 2^3, which exists by Lemma 5.4; an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 10 with a hole of size 2, which exists by Lemma 5.10). □

Lemma 5.29. *For every* $v \equiv 118 \pmod{120}$*, there exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* v*.*

Proof. Let $v = 120k + 118$. Start from a 3-frame of type $10^1 4^{15k+12}$, $k \geq 0$, ([6]) and apply Theorem 5.5 with $h = 2$ to obtain a 2-resolvable 5-fold $(K_4 - e)$-design of order v (the input designs are: a 2-resolvable 5-fold $(K_4 - e)$-design of order 22, which exists by Lemma 5.18; a 2-resolvable 5-fold $(K_4 - e)$-RGDD of type 2^3, which exists by Lemma

5.4; an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 10 with a hole of size 2, which exists by Lemma 5.10). □

6 Main result

The results obtained in the previous sections can be summarized into the following theorem.

Theorem 6.1. *The necessary conditions* $(1) - (3)$ *for the existence of* α*-resolvable* λ*-fold* $(K_4 - e)$*-designs are also sufficient.*

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The endomorphisms of Grassmann graphs*

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Abstract

A graph G is a core if every endomorphism of G is an automorphism. A graph is called a pseudo-core if every its endomorphism is either an automorphism or a colouring. Suppose that $J_q(n,m)$ is a Grassmann graph over a finite field with q elements. We show that every Grassmann graph is a pseudo-core. Moreover, $J_2(4,2)$ is not a core and $J_q(2k+1,2)$ $(k \geq 2)$ is a core.

Keywords: Grassmann graph, core, pseudo-core, endomorphism, maximal clique.

Math. Subj. Class.: 05C60, 05C69

1 Introduction

Throughout this paper, all graphs are finite undirected graphs without loops or multiple edges. For a graph G, we let $V(G)$ denote the vertex set of G. If xy is an edge of G, then x and y are said to be *adjacent*, and denoted by $x \sim y$. Let G and H be two graphs. A *homomorphism* φ from G to H is a mapping $\varphi : V(G) \to V(H)$ such that $\varphi(x) \sim \varphi(y)$ whenever $x \sim y$. If H is the complete graph K_r, then φ is a r-*colouring* of G (*colouring* for short). An *isomorphism* from G to H is a bijection $\varphi : V(G) \to V(H)$ such that $x \sim y$ $\Leftrightarrow \varphi(x) \sim \varphi(y)$. Graphs G and H are called *isomorphic* if there is an isomorphism from

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G to H, and denoted by $G \cong H$. A homomorphism (resp. isomorphism) from G to itself is called an *endomorphism* (resp. *automorphism*) of G.

Recall that a graph G is a *core* if every endomorphism of G is an automorphism. A subgraph H of G is a *core of* G if it is a core and there exists a homomorphism from G to H. Every graph has a core, which is an induced subgraph and is unique up to isomorphism [5]. A graph is called *core-complete* if it is a core or its core is complete.

A graph G is called a *pseudo-core* if every endomorphism of G is either an automorphism or a colouring. Every core is a pseudo-core. Any pseudo-core is core-complete but not vice versa. For more information, see [2, 6, 9].

For a graph G, an important and difficult problem is to distinguish whether G is a core [2, 5, 6, 7, 11, 15]. If G is not a core or we don't know whether it is a core, then we need to judge whether it is a pseudo-core because the concept of pseudo-core is the most close to the core. Recently, Godsil and Royle [6] discussed some properties of pseudo-cores. Cameron and Kazanidis [2] discussed the core-complete graph and the cores of symmetric graphs. The literature [10] showed that every bilinear forms graph is a pseudo-core which is not a core. One of the latest result is from [9], where it was proved that every alternating forms graph is a pseudo-core. Moreover, Orel [13, 12] proved that each symmetric bilinear forms graph (whose diameter is greater than 2) is a core and each Hermitian forms graph is a core.

Suppose that $\mathbb{F}_q$ is the finite field with q elements, where q is a power of a prime. Let V be an n-dimensional row vector space over $\mathbb{F}_q$ and let $\begin{bmatrix} V \\ m \end{bmatrix}$ be the set of all m-dimensional subspaces of V. The *Grassmann graph* $J_q(n, m)$ has the vertex set $\begin{bmatrix} V \\ m \end{bmatrix}$, and two vertices are adjacent if their intersection is of dimension $m - 1$. If $m = 1$, we have a complete graph and hence it is a core. Since $J_q(n, m) \cong J_q(n, n - m)$, we always assume that $4 \leq 2m \leq n$ in our discussion unless specified otherwise. The number of vertices of $J_q(n, m)$ is the Gaussian binomial coefficient:

$$\begin{bmatrix} n \\ m \end{bmatrix} = \prod_{i=1}^{m} \frac{q^{n+1-i} - 1}{q^i - 1}. \tag{1.1}$$

For $J_q(n, m)$, the distance of two vertices X and Y is $d(X, Y) := m - \dim(X \cap Y)$. Any Grassmann graph is distance-transitive [1, Theorem 9.3.3] and connected. By [6, Corollary 4.2], every distance-transitive graph is core-complete, thus every Grassmann graph is core-complete. The Grassmann graph plays an important role in geometry, graph theory, association schemes and coding theory.

Recall that an *independent set* of a graph G is a set of vertices that induces an edgeless graph. The size of the largest independent set is called the *independence number* of G, denoted by $\alpha(G)$. The *chromatic number* $\chi(G)$ of G is the least value of k for which G can be k-colouring. A *clique* of a graph G is a complete subgraph of G. A clique C is maximal if there is no clique of G which properly contains C as a subset. A *maximum clique* of G is a clique with the maximum size. The *clique number* of G is the number of vertices in a maximum clique, denoted by $\omega(G)$.

By [6, p.273], if G is a distance-transitive graph and $\chi(G) > \omega(G)$, then G is a core. Unluckily, applying the eigenvalues or the known results of graph theory for Grassmann graph, to prove the inequality $\chi(G) > \omega(G)$ is difficult. Thus, it is a difficult problem to verify a Grassmann graph being a core. However, there are some Grassmann graphs which are not cores (see Section 4). Therefore, we need to judge whether a Grassmann graph is a pseudo-core. So far, this is an open problem. We solve this problem as follows:

Theorem 1.1. *Every Grassmann graph $J_q(n,m)$ is a pseudo-core.*

The paper is organized as follows. In Section 2, we give some properties of the maximal cliques of Grassmann graphs. In Section 3, we shall prove Theorem 1.1. In Section 4, we discuss cores on Grassmann graphs. We shall show that $J_2(4,2)$ is not a core, $J_q(2k+1,2)$ ($k \geq 2$) is a core.

2 Maximal cliques of Grassmann graph

In this section we shall discuss some properties of the maximal cliques of Grassmann graphs.

We will denote by $|X|$ the cardinal number of a set X. Suppose that V is an n-dimensional row vector space over $\mathbb{F}_q$. For two vector subspaces S and T of V, the *join* $S \vee T$ is the minimal dimensional vector subspace containing S and T. We have the dimensional formula (cf. [8, Lemma 2.1] or [16]):

$$\dim(S \vee T) = \dim(S) + \dim(T) - \dim(S \cap T). \tag{2.1}$$

Throughout this section, suppose that $4 \leq 2m \leq n$. For every $(m-1)$-dimensional subspace P of V, let $[P\rangle_m$ denote the set of all m-dimensional subspaces containing P, which is called a *star*. For every $(m+1)$-dimensional subspace Q of V, let $\langle Q]_m$ denote the set of all m-dimensional subspaces of Q, which is called a *top*. By [4], every maximal clique of $J_q(n,m)$ is a star or a top. For more information, see [14].

By [16, Corollary 1.9],

$$|[P\rangle_m| = \frac{q^{n-m+1}-1}{q-1}, \quad |\langle Q]_m| = \frac{q^{m+1}-1}{q-1}. \tag{2.2}$$

If $n > 2m$, then every maximum clique of $J_q(n,m)$ is a star. If $n = 2m$, then every maximal clique of $J_q(n,m)$ is a maximum clique. By (2.2) we have

$$\omega(J_q(n,m)) = \begin{bmatrix} n-m+1 \\ 1 \end{bmatrix} \ \text{if } n \geq 2m. \tag{2.3}$$

Since $n \geq 2m$, we have

$$|[P\rangle_m| \geq |\langle Q]_m|, \ \text{and} \ |[P\rangle_m| > |\langle Q]_m| \text{ if } n > 2m. \tag{2.4}$$

Lemma 2.1. *If $[P\rangle_m \cap \langle Q]_m \neq \emptyset$, then the size of $[P\rangle_m \cap \langle Q]_m$ is $q+1$.*

Proof. Since $[P\rangle_m \cap \langle Q]_m \neq \emptyset$, one gets $P \subseteq Q$. It follows that $[P\rangle_m \cap \langle Q]_m$ consists of all m-dimensional subspaces containing P in Q. By [16, Corollary 1.9], the desired result follows. $\square$

Lemma 2.2. ([8, Corollary 4.4]) *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two distinct stars (tops). Then $|\mathcal{M}_1 \cap \mathcal{M}_2| \leq 1$.*

Lemma 2.3. *Suppose $[A\rangle_m \neq [B\rangle_m$. Then $[A\rangle_m \cap [B\rangle_m \neq \emptyset$ if and only if $\dim(A \cap B) = m-2$. In this case, $[A\rangle_m \cap [B\rangle_m = \{A \vee B\}$.*

Proof. Since $\dim(A) = \dim(B) = m-1$ and $A \neq B$, one gets $\dim(A \vee B) \geq m$. If $[A\rangle_m \cap [B\rangle_m \neq \emptyset$, then by Lemma 2.2, there exists a vertex C of $J_q(n,m)$ such that $\{C\} = [A\rangle_m \cap [B\rangle_m$. It follows from (2.1) and $A, B \subset C$ that $C = A \vee B$ and $\dim(A \cap B) = m-2$. Conversely, if $\dim(A \cap B) = m-2$, then Lemma 2.2 and (2.1) imply that $C := A \vee B$ is a vertex of $J_q(n,m)$ and hence $\{C\} = [A\rangle_m \cap [B\rangle_m$. $\square$

Lemma 2.4. *Suppose $\langle P]_m \neq \langle Q]_m$. Then $\langle P]_m \cap \langle Q]_m \neq \emptyset$ if and only if $\dim(P \cap Q) = m$. In this case, $\langle P]_m \cap \langle Q]_m = \{P \cap Q\}$.*

Proof. By $\dim(P) = \dim(Q) = m+1$ and $P \neq Q$, we have $\dim(P \cap Q) \leq m$. If $\langle P]_m \cap \langle Q]_m \neq \emptyset$, then Lemma 2.2 implies that there exists a vertex C of $J_q(n,m)$ such that $\{C\} = \langle P]_m \cap \langle Q]_m$. Since $C \subset P \cap Q$, we get that $C = P \cap Q$ and $\dim(P \cap Q) = m$. Conversely, if $\dim(P \cap Q) = m$, then by $P \cap Q \in \langle P]_m \cap \langle Q]_m$ and Lemma 2.2, we have $\{P \cap Q\} = \langle P]_m \cap \langle Q]_m$. □

In the following, let φ be an endomorphism of $J_q(n,m)$ and $\mathrm{Im}(\varphi)$ be the image of φ.

Lemma 2.5. *If $\mathcal{M}$ is a maximal clique, then there exists a unique maximal clique containing $\varphi(\mathcal{M})$.*

Proof. Suppose there exist two distinct maximal cliques $\mathcal{M}'$ and $\mathcal{M}''$ containing $\varphi(\mathcal{M})$. Then $\varphi(\mathcal{M}) \subseteq \mathcal{M}' \cap \mathcal{M}''$. By Lemmas 2.1 and 2.2, $|\mathcal{M}' \cap \mathcal{M}''| \leq q+1$. Since $|\mathcal{M}| = |\varphi(\mathcal{M})|$, by (2.2) we have $|\varphi(\mathcal{M})| > q+1$, a contradiction. □

Lemma 2.6. *Let $\mathcal{M}$ be a star and $\mathcal{N}$ be a top such that $|\varphi(\mathcal{M}) \cap \varphi(\mathcal{N})| > q+1$. Then $\varphi(\mathcal{N}) \subseteq \varphi(\mathcal{M})$.*

Proof. Let $\mathcal{N}'$ be the maximal clique containing $\varphi(\mathcal{N})$. Then $|\varphi(\mathcal{M}) \cap \mathcal{N}'| > q+1$. One gets $\varphi(\mathcal{M}) = \mathcal{N}'$ by Lemmas 2.1 and 2.2. □

Lemma 2.7. *Suppose there exist two distinct stars $[A\rangle_m$ and $[B\rangle_m$ such that*

$$[A\rangle_m \cap [B\rangle_m = \{X\}, \quad \varphi([A\rangle_m) = \varphi([B\rangle_m).$$

If $\varphi([A\rangle_m)$ is a star, then φ is a colouring of $J_q(n,m)$.

Proof. Write $\mathcal{M} := \varphi([A\rangle_m)$. Then $\varphi([B\rangle_m) = \mathcal{M}$ and $\varphi(X) \in \mathcal{M}$. Assume that $\mathcal{M}$ is a star. If $\mathrm{Im}(\varphi) = \mathcal{M}$, then φ is a colouring of $J_q(n,m)$. Now we prove $\mathrm{Im}(\varphi) = \mathcal{M}$ as follows. Suppose that Y is any vertex with $Y \sim X$. Since $G := J_q(n,m)$ is connected, it suffices to show that there exist two distinct stars $[C\rangle_m$ and $[D\rangle_m$ such that

$$\{Y\} = [C\rangle_m \cap [D\rangle_m \quad \text{and} \quad \varphi([C\rangle_m) = \varphi([D\rangle_m) = \mathcal{M}.$$

In fact, if we can prove this point, then we can imply that $\varphi(Z) \in \mathcal{M}$ for all $Z \in V(G)$. We prove it as follows.

Since $X \in \langle X \vee Y]_m \cap [A\rangle_m \cap [B\rangle_m$, using Lemma 2.2 we get $|\langle X \vee Y]_m \cap [A\rangle_m \cap [B\rangle_m| = 1$. By Lemma 2.1 we obtain

$$|\langle X \vee Y]_m \cap [A\rangle_m| = |\langle X \vee Y]_m \cap [B\rangle_m| = q+1.$$

It follows that

$$|\langle X \vee Y]_m \cap ([A\rangle_m \cup [B\rangle_m)| = 2q+1.$$

Observe that

$$\varphi(\langle X \vee Y]_m \cap ([A\rangle_m \cup [B\rangle_m)) \subseteq \varphi(\langle X \vee Y]_m) \cap \varphi([A\rangle_m \cup [B\rangle_m) \subseteq \varphi(\langle X \vee Y]_m) \cap \mathcal{M}.$$

Since the restriction of φ on a clique is injective, one gets

$$|\varphi(\langle X \vee Y]_m) \cap \mathcal{M}| \geq 2q+1 > q+1.$$

Thus, Lemma 2.6 implies that

$$\varphi(\langle X \vee Y]_m) \subseteq \mathcal{M}. \tag{2.5}$$

So $\varphi(Y) \in \mathcal{M}$. Write $C := X \cap Y$. Since every vertex of $[C\rangle_m \setminus \{X\}$ is adjacent to X, by our claim we have $\varphi([C\rangle_m) = \mathcal{M}$.

Pick a vertex Z such that $Z \sim Y$ and the distance from X is 2. Write $D = Y \cap Z$. Since $Y \in [D\rangle_m \cap \langle X \vee Y]_m$, by Lemma 2.1 we have $|[D\rangle_m \cap \langle X \vee Y]_m| = q+1$. It follows from (2.5) that $|\varphi([D\rangle_m) \cap \mathcal{M}| \geq q+1$. Thus Lemma 2.2 implies that $\varphi([D\rangle_m) = \mathcal{M}$. Since $\{Y\} = [C\rangle_m \cap [D\rangle_m$, $[C\rangle_m$ and $[D\rangle_m$ are the desired stars. □

3 Proof of Theorem 1.1

For the proof of Theorem 1.1, we only need to consider the case $4 \leq 2m \leq n$. We divide the proof of Theorem 1.1 into two cases: $n > 2m$ and $n = 2m$.

Lemma 3.1. *If $n > 2m$, then every Grassmann graph $J_q(n, m)$ is a pseudo-core.*

Proof. Suppose that $n > 2m \geq 4$. Then by (2.4), every maximum clique of $J_q(n, m)$ is a star. Let φ be an endomorphism of $J_q(n, m)$. Then the restriction of φ on any clique is injective, so φ transfers stars to stars.

Suppose φ is not a colouring. It suffices to show that φ is an automorphism. Write $G_r := J_q(n, r)$, where $1 \leq r \leq m-1$. By Lemma 2.7, the images under φ of any two distinct and intersecting stars are distinct. Hence by Lemma 2.3, φ induces an endomorphism φ_{m-1} of G_{m-1} such that

$$\varphi([A\rangle_m) = [\varphi_{m-1}(A)\rangle_m.$$

Let X be any vertex of $J_q(n, m)$. Then there exist two vertices X' and X'' of G_{m-1} such that $X = X' \vee X''$. Then $[X'\rangle_m \cap [X''\rangle_m = \{X\}$ and $\varphi(X) \in \varphi([X'\rangle_m) \cap \varphi([X''\rangle_m)$. Since φ is not a colouring, by Lemma 2.7 $\varphi([X'\rangle_m)$ and $\varphi([X''\rangle_m)$ are two distinct stars. By Lemma 2.2, $[\varphi_{m-1}(X')\rangle_m \cap [\varphi_{m-1}(X'')\rangle_m = \{\varphi(X)\}$. Thus Lemma 2.3 implies that

$$\varphi(X) = \varphi_{m-1}(X') \vee \varphi_{m-1}(X''). \tag{3.1}$$

When $m = 2$, G_1 is a complete graph, hence it is a core. We next show that φ_{m-1} is not a colouring of G_{m-1} for $m \geq 3$. For any two vertices A_1 and A_3 of G_{m-1} at distance 2, we claim that

$$\varphi_{m-1}(A_1) \neq \varphi_{m-1}(A_3).$$

There exists an $A_2 \in V(G_{m-1})$ such that $A_1 \sim A_2 \sim A_3$. Write $Y_1 := A_1 \vee A_2$ and $Y_2 := A_2 \vee A_3$. Then $Y_1 \sim Y_2$, so $\varphi(Y_1) \neq \varphi(Y_2)$. By (3.1),

$$\varphi(Y_1) = \varphi_{m-1}(A_1) \vee \varphi_{m-1}(A_2), \quad \varphi(Y_2) = \varphi_{m-1}(A_2) \vee \varphi_{m-1}(A_3).$$

Thus our claim is valid. Otherwise, one has $\varphi(Y_1) = \varphi(Y_2)$, a contradiction.

Pick a star $\mathcal{N}$ of G_{m-1}. Since the diameter of G_{m-1} is at least two, there exists a vertex $A_4 \in V(G_{m-1}) \setminus \mathcal{N}$ that is adjacent to some vertex in $\mathcal{N}$. If $B \in \mathcal{N}$ such that A_4 is not adjacent to B, then $d(A_4, B) = 2$. By our claim, $\varphi_{m-1}(A_4) \neq \varphi(B)$ and hence $\varphi_{m-1}(A_4) \notin \varphi_{m-1}(\mathcal{N})$. Therefore, φ_{m-1} is not a colouring.

By induction, we may obtain induced endomorphism φ_r of G_r for each r. Furthermore,

$$\varphi(X) = \varphi_{k_1}(X_{k_1}) \vee \varphi_{k_2}(X_{k_2}) \vee \cdots \vee \varphi_{k_s}(X_{k_s}), \tag{3.2}$$

where $X = X_{k_1} \vee X_{k_1} \vee \cdots \vee X_{k_s} \in V(G_m)$ and $1 \leq \dim(X_{k_i}) = k_i \leq m-1$.

In order to show that φ is an automorphism, it suffices to show that φ is injective. Assume that X and Y are any two distinct vertices in G_m with $d(X,Y) = s$. Thus $\dim(X \cap Y) = m - s$. If $s = 1$, then $\varphi(X) \neq \varphi(Y)$. Now suppose $s \geq 2$. There are 1-dimensional row vectors $X_i, Y_i, i = 1, \ldots .s$, such that X, Y can be written as $X = (X \cap Y) \vee X_1 \vee \cdots \vee X_s, Y = (X \cap Y) \vee Y_1 \vee \cdots \vee Y_s$. Let $Z = (X \cap Y) \vee X_1 \vee \cdots \vee X_{s-1} \vee Y_s \in V(G_m)$. By $X \sim Z$, $\dim(\varphi(X) \vee \varphi(Z)) = m+1$. Applying (3.2), one has that $\varphi(X) = \varphi_{m-s}(X \cap Y) \vee \varphi_1(X_1) \vee \cdots \vee \varphi_1(X_s)$, $\varphi(Y) = \varphi_{m-s}(X \cap Y) \vee \varphi_1(Y_1) \vee \cdots \vee \varphi_1(Y_s)$ and $\varphi(Z) = \varphi_{m-s}(X \cap Y) \vee \varphi_1(X_1) \vee \cdots \vee \varphi_1(X_{s-1}) \vee \varphi_1(Y_s)$. Therefore, we get $\varphi(X) \vee \varphi(Z) \subseteq \varphi(X) \vee \varphi(Y)$. It follows that $\varphi(X) \neq \varphi(Y)$. Otherwise, one has $\varphi(X) \vee \varphi(Z) \subseteq \varphi(X)$, a contradiction to $\dim(\varphi(X) \vee \varphi(Z)) = m+1$. Hence, φ is an automorphism, as desired.

By above discussion, $J_q(n,m)$ is a pseudo-core when $n > 2m$. □

Lemma 3.2. *If $n = 2m$, then every Grassmann graph $J_q(n,m)$ is a pseudo-core.*

Proof. Suppose that $n = 2m \geq 4$. For a subspace W of V, the *dual subspace $W^\perp$* of W in V is defined by

$$W^\perp = \{v \in V \mid wv^{\mathrm{t}} = 0, \ \forall\, w \in W\},$$

where v^{t} is the transpose of v.

For an endomorphism φ of $J_q(2m,m)$, define the map

$$\varphi^\perp : V(J_q(2m,m)) \longrightarrow V(J_q(2m,m)), \quad A \longmapsto \varphi(A)^\perp.$$

Then $\varphi^\perp$ is an endomorphism of $J_q(2m,m)$. Note that $\varphi^\perp$ is an automorphism (resp. colouring) whenever φ is an automorphism (resp. colouring). For any maximal clique $\mathcal{M}$ of $J_q(2m,m)$, $\varphi(\mathcal{M})$ and $\varphi^\perp(\mathcal{M})$ are of different types.

Next we shall show that $J_q(2m,m)$ is a pseudo-core.

Case 1. There exist $[A\rangle_m$ and $\langle X]_m$ such that $[A\rangle_m \cap \langle X]_m \neq \emptyset$ and $\varphi([A\rangle_m)$, $\varphi(\langle X]_m)$ are of the same type.

By Lemma 2.1, the size of $[A\rangle_m \cap \langle X]_m$ is $q+1$. Then $|\varphi([A\rangle_m) \cap \varphi(\langle X]_m)| \geq q+1$. Since $\varphi([A\rangle_m)$ and $\varphi(\langle X]_m)$ are of the same type, by Lemma 2.2 one gets

$$\varphi([A\rangle_m) = \varphi(\langle X]_m). \tag{3.3}$$

Note that $A \subseteq X$. Pick any $Y \in \begin{bmatrix} V \\ m+1 \end{bmatrix}$ satisfying $A \subseteq Y$ and $\dim(X \cap Y) = m$. Then $\langle Y]_m \cap [A\rangle_m \neq \emptyset$. By Lemma 2.1 we have $|\varphi(\langle Y]_m) \cap \varphi([A\rangle_m)| \geq q+1$. By Lemma 2.2 and (3.3) we obtain either $\varphi(\langle Y]_m) = \varphi(\langle X]_m)$ or $\varphi(\langle Y]_m)$ and $\varphi(\langle X]_m)$ are of different types.

Case 1.1. There exists a $Y \in \begin{bmatrix} V \\ m+1 \end{bmatrix}$ such that $\varphi(\langle Y]_m)$ and $\varphi(\langle X]_m)$ are of different types. For any $B \in \begin{bmatrix} X \cap Y \\ m-1 \end{bmatrix}$, we have that $B \subseteq Y$ and $B \subseteq X$. Since $|[B\rangle_m) \cap \langle X]_m| = |[B\rangle_m) \cap \langle Y]_m| = q+1$, we have similarly

$$|\varphi([B\rangle_m) \cap \varphi(\langle X]_m)| \geq q+1 \quad \text{and} \quad |\varphi([B\rangle_m) \cap \varphi(\langle Y]_m)| \geq q+1.$$

Since $\varphi(\langle Y]_m)$ and $\varphi(\langle X]_m)$ are of different types, Lemma 2.2 implies that $\varphi([B\rangle_m) = \varphi(\langle X]_m)$ or $\varphi([B\rangle_m) = \varphi(\langle Y]_m)$ for any $B \in \begin{bmatrix} X \cap Y \\ m-1 \end{bmatrix}$.

Since the size of $\begin{bmatrix} X \cap Y \\ m-1 \end{bmatrix}$ is at least 3, by above discussion, there exist two subspaces $B_1, B_2 \in \begin{bmatrix} X \cap Y \\ m-1 \end{bmatrix}$ such that $\varphi([B_1\rangle_m) = \varphi([B_2\rangle_m)$. Note that $[B_1\rangle_m \cap [B_2\rangle_m \neq \emptyset$ because

$X \cap Y \in [B_i\rangle_m$ $(i = 1, 2)$. If $\varphi([B_1\rangle_m)$ is a star, then φ is a colouring by Lemma 2.7. Suppose $\varphi([B_1\rangle_m)$ is a top. Then $\varphi^\perp([B_1\rangle_m)$ is a star. By Lemma 2.7 again, $\varphi^\perp$ is a colouring. Hence, φ is also a colouring.

Case 1.2. $\varphi(\langle Y]_m) = \varphi(\langle X]_m)$ for any $Y \in \begin{bmatrix} V \\ m+1 \end{bmatrix}$. Consider a star $[C\rangle_m$ where C satisfies $C \subset X$ and $\dim(C \cap A) = m-2$. Then $(A \vee C) \subseteq X$ and $\dim(A \vee C) = m$. For any $T \in [C\rangle_m$, since $(A \vee C) \subseteq (A \vee T)$ and $m \le \dim(A \vee T) \le m+1$, there exists a subspace $W \in \begin{bmatrix} V \\ m+1 \end{bmatrix}$ such that $(A \vee T) \subseteq W$ and $\dim(W \cap X) \ge m$ (because $(A \vee C) \subseteq W \cap X$).

Since $T \in \langle W]_m$, $\varphi(T) \in \varphi(\langle W]_m)$. By the condition, $\varphi(\langle W]_m) = \varphi(\langle X]_m)$. Then $\varphi(\langle W]_m) = \varphi([A\rangle_m)$ by (3.3). It follows that $\varphi(T) \in \varphi([A\rangle_m)$ for all $T \in [C\rangle_m$, and so $\varphi([C\rangle_m) \subseteq \varphi([A\rangle_m)$. Hence, $\varphi([C\rangle_m) = \varphi([A\rangle_m)$. Since $[C\rangle_m \cap [A\rangle_m \neq \emptyset$, similar to the proof of Case 1.1, φ is a colouring.

Case 2. For any two maximal cliques of different types containing common vertices, their images under φ are of different types.

In this case, φ maps the maximal cliques of the same type to the maximal cliques of the same type.

Case 2.1. φ maps stars to stars. In this case φ maps tops to tops by Lemmas 2.1 and 2.2.

If there exist two distinct stars $\mathcal{M}$ and $\mathcal{M}'$ such that $\mathcal{M} \cap \mathcal{M}' \neq \emptyset$ and $\varphi(\mathcal{M}) = \varphi(\mathcal{M}')$, then φ is a colouring by Lemma 2.7. Now suppose $\varphi(\mathcal{M}) \neq \varphi(\mathcal{M}')$ for any two distinct stars $\mathcal{M}$ and $\mathcal{M}'$ with $\mathcal{M} \cap \mathcal{M}' \neq \emptyset$. By Lemma 2.3, φ induces an endomorphism φ_{m-1} of $J_q(2m, m-1)$ such that $\varphi([A\rangle_m) = [\varphi_{m-1}(A)\rangle_m$. By Lemma 3.1, $J_q(2m, m-1)$ is a pseudo-core. Thus, φ_{m-1} is an automorphism or a colouring.

We claim that φ_{m-1} is an automorphism of $J_q(2m, m-1)$. For any $C \in \begin{bmatrix} V \\ m \end{bmatrix}$ and $B \in \begin{bmatrix} C \\ m-1 \end{bmatrix}$, since $C \in [B\rangle_m$ and $\varphi([B\rangle_m) = [\varphi_{m-1}(B)\rangle_m$, we have $\varphi(C) \in [\varphi_{m-1}(B)\rangle_m$. Then $\varphi_{m-1}(B) \subseteq \varphi(C)$, which implies that $\varphi_{m-1}(\langle C]_{m-1})$ is a top of $J_q(2m, m-1)$. If $m = 2$, our claim is valid. Now suppose $m \ge 3$ and φ_{m-1} is a colouring. Then $\mathrm{Im}(\varphi_{m-1})$ is a star of $J_q(2m, m-1)$. Note that $\varphi_{m-1}(\langle C]_{m-1}) \subseteq \mathrm{Im}(\varphi_{m-1})$ and $|\varphi_{m-1}(\langle C]_{m-1})| > q+1$, contradicting to Lemma 2.1. Hence, our claim is valid. Therefore, φ maps distinct stars onto distinct stars, and φ is an automorphism.

Case 2.2. φ maps stars to tops. In this case φ maps tops to stars by Lemmas 2.1 and 2.2.

Note that $\varphi^\perp$ maps stars to stars. By Case 2.1, $\varphi^\perp$ is an automorphism. Hence, φ is an automorphism.

By above discussion, we have proved that every Grassmann graph $J_q(2m, m)$ is a pseudo-core. □

By Lemmas 3.1 and 3.2, we have proved Theorem 1.1.

4 Cores on Grassmann graphs

In this section, we shall show that $J_2(4, 2)$ is not a core and $J_q(2k+1, 2)$ $(k \ge 2)$ is a core.

It is well-known (cf. [3, Theorem 6.10 and Corollary 6.2]) that the chromatic number of G satisfies the following inequality:

$$\chi(G) \ge \max\{\omega(G),\ |V(G)|/\alpha(G)\}.$$

By [15, Lemma 2.7.2], if G is a vertex-transitive graph, then

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)} \geq \omega(G). \tag{4.1}$$

Lemma 4.1. *Let G be a Grassmann graph. Then G is a core if and only if $\chi(G) > \omega(G)$. In particular, if $\frac{|V(G)|}{\omega(G)}$ is not an integer, then G is a core.*

Proof. By [6, Corollary 4.2], every distance-transitive graph is core-complete, thus G is core-complete. Then, $\chi(G) > \omega(G)$ implies that G is a core. Conversely, if G is a core, then we must have $\chi(G) > \omega(G)$. Otherwise, there exists an endomorphism f of G such that $f(G)$ is a maximum clique of G, a contradiction to G being a core. Thus, G is a core if and only if $\chi(G) > \omega(G)$.

By [2, p.148, Remark], if the core of G is complete, then $|V(G)| = \omega(G)\alpha(G)$. Assume that $\frac{|V(G)|}{\omega(G)}$ is not an integer. Then $|V(G)| \neq \omega(G)\alpha(G)$. Therefore, the core of G is not complete and hence G is a core. $\square$

Denote by $\mathbb{F}_q^{m\times n}$ the set of $m \times n$ matrices over $\mathbb{F}_q$ and $\mathbb{F}_q^n = \mathbb{F}_q^{1\times n}$. Let $G = J_q(n,m)$ where $n > m$. If X is a vertex of G, then $X = [\alpha_1, \dots, \alpha_m]$ is an m-dimensional subspace of the vector space $\mathbb{F}_q^n$, where $\{\alpha_1, \dots, \alpha_m\}$ is a basis of X. Thus, X has a *matrix representation* $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} \in \mathbb{F}_q^{m\times n}$ (cf. [8, 16]). For simpleness, the matrix representation of $X \in V(G)$ is also denoted by X. For matrix representations X, Y of two vertices X and Y, $X \sim Y$ if and only if $\operatorname{rank}\begin{pmatrix} X \\ Y \end{pmatrix} = m+1$. Note that if X is a matrix representation then $X = PX$ (as matrix representation) for any $m \times m$ invertible matrix P over $\mathbb{F}_q$. Then, $V(G)$ has a matrix representation

$$V(G) = \left\{X : X \in \mathbb{F}_q^{m\times n},\ \operatorname{rank}(X) = m\right\}.$$

Now, we give an example of Grassmann graph which is not a core as follows.

Example 4.2. Let $G = J_2(4,2)$. Then G is not a core. Moreover, $\chi(G) = \omega(G) = 7$ and $\alpha(G) = 5$.

Proof. Applying the matrix representation of $V(G)$, $G = J_2(4,2)$ has 35 vertices as follows:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\ A_3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix},\ A_4 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$
$$A_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix},\ A_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},\ A_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix},\ A_8 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$
$$A_9 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix},\ A_{10} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},\ A_{11} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix},\ A_{12} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix},$$
$$A_{13} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix},\ A_{14} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix},\ A_{15} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix},\ A_{16} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix},$$
$$A_{17} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ A_{18} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ A_{19} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ A_{20} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$
$$A_{21} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},\ A_{22} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ A_{23} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},\ A_{24} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix},$$
$$A_{25} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},\ A_{26} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix},\ A_{27} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ A_{28} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$
$$A_{29} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ A_{30} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ A_{31} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ A_{32} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$
$$A_{33} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ A_{34} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ A_{35} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Suppose that $\mathcal{L}_1 = \{A_1, A_{10}, A_{12}, A_{15}, A_{17}\}$, $\mathcal{L}_2 = \{A_2, A_6, A_{20}, A_{19}, A_{34}\}$, $\mathcal{L}_3 = \{A_3, A_8, A_{21}, A_{22}, A_{35}\}$, $\mathcal{L}_4 = \{A_5, A_9, A_{18}, A_{24}, A_{29}\}$, $\mathcal{L}_5 = \{A_7, A_{14}, A_{23}, A_{27}, A_{33}\}$, $\mathcal{L}_6 = \{A_4, A_{13}, A_{25}, A_{28}, A_{30}\}$, and $\mathcal{L}_7 = \{A_{11}, A_{16}, A_{26}, A_{31}, A_{32}\}$. It is easy to see that $V(G) = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \cdots \cup \mathcal{L}_7$ and $\mathcal{L}_1, \ldots, \mathcal{L}_7$ are independent sets. Thus $\chi(G) \leq 7$. On the other hand, (4.1) implies that $\chi(G) \geq \omega(G) = 7$. Therefore, $\chi(G) = \omega(G) = 7$. It follows from Lemma 4.1 that G is not a core. By (4.1) again, we have $\alpha(G) = 5$. □

We believe that $J_q(2k, 2)$ $(k \geq 2)$ is not a core for all q (which is a power of a prime). But this a difficult problem. Next, we give some examples of Grassmann graphs which are cores.

Example 4.3. If $k \geq 2$, then $J_q(2k+1, 2)$ is core.

Proof. When $k \geq 2$, let $G = J_q(2k+1, 2)$. Applying (1.1) and (2.3) we have

$$\frac{|V(G)|}{\omega(G)} = \frac{q^{2k+1}-1}{q^2-1} = \frac{q^{2k+1}-q}{q^2-1} + \frac{1}{q+1}.$$

Thus $\frac{|V(G)|}{\omega(G)}$ is not an integer for any q (which is a power of a prime). By Lemma 4.1, G is a core. □

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On convergence of binomial means, and an application to finite Markov chains*

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Abstract

For a sequence $\{a_n\}_{n\geq 0}$ of real numbers, we define the sequence of its arithmetic means $\{a_n^*\}_{n\geq 0}$ as the sequence of averages of the first n elements of $\{a_n\}_{n\geq 0}$. For a parameter $0 < p < 1$, we define the sequence of p-binomial means $\{a_n^p\}_{n\geq 0}$ of the sequence $\{a_n\}_{n\geq 0}$ as the sequence of p-binomially weighted averages of the first n elements of $\{a_n\}_{n\geq 0}$. We compare the convergence of sequences $\{a_n\}_{n\geq 0}$, $\{a_n^*\}_{n\geq 0}$ and $\{a_n^p\}_{n\geq 0}$ for various $0 < p < 1$, i.e., we analyze when the convergence of one sequence implies the convergence of the other.

While the sequence $\{a_n^*\}_{n\geq 0}$, known also as the sequence of Cesàro means of a sequence, is well studied in the literature, the results about $\{a_n^p\}_{n\geq 0}$ are hard to find. Our main result shows that, if $\{a_n\}_{n\geq 0}$ is a sequence of non-negative real numbers such that $\{a_n^p\}_{n\geq 0}$ converges to $a \in \mathbb{R}\cup\{\infty\}$ for some $0 < p < 1$, then $\{a_n^*\}_{n\geq 0}$ also converges to a. We give an application of this result to finite Markov chains.

Keywords: Sequence, convergence, Cesàro mean, binomial mean, finite Markov chain.

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1 Introduction

For a sequence $\{a_n\}_{n\geq 0}$ of real numbers and for a parameter $0 < p < 1$, define the sequence of its arithmetic means $\{a_n^*\}_{n\geq 0}$ and the sequence of its p-binomial means $\{a_n^p\}_{n\geq 0}$ as

$$a_n^* = \frac{1}{n+1}\sum_{i=0}^{n} a_i \qquad \text{and} \qquad a_n^p = \sum_{i=0}^{n}\binom{n}{i}p^i q^{n-i}a_i,$$

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where $q = 1 - p$. We see that a_n^* is a uniformly weighted average of the numbers $a_0, a_1, \ldots, a_n$ and a_n^p is a binomially weighted average of the numbers $a_0, a_1, \ldots, a_n$.

In this article, we will analyse the relationship between the convergence of sequences $\{a_n\}_{n\geq 0}$, $\{a_n^p\}_{n\geq 0}$ and $\{a_n^*\}_{n\geq 0}$. Our results are presented in the following table.

	$\{a_n\}_{n\geq 0}$	$\{a_n^{p_1}\}_{n\geq 0}$	$\{a_n^{p_2}\}_{n\geq 0}$	$\{a_n^*\}_{n\geq 0}$
$\{a_n\}_{n\geq 0}$		$\Longrightarrow$	$\Longrightarrow$	$\Longrightarrow$
$\{a_n^{p_1}\}_{n\geq 0}$	$\not\Longrightarrow$		$? \overset{a_n\geq 0}{\Longrightarrow}$	$\overset{a_n\geq 0}{\Longrightarrow}$
$\{a_n^{p_2}\}_{n\geq 0}$	$\not\Longrightarrow$	$\Longrightarrow$		$\overset{a_n\geq 0}{\Longrightarrow}$
$\{a_n^*\}_{n\geq 0}$	$\not\Longrightarrow$	$\not\Longrightarrow$	$\not\Longrightarrow$	

Table 1: The table shows whether the convergence of a sequence in the leftmost column implies the convergence of a sequence in the first row, for $0 < p_1 < p_2 < 1$. The symbol $\Longrightarrow$ means that the implication holds, and the symbol $\not\Longrightarrow$ means that there is a counterexample with $a_n \in \{0, 1\}$, for all $n \in \mathbb{N}$. If there is a condition above $\Longrightarrow$, then the implication does not hold in general, but it holds if the condition is true. If there is a ? before the condition, we do not know whether the condition is the right one (an open problem), but the implication does not hold in general.

The sequence $\{a_n^*\}_{n\geq 0}$ is also known as the sequence of Cesàro means and is well studied in the literature [1, 4]. On the other hand, information about the convergence of p-binomial means is hard to find. Also, the notion of p-binomial means is coined especially for the purpose of this article. However, there are a few definitions that are close to ours [1, 4, 5]. First, we have to mention the Hausdorff means [1, 4]: the p-binomial means as well as the arithmetic mean are its special cases. Unfortunately, the Hausdorff means are a bit too general for our purposes in the sense that the known results that are useful for this paper can be quite easily proven in our special cases.

One of the closest notions to the k-binomial mean is the one of k-binomial transform [5]:

$$\tilde{a}_n^k = \sum_{i=0}^{n} \binom{n}{i} k^n a_i,$$

which coincides with $\{a_n^p\}_{n\geq 0}$ for $k = p = 0.5$, but is different for other p and k. Another similar definition is given with Euler means [4, pages 70, 71]:

$$\overline{a}_n = \frac{1}{2^{n+1}} \sum_{i=0}^{n} \binom{n+1}{i+1} a_i.$$

Some results, like the first row and the first column of Table 1, are not hard to prove (Section 3). Other results (Sections 4 and 5) require more careful ideas. This is true especially for the main result of this paper, Theorem 5.1, which proves, using the notation from Table 1, that

$$\{a_n^p\}_{n\geq 0} \overset{a_n\geq 0}{\Longrightarrow} \{a_n^*\}_{n\geq 0}.$$

In Section 6 we give an application of this theorem to finite Markov chains.

2 Preliminaries

Let $\mathbb{N}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ be the sets of non-negative integers, positive real numbers and non-negative real numbers, respectively. For $a \in \mathbb{R}$, let $\lfloor a \rfloor$ be the greatest integer not greater than a and let $\lceil a \rceil$ be the smallest integer not smaller than a. We will allow a limit of a sequence to be infinite and we will write $a < \infty$ (which means exactly $a \in \mathbb{R}$) to emphasize that a is finite.

For functions $f, g : \mathbb{N} \to \mathbb{R}_0^+$ we say that

- $f(n) = \mathrm{O}(g(n))$ if there is some $C > 0$ such that $f(n) \leq Cg(n)$ for all sufficiently large n,
- $f(n) = \Theta(g(n))$ if there are some $C_1, C_2 > 0$ such that $C_1 g(n) \leq f(n) \leq C_2 g(n)$ for all sufficiently large n,
- $f(n) = \mathrm{o}(g(n))$ if $g(n)$ is non-zero for all large enough n and $\lim_{n\to\infty} \dfrac{f(n)}{g(n)} = 0$.

The following lemma will be useful later.

Lemma 2.1. *Let $u : \mathbb{N} \to \mathbb{R}\backslash\{0\}$ and $k : \mathbb{N} \to \mathbb{R}$ be functions such that $\lim_{n\to\infty} u(n)k(n) = \lim_{n\to\infty} u(n) = 0$. Then*

$$\lim_{n\to\infty} \frac{\big(1 + u(n)\big)^{k(n)/u(n)}}{e^{k(n)}} = 1.$$

Proof. Because $e^x = \sum \frac{x^i}{i!}$ and $e^x \geq 1 + x$, there is an analytic function $g : \mathbb{R} \to \mathbb{R}^+$ such that $e^x = 1 + x + g(x)x^2$ and $g(0) = \frac{1}{2}$. Hence, if we omit writing the argument of functions u and k,

$$\lim_{n\to\infty} \frac{(1+u)^{k/u}}{e^k} = \lim_{n\to\infty} \left(\frac{e^u - g(u)u^2}{e^u}\right)^{k/u} = \lim_{n\to\infty} \left(\left(1 - \frac{g(u)u^2}{e^u}\right)^{\frac{e^u}{g(u)u^2}}\right)^{\frac{ukg(u)}{e^u}}.$$

Because $\lim_{n\to\infty} \dfrac{g(u)u^2}{e^u} = 0$ and because $\lim_{x\to 0}(1-x)^{1/x} = e^{-1}$, we have

$$\lim_{n\to\infty} \left(1 - \frac{g(u)u^2}{e^u}\right)^{\frac{e^u}{g(u)u^2}} = e^{-1}.$$

From

$$\lim_{n\to\infty} \frac{ukg(u)}{e^u} = 0,$$

the result follows. □

Some properties of probability mass function of binomial distribution

Let $\mathbf{X}$ be a random variable having a binomial distribution with parameters $p \in (0,1)$ and $n \in \mathbb{N}$. For $q = 1 - p$ and $i \in \mathbb{Z}$, we have by definition

$$\Pr[\mathbf{X} = i] = B_n^i(p) = \begin{cases} \binom{n}{i} p^i q^{n-i} & \text{if } 0 \leq i \leq n \\ 0 & \text{else.} \end{cases}$$

In this subsection, we state and mathematically ground some properties that can be seen from a graph of binomial distribution (see Fig. 1). The results will be nice, some of them folklore, but the proofs will be technical.

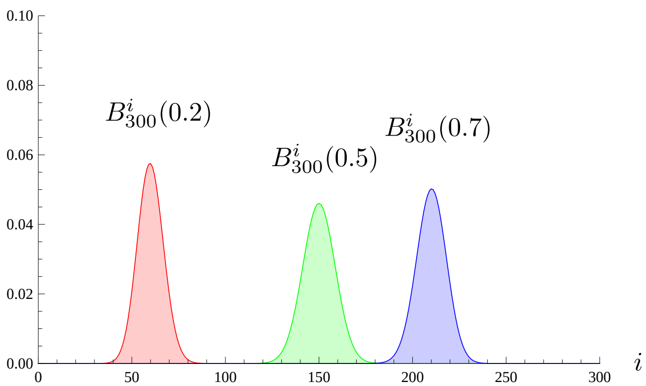

Figure 1: Binomial distribution with $n = 300$ and $p = 0.2$ (red), $p = 0.5$ (green), $p = 0.7$ (blue). The graphs show $B^i_n(p)$ with respect to i.

It is well known (see some basic probability book) that the expected value of $\mathbf{X}$ is $\mathbb{E}(\mathbf{X}) = pn$. First, we will prove that also the "peak" of the probability mass function is roughly at pn.

Lemma 2.2. *For $p \in (0, 1)$, $n \in \mathbb{N}$ and for $0 \leq i \leq n$,*

$$B^i_n(p) \geq B^{i-1}_n(p) \iff i \leq (n + 1)p.$$

Proof. The expression

$$\frac{B^i_n(p)}{B^{i-1}_n(p)} = \frac{(n - i + 1)p}{i(1 - p)}$$

is at least 1 iff $i \leq p(n + 1)$. □

Next, we state a Chernoff bound proven in [3, inequalities (6) and (7)], which explains why the probability mass function for binomial distribution "disappears" (see Fig. 1), when i is far enough from pn.

Theorem 2.3. *Let $\mathbf{X}$ be a binomially distributed random variable with parameters $p \in (0, 1)$ and $n \in \mathbb{N}$. Then for each $\delta \in (0, 1)$,*

$$\Pr\left[|\mathbf{X} - np| \geq np\delta\right] \leq 2e^{-\delta^2 np/3}.$$

We will only use the following corollary of the theorem. It is not difficult to prove and the proof is omitted.

Corollary 2.4. *For $p \in (0, 1)$, let $\alpha : \mathbb{N} \to \mathbb{R}^+$ be some function such that $\alpha(n) < p\sqrt{n}$ for all n. Then, for all $n \in \mathbb{N}$, it holds*

$$\sum_{i:\, |i-np| \geq \sqrt{n}\alpha(n)} B^i_n(p) \leq 2e^{-\alpha^2(n)/(3p)}.$$

This corollary also tells us that, for large n, roughly everything is gathered in an $\mathrm{O}(\sqrt{n})$ neighborhood of np. What is more, the next lemma implies that in $\mathrm{o}(\sqrt{n})$ neighborhood of np, $B_n^i(p)$ does not change a lot.

Lemma 2.5. *Let $p \in (0,1)$ be a parameter and let $\beta(n) : \mathbb{N} \to \mathbb{R}$ be a function such that $|\beta(n)| = \mathrm{O}(\sqrt{n})$ and $\lim_{n\to\infty} |\beta(n)| = \infty$. Then, for all large enough n, it holds*

$$\frac{B_n^{\lfloor np\rfloor}(p)}{B_n^{\lfloor np\rfloor - \lfloor \beta(n)\rfloor}(p)} \leq e^{\frac{1}{p(1-p)} \cdot \frac{\lfloor \beta(n)\rfloor^2}{n}}.$$

Proof. For all large enough n for which $\beta(n) \geq 0$, we have

$$\begin{aligned}
\frac{B_n^{\lfloor np\rfloor}(p)}{B_n^{\lfloor np\rfloor - \lfloor \beta(n)\rfloor}(p)} &= \frac{\binom{n}{\lfloor np\rfloor} p^{\lfloor \beta(n)\rfloor}}{\binom{n}{\lfloor np\rfloor - \lfloor \beta(n)\rfloor}(1-p)^{\lfloor \beta(n)\rfloor}} \\
&= \prod_{i=0}^{\lfloor \beta(n)\rfloor - 1} \frac{(n - \lfloor np\rfloor + \lfloor \beta(n)\rfloor - i)p}{(\lfloor np\rfloor - i)(1-p)} \qquad (2.1) \\
&\leq \prod_{i=0}^{\lfloor \beta(n)\rfloor - 1} \left(1 + \frac{1}{p(1-p)} \cdot \frac{\lfloor \beta(n)\rfloor}{n}\right).
\end{aligned}$$

In the last inequality we used the fact that

$$\frac{(n - \lfloor np\rfloor + \lfloor \beta(n)\rfloor - i)p}{(\lfloor np\rfloor - i)(1-p)} \leq 1 + \frac{1}{p(1-p)} \cdot \frac{\lfloor \beta(n)\rfloor}{n}$$

holds for large enough n, which is true because it is equivalent to

$$(np - \lfloor np\rfloor) + (\lfloor \beta(n)\rfloor - i)p + i(1-p) + \frac{i\lfloor \beta(n)\rfloor}{pn} \leq \frac{\lfloor np\rfloor}{np}\lfloor \beta(n)\rfloor,$$

where

- $np - \lfloor np\rfloor \leq 1$,
- $(\lfloor \beta(n)\rfloor - i)p + i(1-p) \leq \lfloor \beta(n)\rfloor \cdot \max\{p, 1-p\}$, since $i < \lfloor \beta(n)\rfloor$ and
- $\frac{i\lfloor \beta(n)\rfloor}{pn} = \mathrm{O}(1)$, since $\beta(n) = \mathrm{O}(\sqrt{n})$.

Using the fact that $(1+x) \leq e^x$ for all $x \in \mathbb{R}$, we see that

$$\begin{aligned}
\frac{B_n^{\lfloor np\rfloor}(p)}{B_n^{\lfloor np\rfloor - \lfloor \beta(n)\rfloor}(p)} &\leq \prod_{i=0}^{\lfloor \beta(n)\rfloor - 1} \left(1 + \frac{1}{p(1-p)} \cdot \frac{\lfloor \beta(n)\rfloor}{n}\right) \\
&\leq \prod_{i=0}^{\lfloor \beta(n)\rfloor - 1} e^{\frac{1}{p(1-p)} \cdot \frac{\lfloor \beta(n)\rfloor}{n}} \\
&= e^{\frac{1}{p(1-p)} \cdot \frac{\lfloor \beta(n)\rfloor^2}{n}}.
\end{aligned}$$

For all large enough n for which $\beta(n) < 0$, we write $b(n) = |\lfloor\beta(n)\rfloor|$ and we have

$$
\begin{aligned}
\frac{B_n^{\lfloor np\rfloor}(p)}{B_n^{\lfloor np\rfloor-\lfloor\beta(n)\rfloor}(p)} &= \frac{\binom{n}{\lfloor np\rfloor}(1-p)^{b(n)}}{\binom{n}{\lfloor np\rfloor+b(n)}p^{b(n)}} \\
&= \prod_{i=0}^{b(n)-1} \frac{(\lfloor np\rfloor + b(n) - i)(1-p)}{(n-\lfloor np\rfloor - i)p} \\
&\leq \prod_{i=0}^{b(n)-1} \frac{(np + b(n) - i)(1-p)}{(n(1-p) - i)p} \\
&\leq \prod_{i=0}^{b(n)-1} \frac{(n-\lfloor n(1-p)\rfloor + b(n) - i)(1-p)}{(\lfloor n(1-p)\rfloor - i)p},
\end{aligned}
$$

which is the same as (2.1) in the case $\beta(n) \geq 0$, only that p and $(1-p)$ are interchanged. □

Now we know that the values of $B_n^i(p)$ around the peaks in Fig. 1 are close to the value of the peak. The next lemma will tell us that the peak of $B_n^i(p)$ is asymptotically $\frac{1}{\sqrt{2\pi p(1-p)n}}$.

Lemma 2.6. *For $0 < p < 1$, it holds*

$$\lim_{n\to\infty} \sqrt{2\pi p(1-p)n}\,B_n^{\lfloor np\rfloor}(p) = 1.$$

Proof. Using Stirling's approximation

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1,$$

we see that

$$
\begin{aligned}
&\lim_{n\to\infty} \sqrt{2\pi p(1-p)n}\,B_n^{\lfloor np\rfloor}(p) \\
&\quad= \lim_{n\to\infty} \frac{\sqrt{2\pi p(1-p)n}\cdot\sqrt{2\pi n}\left(\frac{n}{e}\right)^n p^{\lfloor np\rfloor}(1-p)^{n-\lfloor np\rfloor}}{\sqrt{2\pi\lfloor np\rfloor}\left(\frac{\lfloor np\rfloor}{e}\right)^{\lfloor np\rfloor}\cdot\sqrt{2\pi(n-\lfloor np\rfloor)}\left(\frac{n-\lfloor np\rfloor}{e}\right)^{n-\lfloor np\rfloor}} \\
&\quad= \lim_{n\to\infty} \frac{n^n p^{\lfloor np\rfloor}(1-p)^{n-\lfloor np\rfloor}}{\lfloor np\rfloor^{\lfloor np\rfloor}\cdot(n-\lfloor np\rfloor)^{n-\lfloor np\rfloor}} \\
&\quad= \lim_{n\to\infty} \left(\frac{np}{\lfloor np\rfloor}\right)^{\lfloor np\rfloor}\cdot\left(\frac{n-np}{n-\lfloor np\rfloor}\right)^{n-\lfloor np\rfloor} \\
&\quad= \lim_{n\to\infty} \left(1+\frac{np-\lfloor np\rfloor}{\lfloor np\rfloor}\right)^{\lfloor np\rfloor}\cdot\left(1-\frac{np-\lfloor np\rfloor}{n-\lfloor np\rfloor}\right)^{n-\lfloor np\rfloor} \\
&\quad= \lim_{n\to\infty} e^{np-\lfloor np\rfloor}\cdot e^{-(np-\lfloor np\rfloor)} = 1,
\end{aligned}
$$

where the last line follows by Lemma 2.1. □

3 Comparing convergence of $\{a_n\}_{n\geq 0}$ with convergence of $\{a_n^p\}_{n\geq 0}$ and $\{a_n^*\}_{n\geq 0}$

In this section we show that the convergence of $\{a_n\}_{n\geq 0}$ implies the convergence of $\{a_n^p\}_{n\geq 0}$ and $\{a_n^*\}_{n\geq 0}$ to the same limit. It is well known [4] that if $\{a_n\}_{n\geq 0}$ converges to $a \in \mathbb{R} \cup \{\infty\}$, then so does $\{a_n^*\}_{n\geq 0}$. The next theorem tells us that in this case, $\{a_n^p\}_{n\geq 0}$ also converges to the same limit.

Theorem 3.1. *If $\{a_n\}_{n\geq 0}$ converges to $a \in \mathbb{R} \cup \{\infty\}$, then $\{a_n^*\}_{n\geq 0}$ and $\{a_n^p\}_{n\geq 0}$ converge to a for all $0 < p < 1$.*

Proof. The case $a = \infty$ is straightforward to handle, so suppose $a < \infty$. Take any $\epsilon > 0$ and such N that $|a_n - a| < \epsilon$ for all $n \geq N$. Then, for $n \geq N$,

$$\begin{aligned} |a_n^* - a| &= \frac{1}{n+1}\left|\sum_{i=0}^{n}(a_i - a)\right| \\ &\leq \frac{1}{n+1}\sum_{i=0}^{n}|a_i - a| \\ &\leq \frac{1}{n+1}\sum_{i=0}^{N}|a_i - a| + \frac{1}{n+1}\cdot \epsilon(n-N). \end{aligned}$$

The last line converges to ϵ when n goes to infinity, which implies that $\{a_n^*\}_{n\geq 0}$ converges to a.

To prove the convergence of binomial means, denote $q = 1 - p$. For $n \geq N$, we get

$$\begin{aligned} |a_n^p - a| &= \left|\sum_{i=0}^{n}\binom{n}{i}p^i q^{n-i}(a_i - a)\right| \\ &\leq \sum_{i=0}^{N}\binom{n}{i}p^i q^{n-i}|a_i - a| + \epsilon\sum_{i=N+1}^{n}\binom{n}{i}p^i q^{n-i} \\ &\leq \sum_{i=0}^{N}\binom{n}{i}p^i q^{n-i}|a_i - a| + \epsilon. \end{aligned}$$

The last line converges to ϵ because $\binom{n}{i}$ grows as a polynomial in n for each fixed value $i \leq N$ and $p^i q^{n-i}$ decreases exponentially. This implies that $\{a_n^p\}_{n\geq 0}$ also converges to a. □

One does not need to go searching for strange examples to see that convergence of $\{a_n^*\}_{n\geq 0}$ or $\{a_n^p\}_{n\geq 0}$ does not imply the convergence of $\{a_n\}_{n\geq 0}$. We state this as a proposition.

Proposition 3.2. *There exists a sequence $\{a_n\}_{n\geq 0}$ of zeros and ones that does not converge, whereas $\{a_n^*\}_{n\geq 0}$ and $\{a_n^p\}_{n\geq 0}$ converge for all $0 < p < 1$.*

Proof. Define

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

Then $\{a_n\}_{n\geq 0}$ does not converge and $\{a_n^*\}_{n\geq 0}$ converges to $\frac{1}{2}$, as can easily be verified. Next, we will prove that $\{a_n^p\}_{n\geq 0}$ converges to $\frac{1}{2}$. First, we see that, for $0 < p < 1$ and $q = 1 - p$, the value of $q - p$ is strictly between -1 and 1, thus $(q-p)^n$ converges to 0 when n goes to infinity. Hence,

$$\sum_{i \text{ is even}}^{n} \binom{n}{i} p^i q^{n-i} - \sum_{i \text{ is odd}}^{n} \binom{n}{i} p^i q^{n-i} = (q-p)^n$$

converges to 0. Because

$$\sum_{i \text{ is even}}^{n} \binom{n}{i} p^i q^{n-i} + \sum_{i \text{ is odd}}^{n} \binom{n}{i} p^i q^{n-i} = 1,$$

we have that $\{a_n^p\}_{n\geq 0}$ converges to $\frac{1}{2}$. □

4 Comparing convergence of binomial means

In this section we compare convergence of sequences $\{a_n^p\}_{n\geq 0}$ for different parameters $p \in (0,1)$. We will see that if $0 < p_1 < p_2 < 1$, then the convergence of $\{a_n^{p_2}\}_{n\geq 0}$ implies the convergence of $\{a_n^{p_1}\}_{n\geq 0}$ to the same limit, while the convergence of $\{a_n^{p_1}\}_{n\geq 0}$ does not imply the convergence of $\{a_n^{p_2}\}_{n\geq 0}$ in general. We leave as an open problem whether for $a_n \geq 0$ it does.

First, let us prove the main lemma in this section, which tells us that the sequence of p_2-binomial means of the sequence of p_1-binomial means of some sequence is the sequence of (p_1p_2)-binomial means of the starting sequence.

Lemma 4.1. *For $0 < p_1, p_2 < 1$ and for a sequence $\{a_n\}_{n\geq 0}$, let $\{b_n\}_{n\geq 0}$ be the sequence of p_1-binomial means of $\{a_n\}_{n\geq 0}$, i.e., $b_n = a_n^{p_1}$ for all n. Then $b_n^{p_2} = a_n^{p_1p_2}$ for all n.*

Proof. Denote $q_1 = 1 - p_1$ and $q_2 = 1 - p_2$. We change the order of summation, consider $\binom{j}{i}\binom{n}{j} = \binom{n}{i}\binom{n-i}{j-i}$ for $i \leq j$ and replace j by $k = j - i$:

$$\begin{aligned}
b_n^{p_2} &= \sum_{j=0}^{n} a_j^{p_1} \binom{n}{j} p_2^j q_2^{n-j} \\
&= \sum_{j=0}^{n} \sum_{i=0}^{j} a_i \binom{j}{i} \binom{n}{j} p_1^i q_1^{j-i} p_2^j q_2^{n-j} \\
&= \sum_{i=0}^{n} a_i \binom{n}{i} p_1^i p_2^i \sum_{j=i}^{n} \binom{n-i}{j-i} q_1^{j-i} p_2^{j-i} q_2^{n-j} \\
&= \sum_{i=0}^{n} a_i \binom{n}{i} p_1^i p_2^i \sum_{k=0}^{n-i} \binom{n-i}{k} (q_1 p_2)^k q_2^{n-i-k} \\
&= \sum_{i=0}^{n} a_i \binom{n}{i} p_1^i p_2^i (q_1 p_2 + q_2)^{n-i} \\
&= \sum_{i=0}^{n} a_i \binom{n}{i} (p_1 p_2)^i (1 - p_1 p_2)^{n-i}.
\end{aligned}$$

The last line equals $a_n^{p_1p_2}$. □

The next theorem will now be trivial to prove.

Theorem 4.2. *For $0 < p_1 < p_2 < 1$ and for a sequence $\{a_n\}_{n\geq 0}$, if $\{a_n^{p_2}\}_{n\geq 0}$ converges to $a \in \mathbb{R} \cup \{\infty\}$, then $\{a_n^{p_1}\}_{n\geq 0}$ also converges to a.*

Proof. From Lemma 4.1 we know that $\{a_n^{p_1}\}_{n\geq 0}$ is the sequence of $\frac{p_1}{p_2}$-binomial means of the sequence $\{a_n^{p_2}\}_{n\geq 0}$. By Theorem 3.1, it converges to a. $\square$

The next proposition tells us that the condition $0 < p_1 < p_2 < 1$ in the above theorem cannot be left out in general.

Proposition 4.3. *For $0 < p_1 < p_2 < 1$, there exists a sequence $\{a_n\}_{n\geq 0}$, such that $\{a_n^{p_1}\}_{n\geq 0}$ converges to 0, but $\{a_n^{p_2}\}_{n\geq 0}$ does not converge.*

Proof. Denote $q_1 = 1 - p_1$ and define $\{a_n\}_{n\geq 0}$ as $a_n = a^n$ for some parameter $a \in \mathbb{R}$. If $a > -1$, $\{a_n\}_{n\geq 0}$ converges (possibly to ∞), so let us examine the case when $a \leq -1$. In this case we have

$$a_n^{p_1} = \sum_{i=0}^{n} \binom{n}{i} a^i p_1^i q_1^{n-i} = (ap_1 + q_1)^n = (p_1(a-1)+1)^n,$$

which converges iff $p_1 < \frac{2}{1-a}$. So we can choose such an a that $p_1 < \frac{2}{1-a} < p_2$, i.e., $1 - \frac{2}{p_1} < a < 1 - \frac{2}{p_2}$. It follows that $\{a_n^{p_1}\}_{n\geq 0}$ converges to 0, but $\{a_n^{p_2}\}_{n\geq 0}$ does not converge. $\square$

The sequence $\{a_n\}_{n\geq 0}$ in the above proof is growing very rapidly in absolute value and the sign of its elements alternates. We think that this is not a coincidence and we state the following open problem.

Open problem 4.4. Let $\{a_n\}_{n\geq 0}$ be a sequence of non-negative real numbers. Is it true that, for all $0 < p_1, p_2 < 1$, the sequence $\{a_n^{p_1}\}_{n\geq 0}$ converges to a iff $\{a_n^{p_2}\}_{n\geq 0}$ converges to a? If the answer is no, is there a counterexample where $a_n \in \{0, 1\}$?

Note that the condition $a_n \geq 0$ is also required for the main result of the paper, Theorem 5.1. If the answer on 4.4 were *yes*, then we would only have to prove Theorem 5.1 in a special case, e.g. for $p = \frac{1}{2}$. The (possibly negative) answer would also make this paper more complete (see Table 1). In the rest of this section we will try to give some insight into this problem and we will present some reasons for why we think it is hard.

Suppose we have $0 < p_1 < p_2 < 1$ and a sequence $\{a_n\}_{n\geq 0}$ of non-negative real numbers such that $\{a_n^{p_1}\}_{n\geq 0}$ converges to $a \in \mathbb{R}$ (the case when $\{a_n^{p_2}\}_{n\geq 0}$ converges is covered by Theorem 4.2). The next lemma implies that $\{a_n\}_{n\geq 0}$ has a relatively low upper bound on how fast its elements can increase, ruling out too large local extremes.

Lemma 4.5. *Let $\{a_n\}_{n\geq 0}$ be a sequence of non-negative real numbers and let $0 < p < 1$. If $\{a_n^p\}_{n\geq 0}$ converges to $a < \infty$, then $a_n = \mathrm{O}(\sqrt{n})$.*

Proof. We know that $a_n^p \geq a_{\lfloor np \rfloor} B_n^{\lfloor np \rfloor}(p)$, where $B_n^{\lfloor np \rfloor}(p) \approx \frac{1}{\sqrt{2\pi p(1-p)n}}$ by Lemma 2.6 and $a_n^p \approx a$ for large n. Hence, $a_{\lfloor np \rfloor} = \mathrm{O}(\sqrt{n})$. $\square$

To see whether $\{a_n^{p_2}\}_{n\geq 0}$ converges, it makes sense to compare $a^{p_1}_{\lfloor n/p_1\rfloor}$ with $a^{p_2}_{\lfloor n/p_2\rfloor}$, since the peaks of the "weights" $B^i_{\lfloor n/p_1\rfloor}(p_1)$ and $B^i_{\lfloor n/p_2\rfloor}(p_2)$ (roughly) coincide at n (see Fig 2). Now the troublesome thing is that, for large n, the peaks are not of the same height, but rather they differ by a factor

$$\sqrt{\frac{1-p_2}{1-p_1}}$$

by Lemma 2.6. Because the weights $B^i_{\lfloor n/p_1\rfloor}(p_1)$ and $B^i_{\lfloor n/p_2\rfloor}(p_2)$ are (really) influential only in the $\mathrm{O}(\sqrt{n})$ neighborhood of n (Corollary 2.4 and Lemma 2.5), where the p_1-weights are only a bit "downtrodden" p_2-weights, it seems that the convergence of $\{a_n^{p_1}\}_{n\geq 0}$ could imply the convergence of $\{a_n^{p_2}\}_{n\geq 0}$.

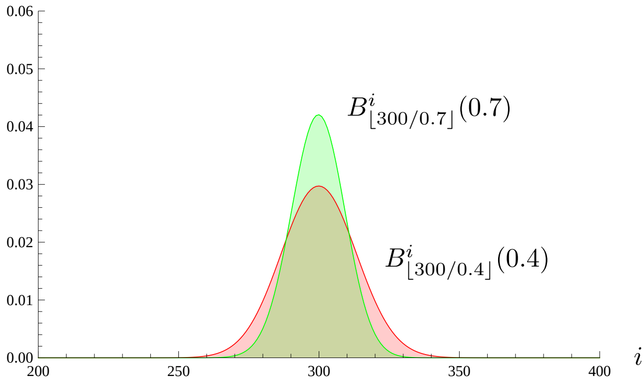

Figure 2: The graphs show $B^i_{\lfloor n/p_1\rfloor}(p_1)$ and $B^i_{\lfloor n/p_2\rfloor}(p_2)$ with respect to i in the neighborhood of n for $n = 300$, $p_1 = 0.4$ (red) and $p_2 = 0.7$ (green).

On the other hand, one could take $a_n = 0$ for all except for some n where there would be outliers of heights $\Theta(\sqrt{n})$. Those outliers would be so far away from each other that the weights $B_n^i(p_1)$ could "notice" two consecutive outliers, while the weights $B_n^i(p_2)$, which are slimmer, could not (in Fig. 2, the two outliers could be at 280 and 320). Then $\{a_n^{p_1}\}_{n\geq 0}$ could converge because there would be a small difference between [when the weights $B_n^i(p_1)$ amplify one outlier] and [when they "notice" two outliers] (these two events seem to be the most opposite). On the other hand, $\{a_n^{p_2}\}_{n\geq 0}$ would not converge. From Chernoff bound (Corollary 2.4) and from Lemma 2.5 it follows that the (horizontal) distance between outliers should be roughly $C\sqrt{n}$ for some C. What C would be the most appropriate?

5 Comparing convergence of $\{a_n^p\}_{n\geq 0}$ with convergence of $\{a_n^*\}_{n\geq 0}$

This section contains the main result of this paper, which is formulated in the next theorem. The proof will be given later.

Theorem 5.1. *Let $\{a_n\}_{n\geq 0}$ be a sequence of non-negative real numbers such that $\{a_n^p\}_{n\geq 0}$ converges to $a \in \mathbb{R}\cup\{\infty\}$ for some $0 < p < 1$. Then $\{a_n^*\}_{n\geq 0}$ converges to a.*

An example of how this theorem can be used is given in Section 6.1. Here we give an example where $\{a_n^p\}_{n\geq 0}$ converges to $a \in \mathbb{R}\cup\{\infty\}$ for all $0 < p < 1$, but $\{a_n^*\}_{n\geq 0}$ does not converge.

Proposition 5.2. *For the sequence* $\{a_n\}_{n\geq 0}$ *given by* $a_n = (-1)^n n$, $\{a_n^p\}_{n\geq 0}$ *converges to 0 for all* $0 < p < 1$ *and* $\{a_n^*\}_{n\geq 0}$ *does not converge.*

Proof. Take $0 < p < 1$ and denote $q = 1 - p$. It holds

$$\begin{aligned} a_n^p &= \sum_{i=0}^{n} (-1)^i i \binom{n}{i} p^i q^{n-i} \\ &= -np \sum_{i=1}^{n} (-1)^{i-1} \binom{n-1}{i-1} p^{i-1} q^{n-1-(i-1)} \\ &= -np(-p+q)^{n-1}. \end{aligned}$$

Because $q - p$ is strictly between -1 and 1, $\{a_n^p\}_{n\geq 0}$ converges to 0.

However, the induction shows that $a_{2n+1}^* = -\frac{1}{2}$ and $a_{2n}^* = \frac{n}{2n+1}$, which implies that $\{a_n^*\}_{n\geq 0}$ does not converge. □

Next, we show that we cannot interchange $\{a_n^p\}_{n\geq 0}$ and $\{a_n^*\}_{n\geq 0}$ in Theorem 5.1.

Proposition 5.3. *There exists a sequence* $\{a_n\}_{n\geq 0}$ *of zeros and ones such that* $\{a_n^*\}_{n\geq 0}$ *converges to 0 and* $\{a_n^p\}_{n\geq 0}$ *diverges for all* $0 < p < 1$.

Proof. Define

$$a_n = \begin{cases} 1 & \text{if there is some } k \in \mathbb{N} \text{ such that } \left|n - 2^{2k}\right| < 2^k k \\ 0 & \text{else.} \end{cases}$$

So $\{a_n\}_{n\geq 0}$ has islets of ones in the sea of zeros. The size of an islet at position N is $\Theta(\sqrt{N}\log(N))$ and the distance between two islets near position N is $\Theta(N)$. It is easy to see that the sequence a_n^* converges to zero.

Now let $0 < p < 1$. By Chernoff bound (Corollary 2.4) we see that $B_n^i(p)$ is concentrated around $i = \lfloor np \rfloor$ and that, for $|i - np| \geq \sqrt{n}\log(n)$, we have roughly nothing left. It is easy (but tedious) to show formally that $\left\{a^p_{\lfloor 2^{2k}/p \rfloor}\right\}_{k\geq 0}$ converges to 1 and that $\left\{a^p_{\lfloor 2^{2k-1}/p \rfloor}\right\}_{k\geq 0}$ converges to 0, which implies that $\{a_n^p\}_{n\geq 0}$ diverges. □

Now we go for the proof of Theorem 5.1. First, for a sequence $\{a_n\}_{n\geq 0}$ and $0 < p < 1$, we **define** $\{a_n^{p*}\}_{n\geq 0}$ as a sequence of arithmetic means of the sequence $\{a_n^p\}_{n\geq 0}$. We get

$$\begin{aligned} a_n^{p*} &= \frac{1}{n+1} \sum_{j=0}^{n} a_j^p \\ &= \frac{1}{n+1} \sum_{j=0}^{n} \sum_{i=0}^{j} a_i \binom{j}{i} p^i q^{j-i} \\ &= \frac{1}{n+1} \sum_{i=0}^{n} a_i \sum_{j=i}^{n} \binom{j}{i} p^i q^{j-i}, \end{aligned}$$

where $q = 1 - p$.

It makes sense to **define** weights $w_n^i(p) = \sum_{j=i}^{n} \binom{j}{i} p^i q^{j-i}$, so that it holds

$$a_n^{p*} = \frac{1}{n+1} \sum_{i=0}^{n} w_n^i(p) a_i.$$

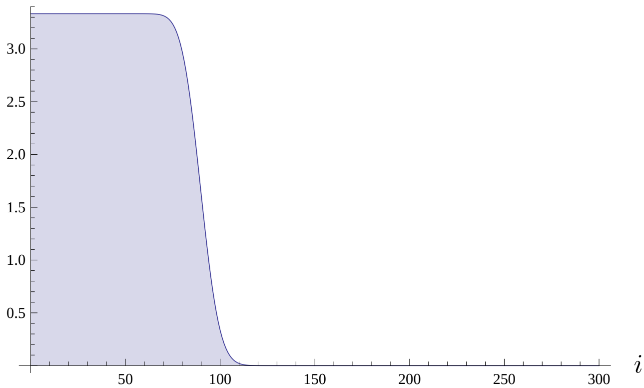

Figure 3: The graph shows $w_{300}^i(0.3)$ with respect to i. We see a steep slope at $i = 90$ plunging from height approximately $\frac{1}{0.3}$ to 0.

We can see in Fig. 3 that the weights $w_n^i(p)$ have a very specific shape. They are very close to $\frac{1}{p}$ for $i < np - \epsilon(n)$ and very close to 0 for $i > np + \epsilon(n)$, for some small $\epsilon(n)$. Such a shape can be well described using the next lemma (and its corollary), which gives another way to compute $w_n^i(p)$.

Lemma 5.4. *For* $0 < p < 1$, $q = 1 - p$, $n \in \mathbb{N}$ *and* $0 \leq i \leq n$, *it holds*

$$w_n^i(p) = \frac{1}{p} \left(1 - \sum_{j=0}^{i} \binom{n+1}{j} p^j q^{n+1-j} \right).$$

Proof. The idea is to use power series centered at q. For a function $f : \mathbb{R} \to \mathbb{R}$, we will write $f^{(i)}: \mathbb{R} \to \mathbb{R}$ for its i-th derivative.

$$
\begin{aligned}
w_n^i(p) &= \sum_{j=i}^{n} \binom{j}{i} p^i q^{j-i} \\
&= \frac{p^i}{i!} \left(\sum_{j=0}^{n} x^j \right)^{(i)} \Bigg|_{x=q} \\
&= \frac{p^i}{i!} \left(\frac{1-x^{n+1}}{1-x} \right)^{(i)} \Bigg|_{x=q} \\
&= \frac{p^{i-1}}{i!} \left(\frac{1-\left(x-q+q\right)^{n+1}}{1-\frac{1}{p}(x-q)} \right)^{(i)} \Bigg|_{x=q} \\
&= \frac{p^{i-1}}{i!} \left[\left(1 - \sum_{k=0}^{n+1} \binom{n+1}{k} (x-q)^k q^{n+1-k} \right) \cdot \left(\sum_{k=0}^{\infty} (x-q)^k p^{-k} \right) \right]^{(i)} \Bigg|_{x=q} \\
&= \frac{p^{i-1} i!}{i!} \left(p^{-i} - \sum_{j=0}^{i} \binom{n+1}{j} q^{n+1-j} p^{j-i} \right) \\
&= \frac{1}{p} \left(1 - \sum_{j=0}^{i} \binom{n+1}{j} p^j q^{n+1-j} \right).
\end{aligned}
$$

□

Define the function $\epsilon : \mathbb{N} \to \mathbb{R}^+$ as

$$
\epsilon(n) = \begin{cases} \sqrt{n}\log(n) & \text{if } n \geq 2 \\ 1 & \text{else.} \end{cases}
$$

Now the following corollary holds.

Corollary 5.5. *For* $0 < p < 1$, $n \in \mathbb{N}$ *and* $0 \leq i \leq n$, *it holds*

$$
\begin{aligned}
w_n^{\lfloor np-\epsilon(n) \rfloor}(p) &\geq \frac{1}{p} - n^{-\Theta(\log(n))}, \\
w_n^{\lfloor np+\epsilon(n) \rfloor}(p) &\leq n^{-\Theta(\log(n))}.
\end{aligned}
$$

Proof. Use the Chernoff bound (Corollary 2.4) on the expression for $w_n^i(p)$ from Lemma 5.4. □

For $0 < p < 1$ and for a sequence $\{a_n\}_{n\geq 0}$, **define** sequences $\{a_n^x(p)\}_{n\geq 0}$, $\{a_n^y(p)\}_{n\geq 0}$

and $\{a_n^z(p)\}_{n\geq 0}$ as

$$a_n^x(p) = \sum_{i=0}^{\lfloor pn-\epsilon(n)\rfloor} w_n^i(p)a_i$$

$$a_n^y(p) = \sum_{i=\lfloor pn-\epsilon(n)\rfloor+1}^{\lfloor pn+\epsilon(n)\rfloor-1} w_n^i(p)a_i$$

$$a_n^z(p) = \sum_{i=\lfloor pn+\epsilon(n)\rfloor}^{n} w_n^i(p)a_i.$$

Hence, we have

$$a_n^{p*} = \frac{1}{n+1}\Big(a_n^x(p) + a_n^y(p) + a_n^z(p)\Big).$$

From Corollary 5.5 we see that the weights in $a_n^x(p)$ are very close to $\frac{1}{p}$, which suggests that $\frac{1}{n+1}a_n^x(p)$ can be very close to $a^*_{\lfloor np\rfloor}$ (see Lemma 5.8 below). From the same corollary we see that $\frac{1}{n+1}a_n^z(p)$ can be very close to 0 (see Lemma 5.7 below). And because we have a sum of only $\Theta(\epsilon(n))$ elements in $a_n^y(p)$, $\frac{1}{n+1}a_n^y(p)$ could also be very close to 0 (see Lemma 5.6 below).

We have just described the main idea for the proof of the main theorem, which we give next. It will use three lemmas just mentioned (one about $a_n^x(p)$, one about $a_n^y(p)$ and one about $a_n^z(p)$), that will be proven later.

Proof of Theorem 5.1. Suppose that $a_n \geq 0$ for all n and suppose that $\{a_n^p\}_{n\geq 0}$ converges to $a \in \mathbb{R}\cup\{\infty\}$ for some $0 < p < 1$. We know that this implies the convergence of $\{a_n^{p*}\}_{n\geq 0}$ to a (Theorem 3.1).

First, we deal with the case $a = \infty$. We can use the fact that $w_n^i(p) \leq \frac{1}{p}$ for all i (see Lemma 5.4), which gives

$$\begin{aligned} a_n^{p*} &= \frac{1}{n+1}\sum_{i=0}^{n} w_n^i(p)a_i \\ &\leq \frac{1}{n+1}\sum_{i=0}^{n}\frac{1}{p}a_i \\ &= \frac{a_n^*}{p}. \end{aligned}$$

Hence, $\{a_n^*\}_{n\geq 0}$ converges to $a = \infty$.

In the case $a < \infty$, we can use Lemma 5.6 and Lemma 5.7 to see that $\left\{\frac{1}{n+1}a_n^y(p)\right\}_{n\geq 0}$ and $\left\{\frac{1}{n+1}a_n^z(p)\right\}_{n\geq 0}$ converge to 0. Hence, $\left\{\frac{1}{n+1}a_n^x(p)\right\}_{n\geq 0}$ converges to a. Lemma 5.8 tells us that in this case $\{a_n^*\}_{n\geq 0}$ also converges to a. □

Now we state and prove the remaining lemmas.

Lemma 5.6. *Let $0 < p < 1$ and let $\{a_n\}_{n\geq 0}$ be a sequence of non-negative real numbers such that $\{a_n^p\}_{n\geq 0}$ converges to $a < \infty$. Then $\left\{\frac{1}{n+1}a_n^y(p)\right\}_{n\geq 0}$ converges to 0.*

Proof. Fix $\tilde{\epsilon} > 0$, define $\delta(n) = \lfloor \log^2(n) \rfloor$ and let $k : \mathbb{N} \to \mathbb{N}$ be such that $pn - \epsilon(n) \leq k(n) \leq pn + \epsilon(n) - \delta(n)$ holds for all n. We claim that

$$\sum_{i=k(n)}^{k(n)+\delta(n)} a_i = \mathrm{O}(\sqrt{n}),$$

where the constant behind the O is independent of k. To prove this, define $N = N(n) = \left\lfloor \frac{k(n)}{p} \right\rfloor$. It follows that $N = n \pm \Theta(\epsilon(n))$. Note that, for large enough n,

$$\sum_{i=k(n)}^{k(n)+\delta(n)} a_i B_N^i(p) \leq \sum_{i=0}^{N} a_i B_N^i(p) < a + \tilde{\epsilon},$$

because $\{a_n^p\}_{n\geq 0}$ converges to a. From Lemma 2.5 which bounds the coefficients $B_N^i(p)$ around $i = pN$ it follows that, for all $k(n) \leq i \leq k(n) + \delta(n)$,

$$B_N^i(p) \geq e^{-\,\mathrm{o}(1)} B_N^{\lfloor Np \rfloor}(p).$$

Using $N = n \pm \Theta(\epsilon(n))$ and the bound

$$B_N^{\lfloor Np \rfloor}(p) = \frac{1}{\Theta(\sqrt{N})}$$

from Lemma 2.6, we get

$$\sum_{i=k(n)}^{k(n)+\delta(n)} a_i < (a + \tilde{\epsilon}) e^{\mathrm{o}(1)}\, \Theta(\sqrt{n}) = \mathrm{O}(\sqrt{n}).$$

Next, we can see that

$$\sum_{i=\lfloor pn-\epsilon(n) \rfloor+1}^{\lfloor pn+\epsilon(n) \rfloor-1} a_i = \mathrm{O}\left(\frac{n}{\log n}\right).$$

Just partition the sum on the left-hand side into $\left\lceil \frac{2\epsilon(n)}{\delta(n)} \right\rceil$ sums of at most $\delta(n)$ elements. Then we have

$$\sum_{i=\lfloor pn-\epsilon(n) \rfloor+1}^{\lfloor pn+\epsilon(n) \rfloor-1} a_i = \mathrm{O}\left(\frac{\epsilon(n)}{\delta(n)}\sqrt{n}\right) = \mathrm{O}\left(\frac{n}{\log n}\right).$$

Now using $w_n^i(p) \leq \frac{1}{p}$ from Lemma 5.4, we get

$$\frac{1}{n+1} a_n^y(p) \leq \frac{1}{(n+1)p} \sum_{i=\lfloor pn-\epsilon(n) \rfloor+1}^{\lfloor pn+\epsilon(n) \rfloor-1} a_i = \frac{1}{(n+1)p}\, \mathrm{O}\left(\frac{n}{\log n}\right),$$

which implies the convergence of $\left\{\frac{1}{n+1} a_n^y(p)\right\}_{n\geq 0}$ to 0. $\square$

Lemma 5.7. *Let $0 < p < 1$ and let $\{a_n\}_{n\geq 0}$ be a sequence of non-negative real numbers such that $\{a_n^p\}_{n\geq 0}$ converges to $a < \infty$. Then $\left\{\frac{1}{n+1}a_n^z(p)\right\}_{n\geq 0}$ converges to 0.*

Proof. From Lemma 5.4 we see that the weights $w_n^i(p)$ decrease with i, so

$$\frac{1}{n+1}a_n^z(p) \leq \frac{w_n^{\lfloor np+\epsilon(n)\rfloor}(p)}{(n+1)} \sum_{\lfloor pn+\epsilon(n)\rfloor}^{n} a_i.$$

Corollary 5.5 gives us $w_n^{\lfloor np+\epsilon(n)\rfloor}(p) \leq n^{-\Theta(\log(n))}$, while Lemma 4.5 implies $a_i = \mathrm{O}(\sqrt{i})$. Hence, $\left\{\frac{1}{n+1}a_n^z(p)\right\}_{n\geq 0}$ converges to 0. □

Lemma 5.8. *Let $0 < p < 1$ and let $\{a_n\}_{n\geq 0}$ be a sequence of non-negative real numbers such that $\left\{\frac{1}{n+1}a_n^x(p)\right\}_{n\geq 0}$ converges to $a < \infty$. Then $\{a_n^*\}_{n\geq 0}$ converges to a.*

Proof. Because the weights $w_n^i(p)$ are bounded from above by $\frac{1}{p}$ (Lemma 5.4), we have

$$\frac{a_n^x(p)}{n+1} \cdot \frac{(n+1)p}{\lfloor pn-\epsilon(n)\rfloor + 1} \leq a^*_{\lfloor pn-\epsilon(n)\rfloor},$$

where the left side converges to a.

Because the weights $w_n^i(p)$ decrease with i (Lemma 5.4) and because $w_n^{\lfloor np-\epsilon(n)\rfloor}(p) \geq \frac{1}{p} - n^{-\Theta(\log(n))}$ (Corollary 5.5), we have

$$a^*_{\lfloor pn-\epsilon(n)\rfloor} \leq \frac{a_n^x(p)}{n+1} \cdot \frac{n+1}{(\frac{1}{p} - n^{-\Theta(\log(n))}) \cdot (\lfloor pn-\epsilon(n)\rfloor + 1)},$$

where the right side converges to a. Hence, $a^*_{\lfloor pn-\epsilon(n)\rfloor}$ is sandwiched between two sequences that converge to a. It follows that $\{a_n^*\}_{n\geq 0}$ converges to a. □

6 Application of Theorem 5.1: a limit theorem for finite Markov chains

For a stochastic matrix[1] P, define the sequence $\{P_n\}_{n\geq 0}$ as $P_n = P^n$. As in the one-dimensional case, we define the sequence $\{P_n^*\}_{n\geq 0}$ as $P_n^* = \frac{1}{n+1}\sum_{i=0}^{n} P_n$. We say that $\{P_n\}_{n\geq 0}$ converges to A if, for all possible pairs (i,j), the sequence of (i,j)-th elements of P_n converges to (i,j)-th element of A. In this section, we will prove the following theorem.

Theorem 6.1. *For any finite stochastic matrix P, the sequence $\{P_n^*\}_{n\geq 0}$ converges to some stochastic matrix A, such that $AP = PA = A$.*

This theorem is nothing new in the theory of Markov chains. Actually, it also holds for (countably) infinite transition matrices P. Although we did not find it formulated this way in literature, it can be easily deduced from the known results. The hardest thing to show

[1] A stochastic matrix is a (possibly infinite) square matrix that has non-negative real entries and for which all rows sum to 1. Each stochastic matrix represents transition probabilities of some discrete Markov chain. No prior knowledge of Markov chains is needed for this paper.

is the convergence of $\{P_n^*\}_{n\geq 0}$ [2, page 32]. After we have it, we can continue as in the proof of Theorem 6.1 below.

We will give a short proof of Theorem 6.1, using only linear algebra and Theorem 5.1. First, we prove a result from linear algebra.

Lemma 6.2. *Let P be a finite stochastic matrix and let $\tilde{P} = \frac{1}{2}\Big(P + I\Big)$. Then*

a) for all eigenvalues λ of $\tilde{P}$, it holds $|\lambda| \leq 1$,

b) for all eigenvalues λ of $\tilde{P}$ for which $|\lambda| = 1$, it holds $\lambda = 1$,

c) the algebraic and geometric multiplicity of eigenvalue 1 of $\tilde{P}$ are the same.

Proof. Since the product and convex combination of stochastic matrices is a stochastic matrix, P^n and $\tilde{P}^n$ are stochastic matrices for each $n \in \mathbb{N}$. First, we will prove by contradiction that, for all eigenvalues λ for P, it holds $|\lambda| \leq 1$. Suppose that there is some eigenvalue λ for P such that $|\lambda| > 1$. Let w be the corresponding eigenvector and let its i-th component be non-zero. Then $|(P^n w)_i| = |\lambda^n| \cdot |w_i|$, where the right side converges to ∞ and the left side is bounded by $\max_j |w_j|$ (since P^n is a stochastic matrix). This gives a contradiction. Hence, for all eigenvalues λ for P, it holds $|\lambda| \leq 1$. Because $\tilde{P}$ is also stochastic, the same holds for $\tilde{P}$.

We see that we can get all eigenvalues of $\tilde{P}$ by adding 1 and dividing by 2 the eigenvalues of P. Because P has all eigenvalues in the unit disc around 0, $\tilde{P}$ has all eigenvalues in a disc centered in $\frac{1}{2}$ of radius $\frac{1}{2}$. Hence, for all eigenvalues λ of $\tilde{P}$, for which $|\lambda| = 1$, it holds $\lambda = 1$.

For the last claim of the lemma, suppose that the algebraic and geometric multiplicity of eigenvalue 1 of $\tilde{P}$ are not the same. Then, by Jordan decomposition, there is an eigenvector v for eigenvalue 1 and a vector w, such that $\tilde{P}w = v + w$. Then, for each $n \in \mathbb{N}$, we have $\tilde{P}^n w = nv + w$. Because v has at least one non-zero component and because all components of $\tilde{P}^n w$ are bounded in absolute value by $\max_j |w_j|$, we have come to contradiction. Hence, the algebraic and geometric multiplicity of eigenvalue 1 of $\tilde{P}$ are the same. □

Proof of Theorem 6.1. For the matrix $\tilde{P} = \frac{1}{2}\Big(P + I\Big)$, let $\tilde{P} = XJX^{-1}$ be its Jordan decomposition. From Lemma 6.2 a) and b) it follows that the diagonal of J consists only of ones and entries of absolute value strictly less than one. From Lemma 6.2 c) it follows that the Jordan blocks for eigenvalue 1 are all 1×1. It follows that J^n converges to some matrix J_0 with only zero entries and some ones on the diagonal. Hence, $\tilde{P}^n$ converges to $A = XJ_0X^{-1}$. Since $\tilde{P}^n$ is a stochastic matrix for all n, the same is true for A. Using $\tilde{P}_n = \tilde{P}^n$, we see that $\{\tilde{P}_n\}_{n\geq 0}$ is just a sequence of 0.5-binomial means of the sequence $\{P_n\}_{n\geq 0}$, hence by Theorem 5.1 $\{P_n^*\}_{n\geq 0}$ also converges to A. Thus, we have

$$\begin{aligned} AP &= \left(\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} P^i\right) P \\ &= \lim_{n\to\infty} \frac{n+2}{n+1}\left(\frac{1}{n+2}\sum_{i=0}^{n+1} P^i - \frac{1}{n+2} I\right) \\ &= A. \end{aligned}$$

The same argument shows also $PA = A$. □

An application of Theorem 6.1 in formal language theory. The following application was suggested by an anonymous reviewer. To each formal language $L \subseteq \Sigma^*$ where Σ is a finite alphabet, we can assign the sequence

$$f_n(L) = \frac{|\Sigma^n \cap L|}{|\Sigma^n|}$$

of relative frequencies of words of length n in L. If this sequence is convergent, then its limit can be taken as a measure for the size of L, which provides interesting information about L. Unfortunately, the sequence $f_n(L)$ can be divergent even if L is a regular language, such as, for example, the language E of all words of even length. But using Theorem 6.1 we can show that $f_n^*(L)$ converges for every regular L as follows. If L is regular, it is recognised by some deterministic finite automaton $(Q, q_0, F, \Sigma, \delta)$ where $Q = \{q_0, q_1, \ldots, q_{m-1}\}$ is the set of states, $q_0 \in Q$ is the starting state, $F \subseteq Q$ is the set of final states, and $\delta : Q \times \Sigma \to Q$ is the transition function. Define the matrix $T \in \mathbb{Q}^{m\times m}$ with elements

$$t_{i,j} = \frac{|\{a \in \Sigma;\ \delta(q_i, a) = q_j\}|}{|\Sigma|}, \qquad i, j = 0, 1, \ldots, m-1.$$

Then T is stochastic and $f_n(L) = \sum_{q_j \in F}(T^n)_{0,j}$, so by Theorem 6.1, $f_n^*(L)$ is convergent and we can define $\mu(L) = \lim_{n\to\infty} f_n^*(L)$ to be the (finitely additive) measure of L. For example, returning to the language E of words of even length, we find that $\mu(E) = 0.5$.

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An infinite class of movable 5-configurations

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Abstract

A geometric 5-configuration is a collection of points and straight lines, typically in the Euclidean plane, in which every point has 5 lines passing through it and every line has 5 points lying on it; that is, it is an (n_5) configuration for some number n of points and lines. Using reduced Levi graphs and two elementary geometric lemmas, we develop a construction that produces infinitely many new 5-configurations which are movable; in particular, we produce infinitely many 5-configurations with one continuous degree of freedom, and we produce 5-configurations with $k-2$ continuous degrees of freedom for all odd $k > 2$.

Keywords: Configurations, incidence geometry.

Math. Subj. Class.: 51A20, 51A45, 51E30, 05B30

A geometric k-configuration is a collection of points and straight lines, typically in the Euclidean plane, where every point lies on k lines and every line passes through k points. Geometric 3-configurations have been studied since the mid-1800s, and geometric 4-configurations since the late 1900s, with the first intelligible drawing of a 4-configuration appearing in a 1990 paper by Grünbaum and Rigby [15]. However, the situation for more *highly incident configurations*, that is, for (p_q, n_k) configurations with at least one of q, $k \geq 4$, is poorly understood in general.

Two constructions that produce infinite families of 5-configurations with a reasonably small number of points and lines are known [7, 9]. The (48_5) configuration shown in Figure 1a

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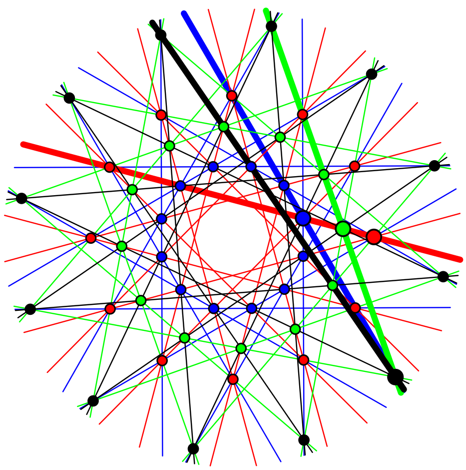
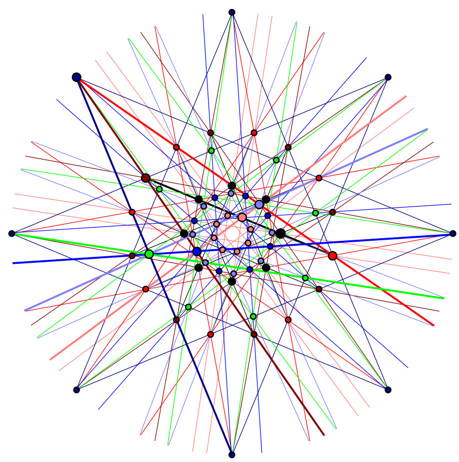
is the smallest known geometric 5-configuration and is the smallest example of the construction in [9]; a reasonably small example of the construction discussed in [7] is shown in Figure 1b (the smallest example is not intelligible at small scale). In his monograph on configurations [14, Section 4.1], Grünbaum spends only 5 pages (mostly pictures) discussing the little that is known about 5-configurations.



(a) A (48_5) configuration with 4 symmetry classes of points and lines



(b) A (64_5) configuration with 8 symmetry classes of points and lines

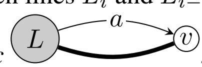
Figure 1: Examples of known small 5-configurations

In this paper, we present a new construction that produces infinitely many new geometric 5-configurations which are *movable*: that is, there is at least one continuous degree of freedom in the construction while fixing 4 points in general position. This construction significantly generalizes the construction presented in [9] and removes the need to complete the construction via a continuity argument, instead providing an entirely ruler-and-compass construction for those configurations, given an initial m-gon. The new construction technique uses two elementary geometric lemmas, the Circumcircle Construction Lemma and the Crossing Spans Lemma, which previously have been used separately in other configuration construction techniques.

1 Definitions; Levi and reduced Levi graphs

Given any (p_q, n_k) configuration, whether geometrically realizable or not, it is possible to construct a corresponding bipartite graph, called a *Levi graph*, which has one white vertex for each point of the configuration and one black vertex for each line of the configuration, with two vertices in the graph incident if and only if the corresponding point and line are incident in the configuration. More details on Levi graphs and configurations may be found in Grünbaum [14, Section 1.4] and Coxeter [12].

We say that a geometric k-configuration is *symmetric* if there exist non-trivial isometries of the Euclidean plane that map the configuration to itself. Note that in other places in the literature, the word 'symmetric' has been used to mean (p_q, n_k) configurations where $q = k$ (and thus $p = n$), i.e., k-configurations. Since we are interested in emphasizing the geometric nature of the configuration, we—following Grünbaum [14, p. 16]—refer to k-configurations as *balanced*, and reserve the word 'symmetric' to refer to the geometric structure. The *symmetry class* of an element (point or line) is the orbit of the element under the symmetry group of the configuration. If a geometric configuration has the property that every symmetry class under some fixed cyclic subgroup of the geometric symmetry group contains the same number of elements, then the configuration is called *polycyclic*; polycyclic configurations were first described by Boben and Pisanski [11].

Given a polycyclic geometric configuration with cyclic symmetry group $\mathbb{Z}_m$, it is possible to construct an edge-labelled bipartite graph, called the *reduced Levi graph*, by associating one vertex of the graph to each symmetry class of points and of lines in the configuration, and connecting two vertices of the graph with an edge precisely when the corresponding elements of the configuration are incident. Suppose the elements of each symmetry class of elements are labelled cyclically counterclockwise, beginning from some chosen 0th element in each class; for example, line class L is labelled $(L)_0, \ldots, (L)_{m-1}$ and vertex class v is labelled $(v)_0, \ldots, (v)_{m-1}$. If for each i, line L_i and vertex v_{i+a} are incident (with indices computed modulo m), the corresponding directed edge in the reduced Levi graph from vertex L to vertex v is labelled a; in the case where L_i and vertex v_i are incident (that is, where $a = 0$), then we use an undirected thick edge. When vertices v_i and v_{i+a} both lie on line L_i, or from an alternate point of view, when lines L_i and L_{i-a} intersect at point v_i, then the reduced Levi graph contains a *double arc* 

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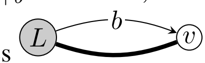
If p and q are any two points, we denote the line L passing through p and q as $p \vee q$. Similarly, if L and M are any two lines, we denote their point of intersection as $L \wedge M$ (possibly at infinity if $L \| M$). Given points $v_0, \ldots, v_{m-1}$ that form the vertices of a regular m-gon centered at $\mathcal{O}$, we say that a line is *span* b if it passes through v_i and v_{i+b} for some i, with all indices computed modulo m; span b lines correspond to double arcs 

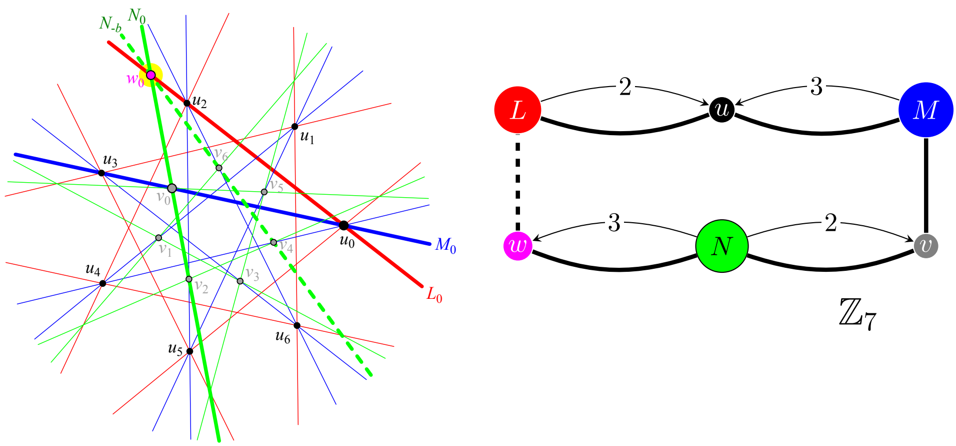
in the reduced Levi graph. A circle C is a *circumcircle of span* b if it passes through $v_i, \mathcal{O}$, and v_{i-b} for some i; to specify which i, we say that C is a *circumcircle of span* b *through* v_d. (Note that span b lines are constructed by moving counterclockwise from the initial point, and span b circumcircles by moving clockwise!)

2 Two construction lemmas

In 2006, one of the authors (LWB) discovered the Crossing Spans Lemma [3] (somewhat restated here):

Lemma 2.1 (Crossing Spans Lemma (CSL))**.** *Given a regular m-gon with vertices cyclically labelled as $u_0, u_1, \ldots, u_{m-1}$ and lines $L_i = u_i \vee u_{i+a}$ of span a and $M_i = u_i \vee u_{i+b}$ of span b, where $1 \leq a \neq b < \frac{m}{2}$, suppose that v_0 is an arbitrary point on M_0 (different from u_0, u_b to avoid degeneracies), and construct other points v_i to be the rotations of v_0 through $\frac{2\pi i}{m}$. Let $N_i = v_i \vee v_{i+a}$ and let $w_i = N_i \wedge N_{i-b}$. Then w_i also lies on L_i.*

Although easy to state and prove, the Crossing Spans Lemma has been used to produce a



(a) Illustrating the Crossing Spans Lemma; $m = 7, a = 2, b = 3$. Only point w_0 in class w has been shown, to better illustrate that the three lines L_i, N_{i-b}, and N_i really do intersect three at a time (that is, no almost-incidences are covered by points).

(b) The reduced Levi graph corresponding to Figure 2; the dashed edge corresponds to the forced incidence.

Figure 2: Illustrating the Crossing Spans Lemma

number of novel constructions for configurations [3, 5, 8, 9]. The Crossing Spans Lemma and its associated reduced Levi graph "gadget" are shown in Figure 2.

In fact, it is straightforward to show (by relabelling symmetry classes and applying duality arguments) that given either of the labelled subgraphs in a reduced Levi graph that are shown in Figure 3, the incidence given by the dashed line is induced, where white nodes correspond to point classes and gray nodes to line classes. These subgraphs, with various choices of labels, are used extensively in the proof of Theorem 4.1.

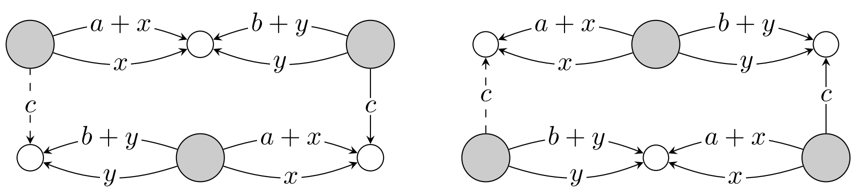
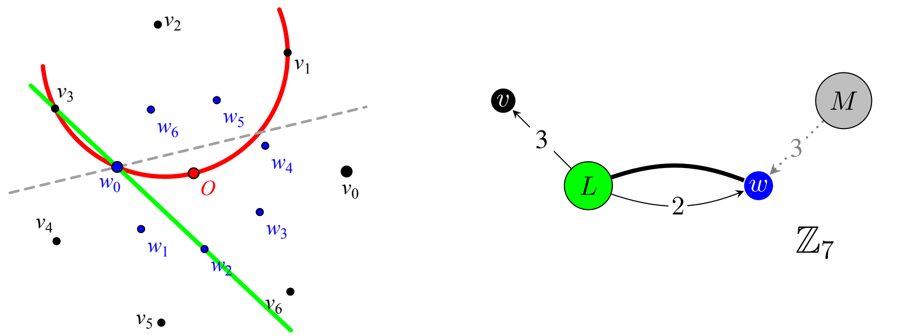


Figure 3: In either of these subgraphs in a reduced Levi graph (over $\mathbb{Z}_m$), the dashed line corresponds to a forced incidence via the CSL; , c, x, y are integers between 0 and $m-1$, and $1 \leq a \neq b < \frac{m}{2}$. Gray vertices correspond to line classes and white vertices to point classes. In the construction in Section 4, we typically take $c = 0$, $x = 0$, and $y = 0$ or δ.

In [14, p. 116–118], Branko Grünbaum described a geometric technique to constructing a certain class of 3-configurations. This technique was extended in [7] to the Circumcircle Construction Lemma. Although the lemma can be stated as a more general incidence theorem [8], we state it as follows in order to facilitate the main construction in Section 4.

Lemma 2.2 (Circumcircle Construction Lemma (CCL))**.** *Let $v_0, v_1, \ldots, v_{m-1}$ and $w_0, w_1, \ldots, w_{m-1}$ form the vertices, labelled cyclically counterclockwise, of two regular convex m-gons centered at $\mathcal{O}$. The point w_0 lies on the circle passing through $v_d, v_{d-b}, \mathcal{O}$ if and only if the points w_0, w_b, v_d are collinear.*

That is, if w_0 lies on the circumcircle of span b through v_d, then the line L_0 of span b through w_0 passes through v_d, and conversely. By symmetry, the line L_{-d} will also pass through the point w_0, and in general, if w_0 is defined to also lie on some other line M_0, then each rotated image w_i will lie on the three lines L_i, L_{i-d} and M_i. The Circumcircle Construction Lemma, along with its reduced Levi graph structure, is illustrated in Figure 4.



(a) Illustrating the Circumcircle Construction Lemma; $m = 7, b = 2, d = 3$. The green line is L_0, and the dashed gray line is a possible other line M_0 passing through w_0 (i.e., w_0 could be defined as the intersection of M_0 and C); other elements of line classes L and M have been suppressed for clarity.

(b) The "gadget" in a reduced Levi graph corresponding to Figure 4a. (The connection between w and M is optional, depending on whether there happens to be a line M_0 passing through w_0; this is the typical situation in applications of the CCL.)

Figure 4: The Circumcircle Construction Lemma.

3 Celestial 4-configurations

The building blocks for the new construction of 5-configurations presented in Section 4 are the *celestial 4-configurations*, which are configurations that have the property that every point has two lines from each of two symmetry classes of lines passing through it, and every line has two points from each of two symmetry classes of points lying on it. An example of such a configuration is shown in Figure 5, along with a general reduced Levi graph. Celestial 4-configurations were first described in detail (aside from a handful of examples, e.g., [15, 16]) in Boben and Pisanski's article *Polycyclic Configurations* [11], as the main class of 4-configurations analyzed in that paper. Their description was expanded in Grünbaum's monograph *Configurations of Points and Lines* [14, Sections 3.5–3.8], although in that chapter, he unfortunately called them *k-astral* configurations (even though as he defined previously [14, p. 34], a k-astral configuration is simply a configuration with k symmetry classes of points and of lines, and there exist k-astral 4-configurations that are not k-celestial [13]).

Every k-celestial 4-configuration can be described by a celestial symbol

$$m\#(s_1, t_1; \ldots; s_k, t_k)$$

that satisfies four axioms:

Axiom 1: (order condition) $s_i \neq t_i \neq s_{i+1}$ (with indices taken modulo m)

Axiom 2: (even condition) $\sum_{i=1}^{k}(s_i - t_i) = 2\delta$ for some integer δ

Axiom 3: (cosine condition) $\prod_{i=1}^{k} \cos\left(\frac{s_i\pi}{m}\right) = \prod_{i=1}^{k} \cos\left(\frac{t_i\pi}{m}\right)$

Axiom 4: (substring condition) no substring $s_i, t_i; \ldots; s_j, t_j$ or $t_i; s_{i+1}, \ldots, t_j; s_{j+1}$ satisfies the previous axioms.

A symbol satisfying the 4 axioms is said to be *valid*. Although celestial 4-configurations are probably the most well-understood class of 4-configuration, they are still poorly understood in general. The collection of 2-celestial configurations is completely classified ([2], with a clearer proof in [14, p. 210-211]), but general k-celestial configurations are not completely classified, and the problem appears to be non-tractable (since it depends on being able to solve certain trigonometric diophantine equations). However, some known families of valid k-celestial configurations, primarily for $k = 3, 4$, were presented in [1].

Given a valid symbol, there is a corresponding *cohort* $m\#S; T$, where $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ (as sets), which corresponds to a collection of valid symbols; in particular, the sets in a cohort must satisfy the even and cosine conditions, and it must be possible to find an ordering of the s_i and t_i that satisfies the order condition.

To construct a k-celestial 4-configuration $m\#(s_1, t_1; \ldots; s_k, t_k)$ with k point classes $v_1, \ldots, v_k$ and k line classes $L_1, \ldots, L_k$, do the following:

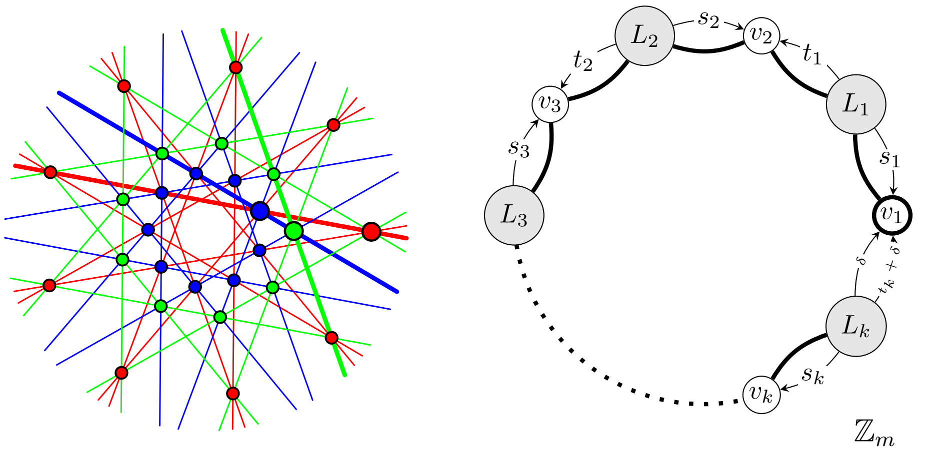
Algorithm 1 (Constructing a celestial 4-configuration)**.**

Input: A valid celestial symbol $m\#(s_1, t_1; \ldots; s_k, t_k)$.

1. Construct the vertices of a regular m-gon centered at $\mathcal{O}$, labelled $(v_1)_0, \ldots (v_1)_{m-1}$.
2. Let L_1 be the collection of lines of span s_1 with respect to point class v_1: that is, let $(L_1)_i = (v_1)_i \vee (v_1)_{i+s_1}$.
3. Construct point class v_2 to be the set of t_1-st intersection points of the lines L_1: that is, $(v_2)_i = (L_1)_i \wedge (L_1)_{i-t_1}$.
4. Continue in this fashion; line class L_2 is the set of lines of span s_2 with respect to point class v_2, point class v_3 is the set of t_2-nd intersection points of the lines L_2, etc., stopping after the construction of line class L_k.

Because the symbol $m\#(s_1, t_1; \ldots; s_k, t_k)$ is valid, the point class v_{k+1} corresponds, as a set, to point class v_1, and in particular, $(v_{k+1})_0 = (v_1)_\delta$, where $2\delta = \sum_{i=1}^{k}(s_i - t_i)$.

The general reduced Levi graph for the configuration $m\#(s_1, t_1; \ldots; s_k, t_k)$ is shown in Figure 5b; δ, the "twist" [11], is guaranteed to be an integer by the even condition. In general, the underlying graph for every reduced Levi graph of a celestial 4-configuration is a *double cycle* of even length; that is, an even cycle in which every edge is replaced by a pair of parallel edges.



(a) The celestial 4-configuration $9\#(4, 3; 2, 3; 1, 3)$. The 0th element of each symmetry class is shown larger (points) or thicker (lines), and elements in different symmetry classes are distinguished by color (class 1 is red, class 2 is blue, and class 3 is green).

(b) The reduced Levi graph, a *double cycle*, for a general celestial 4-configuration, where $\delta = \frac{1}{2}\sum_{i=1}^{k}(s_i - t_i)$.

Figure 5: Celestial 4-configurations

4 Constructing movable 5-configurations

The general idea of the construction is to produce a 5-configuration whose reduced Levi graph consists of concentric double cycles, each of which corresponds to a particular celestial 4-configuration, where the double cycles are successively linked by single edges by applying the CSL, and finally, the innermost cycle is linked to the outermost cycle using the CCL; if $k > 2$ the construction will produce a movable 5-configuration. The reduced Levi graph is shown in Figure 6.

More specifically, the reduced Levi graph contains k concentric double cycles, each of which corresponds to a k-celestial 4-configuration with cohort $m\#S;T$ where $S \cap T = \varnothing$. If the outermost cycle corresponds to the configuration with symbol

$$m\#(s_1, t_1; s_2, t_2; \ldots; s_{k-1}, t_{k-1}; s_k, t_k),$$

then each successive cycle has the s_i's permuted cyclically one step while the t_i's remain fixed: that is, the second cycle has symbol

$$m\#(s_2, t_1; s_3, t_2; \ldots; s_k, t_{k-1}; s_1, t_k),$$

the third has symbol

$$m\#(s_3, t_1; s_4, t_2; \ldots; s_1, t_{k-1}; s_2, t_k),$$

and so on, so that the innermost cycle has symbol

$$m\#(s_k, t_1; s_1, t_2; \dots; s_{k-2}, t_{k-1}; s_{k-1}, t_k).$$

The point classes of the celestial configuration corresponding to cycle j are labelled $v_1^j, \dots v_k^j$ and the line classes $L_1^j, \dots L_k^j$; that is, the superscript indicates the cycle, and the subscript the symmetry class in the celestial configuration. In Figure 6, the first point class of each celestial configuration is highlighted.

Given a valid configuration symbol $m\#(s_1, t_1; \dots; s_k, t_k)$ with cohort $m\#S; T$ with the property that $S \cap T = \varnothing$, the geometric construction algorithm to produce a 5-configuration with $k-2$ continuous degrees of freedom is given in Algorithm 2. If $k = 2$ the configuration is static and has been described previously in [9]; however, the construction algorithm given here, which uses the CCL to complete the construction, eliminates the need for completing the configuration via a continuity argument as described in that paper.

Algorithm 2 (Constructing a 5-configuration)**.**

Input: A valid celestial symbol $m\#(s_1, t_1; \dots; s_k, t_k)$ with the property that $S \cap T = \varnothing$.

1. Construct the first k-celestial 4-configuration with symbol $m\#(s_1, t_1; \dots; s_k, t_k)$, with point classes $v_1^1, \dots, v_k^1$ and line classes $L_1^1, \dots, L_k^1$.
2. If $k > 2$, for $j = 2, \dots, k-1$:
   (a) Place a new point $(v_1^j)_0$ arbitrarily on line $(L_1^{j-1})_0$, and construct the rest of the points $(v_1^j)_i$ in point class v_1^2 by rotating $(v_1^1)_0$ by $\frac{2\pi i}{m}$ for $i = 0, \dots m-1$.
   (b) Using the point class v_1^j as the starting m-gon, construct the configuration

   $$m\#(s_j, t_1; s_{j+1}, t_2; \dots; s_{j-2}, t_{k-1}; s_{j-1}, t_k)$$

   (where the sequence $s_1, s_2, \dots, s_{k-1}, s_k$ has been cyclically permuted j steps but the sequence $t_1, \dots, t_k$ remains fixed).
3. To construct the k-th celestial configuration:
   (a) Construct a circumcircle $\mathcal{C}$ of span s_k through $(v_1^1)_c$, choosing c (and varying continuous parameters if possible/necessary) so that $\mathcal{C}$ intersects line $(L_1^{k-1})_0$.
   (b) Let $(v_1^k)_0$ be the intersection of $\mathcal{C}$ with line $(L_1^{k-1})_0$, and let $(v_1^k)_i$ be the rotation of $(v_1^k)_0$ through $\frac{2\pi i}{m}$ about $\mathcal{O}$.
   (c) Construct configuration

   $$m\#(s_k, t_1; s_1, t_2; \dots; s_{k-2}, t_{k-1}; s_{k-1}, t_k)$$

   using the points $(v_1^k)_i$ as the initial set of points.

Theorem 4.1. *Algorithm 2, beginning with $m\#(s_1, t_1; \dots; s_k, t_k)$, creates a valid 5-configuration with mk^2 points, mk^2 lines and $k-2$ continuous degrees of freedom.*

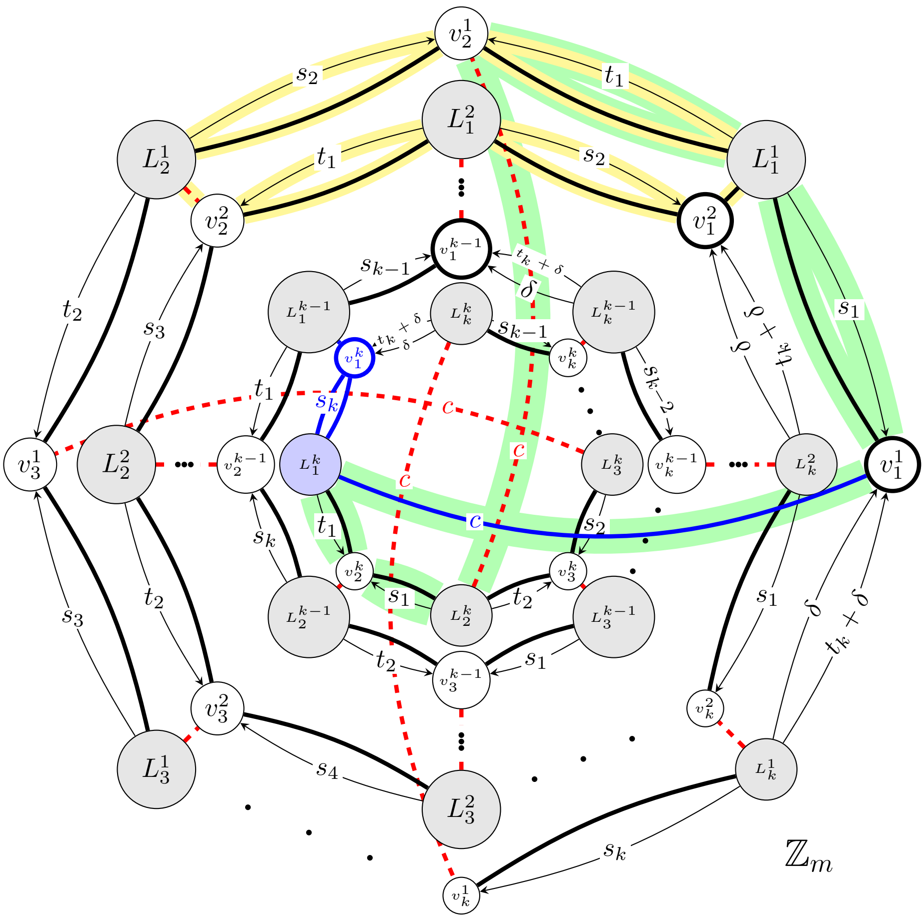


Figure 6: The reduced Levi graph, over $\mathbb{Z}_m$, for a movable 5-configuration with k^2 point classes and k^2 line classes. It consists of k concentric double cycles, each corresponding to a particular celestial 4-configuration, with the double cycles linked by arcs. The arcs shown red and dashed are induced by the Crossing Spans Lemma, with example CSL gadgets inducing the dashed edges highlighted in yellow and green, while the structure shown in blue is constructed via the Circumcircle Construction Lemma.

Proof. First, note that Algorithm 2 constructs k celestial configurations; each celestial configuration contains k symmetry classes of points and of lines, and each symmetry class contains m elements, for a total of mk^2 points and mk^2 lines.

Second, for $j = 2, \ldots, k-1$, the point $(v_1^j)_0$ is placed arbitrarily on line $(L_1^{j-1})_0$, for $(k-1) - 2 + 1 = k - 2$ continuous degrees of freedom.

Thus, the nontrivial part of the proof is to show that every point lies on 5 lines, and every line passes through 5 points.

Recall that the symbol for celestial configuration j is

$$m\#(s_j, t_1; s_{j+1}, t_2; \ldots; s_{j+\ell}, t_\ell; \ldots; s_{j-1}, t_k).$$

By construction, for each $j = 1, \ldots, k-1$, each line $(L_1^j)_i$ passes through the point $(v_1^{j+1})_i$ (that is, the first symmetry class of points in celestial configuration $j+1$ lies on the first symmetry class of lines in celestial configuration j), as well as through points $(v_1^j)_i$, $(v_1^j)_{i+s_j}$, $(v_2^j)_i$, and $(v_2^j)_{i+t_1}$ from celestial configuration j.

By careful choice of labels and the Crossing Spans Lemma, it follows that for all $\ell = 2, \ldots, k-1$ (with ℓ indexing the symmetry classes in the celestial configuration j), each line $(L_\ell^j)_i$ passes through point $(v_\ell^{j+1})_i$, as well as through points $(v_\ell^j)_i$, $(v_\ell^j)_{i+s_{j+\ell}}$, $(v_{\ell+1}^j)_i$ and $(v_{\ell+1}^j)_{i+t_\ell}$ from celestial configuration j.

A CSL gadget showing that points v_2^2 are incident with lines L_2^1 (dashed red line) beginning with the input that points v_1^2 are constructed incident with lines L_2^1 (solid black line) is highlighted in Figure 6 in yellow.

Finally, again by the CSL, line $(L_k^j)_i$ passes through point $(v_k^{j+1})_i$, as well as through points $(v_k^j)_i$, $(v_k^j)_{i+s_{j-1}}$, $(v_1^j)_{i+\delta}$ and $(v_1^j)_{i+\delta+t_k}$ from the completion of the celestial configuration j.

Thus, for $j = 1, \ldots, k-1$ (indexing the celestial configuration), $\ell = 1, \ldots k$ (indexing the symmetry class in the celestial configuration) and $i = 0, \ldots m-1$ (indexing the elements of the symmetry class) each line $(L_\ell^j)_i$ has 5 points lying on it. By inspection of the previous incidences, for $j = 2, \ldots, k-1$, each point $(v_\ell^j)_i$ has 5 lines passing through it; however, points $(v_\ell^1)_i$ only have 4 lines passing through them so far.

However, in step 3, we constructed $(v_1^k)_0$ be the intersection of $\mathcal{C}$ with line $(L_1^{k-1})_0$, where $\mathcal{C}$ is a circle of span s_k through $(v_1^1)_c$. By the Circumcircle Construction Lemma it follows that points $(v_1^k)_0$, $(v_1^k)_{s_k}$ and $(v_1^1)_c$ are collinear; that is line $(L_1^k)_0$, which is span s_k with respect to the points v_1^k by construction, passes through point $(v_1^1)_c$. By symmetry, it follows that line $(L_1^k)_i$ passes through $(v_1^1)_{i+c}$ for $i = 0, \ldots, m-1$. (This is represented by the thick blue line connecting the inner and outer rings in Figure 6.) By construction of the kth celestial configuration, it follows that line $(L_1^k)_i$ also passes through points $(v_1^k)_i$, $(v_1^k)_{i+s_k}$, $(v_2^k)_i$ and $(v_2^k)_{i+t_k}$.

A final application of the Crossing Spans Lemma on gadgets connecting the inner and outer ring shows that symmetry class L_ℓ^k in the k-th celestial configuration is incident with symmetry class v_ℓ^1 in the first celestial configuration. The CSL gadget showing that L_2^k is incident with v_2^1 (dashed red curve), beginning with the fact that L_1^k is incident with v_1^1

(thick blue curve) is highlighted in green in Figure 6. Specifically, for $\ell = 2, \ldots k-1$, $(L_\ell^k)_i$ passes through $(v_\ell^1)_i$, $(v_\ell^k)_i$, $(v_\ell^k)_{i+s_{\ell-1}}$, $(v_\ell^k)_i$, $(v_\ell^k)_{i+t_\ell}$. Finally, $(L_k^k)_i$ passes through $(v_k^1)_i$, $(v_k^k)_i$, $(v_k^k)_{i+s_{k-1}}$, $(v_1^k)_{i+\delta}$, and $(v_1^k)_{i+\delta+t_k}$. Thus, every point lies on 5 lines, and every line passes through 5 points.

□

5 Some valid inputs for Algorithm 2

Proposition 5.1. *The smallest movable 5-configuration produced by Algorithm 2 uses* $9\#(4,3;2,3;1,3)$ *(or another configuration with the same cohort) as its input and has 81 points and lines.*

Proof. If $k = 2$, Algorithm 2 produces static configurations. Inspection of a list of all valid symbols for small 3-celestial configurations (e.g., from [14, Table 3.7.1] or from the personal list of one of the the authors (LWB)) shows that the cohort $9\#\{4,2,1\};\{3,3,3\}$ is the smallest cohort with disjoint sets. □

This configuration is shown in Figure 7.

Theorem 5.2. *There exist infinitely many 5-configurations with one continuous degree of freedom.*

Proof. From [1] we know that

$$2q\#\{q-p, p, q-2r\};\{q-r, r, q-2p\}, \text{ for } q \geq 4 \text{ and } 0 < p, r < q$$

is a valid family of celestial 4-configuration cohorts.

Suppose that $r \neq p$, $r \neq \frac{q}{3}$, $p \neq \frac{q}{3}$ and $p + r \neq q$. Under these conditions, the sets S and T will always be disjoint. To see this, first note that $q - p \neq q - r$, because $p \neq r$; $q - p \neq r$, because $p + r \neq q$; and $q - p \neq q - 2p$ because $p \neq 0$. Next, $p \neq q - r$ because $p + r \neq q$; $p \neq r$ by hypothesis; and $p \neq q - 2p$ since $p \neq q/3$. Finally, $q - 2r \neq q - r$ because $r \neq 0$; $q - 2r \neq r$ since $r \neq q/3$; and $q - 2r \neq q - 2p$ because $r \neq p$. Thus, the sets are disjoint. Hence the cohort is valid as input for Algorithm 2.

In particular, $p = 1$ and $r = 2$ produces the valid input cohort $2q\#\{q-1, 1, q-4\};\{q-2, 2, q-2\}$ for any $q \geq 4$. □

Lemma 5.3. *The cohort* $3q\#\{1, 2, \ldots, 2^{k-1}\};\{\underbrace{q, q, \ldots, q}_{k}\}$ *for* $q = \dfrac{2^k+1}{3}$, *k odd and $k > 2$ is a valid celestial cohort.*

Proof. Note that the cohort $9\#\{1,2,4\};\{3,3,3\}$ can be viewed as the case $k = 3$ of this cohort.

To show the cohort is valid, we need to show that $q = \dfrac{2^k+1}{3}$ is an integer and that the cohort satisfies the cosine and even conditions.

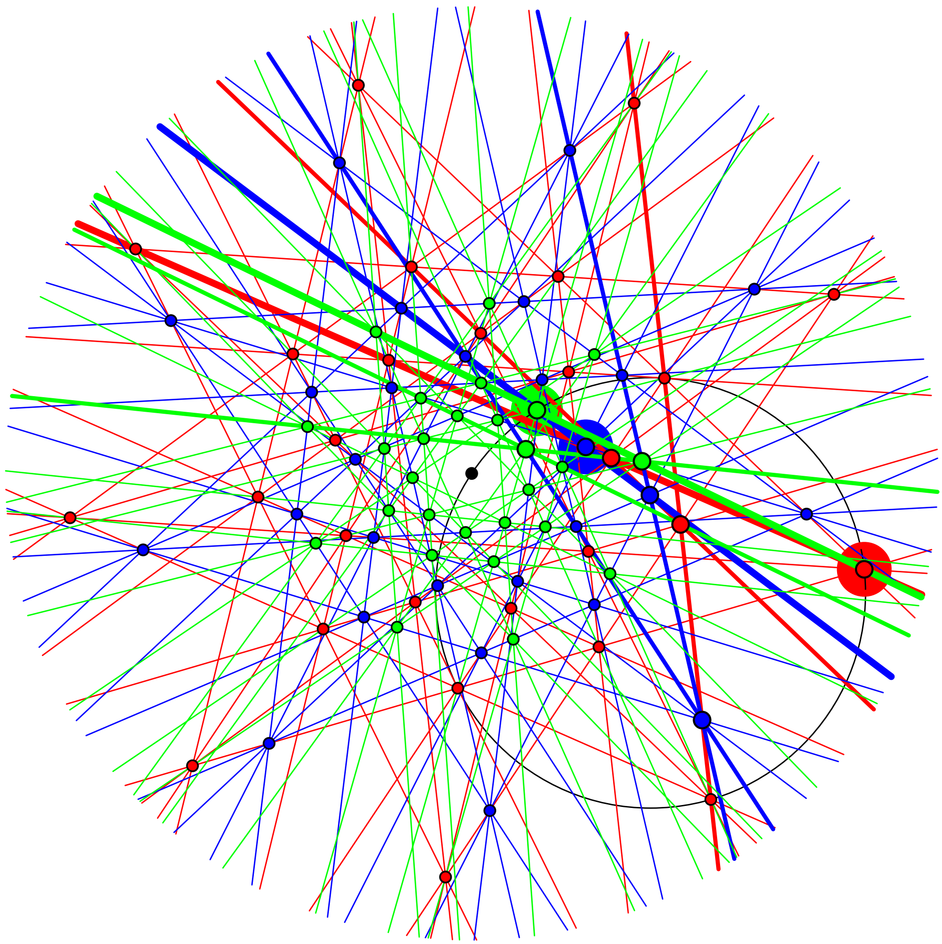


Figure 7: The smallest movable 5-configuration produced by Algorithm 2, an (81_5) configuration, with initial celestial configuration $9\#(4,3;2,3;1,3)$ shown in red, second celestial configuration $9\#(2,3;1,3;4,3)$ shown in blue, and final celestial configuration $9\#(1,3;4,3;2,3)$ shown in green. The point $(v_1^1)_0$ is highlighted in red, the line $(L_1^1)_0$ is the thickest red line, the point $(v_1^2)_0$ is highlighted in blue, and the line $(L_1^2)_0$ is the thickest blue line. The point $(v_1^3)_0$, which was constructed via the intersection of $(L_1^2)_0$ with the black circumcircle of span 1 through $(v_1^1)_0$, is highlighted in green, and $(L_1^3)_0$ is the thickest green line. Other 0th elements of symmetry classes are shown at medium weights. Already we have reached the limits of intelligibility of a small-scale diagram.

If $k = 2j + 1$ for some integer j, it is straightforward to show that

$$2^k + 1 = 2^{2j+1} + 1 = (2+1)\sum_{i=0}^{2j}(-1)^i 2^i,$$

so $2^{2j} + 1$ is clearly divisible by 3, and $q = \sum_{i=0}^{2j}(-1)^i 2^i$, which is odd.

Moreover, if $s_i = 2^{i-1}$, then $\sum_{i=1}^{k} 2^{i-1} = 2^k + 1$. Thus, if $t_i = q$ for $i = 1, \ldots, k$, then

$$\sum_{i=1}^{k}(s_i - t_i) = \left(2^k + 1\right) - (2j+1)q$$

is even, since both terms are odd.

It remains to show the cosine condition is fulfilled: that is, we need to show that for $q = \frac{2^k+1}{3}$,

$$\prod_{i=1}^{k}\cos\left(\frac{2^{i-1}\pi}{3q}\right) = \prod_{i=1}^{k}\cos\left(\frac{q\pi}{3q}\right). \tag{5.1}$$

The right-hand side of equation (5.1) clearly evaluates to $\frac{1}{2^k}$. To see the left-hand side also evaluates to $\frac{1}{2^k}$, we use the following trigonometric identity, which can be proved using the identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and induction (see [10]):

$$2^k \prod_{j=0}^{k-1}\cos\left(2^j a\right) = \frac{\sin\left(2^j a\right)}{\sin(a)}. \tag{5.2}$$

Applying this identity to the left-hand side of (5.1), we see that

$$\begin{aligned}
\prod_{i=1}^{k}\cos\left(\frac{2^{i-1}\pi}{3q}\right) = \prod_{i=1}^{k}\cos\left(\frac{2^{i-1}\pi}{2^k+1}\right) &= \frac{1}{2^k}\left(\frac{\sin\left(\frac{2^k\pi}{2^k+1}\right)}{\sin\left(\frac{\pi}{2^k+1}\right)}\right)\\
&= \frac{1}{2^k}\sin\left(\pi - \frac{\pi}{2^k+1}\right)\csc\left(\frac{\pi}{2^k+1}\right)\\
&= \frac{1}{2^k}\left(\sin(\pi)\cos\left(\frac{\pi}{2^k+1}\right) - \cos(\pi)\sin\left(\frac{\pi}{2^k+1}\right)\right)\csc\left(\frac{\pi}{2^k+1}\right)\\
&= \frac{1}{2^k}\left(0 - (-1)\sin\left(\frac{\pi}{2^k+1}\right)\right)\csc\left(\frac{\pi}{2^k+1}\right)\\
&= \frac{1}{2^k},
\end{aligned}$$

so the cosine condition is satisfied. □

Theorem 5.4. *There exists at least one 5-configuration with s continuous degrees of freedom, for infinitely many values of s.*

Proof. Use the cohort $3q\#\{1,2,\ldots,2^{k-1}\}; \{\underbrace{q,q,\ldots,q}_{k}\}$ for $q = \frac{2^k+1}{3}$, k odd and $k > 2$ from Lemma 5.3; clearly, the sets S and T are disjoint. This produces a movable 5-configuration with $k-2$ degrees of freedom for all odd $k \geq 3$. □

6 Open Questions

Question 1. In [8], the Crossing Spans Lemma is generalized to allow larger and differently labelled subgraphs, as the Extended Crossing Spans Lemma. Are there interesting movable configurations that can be constructed from this generalization?

Question 2. This construction depends on two very simple geometric lemmas, which are straightforward to prove using basic Euclidean geometry. Are there other such useful lemmas? What techniques can be used, and which incidence theorems, to construct new configurations from known configurations while retaining useful symmetry properties?

Question 3. Finding movable 3-configurations is easy [6], and there are a number of known classes of movable 4-configurations [3, 4, 8, 14]. This paper presents a class of movable 5-configurations. Are there movable k-configurations for any $k > 5$? For all $k > 5$? In particular, are there movable 6-configurations?

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Odd automorphisms in vertex-transitive graphs

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Abstract

An automorphism of a graph is said to be *even/odd* if it acts on the set of vertices as an even/odd permutation. In this article we pose the problem of determining which vertex-transitive graphs admit odd automorphisms. Partial results for certain classes of vertex-transitive graphs, in particular for Cayley graphs, are given. As a consequence, a characterization of arc-transitive circulants without odd automorphisms is obtained.

Keywords: Graph, vertex-transitive, automorphism group, even permutation, odd permutation.

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1 Introduction

Apart from being a rich source of interesting mathematical objects in their own right, vertex-transitive graphs provide a perfect platform for investigating structural properties of transitive permutation groups from a purely combinatorial viewpoint. The recent outburst

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of research papers on this topic should therefore come as no surprise. Most of these papers have arisen as direct attempts - by developing consistent theories and strategies – to solve open problems in vertex-transitive graphs; the hamiltonicity problem [17], for example, being perhaps the most popular among them.

In this context knowing the full (or as near as possible) automorphism group of a vertex-transitive graph is important because it provides the most complete description of its structure. While some automorphisms are obvious, often part of the defining properties, there are others, not so obvious and hence more difficult to find.

Consider for example bicirculants, more precisely, n-bicirculants, that is, graphs admitting an automorphism ρ with two orbits of size $n \geq 2$ and no other orbits. There are three essentially different possibilities for such a graph to be vertex-transitive depending on whether its automorphism group contains a swap and/or a mixer, where a *swap* is an automorphism interchanging the two orbits of ρ, and a *mixer* is an automorphism which neither fixes nor interchanges the two orbits of ρ. For example, the Petersen graph has swaps and mixers, prisms (except for the cube) have only swaps, while the dodecahedron has only mixers. Clearly, swaps are the "obvious" automorphisms and mixers are "not so obvious" ones (see Figure 1).

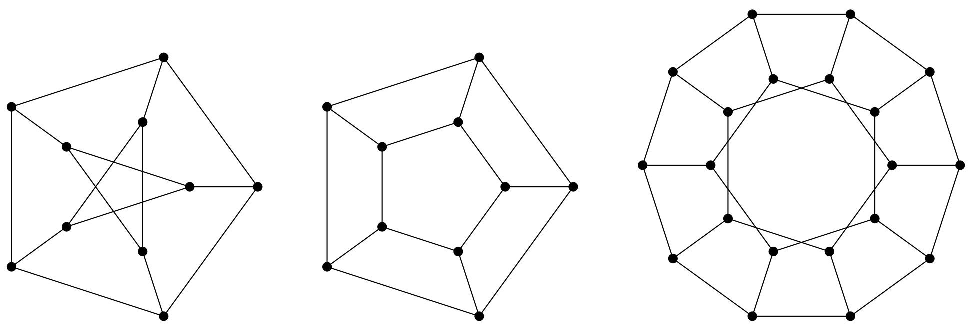

Figure 1: The Petersen graph, the 5-prism and the dodecahedron – the first two admit a swap, while the third one does not.

In this paper we propose to approach the sometimes elusive separation line between the obvious and not so obvious automorphisms via the even/odd permutations dichotomy. Let us call an automorphism of a graph *even/odd* if it acts on the vertex set as an even/odd permutation. Further, a graph is said to be *even-closed* if all of its automorphisms are even. The Petersen graph and odd prisms have odd automorphisms, the swaps being such automorphisms. On the other hand, the dodecahedron has only even automorphisms [13]. Furthermore, consider the two cubic $2k$-bicirculants, $k > 1$, shown in Figure 2 for $k = 4$. Both have swaps which are even automorphisms. More precisely, all of the automorphisms of the $2k$-prism on the left-hand side are even. As for the graph on the right-hand side – the Cayley graph $\mathrm{Cay}(\mathbb{Z}_{4k}, \{\pm 1, 2k\})$ on the cyclic group $\mathbb{Z}_{4k} = \langle 1 \rangle$ – any generator of the left regular representation of $\mathbb{Z}_{4k}$ is an odd automorphism (note that the bicirculant structure of this graph arises from the action of the square of any generator of the left regular representation of $\mathbb{Z}_{4k}$).

This brings us to the following natural question: Given a transitive group of even automorphisms H of a graph X, is there a group $G \leq \mathrm{Aut}(X)$ containing odd automorphisms

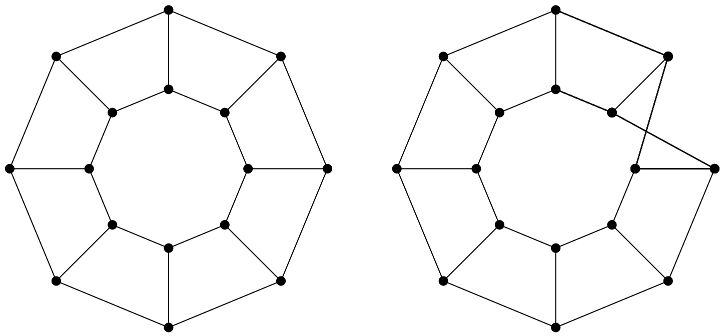

Figure 2: Two examples of cubic $2k$-bicirculants for $k = 4$, one with and one without odd automorphisms.

of X and H as a subgroup? In particular, we would like to focus on the following problem.

Problem 1.1. Which vertex-transitive graphs admit odd automorphisms?

Of course, in some cases, the answer to the above problem will be purely arithmetic. Such is for example the case with cycles. Clearly, all cycles of even length admit odd automorphisms, while cycles of odd length $2k + 1$ admit odd automorphisms if and only if k is odd. The answer for some of the well studied classes of graphs, however, suggest that the above even/odd question goes beyond simple arithmetic conditions and is likely to uncover certain more complex structural properties. For example, while the general distinguishing feature for cubic symmetric graphs (with respect to the above question) is their order $2n$, n even/odd, there are exceptions on both sides. Namely, there exist cubic symmetric graphs without odd automorphisms for n odd, and with odd automorphisms for n even, see [13].

In this paper a special emphasis is given to certain classes of Cayley graphs (see Section 3), such as circulants for example. Theorem 3.15 gives a necessary and sufficient condition for a normal circulant to be even-closed. This result combined together with certain other results of this section then leads to a characterization of even-closed arc-transitive circulants, see Theorem 3.16. In Section 4 the even/odd question is discussed in the more general context of vertex-transitive graphs.

2 Preliminaries

Here we bring together definitions, notation and some results that will be needed in the remaining sections.

For a finite simple graph X let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ be its vertex set, its edge set, its arc set and its automorphism group, respectively. A graph is said to be *vertex-transitive*, *edge-transitive* and/or *arc-transitive* (also *symmetric*) if its automorphism group acts transitively on the set of vertices, the set of edges, and/or the set of arcs of the graph, respectively. A non-identity automorphism is *semiregular*, in particular (m, n)-*semiregular* if it has m cycles of equal length n in its cycle decomposition, in other words m orbits of equal length n. An n-*circulant* (*circulant*, in short) is a graph admitting a $(1, n)$-semiregular automorphism, and an n-*bicirculant* (*bicirculant*, in short) is a graph admitting a $(2, n)$-semiregular automorphism.

Given a group G and a symmetric subset $S = S^{-1}$ of $G \setminus \{1\}$, the *Cayley graph* $X = \mathrm{Cay}(G, S)$ has vertex set G and edges of the form $\{g, gs\}$ for all $g \in G$ and $s \in S$. Every Cayley graph is vertex-transitive but there exist vertex-transitive graphs that are not Cayley, the Petersen graph being the smallest such graph. Cayley graphs are characterized in the following way. A graph is a Cayley graph of a group G if and only if its automorphism group contains a regular subgroup G_L, referred to as the *left regular representation* of G, isomorphic to G, see [19]. Using the terminology and notation of Cayley graphs, note that an n-circulant is a Cayley graph $\mathrm{Cay}(G, S)$ on a cyclic group G of order n relative to some symmetric subset S of $G \setminus \{\mathrm{id}\}$, usually denoted by $\mathrm{Circ}(n, S)$.

The first of the two group-theoretic observations below reduces the question of existence of odd automorphisms to Sylow 2-subgroups of the automorphism group.

Proposition 2.1. *A permutation group G contains an odd permutation if and only if its Sylow 2-subgroups contain an odd permutation.*

Proof. Since any odd permutation α is of even order, we can conclude that α^k, where k is the largest odd number dividing the order of α, is a non-trivial odd permutation belonging to a Sylow 2-subgroup of G. □

Proposition 2.2. *A permutation group G acting semiregularly with an odd number of orbits admits odd permutations if and only if its Sylow 2-subgroups are cyclic and non-trivial.*

Proof. Note that any Sylow 2-subgroup of G must also have an odd number of orbits. Thus if a Sylow 2-subgroup is cyclic and non-trivial, the corresponding generators are odd permutations. On the other hand, if a Sylow 2-subgroup J is not cyclic (or is trivial) then the semiregularity of G implies that all of the elements of J must be even permutations. By Proposition 2.1 G itself consists solely of even permutations. □

As a consequence of Proposition 2.2, for some classes of graphs the existence of odd automorphisms is easy to establish. For instance, in Cayley graphs the corresponding regular subgroup contains odd automorphisms if and only if its Sylow 2-subgroup is cyclic and non-trivial. When a Sylow 2-subgroup is not cyclic, however, the search for odd automorphisms has to be done outside this regular subgroup, raising the complexity of the problem.

3 Cayley graphs

In this section we give some general results about the existence of odd automorphisms in Cayley graphs and discuss the problem in detail for circulants. The first proposition, a corollary of Proposition 2.2, gathers straightforward facts about the existence of odd automorphisms in Cayley graphs. (A graph is said to be a *graphical regular representation*, or a *GRR*, for a group G if its automorphism group is isomorphic to G and acts regularly on the vertex set of the graph.)

Proposition 3.1. *A Cayley graph on a group G admits an odd automorphism in G_L if and only if G has cyclic Sylow 2-subgroups. In particular,*

- *a Cayley graph of order $2 \pmod 4$ admits odd automorphisms,*
- *a GRR admits an odd automorphism if and only if the Sylow 2-subgroups of G are cyclic.*

By Proposition 3.1, Cayley graphs of order twice an odd number admit odd automorphisms (they exist in a regular subgroup of the automorphism group). As for Cayley graphs whose order is odd or divisible by 4 the answer is not so simple. The next proposition answers the question of existence of odd automorphisms in particular subgroups of automorphisms of Cayley graphs on abelian groups.

Proposition 3.2. *Let $X = Cay(G, S)$ be a Cayley graph on an abelian group G and let $\tau \in Aut(G)$ be such that $\tau(i) = -i$. Then $\langle G_L, \tau\rangle \le Aut(X)$, and there exists an odd automorphism in $\langle G_L, \tau\rangle$ if and only if one of the following holds:*

(i) $|G| \equiv 3 \pmod 4$ *(in which case τ is an odd automorphism),*

(ii) $|G| \equiv 2 \pmod 4$*,*

(iii) $|G| \equiv 0 \pmod 4$ *and a Sylow 2-subgroup of G is cyclic.*

Proof. First recall that the mapping $\tau\colon G \to G$ defined by $\tau(i) = -i$ is an automorphism of the group G if and only if G is abelian. Moreover, since $S = -S$ it is easy to see that $\tau \in \mathrm{Aut}(X)$.

Clearly, when $|G| \equiv 1 \pmod 4$ there are no odd automorphisms in $\langle G_L, \tau\rangle$. Suppose now that $|G| \not\equiv 1 \pmod 4$. If $|G| \equiv 3 \pmod 4$ then the involution τ has $2k + 1$ cycles of length 2 and one fixed vertex in its cyclic decomposition, and so it is an odd automorphism. If $|G| \equiv 2 \pmod 4$ then there exist odd automorphisms in $G_L \le \langle G_L, \tau\rangle$ by Proposition 3.1.

We are therefore left with the case $|G| \equiv 0 \pmod 4$. Hence suppose that G is of such order. If a Sylow 2-subgroup J of G_L is cyclic then a generator of J is a product of an odd number $|G|/|J|$ of cycles of length $|J|$, and is thus an odd automorphism. On the other hand, if J is not cyclic then every element of J has an even number of cycles in its cyclic decomposition. As for τ, an element of G is fixed by τ if and only if it is an involution. In other words, it fixes the largest elementary abelian 2-group T inside the Sylow 2-subgroup J, say of order 2^k. Consequently, the number of transpositions in the cyclic decomposition of τ equals $|G|/2 - 2^k$, which is an even number if and only if $k \ge 1$. Consequently, τ is an odd automorphism if and only if $T \cong \mathbb{Z}_2$ and hence J is cyclic. □

Corollary 3.3. *Let $X = Circ(n, S)$, where S is a symmetric subset of $\mathbb{Z}_n$, and either n is even or $n \equiv 3 \pmod 4$. Then X admits odd automorphisms.*

When $n \equiv 1 \pmod 4$ the situation is more complex. For example, cycles $C_{4k+1} = \mathrm{Circ}(4k + 1, \{\pm 1\})$ admit only even automorphisms. On the other hand, the circulant $\mathrm{Circ}(13, \{\pm 1, \pm 5\})$ is an example of a $(4k + 1)$-circulant admitting odd automorphisms. Namely, one can easily check that the permutation $(0)(1, 5, 12, 8)(2, 10, 11, 3)(4, 7, 9, 6)$ arising from the action of $5 \in \mathbb{Z}_{13}^*$ is one of its odd automorphisms. (For a positive integer n we use $\mathbb{Z}_n^*$ to denote the multiplicative group of units of $\mathbb{Z}_n$.) We therefore propose the following problem.

Problem 3.4. Classify even-closed circulants of order $n \equiv 1 \pmod 4$.

A partial answer to this problem is given at the end of this section, see Corollary 3.11 and Theorem 3.16. We start with the class of connected arc-transitive circulants. The classification of such circulants, obtained independently by Kovács [12] and Li [16], is essential to this end. In order to state the classification let us recall the concept of normal Cayley graphs and certain graph products.

Let X and Y be graphs. The *wreath (lexicographic) product* $X[Y]$ of X by Y is the graph with vertex set $V(X) \times V(Y)$ and edge set $\{\{(x_1, y_1), (x_2, y_2)\} \colon \{x_1, x_2\} \in E(X)$, or $x_1 = x_2$ and $\{y_1, y_2\} \in E(Y)\}$. The *deleted wreath (deleted lexicographic) product* $X \wr_d Y$ of X by Y is the graph with vertex set $V(X) \times V(Y)$ and edge set $\{\{(x_1, y_1), (x_2, y_2)\} \colon \{x_1, x_2\} \in E(X)$ and $y_1 \neq y_2$, or $x_1 = x_2$ and $\{y_1, y_2\} \in E(Y)\}$. If $Y = \overline{K_b} = bK_1$ then the deleted lexicographic product $X \wr_d Y$ is denoted by $X[\overline{K_b}] - bX$.

Let $X = \mathrm{Cay}(G, S)$ be a Cayley graph on a group G. Denote by $\mathrm{Aut}(G, S)$ the set of all automorphisms of G which fix S setwise, that is,

$$\mathrm{Aut}(G, S) = \{\sigma \in \mathrm{Aut}(G) | S^\sigma = S\}.$$

It is easy to check that $\mathrm{Aut}(G, S)$ is a subgroup of $\mathrm{Aut}(X)$ and that it is contained in the stabilizer of the identity element id $\in G$. Following Xu [25], $X = \mathrm{Cay}(G, S)$ is called a *normal Cayley graph* if G_L is normal in $\mathrm{Aut}(X)$, that is, if $\mathrm{Aut}(G, S)$ coincides with the vertex stabilizer id $\in G$. Moreover, if X is a normal Cayley graph, then $\mathrm{Aut}(X) = G_L \rtimes \mathrm{Aut}(G, S)$ (see [11]).

Proposition 3.5. [12, 16] *Let X be a connected arc-transitive circulant of order n. Then one of the following holds:*

(i) $X \cong K_n$;

(ii) $X = Y[\overline{K_d}]$, *where $n = md$, $m, d > 1$ and Y is a connected arc-transitive circulant of order m;*

(iii) $X = Y[\overline{K_d}] - dY$, *where $n = md$, $d > 3$, $gcd(d, m) = 1$ and Y is a connected arc-transitive circulant of order m;*

(iv) *X is a normal circulant.*

The proof of the next proposition is straightforward.

Proposition 3.6. *Complete graphs K_n and their complements $\overline{K_n}$, $n \geq 2$, admit odd automorphisms.*

Propositions 3.7, 3.8, 3.9, and 3.10 deal with the existence of odd automorphisms in the framework of (deleted) lexicographic products of graphs.

Proposition 3.7. *Let Z be a graph admitting an odd automorphism. Then a lexicographic product $Y[Z]$ of the graph Z by a graph Y admits odd automorphisms. In particular, $Y[\overline{K_d}]$, $d > 1$, admits odd automorphisms.*

Proof. An odd automorphism is constructed by taking a map that acts trivially on all blocks (that is, copies of the graph Z) but one, where it acts as an odd automorphism of the graph Z. By Proposition 3.6, $\overline{K_d}$ admits an odd automorphism, so such a map exists when $Z = \overline{K_d}$. □

Proposition 3.8. *Let X be the deleted lexicographic product $X = Y \wr_d Z$ of a graph Y by a graph Z, where Z has odd automorphisms and Y is of odd order. Then X admits odd automorphisms.*

Proof. An odd automorphism is constructed by taking a map that acts as the same odd automorphism on each of the odd number of copies of the graph Z. □

Proposition 3.9. *Let X be the deleted lexicographic product $X = Y \wr_d Z$ of a graph Y by a graph Z, where Z is of odd order and Y has odd automorphisms. Then X admits odd automorphisms.*

Proof. Let α' be an odd automorphism of Y. Let $\alpha : V(X) \to V(X)$ be defined with $\alpha((y, z)) = (\alpha'(y), z)$. It is easy to see that $\alpha \in \mathrm{Aut}(X)$, and the fact that $|V(Z)|$ is odd implies that α is an odd automorphism of X. □

Propositions 3.8 and 3.9 combined together imply existence of odd automorphisms in arc-transitive circulants belonging to the family given in Proposition 3.5(iii).

Proposition 3.10. *Let X be an arc-transitive circulant isomorphic to the deleted lexicographic product $Y[dK_1] - dY$, where Y is an arc-transitive circulant of order coprime with $d > 1$. Then X has an odd automorphism.*

Proof. Suppose first that Y is of odd order. Then, since, by Proposition 3.6, dK_1 admits odd automorphisms, the existence of odd automorphisms in $\mathrm{Aut}(X)$ follows from Proposition 3.8. Suppose now that Y is of even order. Then any generator of a regular cyclic subgroup of $\mathrm{Aut}(Y)$ is an odd automorphism. Since in this case d is odd the existence of odd automorphisms in $\mathrm{Aut}(X)$ follows from Proposition 3.9. □

Corollary 3.3 and Propositions 3.6, 3.7 and 3.10 combined together imply that even-closed arc-transitive circulants may only exist amongst normal arc-transitive circulants of order $1 \pmod 4$. In all other cases an arc-transitive circulant admits an odd automorphism.

Corollary 3.11. *An even-closed arc-transitive circulant is normal and has order $1 \pmod 4$.*

For the rest of this section we may, in our search for odd automorphisms, therefore restrict ourselves to normal circulants. Let $X = \mathrm{Circ}(n, S)$ be a normal arc-transitive circulant of order order $1 \pmod 4$ and let $s \in S$. Then for any $s' \in S$ there must be an automorphism α of G such that $\alpha(s) = s'$, and so s and s' are of the same order. Thus if s is not of order n then $\mathrm{Circ}(n, S)$ is not connected. Hence it has at least three components (since n is not even), and has an automorphism that fixes all but one component while rotating that component, but this automorphism does not normalize the regular cyclic subgroup of $\mathrm{Aut}(X)$. This shows that we may assume that $1 \in S$ (note that additive notation is used for $\mathbb{Z}_n$). This fact is used throughout this section.

The following lemma about the action of the multiplicative group of units is needed in this respect. For a positive integer n we use n_p to denote the highest power of p dividing n.

Lemma 3.12. *Let p be an odd prime, and let $k \geq 1$ be a positive integer. Then $\mathbb{Z}_{p^k}^*$, in its natural action on $\mathbb{Z}_{p^k}$, admits an odd permutation if and only if k is odd.*

Proof. By Proposition 2.1 it suffices to consider the Sylow 2-subgroup J of $\mathbb{Z}_{p^k}^*$. Since $\mathbb{Z}_{p^k}^*$ is a cyclic group, J is cyclic too. Let α be a generator of J. We claim that $\langle \alpha \rangle$ acts semiregularly on $\mathbb{Z}_{p^k} \setminus \{0\}$. Suppose on the contrary that this is not the case. Then there exist $m \in \mathbb{N}$ such that $\alpha^m \neq 1$ and $\alpha^m(x) = x$ for some $x \in \mathbb{Z}_{p^k} \setminus \{0\}$. This is equivalent to

$$(\alpha^m - 1)x \equiv 0 \pmod{p^k}.$$

The above equation admits a non-trivial solution if and only if $\alpha^m - 1$ is divisible by p. Suppose that $j < k$ is such that $p^j \parallel \alpha^m - 1$. There exists $A \in \mathbb{Z}$ such that $\alpha^m = Ap^j + 1$ and $(A, p) = 1$. Since $\alpha^m \in J$ there exists $s \in \mathbb{N}$ such that $(\alpha^m)^{2^s} \equiv 1 \pmod{p^k}$. It follows that $(Ap^j + 1)^{2^s} \equiv 1 \pmod{p^k}$, and so

$$\sum_{i=1}^{2^s} \binom{2^s}{i} (Ap^j)^i \equiv 0 \pmod{p^k}.$$

For each $i > 1$ the number $\binom{2^s}{i}(Ap^j)^i$ is divisible by p^{j+1}. Consequently, $2^s Ap^j$ is divisible by p^{j+1}, and so we conclude that p divides $2^s A$, a contradiction.

As claimed above, this shows that α acts semiregularly on $\mathbb{Z}_{p^k} \setminus \{0\}$ with

$$\frac{p-1}{(p^k - p^{k-1})_2} \cdot \frac{p^k - 1}{p - 1} = (1 + p + \ldots + p^{k-1}) \frac{p-1}{(p^k - p^{k-1})_2}$$

cycles of even length $(p^k - p^{k-1})_2 = (p-1)_2$ in its cycle decomposition (since α is a generator of J). Since the parity of $1 + p + \ldots + p^{k-1}$ depends on whether k is even or odd, it follows that α is an odd permutation if and only if k is odd. The result follows. □

Corollary 3.13. *Let p be an odd prime, and let $k \geq 1$ be a positive integer such that $p^k \equiv 1 \pmod 4$. Then a normal arc-transitive circulant $X = Cay(\mathbb{Z}_{p^k}, S)$ admits an odd automorphism if and only if k is odd and S contains the Sylow 2-subgroup of $\mathbb{Z}_{p^k}^*$.*

Proof. Recall that $\mathrm{Aut}(X) \cong \mathbb{Z}_{p^k} \rtimes S$, and thus X admits odd automorphisms if and only if S contains an element giving rise to an odd permutation on $\mathbb{Z}_{p^k}$ (generators of Sylow 2-subgroups of $\mathbb{Z}_{p^k}^*$ are odd permutations on $\mathbb{Z}_{p^k}$). The result is thus obtained by combining together Proposition 2.1 and Lemma 3.12. □

Lemma 3.14. *Let $n = p_1^{2k_1+1} \cdots p_a^{2k_a+1} q_1^{2l_1} \ldots q_b^{2l_b}$ be a prime decomposition of an odd integer n, and let $\mathbb{Z}_n \cong P_1 \oplus \cdots \oplus P_a \oplus Q_1 \oplus \cdots \oplus Q_b$, where $P_i \cong \mathbb{Z}_{p_i^{2k_i+1}}$, $i \in \{1, \ldots, a\}$, and $Q_i \cong \mathbb{Z}_{q_i^{2l_i}}$, $i \in \{1, \ldots, b\}$. Further, let α_i and β_i, respectively, be generators of the Sylow 2-subgroup of P_i^* and the Sylow 2-subgroup of Q_i^*. Then, for each i, we have that α_i is an odd permutation on $\mathbb{Z}_n$, and β_i is an even permutation on $\mathbb{Z}_n$.*

Proof. Observe that each cycle in the cycle decomposition of $\alpha_i \in P_i$ (considered as a permutation of $\mathbb{Z}_{p_i^{2k_i+1}}$) is lifted to $n/p_i^{2k_i+1}$ cycles of the same length in the cycle decomposition of α_i (when considered as a permutation of $\mathbb{Z}_n$). By Lemma 3.12, α_i is an odd permutation on $\mathbb{Z}_{p_i^{2k_i+1}}$ for each i. Similarly, β_i is an even permutation on $\mathbb{Z}_{q_i^{2l_i}}$ for each i. Since n is odd, the result follows. □

We introduce the following notation. Let $n = p_1^{k_1} \cdots p_a^{k_a}$ be a prime decomposition of a positive integer n, let

$$\mathbb{Z}_n \cong \oplus_{i=1}^a P_i, \text{ where } P_i \cong \mathbb{Z}_{p_i^{k_i}},$$

and let $J(p_i)$ be the Sylow 2-subgroup of P_i^*. In the next theorem a necessary and sufficient condition for a normal circulant to be even-closed is given. One of the immediate consequences is, for example, that a normal circulant of order n^2, n odd, is even-closed.

Theorem 3.15. *Let $n = p_1^{k_1} \cdots p_a^{k_a}$ be a prime decomposition of a positive integer n, and let $X = Circ(n, S)$ be a normal arc-transitive circulant on $\mathbb{Z}_n \cong \oplus_{i=1}^a P_i$. Then X is even-closed if and only if $n \equiv 1 \pmod 4$ and for every $\alpha = \oplus_{i=1}^a \alpha_i \in S$ we have*

$$\sum_{i=1}^{a} \theta_i(\alpha) \equiv 0 \pmod 2, \text{ where } \theta_i(\alpha) = \begin{cases} 1; & \text{if } J(p_i) \leq \langle \alpha_i \rangle \text{ and } k_i \text{ is odd} \\ 0; & \text{otherwise} \end{cases}.$$

Proof. By Corollary 3.3 for $n \not\equiv 1 \pmod 4$ the graph X admits odd automorphisms. We may therefore assume that $n \equiv 1 \pmod 4$. By Lemma 3.14, the existence of odd automorphisms in X depends solely on the parity of the exponents k_i and the containment of the generators of the corresponding Sylow 2-subgroups in S, and the result follows. (Recall, that we are using the assumption that $1 \in S$ and the fact that in a normal arc-transitive circulant, every element of S is conjugate, so every $\alpha \in S$ is odd if and only if X has an odd automorphism.) □

Combined together with Corollary 3.11 and Theorem 3.15 we have the following characterization of even-closed arc-transitive circulants.

Theorem 3.16. *Let X be an even-closed arc-transitive circulant of order n and let $n = p_1^{k_1} \cdots p_a^{k_a}$ be a prime decomposition of n. Then X is a normal circulant $X = Circ(n, S)$ on $\mathbb{Z}_n \cong \oplus_{i=1}^a P_i$, $n \equiv 1 \pmod 4$ and for every $\alpha = \oplus_{i=1}^a \alpha_i \in S$ we have*

$$\sum_{i=1}^{a} \theta_i(\alpha) \equiv 0 \pmod 2, \text{ where } \theta_i(\alpha) = \begin{cases} 1; & \text{if } J(p_i) \leq \langle \alpha_i \rangle \text{ and } k_i \text{ is odd} \\ 0; & \text{otherwise} \end{cases}.$$

4 Vertex-transitive graphs

It is known that every finite transitive permutation group contains a fixed-point-free element of prime power order (see [9, Theorem 1]), but not necessarily a fixed-point-free element of prime order and, hence, a semiregular element (see for instance [3, 9]). In 1981 the third author asked if every vertex-transitive digraph with at least two vertices admits a semiregular automorphism (see [20, Problem 2.4]).

Despite considerable efforts by various mathematicians the problem remains open, with the class of vertex-transitive graphs having a solvable automorphism group being the main obstacle. The most recent result on the subject is due to Verret [24] who proved that every arc-transitive graph of valency 8 has a semiregular automorphism, which was the smallest open valency for arc-transitive graphs (see [7, 10, 23] and [15] for an overview of the status of this problem). While the existence of such automorphisms in certain vertex-transitive graphs has proved to be an important building block in obtaining at least partial solutions in many open problems in algebraic graph theory, such as for example the hamiltonicity problem (see [14, 2, 17]), the connection to the even/odd problem is straightforward.

Proposition 4.1. *An even-closed vertex-transitive graph does not have even order semiregular automorphisms with an odd number of orbits.*

This suggest that in a search for odd automorphisms a special attention should be given to semiregular automoprhisms of even order.

Furthermore, for those classes of vertex-transitive graphs for which a complete classification (together with the corresponding automorphism groups) exists, the answer to

Problem 1.1 is, at least implicitly, available right there – in the classification. Such is, for example, the case of vertex-transitive graphs of order a product of two primes, see [6, 8, 21, 22, 18], and the case of vertex-transitive graphs which are graph truncations, see [1]. The hard work needed to complete these classifications suggest that the even/odd question is by no means an easy one. Let us consider, for example, the class of all vertex-transitive graphs of order $2p$, p a prime. In the completion of the classification of these graphs, the classification of finite simple groups is an essential ingredient in handling the case of primitive automorphism groups. We know, by this classification, that the Petersen graph and its complement are the only such graphs with a primitive automorphism group. Of course, they both admit odd automorphisms. As for imprimitive automorphism groups, it all depends on the arithmetic of p. When $p \equiv 3 \pmod 4$, the graphs are necessarily Cayley graphs (of dihedral groups) and hence must admit odd automorphisms. (Namely, reflections interchanging the two orbits of the rotation in the dihedral group are odd automorphisms.) When $p \equiv 1 \pmod 4$, then it follows by the classification of these graphs [20] that there is an automorphism of order 2^k, $k \geq 1$, interchanging the two blocks of imprimitivity of size p, having one orbit of size 2 and $2(p-1)/2^k$ orbits of size 2^k, thus an odd number of orbits in total (since 2^k divides $p-1$). We have thus shown that every vertex-transitive graph of order twice a prime number admits an odd automorphism. However, no “classification of finite simple groups free” proof of the above fact is known to us.

In conclusion we would like to make the following comment with regards to cubic vertex-transitive graphs. Recall that the class of cubic vertex-transitive graphs decomposes into three subclasses depending on the number of orbits of the vertex-stabilizer on the set of neighbors of a vertex. These subclasses are arc-transitive graphs (one orbit), graphs with vertex-stabilizers being 2-groups (two orbits) and GRR graphs (three orbits), see [4]. (Note that there are two types of cubic GRR graphs, those with connecting set consisting of three involutions and those with connecting set consisting of an involution, a non-involution and its inverse, see [5].) For the first and third subclass the answer to Problem 1.1 is given in [13] and Proposition 3.1, respectively, while the problem is still open for the second subclass. Examples given in Section 1 (see also Figure 2) show, however, that this second subclass contains infinitely many even-closed graphs as well as infinitely many graphs admitting odd automorphisms.

Problem 4.2. Classify cubic vertex-transitive graphs with vertex-stabilizers being 2-groups that admit odd automorphisms.

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- Dragan Marušič (Algebra and Discrete Mathematics);
- Enes Pasalic (Cryptography).

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The positions are for 4 years and include a tuition fee waiver. More information available at: http://www.famnit.upr.si/sl/novice/javni-razpis-za-mlad-3

The deadline for applications is July 29, 2016. Applicants should send a letter of interest with CV and two recommendation letters to:

"Young Researcher position"
University of Primorska, UP IAM
Muzejski trg 2
6000 Koper
Slovenia

The application should also be sent by e-mail to the address manca.drobne@upr.si.

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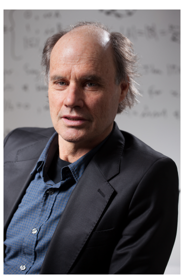


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