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On Graphs with the Smallest Eigenvalue at Least $-1-\sqrt{2}$, part I

Tetsuji Taniguchi
Graduate School of Mathematics, Kyushu University, Fukuoka 812-81, Japan

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Abstract

There are many results on graphs with the smallest eigenvalue at least -2. As a next step, A. J. Hoffman proposed to study graphs with the smallest eigenvalue at least $-1-\sqrt{2}$. In order to deal with such graphs, R. Woo and A. Neumaier introduced the concept of a Hoffman graph, and defined a new generalization of line graphs which depends on a family of Hoffman graphs. They proved a theorem analogous to Hoffman's, using a particular family consisting of four isomorphism classes.

In this paper, we deal with a generalization based on a family $\mathscr{H}$ smaller than the one which they dealt with, yet including generalized line graphs in the sense of Hoffman. The main result is that the cover of an $\mathscr{H}$-line graph with at least 8 vertices is unique.

Keywords: Generalized line graph, Spectrum.

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1 Introduction

In [1], P. J. Cameron, J. M. Goethals, J. J. Seidel, and E. E. Shult have shown that graphs with the smallest eigenvalue at least -2 are represented by a subset of the root system A_n, D_n or E_8. A graph represented by a subset of A_n is the line graph of a bipartite graph. A graph represented by a subset of D_n is a generalized line graph. A graph represented by a subset of E_8 has at most 36 vertices, and its maximum degree is at most 28. A graph is said to be exceptional if it is connected, has the smallest eigenvalue at least -2, and is not a generalized line graph. Such graphs are represented by a subset of E_8, hence the number of exceptional graphs is finite. The 473 maximal exceptional graphs have been constructed theoretically in [3] (although first found by computer).

Let $\lambda_{\min}(G)$ be the smallest eigenvalue of a graph G. Let Λ^1 be the set of all real numbers λ such that $\lambda = \lambda_{\min}(G)$ for some graph G. Let $\delta(G)$ denote the minimum degree of a graph G. In [6], A. J. Hoffman has shown the following theorem:

E-mail address: tetsuzit@math.kyushu-u.ac.jp (Tetsuji Taniguchi)

Theorem 1 (A. J. Hoffman)**.** *There exists an integer valued function f, defined on the intersection of Λ^1 with the half-open interval $(-1-\sqrt{2},\ -1]$, such that*

(1) if G is a connected graph with $-2 < \lambda_{\min}(G) \leq -1$, $\delta(G) \geq f(\lambda_{\min}(G))$, then G is a clique and $\lambda_{\min}(G) = -1$;

(2) if G is a connected graph with $-1-\sqrt{2} < \lambda_{\min}(G) \leq -2$, $\delta(G) \geq f(\lambda_{\min}(G))$, then G is a generalized line graph and $\lambda_{\min}(G) = -2$.

Obviously, we are interested in making the values of the function f as small as possible. Since $\lambda_{\min}(G) \leq -1$ if G has at least one edge, and $\lambda_{\min}(G) = -1$ if and only if G is a clique, we may take $f(-1) = 1$. Since an exceptional graph has the maximum degree at most 28 as mentioned above, if a graph has the smallest eigenvalue at least -2 and the maximum degree at least 29, then it is a generalized line graph. Hence we may take $f(-2) = 29$. In order to state the extension of Theorem 1.1 by Woo and Neumaier [8], we need to introduce several concepts.

Definition 2. A *Hoffman graph* is a graph H with vertex labeling $V(H) \to \{s, f\}$, satisfying the following conditions:

1. every vertex with label f is adjacent to at least one vertex with label s;
2. vertices with label f are pairwise non-adjacent.

We call a vertex with label s a *slim vertex*, and a vertex with label f a *fat vertex*. We denote by $V_s(H)$ ($V_f(H)$) the set of slim (fat) vertices of H. An ordinary graph without labeling can be regarded as a Hoffman graph without fat vertex. Such a graph is called a *slim graph*. The subgraph of a Hoffman graph H induced on $V_s(H)$ is called the *slim subgraph* of H. We draw Hoffman graphs by depicting vertices as large (small) black dots if they are fat (slim).

We denote by $[H]$ the isomorphism class of Hoffman graphs containing H. In the following, all graphs considered are Hoffman graphs and all subgraphs considered are induced subgraphs.

Definition 3. Let H be a Hoffman graph, and let H^i $(i = 1, 2, \ldots, n)$ be a family of subgraphs of H. The graph H is said to be the *sum* of H^i $(i = 1, 2, \ldots, n)$, denoted

$$H = \biguplus_{i=1}^{n} H^i, \tag{1}$$

if the following conditions are satisfied:

(i) $V(H) = \bigcup_{i=1}^{n} V(H^i)$;

(ii) $V_s(H^i) \cap V_s(H^j) = \emptyset$ if $i \neq j$;

(iii) if $x \in V_s(H^i)$ and $y \in V_f(H)$ are adjacent, then $y \in V(H^i)$;

(iv) if $x \in V_s(H^i)$, $y \in V_s(H^j)$ and $i \neq j$, then x and y have at most one common fat neighbour, and they have one if and only if they are adjacent.

One can show easily that the sum defined above is associative, in the sense that one of $H = H^1 \uplus H^2 \uplus H^3$, $H = (H^1 \uplus H^2) \uplus H^3$ implies the other.

Example 4. The smallest fat Hoffman graph is the graph H_1 consisting of two adjacent vertices, one of which is slim and the other fat. Let H be a connected Hoffman graph which is the sum (1), where $H^i \cong H_1$ for all $i = 1, 2, \ldots, n$. Then the slim subgraph of H is the complete graph on n vertices.

Example 5. Let H_2 be the Hoffman graph consisting of two non-adjacent fat vertices and one slim vertex, where both fat vertices are adjacent to the slim vertex. Let H be a connected Hoffman graph which is the sum (1), where $H^i \cong H_2$ for all $i = 1, 2, \ldots, n$. Then the slim subgraph of H is the line graph of a graph with n edges.

Example 6. Let H_3 be the Hoffman graph consisting of two non-adjacent slim vertices and one fat vertex, where both slim vertices are adjacent to the fat vertex. Let H be a connected Hoffman graph which is the sum (1), where $H^i \cong H_3$ for all $i = 1, 2, \ldots, n$. Then the slim subgraph of H is the cocktail party graph on $2n$ vertices.

Example 7. Let H be a connected Hoffman graph which is the sum (1), where $H^i \cong H_2$ or $H^i \cong H_3$ for all $i = 1, 2, \ldots, n$. Then the slim subgraph of H is a generalized line graph.

Definition 8. For a Hoffman graph H, let A be its adjacency matrix,

$$A = \begin{bmatrix} A_s & C \\ C^T & O \end{bmatrix}$$

in a labeling in which the fat vertices come last. *Eigenvalues* of H are the eigenvalues of the real symmetric matrix $A_s - CC^T$. Let $\lambda_{\min}(H)$ denote the smallest eigenvalue of H. For a connection of $\lambda_{\min}(H)$ with the smallest eigenvalue of slim graphs, we refer [8, Proposition 5.3].

Hoffman claims that it is reasonable to believe that there is a sequence of numbers $\alpha_1 = -1 > \alpha_2 = -2 > \alpha_3 = -1-\sqrt{2} > \alpha_4 > \cdots$ tending to some limit $\bar{\alpha}$, satisfying the following condition: for each $\lambda \in \Lambda^1$ with $\alpha_i \geq \lambda > \alpha_{i+1}$, there is a positive integer $f(\lambda)$ such that, $\lambda_{\min}(G) = \lambda$ and $\delta(G) \geq f(\lambda)$ imply $\lambda = \alpha_i$. In order to deal with graphs with $\lambda_{\min}(G) < \alpha_2$, in particular, those graphs G with $\alpha_3 \leq \lambda_{\min}(G) < \alpha_2$, R. Woo and A. Neumaier defined a new generalization of line graphs in [8], introducing the concept of a Hoffman graph. The definition can be rewritten as follows using the notation introduced in Definition 3.

Definition 9. Let $\mathscr{H}$ be a family of isomorphism classes of Hoffman graphs. An $\mathscr{H}$-line graph Γ is a subgraph of a graph $H = \biguplus_{i=1}^{n} H^i$ such that $[H^i] \in \mathscr{H}$ for all $i \in \{1, 2, \ldots, n\}$. In this case, we call H an $\mathscr{H}$-cover graph of Γ. If $V_s(\Gamma) = V_s(H)$, then we call H a strict $\mathscr{H}$-cover graph of Γ.

Two strict $\mathscr{H}$-covers K and L of Γ are called equivalent, if there exists an isomorphism $\varphi : K \to L$ such that $\varphi|_\Gamma$ is the identity automorphism of Γ.

R. Woo and A. Neumaier suggested α_4 (≈ -2.4812) to be the smallest root of the polynomial $x^3 + 2x^2 - 2x - 2$ in [8]. In their paper, some Hoffman graphs with the smallest eigenvalue at least $-1-\sqrt{2}$ are given. They are listed in Figure 1 together with their smallest eigenvalues $\lambda_{\min}$. Actually some of these graphs are not used in this paper, but we use the same symbols as in [8] to avoid confusion.

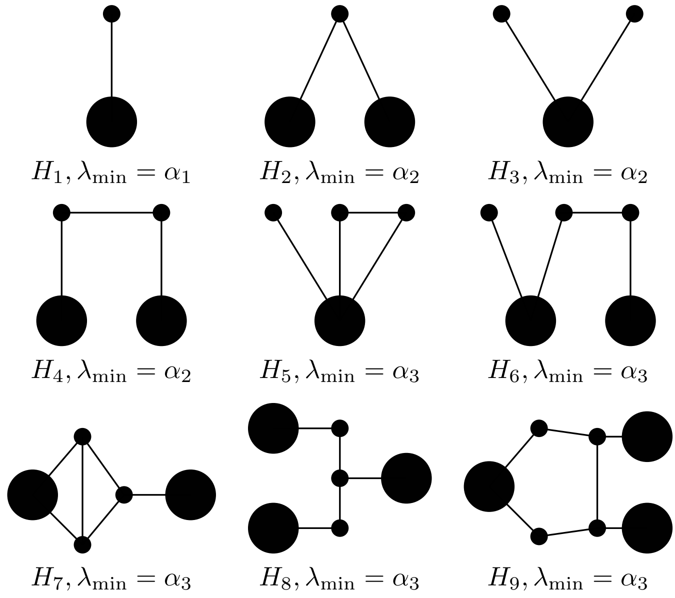

Figure 1:

Theorem 10 (R. Woo and A. Neumaier [8])**.** *Let α_4 ($\approx\ -2.4812$) be the smallest root of the polynomial $x^3 + 2x^2 - 2x - 2$. Then there exists an integer valued function f, defined on the intersection of Λ^1 with the half-open interval $(\alpha_4,\ -1-\sqrt{2}]$, such that if G is a connected graph with $\alpha_4 < \lambda_{\min}(G) \leq -1-\sqrt{2}$, $\delta(G) \geq f(\lambda_{\min}(G))$, then G is an $\{[H_2],\ [H_5],\ [H_7],\ [H_9]\}$-line graph.*

In particular, $\lambda_{\min}(G) = -1-\sqrt{2}$.

In [8], R. Woo and A. Neumaier consider it one of the problems to find a complete list of minimal forbidden subgraphs for slim (or arbitrary) $\{[H_2],\ [H_5],\ [H_7],\ [H_9]\}$-line graphs, where $H_2,\ H_5,\ H_7$ and H_9 are graphs shown in Figure 1. If one finds such a list to be finite, then it is expected that there exist finitely many maximal slim graphs, which have the smallest eigenvalue at least $-1-\sqrt{2}$ and are not $\{[H_2],\ [H_5],\ [H_7],\ [H_9]\}$-line graphs, by applying the star complement technique to the list (cf. [3]). This might lead to the determination of the value $f(-1-\sqrt{2})$. But it seems very difficult to find all minimal forbidden subgraphs in this case.

L. W. Beineke proved that line graphs are characterized by a collection of only nine forbidden subgraphs (see [4]). Similarly, minimal forbidden subgraphs for the class of generalized line graphs were determined in [2]. In view of Theorem 10, it would be desirable to determine minimal forbidden subgraphs for the class of slim $\{[H_2], [H_5], [H_7], [H_9]\}$-line graphs. However, as the number of such minimal forbidden subgraphs seems very large, it may make sense to consider first the smaller class of $\{[H_2], [H_5]\}$-line graphs.

A generalized cocktail party graph is a graph isomorphic to a clique with independent edges removed. In [2], D. Cvetković, M. Doob and S. Simić showed the following theorem:

Theorem 11 (D. Cvetković, M. Doob and S. Simić)**.** *If G is a connected generalized line graph with more than six vertices, then there exists a unique partition of the edges of G into generalized cocktail party graphs satisfying the following conditions:*

(i) *each vertex is in at most two generalized cocktail party graphs;*

(ii) *two generalized cocktail party graphs have at most one common vertex;*

(iii) *if two generalized cocktail party graphs have a common vertex, then it is adjacent to all other vertices in both of them.*

As this theorem was used to determine the minimal forbidden subgraphs for the class of generalized line graphs, we intend to prove an analogous theorem for $\{[H_2],[H_5]\}$-line graphs, then to determine the minimal forbidden subgraphs for the class of $\{[H_2],[H_5]\}$-line graphs.

In this paper and the subsequent paper [9], we completely determine minimal forbidden subgraphs for the class of slim $\{[H_2],[H_5]\}$-line graphs. First, in this paper, we show that $\{[H_2],[H_5]\}$-line graphs are the same as the $\{[H_2],[H_3],[H_5]\}$-line graphs. The latter family turned out to be easier when dealing with cover graphs, and we prove the uniqueness of strict $\{[H_2],[H_3],[H_5]\}$-cover graphs. This result will then play a crucial role in obtaining an upper bound on the number of vertices in a minimal forbidden subgraph. The details will be given in the subsequent paper [9].

2 Some results on Hoffman graphs

For a Hoffman graph H, let $E(H)$ denote the set of all edges of H. For a vertex v of a Hoffman graph H, we denote by $N_H^s(v)$ (resp. $N_H^f(v)$) the set of all slim (resp. fat) neighbours of v, and by $N_H(v)$ the set of all neighbours of v, i.e., $N_H(v) = N_H^s(v) \cup N_H^f(v)$. We write $G \subset H$ if G is an induced subgraph of H. In particular, if $V_s(G) = V_s(H)$, then we write $G \subset^* H$. We denote by $\langle S\rangle_H$ the subgraph of H induced on a set of vertices S. If a vertex x is adjacent to a vertex y, then we write $x \sim y$. For a Hoffman graph H and a subset $S \subset V_s(H)$, let $\langle\langle S\rangle\rangle_H$ denote the subgraph

$$\langle\langle S\rangle\rangle_H = \langle S \cup (\bigcup_{z\in S} N_H^f(z))\rangle_H.$$

Also, define $H-S$, $H-x$ by $H-S = \langle\langle V_s(H)\setminus S\rangle\rangle_H$, $H-x = H-\{x\}$, respectively, where $x \in V_s(H)$. Let $\emptyset$ be an empty set, and let ϕ be an empty graph. For graphs G and H, we denote by $G+H$ the disjoint union of G and H, and by nG the disjoint union $G+G+\cdots+G$ of n copies of G. Let $G*H$ denote the graph consisting of the disjoint union $G+H$, together with edges uv, for all $u\in V(G)$ and all $v \in V(H)$.

For the remainder of this section, we let $H = \biguplus_{i=0}^n H^i$ be a graph for some family of Hoffman subgraphs H^i $(i = 0,1,\ldots,n)$. If $I \subset \{0,1,2,\ldots,n\}$ and $H_I = \langle\bigcup_{i\in I} V(H^i)\rangle_H$, then obviously $H_I = \biguplus_{i\in I} H^i$ holds.

Lemma 12. *Let S be a subset of $V_s(H)$. Then*

$$\langle\langle S\rangle\rangle_H = \biguplus_{i=0}^{n}(\langle\langle S\cap V_s(H^i)\rangle\rangle_H).$$

In particular, if $x \in V_s(H^0)$, then

$$H-x = (H^0-x) \uplus (\biguplus_{i=1}^{n} H^i).$$

Proof. We verify the conditions (i)–(iv) of Definition 3. The conditions (i)–(iii) are obvious. To verify the condition (iv), suppose $x \in V_s(\langle\langle S \cap V_s(H^i)\rangle\rangle_H) = S \cap V_s(H^i)$, $y \in S \cap V_s(H^j)$ and $i \neq j$. Then x and y have at most one common fat neighbour in H, and they have one iff they are adjacent. Since the fat neighbours of x and y are all contained in $\langle\langle S\rangle\rangle_H$, x and y have at most one common fat neighbour in $\langle\langle S\rangle\rangle_H$ iff they are adjacent. □

Lemma 13. *If we construct a graph K containing H and its subgraph $\tilde{H}^0$ containing H^0, by one of the following ways, then*

$$K = \tilde{H}^0 \uplus (\biguplus_{i=1}^{n} H^i) \tag{2}$$

holds:

(i) *add a new slim vertex x and edges joining x to some of the vertices of $V_s(H^0)$; define $\tilde{H}^0$ to be the subgraph induced on $V(H^0) \cup \{x\}$,*

(ii) *in addition to* (i)*, add an edge joining x to a fixed fat vertex α of H^0, and the edges xy ($y \in V_s(H^i)$, $1 \leq i \leq n$, $\alpha \sim y$); define $\tilde{H}^0$ to be the subgraph induced on $V(H^0) \cup \{x\}$,*

(iii) *add a new fat vertex β and edges joining β to some of the vertices of $V_s(H^0)$; define $\tilde{H}^0$ to be the subgraph induced on $V(H^0) \cup \{\beta\}$,*

Proof. In each of the cases (i)–(iii), we verify the conditions (i)–(iv) of Definition 3. The conditions (i)–(ii) of Definition 3 are obvious for each of the cases (i)–(iii). The condition (iii) of Definition 3 is obvious for the case (i). It is also easy to see for the cases (ii)–(iii), since all the newly added edges joining slim and fat vertices are contained in $\tilde{H}^0$. The condition (iv) of Definition 3 is obvious for the case (i), since the new slim vertex x has no fat neighbour. It is easy to see for the case (ii), since α is the unique fat neighbour of x. It is also easy for the case (iii), since all the slim neighbours of β are in $\tilde{H}^0$. □

Lemma 14. *Let K^0 be a graph containing a subgraph G isomorphic to H^0. Suppose that for each $x \in V_s(K^0) \setminus V_s(G)$, $|N_{K^0}(x) \cap V_f(G)| \leq 1$. Then there exists a graph K with*

$$K = \biguplus_{i=0}^{n} K^i \tag{3}$$

containing a subgraph isomorphic to H and $K^i \cong H^i$ for $i = 1, 2, \ldots, n$.

Proof. We first prove the special case where $V_f(K^0) = V_f(G)$, by induction on $|V_s(K^0)| - |V_s(G)|$. The assertion is trivial when $|V_s(K^0)| - |V_s(G)| = 0$, since we may take $K = H$. Suppose $|V_s(K^0)| - |V_s(G)| > 0$. Pick a vertex $x \in V_s(K^0) \setminus V_s(G)$, and put $L^0 = K^0 - x$. Then $L^0 \supset G$ and, for each $y \in V_s(L^0) \setminus V_s(G)$, $|N_{L^0}(y) \cap V_f(G)| = |N_{K^0}(y) \cap V_f(G)| \leq 1$ by the assumption. Thus, by inductive hypothesis, there exists a graph L with $L = \biguplus_{i=0}^{n} L^i$ containing a subgraph isomorphic to H, $L^i \cong H^i$ ($1 \leq i \leq n$). We construct a graph K from L by adding the vertex x as follows. First, we add edges joining x to vertices of L^0 in such a way that the subgraph induced on $L^0 \cup \{x\}$ is isomorphic to K^0. If $|N_{K^0}(x) \cap V_f(G)| = 0$, then $N^f_{K^0}(x) = \emptyset$, so by Lemma 13(i), (3) holds. If $|N_{K^0}(x) \cap V_f(G)| = 1$, then $N^f_{K^0}(x) = \{\alpha\}$ for some fat vertex α. We then add the edges xy ($y \in N^s_L(\alpha) \setminus V_s(L^0)$). By Lemma 13(ii), (3) holds.

Next we consider the general case. Let L^0 be the subgraph of K^0 induced on $V_s(K^0) \cup V(G)$. Then by the special case above, there exists a graph L with $L = \biguplus_{i=0}^n L^i$ containing a subgraph isomorphic to H, and $L^i \cong H^i$ $(1 \le i \le n)$. It remains to construct a graph K from L by adding the fat vertices $V(K^0) \setminus V(L^0) = V_f(K^0) \setminus V_f(L^0)$ to L^0. These fat vertices can be added one by one, by Lemma 13(iii), in such a way that the resulting graph is isomorphic to K^0. Then (3) holds, and K has a subgraph isomorphic to L, and $K^i \cong L^i$ $(1 \le i \le n)$. Thus K has a subgraph isomorphic to H, and $K^i \cong H^i$ $(1 \le i \le n)$. □

Lemma 15. *Suppose that H is connected, H^0 has a unique fat vertex α, and α is adjacent to all the slim vertices of H^0. Then $\langle \bigcup_{i=1}^n V(H^i) \rangle_H$ is connected.*

Proof. Let $H' = \langle \bigcup_{i=1}^n V(H^i) \rangle_H$. Since H is connected, α must have a slim neighbour in H', and this implies $\alpha \in V_f(H')$. Then it suffices to show that every slim vertex x of H' can be connected to α by a path in H'. Since H is connected, then there exists a path $x = v_0, v_1, \dots, v_r = \alpha$ in H. If $v_i \notin V_s(H^0)$ for all i, then this path is entirely contained in H'. Otherwise, put

$$i = \min\{i \mid v_i \in V_s(H^0)\} - 1.$$

Then $v_0, v_1, \dots, v_i \in V(H')$, $v_i \in \{\alpha\} \cup V_s(H')$, $v_{i+1} \in V_s(H^0)$. If $v_i = \alpha$, then $x = v_0, v_1, \dots, v_i = \alpha$ is a path in H'. If $v_i \in V_s(H')$, then v_i and v_{i+1} have a common fat neighbour which must be α. Thus $x = v_0, v_1, \dots, v_i, \alpha$ is a path in H'. □

3 A new generalization of line graphs

In this section, we let $\mathscr{H}$ be a family of isomorphism classes of graphs, except examples where $\mathscr{H}$ is explicitly defined.

Lemma 16. *Suppose $H = \biguplus_{i=0}^n H^i$, and assume that there exists an $\mathscr{H}$-cover graph K^i of H^i such that*

$$|N_{K^i}(x) \cap V_f(H^i)| \le 1 \quad \forall x \in V_s(K^i) \setminus V_s(H^i) \tag{4}$$

for each i. Then H is an $\mathscr{H}$-line graph.

Proof. We prove the assertion by induction on $k = |\{i \mid [H^i] \notin \mathscr{H}\}|$. The assertion is trivial when $k = 0$, so assume $k > 0$. Then without loss of generality, we may assume $[H^0] \notin \mathscr{H}$. By the assumption, there exists an $\mathscr{H}$-cover graph $K^0 = \biguplus_{j=1}^m L^j$ of H^0, $[L^j] \in \mathscr{H}$, such that

$$|N_{K^0}(x) \cap V_f(H^0)| \le 1 \quad \forall x \in V_s(K^0) \setminus V_s(H^0).$$

By Lemma 14, there exists a graph $\tilde{K}$ with $\tilde{K} = \biguplus_{i=0}^n \tilde{K}^i$ containing a subgraph G isomorphic to H, $\tilde{K}^0 = K^0$, and $\tilde{K}^i \cong H^i$ for $i = 1, 2, \dots, n$. Hence

$$\tilde{K} = (\biguplus_{j=1}^m L^j) \uplus (\biguplus_{i=1}^n \tilde{K}^i).$$

Now, by inductive hypothesis, we conclude that $\tilde{K}$ is an $\mathscr{H}$-line graph, and so is its subgraph G. Since $H \cong G$, H is an $\mathscr{H}$-line graph. □

Lemma 17. *Suppose that $\mathscr{H}$ satisfies the following condition:*

$$[H] \in \mathscr{H},\ H \not\cong H_2 \implies |N_H^f(x)| \le 1 \quad \forall x \in V_s(H). \tag{5}$$

If G is an $\mathscr{H}$-line graph, then there exists an $\mathscr{H}$-cover graph K of G satisfying the condition

$$|N_K(x) \cap V_f(G)| \le 1 \quad \forall x \in V_s(K) \setminus V_s(G). \tag{6}$$

Proof. We may assume without loss of generality that G is connected. Since G is an $\mathscr{H}$-line graph, there exists an $\mathscr{H}$-cover graph L of G, with $L = \biguplus_{j=1}^m L^j$, $[L^j] \in \mathscr{H}$. Put

$$J_0 = \{j \mid L^j \cong H_2 \text{ and } V_s(L^j) \not\subset V_s(G)\},$$
$$J = \{1, 2, \ldots, m\} \setminus J_0,$$

and let K be the subgraph of L induced on the set $\bigcup_{j \in J} V(L^j)$. Then we have

$$K = \biguplus_{j \in J} L^j. \tag{7}$$

We claim $V(G) \subset V(K)$. Indeed, if $x \in V_s(G)$, then $x \in V_s(L^j)$ for some j. If $L^j \cong H_2$, then $V_s(L^j) = \{x\} \subset V_s(G)$. This implies $j \in J$, and hence $x \in V_s(K)$. If $\alpha \in V_f(G)$, then, as G is connected and $V(G) \ne \{\alpha\}$, α has a slim neighbour $x \in V_s(G)$. Then $x \in V_s(L^j)$ for some $j \in J$ by the previous case. This forces $\alpha \in V_f(L^j)$ by the condition (iii) of Definition 3. Hence $\alpha \in V_f(K)$. Therefore, we have shown $V(G) \subset V(K)$, and K is an $\mathscr{H}$-cover graph of G.

It remains to prove the assertion (6). Suppose $x \in V_s(K) \setminus V_s(G)$. Then $x \in V_s(L^j)$ for some $j \in J$ and, by (7), we have

$$|N_K(x) \cap V_f(G)| = |N_{L^j}(x) \cap V_f(G)| \le |N_{L^j}^f(x)|.$$

Since $j \in J$ and $x \in V_s(L^j) \setminus V_s(G)$, we must have $L^j \not\cong H_2$. Since $[L^j] \in \mathscr{H}$, the assumption (5) implies $|N_{L^j}^f(x)| \le 1$. Therefore $|N_K(x) \cap V_f(G)| \le 1$. □

Lemma 18. *Suppose that $\mathscr{H}$ satisfies the condition (5). If $\mathscr{H}'$ is a family of isomorphism classes of $\mathscr{H}$-line graphs, then every $\mathscr{H}'$-line graph is an $\mathscr{H}$-line graph.*

Proof. Suppose $H = \biguplus_{i=0}^n H^i$, $[H^i] \in \mathscr{H}'$. We aim to show that H is an $\mathscr{H}$-line graph. Since H^i is an $\mathscr{H}$-line graph for each i, Lemma 17 implies that there exists an $\mathscr{H}$-cover graph K^i of H^i satisfying the condition (4). By Lemma 16, we conclude that H is an $\mathscr{H}$-line graph. □

Example 19. Let $\mathscr{H} = \{[H_2], [H_3], [H_5]\}$. Let $\mathscr{H}' = \{[H_2], [H_5]\}$. Then $\mathscr{H}$ satisfies the condition (5). Since H_3 is a subgraph of H_5, H_3 is an $\mathscr{H}$-line graph. By Lemma 18, every $\mathscr{H}$-line graph is an $\mathscr{H}'$-line graph. In particular, a slim generalized line graph is an $\mathscr{H}$-line graph.

Lemma 20. *Suppose $H = \biguplus_{i=0}^n H^i$, and assume that there exists a strict $\mathscr{H}$-cover graph K^i of H^i for each i. Then H has a strict $\mathscr{H}$-cover graph.*

Proof. We prove the assertion by induction on $k = |\{i \mid [H^i] \notin \mathscr{H}\}|$. The assertion is trivial when $k = 0$, so assume $k > 0$. Then without loss of generality, we may assume $[H^0] \notin \mathscr{H}$. By the assumption, there exists a strict $\mathscr{H}$-cover graph $K^0 = \biguplus_{j=1}^m L^j$ of H^0, $[L^j] \in \mathscr{H}$. By Lemma 14, there exists a graph $\tilde{K}$ with $\tilde{K} = \biguplus_{i=0}^n \tilde{K}^i$ containing a subgraph G isomorphic to H, $\tilde{K}^0 = K^0$, and $\tilde{K}^i \cong H^i$ for $i = 1, \dots, n$. Then we obtain

$$\tilde{K} = (\biguplus_{j=1}^{m} L^j) \uplus (\biguplus_{i=1}^{n} \tilde{K}^i).$$

Now, by inductive hypothesis, we conclude that $\tilde{K}$ has a strict $\mathscr{H}$-cover graph. Since $V_s(\tilde{K}) = V_s(G)$ and $H \cong G$, H has a strict $\mathscr{H}$-cover graph. □

Lemma 21. *Assume that, for any $[G] \in \mathscr{H}$, and for any Hoffman subgraph K of G, there exists a strict $\mathscr{H}$-cover graph of K. Then every $\mathscr{H}$-line graph has a strict $\mathscr{H}$-cover graph.*

Proof. Let Γ be an $\mathscr{H}$-line graph. Let $H = \biguplus_{i=0}^n H^i$ be an $\mathscr{H}$-cover graph of Γ. Let $\Delta = \langle\langle V_s(\Gamma)\rangle\rangle_H$. then by Lemma 12,

$$\Delta = \biguplus_{i=0}^{n} \langle\langle V_s(\Gamma) \cap V_s(H^i)\rangle\rangle_H.$$

Since $\langle\langle V_s(\Gamma) \cap V_s(H^i)\rangle\rangle_H$ is a Hoffman subgraph of H^i and $[H^i] \in \mathscr{H}$, there exists a strict $\mathscr{H}$-cover graph K^i of $\langle\langle V_s(\Gamma) \cap V_s(H^i)\rangle\rangle_H$. By Lemma 20, Δ has a strict $\mathscr{H}$-cover graph, which is a strict $\mathscr{H}$-cover graph of Γ, since $\Gamma \subset \Delta$ and $V_s(\Gamma) = V_s(\Delta)$. □

Example 22. Let $\mathscr{H} = \{[H_2], [H_3], [H_5]\}$. Then $\mathscr{H}$ satisfies the condition (5). Moreover, for any $[G] \in \mathscr{H}$, and for any Hoffman subgraph K of G, there exists a strict $\mathscr{H}$-cover graph of K. Indeed, the only nontrivial case is the graph obtained from H_5 by deleting the unique slim vertex without slim neighbours. This graph has a strict $\mathscr{H}$-cover graph which is the join $H_2 \uplus H_2$ of two copies of H_2 sharing a fat vertex. Therefore, by Lemma 21, every $\mathscr{H}$-line graph has a strict $\mathscr{H}$-cover graph. In particular, every slim $\mathscr{H}$-line graph has a strict $\mathscr{H}$-cover graph.

Lemma 23. *Let $\mathscr{H}$ be a family of isomorphism classes of Hoffman graphs, satisfying the condition (5). Let H be an $\mathscr{H}$-line graph. Then,*

(i) *if $u \in V_s(H)$, then $|N_H^f(u)| \le 2$,*

(ii) *if u, v are distinct slim vertices of H, then $|N_H^f(u) \cap N_H^f(v)| \le 1$.*

Proof. Since H is an $\mathscr{H}$-line graph, there exists an $\mathscr{H}$-cover graph $K = \biguplus_{i=1}^n K^i$ of H, $[K^i] \in \mathscr{H}$.

(i) Suppose $|N_H^f(u)| \ge 3$. If $u \in V_s(K^i)$, then $N_H^f(u) \subset V_f(K^i)$ from Definition 3(iii). Since $|N_H^f(u)| \ge 3$, $K^i \not\cong H_2$. Hence, by (5), $|N_H^f(u)| \le 1$. But this contradicts the hypothesis.

(ii) Now $N_H^f(u) \cap N_H^f(v) \subset N_K^f(u) \cap N_K^f(v)$ holds. If $u, v \in V_s(K^i)$ for some i, then by Definition 3(iii), we have $N_K^f(u) \cap N_K^f(v) \subset N_{K^i}^f(u) \cap N_{K^i}^f(v)$. Thus the result follows from the condition (5). If $u \in V_s(K^i)$ and $v \in V_s(K^j)$ for some distinct i, j, then the $|N_K^f(u) \cap N_K^f(v)| \le 1$ by Definition 3(iv). □

Lemma 24. *Let $\mathscr{H}$ be a family of isomorphism classes of Hoffman graphs, satisfying the condition (5). Let K and L be Hoffman graphs, satisfying the following conditions:*

(i) $K = K^0 \uplus K'$, $L = L^0 \uplus L'$,

(ii) $\langle V_s(K)\rangle_K$ *and* $\langle V_s(K')\rangle_{K'}$ *are connected,*

(iii) $[K^0] = [L^0] \in \mathscr{H}$,

(iv) $K' = \biguplus_{i=1}^{m_1} K^i$, $[K^i] \in \mathscr{H}$,

(v) $L' = \biguplus_{i=1}^{m_2} L^i$, $[L^i] \in \mathscr{H}$,

(vi) $\varphi : \langle V_s(K)\rangle_K \to \langle V_s(L)\rangle_L$*: isomorphism,*

(vii) $\varphi' : K' \to L'$*: isomorphism,*

(viii) $\varphi|_{V_s(K')} = \varphi'|_{V_s(K')}$,

(ix) $|V_f(K^0)| = 1$ *if* $K^0 \not\cong H_2$,

(x) $|V_s(K)| \geq 6$.

Then there exists an isomorphism $\psi : K \to L$ which is an extension of φ.

Proof. By (vii) and (viii)

$$\varphi(V_s(K')) = V_s(L'). \tag{8}$$

Thus

$$\varphi(V_s(K^0)) = \varphi(V_s(K) \setminus V_s(K')) = V_s(L) \setminus V_s(L') = V_s(L^0). \tag{9}$$

By (ii), K is connected. Thus $V_f(K^0) \cap V_f(K') \neq \emptyset$. Also, (ii) and (vi) imply that $\langle V_s(L)\rangle_L$ is connected. Thus L is connected. Hence $V_f(L^0) \cap V_f(L') \neq \emptyset$. Observe that the conditions are symmetric with respect to K and L.

Claim:

If $u \in V_s(K')$ and $v \in V_s(K^0)$ are adjacent, then $N_L^f(\varphi(u)) \cap N_L^f(\varphi(v)) \neq \emptyset$. If $u \in V_s(L')$ and $v \in V_s(L^0)$ are adjacent, then $N_K^f(\varphi^{-1}(u)) \cap N_K^f(\varphi^{-1}(v)) \neq \emptyset$.

Proof of Claim. Suppose $u \in V_s(K')$ and $v \in V_s(K^0)$ are adjacent. Then

$$\varphi(u) \in \varphi(V_s(K')) = V_s(L') \quad \text{and} \quad \varphi(v) \in V_s(L^0),$$

by (8) and (9), respectively. Hence the claim follows from Definition 3(iv). Switching K, L and replacing φ by φ^{-1} etc., we obtain the second statement. □

Case 1:

$|V_f(K^0) \cap V_f(K')| \geq 2$ *or* $|V_f(L^0) \cap V_f(L')| \geq 2$.

Without loss of generality, we may suppose $|V_f(K^0) \cap V_f(K')| \geq 2$. By (iii) and (ix), $K^0 \cong L^0 \cong H_2$. Let $V_s(K^0) = \{x\}$, $V_f(K^0) = \{f_1, f_2\} \subset V_f(K')$. By (9), $V_s(L^0) = \{\varphi(x)\}$. Let $V_f(L^0) = \{g_1, g_2\}$.

If $\{\varphi'(f_1), \varphi'(f_2)\} = \{g_1, g_2\}$, then the common extension of φ and φ' is a desired isomorphism ψ.

Next suppose $|\{\varphi'(f_1), \varphi'(f_2)\} \cap \{g_1, g_2\}| = 1$. Then we may assume $\varphi'(f_1) \notin \{g_1, g_2\}$ and $\varphi'(f_2) = g_2$. Let $u, u' \in N^s_{K'}(f_1) \subset V_s(K')$. By Lemma 23(i), $|N^f_{K'}(u)| \leq 2$ and $|N^f_{K'}(u')| \leq 2$. Hence $|N^f_{L'}(\varphi(u))| \leq 2$ and $|N^f_{L'}(\varphi(u'))| \leq 2$. Since u and u' are adjacent to x in L', $\left(N^f_{L'}(\varphi(u)) \cup N^f_{L'}(\varphi(u'))\right) \setminus \{\varphi'(f_1)\} \subset \{g_1, g_2\}$. By Lemma 23(ii), $N^f_{L'}(\varphi(u)) \cup N^f_{L'}(\varphi(u')) = \{\varphi'(f_1), g_1, g_2\}$. This implies $f_2 \in N^f_{K'}(u) \cup N^f_{K'}(u')$. Hence $|N^f_{K'}(u) \cup N^f_{K'}(u')| \geq |\{f_1, f_2\}| = 2$. But this contadicts Lemma 23(ii). Thus $|N^s_{K'}(f_1)| = 1$. Let $\{u_1\} = N^s_{K'}(f_1)$. Since $\mathscr{H}$ is assumed to satisfy (5), $K^1 = \langle\{u_1, f_1, \varphi'^{-1}(g_1)\}\rangle_{K'} \cong H_2$ constitutes one of the family of subgraphs covering K'. In particular, u_1 is an isolated vertex in $\langle V_s(K')\rangle_{K'}$. By (ii), we conclude $V_s(K') = \{u_1\}$, and therefore $K' = K^1$. But then $f_2 \in V_f(K')$ is an isolated fat vertex. This is a contradiction. Hence $\{\varphi'(f_1), \varphi'(f_2)\} \cap \{g_1, g_2\} = \emptyset$.

Let $u \in N^s_K(x)$. Then $|N^f_K(u) \cap \{f_1, f_2\}| = 1$. Since $\varphi(u) \in N^s_L(\varphi(x))$, we also have $|N^f_L(\varphi(u)) \cap \{g_1, g_2\}| = 1$. This implies that there is a mapping

$$\sigma : N^s_K(x) \to \{f_1, f_2\} \times \{g_1, g_2\}$$

defined by $\sigma(u) = (\sigma_1(u), \sigma_2(u))$, $\{\sigma_1(u)\} = N^f_K(u) \cap N^f_K(x)$, $\{\sigma_2(u)\} = N^f_L(\varphi(u)) \cap N^f_L(\varphi(x))$. By Lemma 23, $\{\varphi'(\sigma_1(u)), \sigma_2(u)\} = N^f_L(\varphi(u))$. Hence σ is injective. Thus $|N^s_K(x)| \leq 4$. Moreover, every vertex $u \in N^s_K(x)$ has two fat neighbours, so $\langle\{\varphi(u), \varphi'(\sigma_1(u)), \sigma_2(u)\}\rangle_{L'} \cong H_2$ constitutes one of the family of subgraphs covering L'. Thus, if $u' \sim u$ for some $u \in N^s_K(x)$, then $\varphi(u') \in N^s_L(\varphi'(f_1)) \cup N^s_L(\varphi'(f_2)) \cup N^s_L(g_1) \cup N^s_L(g_2)$. Then $\varphi(u') \in N^s_L(\varphi(x))$, hence $u' \in N^s_K(x)$. Therefore, $V_s(K) = \{x\} \cup N^s_K(x)$ has at most 5 elements. But this contradicts the condition (x).

Case 2:

$|V_f(K^0) \cap V_f(K')| = 1$ *and* $|V_f(L^0) \cap V_f(L')| = 1$.

First suppose $K^0 \not\cong H_2$. By (iii) and (ix), $|V_f(K^0)| = |V_f(L^0)| = 1$, so let $V_f(K^0) = \{f\}$, $V_f(L^0) = \{g\}$. If $\varphi'(f) = g$, then the common extension of φ and φ' is a desired isomorphism ψ. Hence we may suppose $\varphi'(f) \neq g$, i.e., $\varphi'(f) \notin V_f(L^0)$. Suppose $|N^s_{K'}(f)| \geq 2$, and let $u, u' \in N^s_{K'}(f)$ be distinct. Let $v \in N^s_{K^0}(f)$. Since $v \sim f$ in K^0, u and u' are adjacent to v in K. Hence, for each $s \in \{u, u'\}$, $N^f_L(\varphi(s)) \cap N^f_L(\varphi(v)) \neq \emptyset$ by Claim, and therefore $g \in N^f_{L'}(\varphi(s)) \cap N^f_{L^0}(\varphi(v)) \subset N^f_{L'}(\varphi(s))$ since $\varphi(v) \in V(L^0)$, $\varphi(s) \in V(L')$ and $V_f(L^0) \cap V_f(L') = \{g\}$. Hence $g \in N^f_{L'}(\varphi(u)) \cap N^f_{L'}(\varphi(u'))$. Also, $\varphi'(f) \in N^f_{K'}(\varphi(u)) \cap N^f_{K'}(\varphi(u'))$. This contradicts Lemma 23. Hence $|N^s_{K'}(f)| = 1$, so let $N^s_{K'}(f) = \{u\}$. Similarly, switching K, L and replacing φ by φ^{-1} etc., $|N^s_{L'}(g)| = 1$, hence $N^s_{L'}(g) = \{\varphi(u)\}$. Since $\varphi'^{-1}(N^s_{L'}(g)) = N^s_{K'}(\varphi'^{-1}(g))$, $N^s_{K'}(\varphi'^{-1}(g)) = \{u\}$. Hence $K' = \langle\{u, f, \varphi'^{-1}(g)\}\rangle_{K'} \cong H_2$. Let ρ be the automorphism of K' which switches the two fat vertices f and $\varphi'^{-1}(g)$. Then $\varphi' \circ \rho(f) = g$. Replacing φ' by $\varphi' \circ \rho$, the conditions (vii) and (viii) are satisfied, so the proof reduces to the previously treated case where $\varphi'(f) = g$.

Next suppose $K^0 \cong H_2$. Then $L^0 \cong H_2$. Let $f' \in V_f(K^0) \setminus V_f(K')$, $g' \in V_f(L^0) \setminus V_f(L')$. The common extension ψ of φ and φ' defined by $\psi(f') = g'$ is a desired isomorphism. □

Remark 25. The hypothesis $|V_s(K)| \geq 6$ in Lemma 24 is necessary. Indeed, let $K = L = \biguplus_{i=0}^{4} K^i$ be the graph defined as follows:

$$K^i = L^i \cong H_2, \quad V_s(K^i) = \{u_i\} \quad (0 \leq i \leq 4),$$
$$V_f(K^0) = \{f_0, f_1\},\ V_f(K^1) = \{f_0, f_2\},\ V_f(K^2) = \{f_1, f_2\},$$
$$V_f(K^3) = \{f_1, f_3\},\ V_f(K^4) = \{f_3, f_0\}.$$

Let $K' = L' = \biguplus_{i=1}^{4} K^i$. Then $\langle V_s(K') \rangle_{K'} \cong C_4$ and $\langle V_s(K) \rangle_K \cong K_1 * C_4$. Let φ be the automorphism of $\langle V_s(K) \rangle_K$ which switches u_2 and u_4, leaving all the other vertices fixed. Let φ' be the automorphism of K' which switches (u_2, u_4), (f_0, f_2) and (f_1, f_3), leaving all the other vertices fixed. Then all but the last hypotheses of Lemma 24 holds. If there exists an automorphism ψ of K which is an extension of φ, then

$$\psi(f_0) \in \psi(N_K^f(u_0) \cap N_K^f(u_1) \cap N_K^f(u_4)) = N_K^f(u_0) \cap N_K^f(u_1) \cap N_K^f(u_2) = \emptyset.$$

This is a contradiction.

4 The uniqueness of $\mathscr{H}$-cover graphs

In this section, we let $\mathscr{H} = \{[H_2], [H_3], [H_5]\}$. By Example 22, every slim $\mathscr{H}$-line graph has a strict $\mathscr{H}$-cover graph, and we shall show that a strict $\mathscr{H}$-cover graph of a slim $\mathscr{H}$-line graph is unique up to equivalence, if the number of slim vertices is at least 8.

The following lemma is well known in the graph theory:

Lemma 26. *Let Γ be a connected (slim) graph. Then there exists a (slim) vertex x in $V(\Gamma)$ such that $\Gamma - x$ is connected.*

Proof. See Problem 6(a) in section 6 of [7]. □

Lemma 27. *If $H = \biguplus_{i=0}^{n} H^i$ is a connected graph with $[H^i] \in \mathscr{H}$ and $n > 0$, then $\Gamma = \langle V_s(H) \rangle_H$ is connected.*

Proof. We prove the assertion by induction on n. The assertion is easy to verify when $n = 1$. Suppose $n > 1$, and let $H' = \langle \bigcup_{i=1}^{n} V(H^i) \rangle_H$. Since H is connected, there exists $\alpha \in V_f(H^0) \cap V_f(H')$. Since $[H^0] \in \mathscr{H}$, every slim vertex of H^0 is adjacent to α, and hence every slim vertex of H^0 has a slim neighbour in H'. Since $H' = \biguplus_{i=1}^{n} H^i$ is connected by inductive hypothesis, we see that Γ is connected. □

Example 28. The cocktail party graph $CP(3)$ which is the unique 4-regular graph on 6 (slim) vertices, has two non-isomorphic strict $\mathscr{H}$-cover graphs (See Figure 2).

Example 29. All strict $\mathscr{H}$-cover graphs of $K_1 * 2(K_1 + K_2)$ are isomorphic, but not unique up to equivalence (See Figure 3).

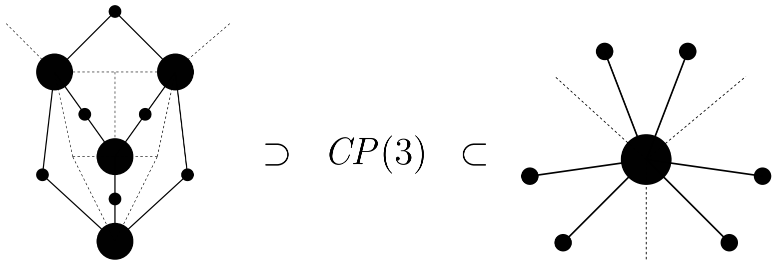

Figure 2:

We verified by computer that the graph $K_1 * 2(K_1 + K_2)$ is the only slim graph with 7 vertices whose strict $\mathscr{H}$-cover graph is not unique up to equivalence. We also verified that all slim graphs with 8 vertices have a unique strict $\mathscr{H}$-cover graph up to equivalence.

Example 30. Let x be a slim vertex of a graph X with $[X] \in \mathscr{H}$. Then there exists a graph $\tilde{X}$ such that $X - x \subset^* \tilde{X}$. Table 1 shows $\tilde{X} = \tilde{X}_1 \uplus \tilde{X}_2$, for some graphs $\tilde{X}_1$ and $\tilde{X}_2$ satisfying $[\tilde{X}_1], [\tilde{X}_2] \in \mathscr{H} \cup \{[\phi]\}$. In Table 1, the fat vertices of $\tilde{X}$ not in X are coloured by white.

We shall now prove our main result.

Theorem 31. *Let $\mathscr{H} = \{H_2, H_3, H_5\}$. Let Γ be a connected slim $\mathscr{H}$-line graph with at least 8 vertices. Then there exists a strict $\mathscr{H}$-cover graph of Γ, and it is unique up to equivalence.*

Proof. From Example 22, every slim $\mathscr{H}$-line graph has a strict $\mathscr{H}$-cover graph. We prove the uniqueness by induction on $|V(\Gamma)|$. Let K and L be strict $\mathscr{H}$-cover graphs of Γ. As we have mentioned above, the uniqueness of a strict $\mathscr{H}$-cover graph of a connected slim $\mathscr{H}$-line graph has been verified by computer. Suppose $|V(\Gamma)| \geq 9$.

Now, by Lemma 26, there exists a slim vertex x in $V(\Gamma)$ such that $\Gamma - x$ is connected. Let $K = \biguplus_{i=0}^{m_1} K^i$, $[K^i] \in \mathscr{H}$, and $L = \biguplus_{i=0}^{m_2} L^i$, $[L^i] \in \mathscr{H}$, be strict $\mathscr{H}$-cover graphs of Γ. Without loss of generality, we may assume $x \in V_s(K^0) \cap V_s(L^0)$. We put $K' = \biguplus_{i=1}^{m_1} K^i$ and $L' = \biguplus_{i=1}^{m_2} L^i$. From Lemma 15, K' is connected. Hence, from Lemma 27, $\langle V_s(K') \rangle_{K'}$ is connected. Therefore, we have shown that the hypothesis (ii) of Lemma 24 holds. Let

$$\varphi : \langle V_s(K) \rangle_K \to \langle V_s(L) \rangle_L$$

be the identity automorphism of $\Gamma = \langle V_s(K) \rangle_K = \langle V_s(L) \rangle_L$. Since $9 \leq |V(\Gamma)| \leq 3(m_j + 1)$, we have $m_j \geq 2$ $(j = 1, 2)$. We construct a graph $\tilde{K}^0$ from K^0 according to Table 1. Then either $\tilde{K}^0 = K^0 - x$ or $\tilde{K}^0$ is obtained from $K^0 - x$ by the operation described in Lemma 13(iii). It then follows from Lemma 12 or Lemma 13 that we obtain a graph $\tilde{K} = \tilde{K}^0 \uplus K'$ which is a strict $\mathscr{H}$-cover graph of $K - x$. Similarly, there exists a strict $\mathscr{H}$-cover graph $\tilde{L} = \tilde{L}^0 \uplus L'$ of $L - x$. Then by induction, the strict $\mathscr{H}$-cover graphs $\tilde{K}$ and $\tilde{L}$ of $\Gamma - x$ are equivalent. Hence, there exists an isomorphism $\tilde{\varphi}$: $\tilde{K} \to \tilde{L}$ such that $\tilde{\varphi}|_{\Gamma - x} = \mathbf{1}_{\Gamma - x}$.

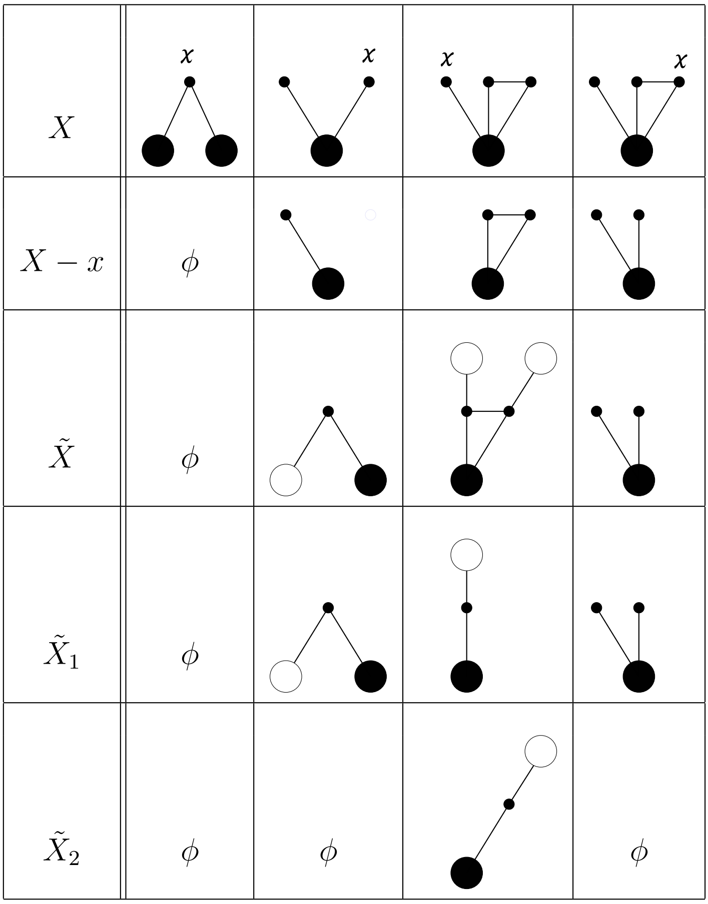

Table 1:

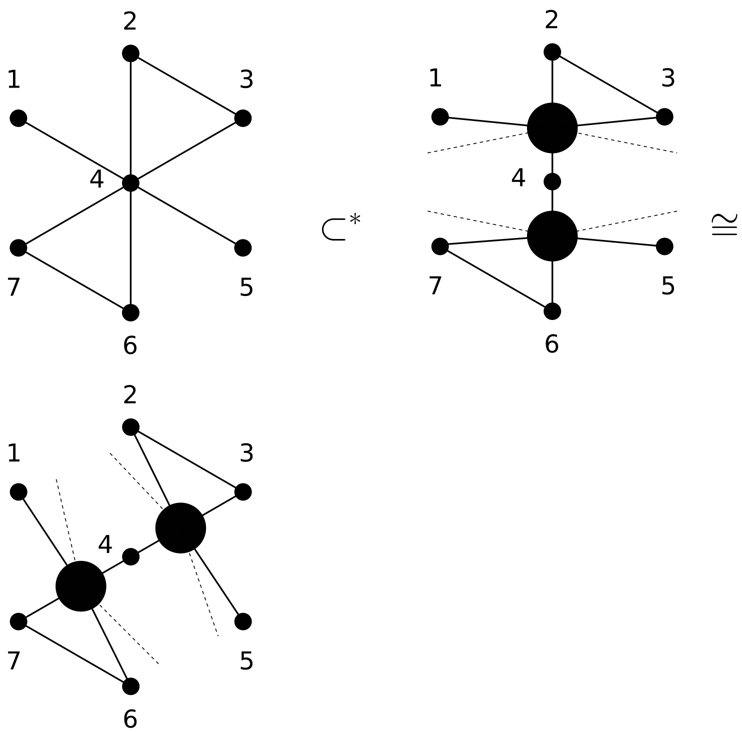

Figure 3:

Claim 1:

If $s \in V_s(K)\setminus\{x\}$*,* $f \in V_f(K)$ *are adjacent, then either* $s\tilde{\varphi}(f) \in E(L^0)$ *or* $s\tilde{\varphi}(f) \in E(L')$*. If, in addition,* $s \not\sim x$*, then* $s\tilde{\varphi}(f) \in E(L')$ *implies* $\tilde{\varphi}(f) \notin V(L^0)$*.*

Proof of Claim 1. Since $\tilde{\varphi}$ is an isomorphism which is the identity on $\Gamma - x$, $s = \tilde{\varphi}(s) \sim \tilde{\varphi}(f)$. Thus we have either $s\tilde{\varphi}(f) \in E(L^0)$ or $s\tilde{\varphi}(f) \in E(L')$ by Definition 3(iii). Suppose $s \not\sim x$ and $s\tilde{\varphi}(f) \in E(L')$. Then $N_L^f(s) \cap N_L^f(x) = \emptyset$ by Definition 3(iv). In particular, $\tilde{\varphi}(f) \not\sim x$. Since $[L^0] \in \mathscr{H}$, this implies $\tilde{\varphi}(f) \notin V(L^0)$. □

Claim 2:

If $K^0 \cong H_3$ *or* H_5*, then* $V_s(K^0) \subset V_s(L^0)$*.*

Proof of Claim 2. Let $V_f(K^0) = \{f\}$, and let $y \in V_s(K^0) \setminus \{x\}$ be such that $x \not\sim y$. Then $y \sim f$ since $[K^0] \in \mathscr{H}$. By Claim 1, either $y\tilde{\varphi}(f) \in E(L^0)$ or $y\tilde{\varphi}(f) \in E(L')$ and $\tilde{\varphi}(f) \notin V(L^0)$.

Suppose $y\tilde{\varphi}(f) \in E(L^0)$. If $K_0 \cong H_3$, then $V_s(K^0) = \{x, y\} \subset V_s(L^0)$. If $K_0 \cong H_5$, then let $V_s(K^0) = \{x, y, z\}$. If $z \in V_s(L')$, then we have $y \sim z$ since $\tilde{\varphi}(f)$ is a common fat neighbour of $y \in V_s(L^0)$ and $z = \tilde{\varphi}(z) \in V_s(L')$, and also $x \sim z$ by the same reason.

Since $K_0 \cong H_5$ has only one edge among its slim vertices, this is a contradiction. Therefore, $z \in V_s(L^0)$, and $V_s(K^0) = \{x, y, z\} \subset V_s(L^0)$.

Next suppose $y\tilde{\varphi}(f) \in E(L')$ and $\tilde{\varphi}(f) \notin V(L^0)$. We put $K^{(1)} = \langle\langle N^s_{K'}(f)\rangle\rangle_{K'}$. Since K is connected, $V_s(K^{(1)}) \neq \emptyset$. Let $u \in V_s(K^{(1)})$. Then $u \sim x$ in K, i.e., in L. By Claim 1, either $u\tilde{\varphi}(f) \in E(\tilde{L}^0)$ or $u\tilde{\varphi}(f) \in E(L')$. Since $\tilde{\varphi}(f) \notin V_f(L^0)$, $\tilde{\varphi}(f) \in V_f(L')$, and therefore $u \in V_s(L')$. By Lemma 23(i), $|N^f_{K^{(1)}}(u)| \leq 2$, i.e., $|N^f_{K^{(1)}}(u) \setminus \{f\}| \leq 1$. Since $u \sim x$ in L, there exists $g \in N^f_{L'}(u) \cap N^f_{L^0}(x)$. Then obviously $u\tilde{\varphi}^{-1}(g) \in E(K')$. Hence $N^f_{K^{(1)}}(u) \setminus \{f\} = \{\tilde{\varphi}^{-1}(g)\}$. Thus $\tilde{\varphi}(N^f_{K^{(1)}}(u) \setminus \{f\}) = \{g\} \subset V_f(L^0)$, and

$$\tilde{\varphi}(V_f(K^{(1)}) \setminus \{f\}) \subset V_f(L^0). \tag{10}$$

Let $u' \in V_s(K^{(1)}) \setminus \{u\}$. Then, similarly, there exists a unique fat vertex $g' \in N^f_{L^0}(x) \cap N^f_{L'}(u')$. Then $\tilde{\varphi}^{-1}(g') \in N^f_{K'}(u')$. Since u and u' have a unique common fat neighbour f from Lemma 23(ii), $\tilde{\varphi}^{-1}(g) \neq \tilde{\varphi}^{-1}(g')$, i.e., $g \neq g'$. This implies

$$|V_s(K^{(1)})| = |N^f_{L^0}(x) \cap \tilde{\varphi}(V_f(K^{(1)}))|. \tag{11}$$

Hence, since $|V_f(L^0)| \leq 2$, $|V_s(K^{(1)})| \leq 2$. Thus

$$|V_s(K)| - |V_s(K^0)| - |V_s(K^{(1)})| \geq 9 - 3 - 2 > 0. \tag{12}$$

We put

$$K^{(2)} = \langle\langle(\bigcup_{\substack{\alpha \in V_f(K^{(1)}) \\ \alpha \neq f}} N^s_{K'}(\alpha)) \setminus V_s(K^{(1)})\rangle\rangle_{K'}.$$

By (12), $V_s(K^{(2)}) \neq \emptyset$. Let $v \in V_s(K^{(2)})$. From Lemma 23(ii), $|N^f_{K'}(u) \cap N^f_{K'}(v)| \leq 1$ for any $u \in V_s(K^{(1)})$. Hence $v \not\sim f$ in K, i.e., $v \not\sim x$ in Γ. Moreover $\tilde{\varphi}(N^f_{K'}(v) \cap V_f(K^{(1)})) \subset \tilde{\varphi}(V_f(K^{(1)}) \setminus \{f\}) \subset V_f(L^0)$ by (10). Hence, by Claim 1, $v \in V_s(L^0)$, i.e.,

$$V_s(K^{(2)}) \subset V_s(\tilde{L}^0). \tag{13}$$

This implies $L^0 \cong H_3$ or H_5. Hence, by (13),

$$|V_s(K^{(2)})| \leq |V_s(L^0) \setminus \{x\}| \leq 2. \tag{14}$$

Moreover, by (13) and Definition 3(iii), $\tilde{\varphi}(E(K^{(2)})) \subset E(\tilde{L}^0)$, i.e.,

$$\tilde{\varphi}(K^{(2)}) \subset \tilde{L}^0. \tag{15}$$

Let $V_f(L^0) = \{g\}$. By (10), $\tilde{\varphi}(V_f(K^{(1)}) \setminus \{f\}) = \{g\}$. Hence $V_f(K^{(1)}) = \{f, \tilde{\varphi}^{-1}(g)\}$. Moreover, by (11),

$$|V_s(K^{(1)})| = |\{g\} \cap \tilde{\varphi}(\{f, \tilde{\varphi}^{-1}(g)\})| = |\{g\}| = 1. \tag{16}$$

Moreover,

$$V_s(\tilde{L}^0) = N^s_{\tilde{L}^0}(g) = \tilde{\varphi}^{-1}(N^s_{\tilde{L}^0}(g)) \subset N^s_{\tilde{K}}(\tilde{\varphi}^{-1}(g)).$$

Since $u \notin V_s(L^0)$, $V_s(\tilde{L}^0) \subset N^s_{\tilde{K}}(\tilde{\varphi}^{-1}(g)) \setminus \{u\}$. Hence, since $N^s_{\tilde{K}}(\tilde{\varphi}^{-1}(g)) = V_s(K^{(1)}) \cup V_s(K^{(2)}) = \{u\} \cup V_s(K^{(2)})$, $V_s(\tilde{L}^0) \subset V_s(K^{(2)})$. Thus, by (15),

$$\bigcup_{\substack{\alpha\in V_f(K^{(2)})\\ \alpha\neq\tilde{\varphi}^{-1}(g)}} N^s_{\tilde{K}}(\alpha) = \bigcup_{\substack{\beta\in\tilde{\varphi}(V_f(K^{(2)}))\\ \beta\neq g}} N^s_{\tilde{\varphi}(\tilde{K})}(\beta) \subset \bigcup_{\substack{\beta\in V_f(\tilde{L}^0)\\ \beta\neq g}} N^s_{\tilde{L}}(\beta) = V_s(\tilde{L}^0) \subset V_s(K^{(2)}).$$

Hence, since K' is connected, $K' = K^{(1)} \uplus K^{(2)}$. Thus $|V_s(\tilde{K})| = |V_s(\tilde{K}^0)| + |V_s(K^{(1)})| + |V_s(K^{(2)})|$. Hence, by (15) and (16), $|V(\Gamma)| \leq 6$ since $|V_s(\tilde{K}^0)| \leq 2$. But this contradicts the hypothesis. □

Claim 3:

$V_s(K^0) = V_s(L^0)$.

Proof of Claim 3. Suppose $K^0 \not\cong H_2$. By Claim 2, $V_s(K^0) \subset V_s(L^0)$. Since $|V_s(K^0)| \geq 2 > |V_s(H_2)|$, we have $L^0 \not\cong H_2$, and hence the same argument yields $V_s(L^0) \subset V_s(K^0)$. Therefore, $V_s(K^0) = V_s(L^0)$. Similarly, $L^0 \not\cong H_2$ implies $V_s(K^0) = V_s(L^0)$. If $K^0 \cong L^0 \cong H_2$, then $V_s(K^0) = \{x\} = V_s(L^0)$. □

Obviously the hypotheses (i), (iv), (v), (vi), (ix) and (x) of Lemma 24 hold. The hypothesis (ii) holds as mentioned above. Moreover, since $[K^0], [L^0] \in \mathscr{H}$, by Claim 3, it follows that $K^0 \cong L^0$, i.e., the hypothesis (iii) of Lemma 24 holds.

Since $V_s(\tilde{K}^0) \subset V(\Gamma - x)$, we have $\tilde{\varphi}(V_s(\tilde{K}^0)) = \mathbf{1}_{\Gamma-x}(V_s(\tilde{K}^0)) = V_s(\tilde{K}^0) = V_s(K^0)\setminus\{x\} = V_s(L^0)\setminus\{x\} = V_s(\tilde{L}^0)$. Now $\tilde{\varphi}(\tilde{K}) - V_s(\tilde{L}^0) = \tilde{\varphi}(\tilde{K}^0 \uplus K') - \tilde{\varphi}(V_s(\tilde{K}^0)) = \tilde{\varphi}(\tilde{K}^0 \uplus K' - V_s(\tilde{K}^0)) = \tilde{\varphi}(K')$, and $\tilde{L} - V_s(\tilde{L}^0) = \tilde{L}^0 \uplus L' - V_s(\tilde{L}^0) = L'$. Since $\tilde{\varphi}(\tilde{K}) = \tilde{L}$, we obtain $\tilde{\varphi}(K') = L'$. Moreover, since $\tilde{\varphi}|_{\Gamma-x} = \mathbf{1}_{\Gamma-x}$ and $V_s(K') \subset V_s(\Gamma - x)$, $\tilde{\varphi}|_{V_s(K')} = \mathbf{1}_{V_s(K')} = \varphi|_{V_s(K')}$. Therefore the isomorphism $\tilde{\varphi}|_{K'} : K' \to L'$ satisfies $\varphi|_{V_s(K')} = \tilde{\varphi}|_{V_s(K')}$. By putting $\varphi' = \tilde{\varphi}|_{K'}$, we obtain the hypotheses (vii) and (viii) of Lemma 24.

Hence the hypotheses (i)–(x) of Lemma 24 hold, and therefore there exists an isomorphism ψ: $K \to L$ which is an extension of the identity mapping φ. □

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References

[1] P. J. Cameron, J. M. Goethals, J. J. Seidel, and E. E. Shult, Line graphs, root systems and elliptic geometry, *J. Algebra* **43** (1976), 305–327.

[2] D. Cvetković, M. Doob and S. Simić, Generalized line graphs, *J. Graph Theory* **5** (1981), 385–399.

[3] D. Cvetković, M. Lepović, P. Rowlinson and S. K. Simić, The maximal exceptional graphs, *J. Combin. Theory Ser. B* **86** (2002), no. 2, 347–363.

[4] D. Cvetković, P. Rowlinson, S. K. Simić, *Spectral Generalizations of Line Graphs — On Graphs with Least Eigenvalue* -2, Cambridge Univ Press, 2004.

[5] D. Cvetković, P. Rowlinson, S. K. Simić, *Constructions of the maximal exceptional graphs with largest degree 28*, Department of Computing Science and Mathematics, University of Stirling, Scotland, Technical Report CSM-156, Stirling, 2000.

[6] A. J. Hoffman, On graphs whose least eigenvalue exceeds $-1-\sqrt{2}$, *Linear Algebra Appl.* **16** (1977), 153–165.

[7] L. Lovász, *Combinatorial Problems and Exercises*, 2nd edition, North-Holland, 1993.

[8] R. Woo and A. Neumaier, On graphs whose smallest eigenvalue is at least $-1-\sqrt{2}$, *Linear Algebra Appl.* **226/228** (1995), 577–591.

[9] T. Taniguchi, On graphs with the smallest eigenvalue at least $-1-\sqrt{2}$, part II, in preparation.

[10] The Magma Computational Algebra System for Algebra, Number Theory and Geometry. URL: http://magma.maths.usyd.edu.au/magma/