Also available at http://amc.imfm.si
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 4 (2011) 25–27

Graphs with chromatic numbers strictly less than their colouring numbers*

Xuding Zhu

Department of Mathematics, Zhejiang Normal University, Zhejiang, P. R. China, and Department of Applied Mathematics, National Sun Yat-sen University, Taiwan

Received 26 April 2010, accepted 26 August 2010, published online 18 October 2010

Abstract

The colouring number of a graph G, defined as $\mathrm{col}(G) = 1 + \max_{H \subseteq G} \delta(H)$, is an upper bound for its chromatic number. In this note, we prove that it is NP-complete to determine whether an arbitrary graph G has chromatic number strictly less than its colouring number.

Keywords: Chromatic number, colouring number, Szekeres-Wilf inequality, NP-completeness.

Math. Subj. Class.: 05C15

1 Main result

An easy upper bound for the chromatic number of a graph G is that $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. This upper bound is sharp; however, Brooks' Theorem [1] shows that the bound is only attained by complete graphs and odd cycles. The colouring number $\mathrm{col}(G)$ of G is defined as $\mathrm{col}(G) = 1 + \max_{H \subseteq G} \delta(H)$, where $\delta(H)$ is the minimum degree of H. The Szekeres-Wilf inequality $\chi(G) \leq \mathrm{col}(G)$ gives a better upper bound for $\chi(G)$ [3]. This upper bound is also an easy bound, as the colouring number of G can be calculated in linear time as follows: Assume G has n vertices. Let $G_0 = G$, and for $1 \leq i \leq n-1$, let $G_i = G_{i-1} - v_i$, where v_i is a vertex of minimum degree in G_{i-1}. Then $\mathrm{col}(G) = \max \delta(G_i) + 1$. One naturally wonders if there is an analog of Brooks' Theorem that gives a simple characterization of all the graphs G for which the Szekeres-Wilf inequality holds with equality. This note shows that it is unlikely to have a simple characterization for such graphs, as it is NP-complete to decide whether $\chi(G) < \mathrm{col}(G)$ for an arbitrary graph G.

*In memory of Michael O. Albertson.

E-mail address: zhu@math.nsysu.edu.tw (Xuding Zhu)

Theorem 1.1. *The following decision problem is NP-complete:*
Instance: A graph G.
Question: Is $\chi(G) < \mathrm{col}(G)$?

Proof. As $\mathrm{col}(G)$ can be computed in linear time, it is obvious that the problem is in NP. In the following, we reduce the well-known NP-complete 3-colourability problem to the above decision problem.

Suppose we need to decide whether a given graph G is 3-colourable. If $\mathrm{col}(G) \leq 3$, then $\chi(G) \leq \mathrm{col}(G) \leq 3$ and G is 3-colourable.

Assume $\mathrm{col}(G) = k \geq 4$. We construct a new graph G' as follows: Take a copy of G. For each 4-subset X of $V(G)$, add a set U_X of $k-4$ new vertices. Add edges to connect every pair of vertices in U_X (so that U_X induces a copy of K_{k-4}), and connect each vertex of U_X to every vertex of X (the vertices in X are 'old' vertices in $V(G)$). For different 4-subsets X, X' of $V(G)$, U_X and $U_{X'}$ are disjoint. Also U_X is disjoint from $V(G)$. So if G has n vertices, then G' has $n + \binom{n}{4} \times (k-4) \leq n^5$ vertices. Note that if $k = 4$, then $G' = G$.

We shall show that G is 3-colourable if and only if $\chi(G') < \mathrm{col}(G')$. Since all the new vertices (i.e., vertices not in $V(G)$) have degree $k-1$, we know that $\mathrm{col}(G') = \mathrm{col}(G) = k$. If $k = 4$, then $G = G'$ and $\chi(G') < \mathrm{col}(G') = 4$ is equivalent to $G = G'$ is 3-colourable. Assume $k \geq 5$. If G has a 3-colouring f, then we can extend f to a $(k-1)$-colouring of G'. This is so, because if v is an added vertex, then $v \in U_X$ for some 4-subset X of $V(G)$. The vertex v has $k-1$ neighbours, and at least two of the neighbours of v in X are coloured by the same colour. So we can choose a colour for v which is not used by any of its neighbours.

Conversely, assuming G is not 3-colourable, we shall show that G' is not $(k-1)$-colourable. Assume to the contrary that f is a $(k-1)$-colouring of G'. Since G is not 3-colourable, the restriction of f to $V(G)$ uses at least 4 colours. So there is a 4-subset X of $V(G)$ such that $|f(X)| = 4$. As each vertex of X is adjacent to all the vertices in U_X, none of the 4 colours in $f(X)$ can be used by any vertex in U_X. So the number of colours that can be used on the vertices of U_X is $|U_X| - 1$. This is impossible, as U_X induces a complete graph.

So the problem of deciding whether G is 3-colourable is reduced to the problem of deciding whether $\chi(G') = \mathrm{col}(G')$. As $|V(G')| \leq |V(G)|^5$, the reduction is polynomial. So it is NP-complete to decide whether an arbitrary graph G' satisfies the strict inequality $\chi(G') < \mathrm{col}(G')$. □

Acknowledgement

The author thanks Professor Wilf for asking this question during the coffee break at the conference "CoNE Revisited: Celebrating the Inspirations of Michael O. Albertson".

References

[1] R. L. Brooks, On colouring the nodes of a network, *Proc. Cambridge Philos. Soc.* **37** (1947), 194–197.

[2] M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W. H. Freeman, New York, 1979.

[3] G. Szekeres and H. S. Wilf, An inequality for the chromatic number of a graph, *J. Comb. Theory* **4** (1968), 1–3.