

# On $\gamma$ –hyperellipticity of graphs

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## Abstract

The basic objects of research in this paper are graphs and their branched coverings. By a graph, we mean a finite connected multigraph. The genus of a graph is defined as the rank of the first homology group. A graph is said to be  $\gamma$ -hyperelliptic if it is a two fold branched covering of a genus  $\gamma$  graph. The corresponding covering involution is called  $\gamma$ -hyperelliptic.

The aim of the paper is to provide a few criteria for the involution  $\tau$  acting on a graph  $X$  of genus  $g$  to be  $\gamma$ -hyperelliptic. If  $\tau$  has at least one fixed point then the first criterium states that there is a basis in the homology group  $H_1(X)$  whose elements are either invertible or split into  $\gamma$  interchangeable pairs under the action of  $\tau_*$ . The second criterium is given by the formula  $\text{tr}_{H_1(X)}(\tau_*) = 2\gamma - g$ . Similar results are also obtained in the case when  $\tau$  acts fixed point free.

*Keywords:* Graph, hyperelliptic graph, homology group, Riemann–Hurwitz formula, Schreier formula.

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## 1 Introduction

The theory of Riemann surfaces was founded in classical works by B. Riemann and A. Hurwitz. A Riemann surface was originally defined in terms of branched coverings over the Riemann sphere. An important class of surfaces consists of hyperelliptic surfaces, which are defined as branched double coverings of the Riemann sphere.

It is well known that any surface of genus 2 is hyperelliptic. The Farkas-Accola theorem ([1], [6]) states that any unbranched two fold covering of a surface of genus 2 is also a hyperelliptic surface. In [2] Accola showed that an irregular three fold covering of a Riemann surface of genus 2 is also hyperelliptic, while its regular three fold covering is a two fold covering of the torus. Define a Riemann surface be  $\gamma$ -hyperelliptic if it is a two fold branched covering of a genus  $\gamma$  surface. The basic properties of  $\gamma$ -hyperelliptic surfaces and their automorphism groups are investigated in the papers ([1], [4], [13]).

Over the last decade, discrete versions of the theory of Riemann surfaces have been addressed in numerous studies. In these theories, finite graphs play the role of Riemann surfaces, while holomorphic mappings are replaced by harmonic ones. Harmonic mappings of graphs are also known as wrapped quasi-coverings or just branched coverings of graphs. The foundation of this theory was done in the paper [12] in terms of dual voltage assignment. Later, the theory was developed from different points of view by H. Urakawa [14], M. Baker and S. Norine [3], S. Corry [5] and others.

The discrete theory found effective applications in coding theory, stochastic theory, and financial mathematics. References concerning this subject can be found in [3].

The basic objects of research in this paper are graphs and their coverings. By a graph, we mean a finite connected multigraph  $X$ , possibly with loops.

Denote by  $H_1(X)$  the integer homology group of  $X$ . The genus  $g$  of a graph  $X$  is defined as the rank of  $H_1(X)$  (that is, as the Betti number or the cyclomatic number of a graph). Denote by  $V$  and  $E$  the number of vertices and edges of  $X$  respectively. Then

$$g = 1 - V + E. \quad (1.1)$$

A graph is said to be hyperelliptic if it is a double branched covering of a tree. Note that any two-edge connected graph of genus 2 is hyperelliptic [3].

In this paper we introduce a notion of  $\gamma$ -hyperelliptic graph. A graph is said to be  $\gamma$ -hyperelliptic if it is a two fold branched covering of a genus  $\gamma$  graph. The corresponding covering involution is called  $\gamma$ -hyperelliptic.

The aim of the paper is to provide some criteria for the involution  $\tau$  acting on a graph  $X$  of genus  $g$  to be  $\gamma$ -hyperelliptic. We suppose that  $\tau$  acts freely and without edge inversion on the set of directed edges of  $X$ .

The main results of the paper are Theorems 4.1 and 4.3. Theorem 4.1 gives two criteria for an involution acting on a graph  $X$  of genus  $g$  with fixed points to be  $\gamma$ -hyperelliptic. Theorem 4.3 provides necessary and sufficient conditions for a  $\gamma$ -hyperelliptic involution to act without fixed points.

## 2 Preliminary results

In this paper, by a *graph*  $X$  we mean a finite connected multigraph, possibly with loops. See, for example, paper [9] for a formal definition of the graph with multiple edges and loops. All edges of  $X$ , including loops, are provided by two possible orientations. Denote by  $\mathcal{V}(X)$  the set of vertices of  $X$  and by  $\mathcal{E}(X)$  the set of directed edges of  $X$ . We introduce

two maps  $s, t : \mathcal{E}(X) \rightarrow \mathcal{V}(X)$  sending an edge  $e \in \mathcal{E}(X)$  to its *source* and *terminate* vertices  $s(e)$  and  $t(e)$  respectively. We will use also a fixed point free involution  $e \rightarrow \bar{e}$  of  $\mathcal{E}(X)$  (reversal of orientation) such that  $s(\bar{e}) = t(e)$  and  $s(e) = t(\bar{e})$ .

We put  $\text{St}(x) = \text{St}^X(x) = \{e \in \mathcal{E}(X) : s(e) = x\}$  for the *star* of  $x$  and call  $\text{deg}(x) = |\text{St}(x)|$  the *degree* (or *valency*) of  $x$ .

### 2.1 Morphisms of graphs

A morphism of graphs  $\varphi : X \rightarrow Y$  sends vertices to vertices, edges to edges, and, for any  $e \in \mathcal{E}(X)$ ,  $\varphi(s(e)) = s(\varphi(e))$ ,  $\varphi(t(e)) = t(\varphi(e))$ , and  $\varphi(\bar{e}) = \overline{\varphi(e)}$ . For  $x \in X$  we then have the local map

$$\varphi_x : \text{St}^X(x) \rightarrow \text{St}^Y(\varphi(x)).$$

We call  $\varphi$  *locally surjective or bijective*, respectively, if  $\varphi_x$  has this property for all  $x \in X$ . We call  $\varphi$  a *covering* if  $\varphi$  is surjective and locally bijective. A bijective morphism is called an *isomorphism*, and an isomorphism  $\varphi : X \rightarrow X$  is called an *automorphism*.

### 2.2 Harmonic morphisms

Let  $X, X'$  be graphs. Let  $\varphi : X \rightarrow X'$  be a morphism of graphs. We now come to one of the key definitions in this paper.

A morphism  $\varphi : X \rightarrow X'$  is said to be *harmonic* (or *branched covering*) if, for all  $x \in \mathcal{V}(X)$ ,  $y \in \mathcal{V}(X')$  such that  $y = \varphi(x)$ , the quantity  $|\{e \in \mathcal{E}(X) : x = s(e), \varphi(e) = e'\}|$  is the same for all edges  $e' \in \mathcal{E}(X')$  such that  $y = s(e')$ .

We note that an arbitrary covering of graphs is a harmonic morphism. Let  $\varphi : X \rightarrow X'$  be harmonic and let  $x \in \mathcal{V}(X)$ . Define the *multiplicity* of  $\varphi$  at  $x$  by

$$m_\varphi(x) = |\{e \in \mathcal{E}(X) : x = s(e), \varphi(e) = e'\}| \tag{2.1}$$

for any edge  $e' \in \mathcal{E}(X')$  such that  $\varphi(x) = s(e')$ . By the definition of a harmonic morphism,  $m_\varphi(x)$  is independent of the choice of  $e'$ .

If  $\text{deg}(x)$  denotes the degree of a vertex  $x$ , we have the following formula relating degrees and the multiplicity:

$$\text{deg}(x) = \text{deg}(\varphi(x))m_\varphi(x). \tag{2.2}$$

We define the degree of a harmonic morphism  $\varphi : X \rightarrow X'$  by the formula

$$\text{deg}(\varphi) = |\{e \in \mathcal{E}(X) : \varphi(e) = e'\}| \tag{2.3}$$

for any edge  $e' \in \mathcal{E}(X')$ . By the following lemma (see [3], Lemma 2.2) the right-hand side of (2.3) does not depend on the choice of  $e'$  (and therefore  $\text{deg}(\varphi)$  is well defined):

**Lemma 2.1.** *The quantity  $|\{e \in \mathcal{E}(X) : \varphi(e) = e'\}|$  is independent of the choice of  $e' \in \mathcal{E}(X')$ .*

### 2.3 The $\gamma$ -hyperelliptic graphs and involutions

A graph  $X$  is said to be  $\gamma$ -*hyperelliptic* if there is a two fold harmonic map  $\varphi : X \rightarrow X'$  on a graph  $X'$  of genus  $\gamma$ . That is, a graph is  $\gamma$ -hyperelliptic if it is a two fold branched covering of a genus  $\gamma$  graph. The corresponding covering involution  $\tau$  is called  $\gamma$ -*hyperelliptic*.

Recall that the covering involution of  $\varphi$  is an order two automorphism  $\tau$  of graph  $X$  satisfying  $\varphi \circ \tau = \varphi$ .

Since morphism  $\varphi$  is harmonic, by Lemma 2.1 each directed edge of  $X'$  has exactly two preimages in the set  $\mathcal{E}(X)$  of directed edges of  $X$ . Hence,  $\tau$  permutes these preimages and, consequently, acts freely on the set  $\mathcal{E}(X)$ . In the category of graphs we deal with, all morphisms send vertices to vertices, edges to edges and loops to loops. In particular, this implies that the covering involution  $\tau$  acts on  $X$  without edge inversion. That is, for every edge  $e$  of  $X$  we have  $\tau(e) \neq \bar{e}$ .

On the other hand, if  $\tau$  is an order two automorphism of a graph  $X$  acting freely on the set of directed edges  $\mathcal{E}(X)$  and without edge inversion, then the factor space  $X' = X/\langle\tau\rangle$  is a graph and the canonical map  $X \rightarrow X' = X/\langle\tau\rangle$  is harmonic.

Summarizing, we characterize a  $\gamma$ -hyperelliptic involution on graph  $X$  as an involution acting on the set  $\mathcal{E}(X)$  freely, without edge inversion, and such that genus of the factor graph  $X/\langle\tau\rangle$  is  $\gamma$ .

The case  $\gamma = 0$  corresponds to hyperelliptic graphs and hyperelliptic involutions defined earlier in [3] in a more general aspect.

### 2.4 The Riemann-Hurwitz formula for $\gamma$ -hyperelliptic involution

Let  $\tau$  be a  $\gamma$ -hyperelliptic involution acting on a graph  $X$  of genus  $g$ . Then genus of the factor graph  $X' = X/\langle\tau\rangle$  is  $\gamma$ . Consider the induced harmonic morphism  $\varphi : X \rightarrow X'$ . By the previous section,  $\tau$  has neither fixed nor invertible edges, but it may have fixed vertices. For any vertex  $a \in \mathcal{V}(X)$  the multiplicity  $m_\varphi(x)$  of  $\varphi$  at  $x$  is equal to 1 or 2. If  $m_\varphi(x) = 1$  then the local map  $\varphi_x : \text{St}^X(x) \rightarrow \text{St}^{X'}(\varphi(x))$  is bijective. Then the vertex  $\varphi(x)$  has two preimages  $x$  and  $\tau(x)$ . So,  $x$  is not fixed by  $\tau$ . If  $m_\varphi(x) = 2$  then the local map  $\varphi_x : \text{St}^X(x) \rightarrow \text{St}^{X'}(\varphi(x))$  is two-to-one on the edges and  $\varphi(x)$  has only one preimage  $x = \tau(x)$ . That is,  $x$  is a fixed point of  $\tau$ .

Denote by  $V$  and  $E$  the number of vertices and undirected edges of  $X$  respectively. Define in a similar way the numbers  $V'$  and  $E'$  for the graph  $X'$ . Since  $\tau$  acts freely on edges we have  $E = 2E'$ . Let  $\tau$  have  $r$  fixed points on  $X$ . Then there are exactly  $r$  vertices of  $X'$  with the unique preimage under  $\varphi$ . Hence,  $V = 2V' - r$ . By formula (1.1) we have  $g - 1 = E - V$  and  $\gamma - 1 = E' - V'$ . Finally, we obtain

$$g - 1 = 2(\gamma - 1) + r. \tag{2.4}$$

This is a discrete version of the classical Riemann-Hurwitz formula from the theory of Riemann surfaces. More general statement of the Riemann-Hurwitz formula for the groups acting on a graph with fixed and invertible edges one can find in [10].

### 3 Homological basis adapted to the action of an involution

The main results of this section are Theorems 3.2 and 3.5. They can be considered as discrete versions of the results attained earlier for Riemann surfaces by Jane Gilman [7]. We start with the following definition.

**Definition 3.1.** Let  $X$  be a finite connected graph and  $\tau$  be an involution acting freely and without edge inversion on the set of directed edges of  $X$ . Suppose that  $\tau$  has at least one fixed vertex on  $X$ . A homological basis  $\mathcal{B}$  in  $H_1(X)$  is said to be *adapted to*  $\tau$  if it consists

of the elements  $B_i, C_i, i = 1, \dots, s, D_j, j = 1, \dots, t$  such that

$$\tau_*(B_i) = C_i, \tau_*(C_i) = B_i \text{ and } \tau(D_j) = -D_j.$$

The following theorem yields the conditions on  $\tau$  to get an adapted homological basis.

**Theorem 3.2.** *Let  $X$  be a finite connected graph and  $\tau$  is an involution acting freely and without edge inversion on the set of directed edges of  $X$ . Suppose that  $\tau$  has at least one fixed vertex on  $X$ . Then  $H_1(X)$  has a basis adapted to  $\tau$ .*

*Proof.* We prove the theorem using induction by genus  $g = g(X)$ . If  $g = 0$  then  $X$  is a tree,  $H_1(X) = \{0\}$  is a trivial group and  $\mathcal{B} = \{0\}$  is a homological basis adapted to  $\tau$ . For  $g > 1$  graph  $X$  has at least one nontrivial cycle  $E$ . Provide  $E$  one of its two possible orientations  $\vec{E}$ . Let  $-\vec{E}$  be the same cycle with the opposite orientation. There are three possibilities:

- (i)  $\tau(\vec{E}) \neq \pm\vec{E}$ ,
- (ii)  $\tau(\vec{E}) = \vec{E}$ ,
- (iii)  $\tau(\vec{E}) = -\vec{E}$ .

In the case (i) we set  $E' = \tau(E)$  and consider an edge  $e \in E \setminus E'$ . Let  $e' = \tau(e)$  then  $e' \in E' \setminus E$ . Denote by  $s(e)$  and  $t(e)$  the source and terminate vertices of edge  $e$ . Consider the graph  $X' = X \setminus \{e, e'\}$ . Since  $e$  and  $e'$  are not in  $E$ , the set  $E \setminus e$  is a path in  $X'$  from  $t(e)$  to  $s(e)$ . In a similar way,  $E' \setminus e'$  is a path in  $X'$  connecting vertices  $t(e')$  and  $s(e')$ . That is,  $X'$  is a connected graph of genus  $g(X') = g(X) - 2$ . Note that  $\tau$  leaves the graph  $X'$  invariant. Moreover,  $\tau$  acts without edge inversion on the set of directed edges of  $X'$  and has the same set of fixed vertices as before. By induction,  $X'$  already has a homological basis  $\mathcal{B}'$  adapted to  $\tau$ . Recall that  $\mathcal{B}'$  consist of  $g(X')$  elements. Consider the cycles  $E$  and  $E'$  as elements of  $H_1(X)$ . We add  $E$  and  $E'$  to  $\mathcal{B}'$  to form the set  $\mathcal{B} = \mathcal{B}' \cup \{E, E'\}$  of homological cycles in  $H_1(X)$ . Since edges  $e$  and  $e'$  are included only in homological cycles  $E$  and  $E'$  respectively, the set  $\mathcal{B}$  consists of  $g(X') + 2 = g(X)$  linear independent elements of  $H_1(X)$ . Therefore,  $\mathcal{B}$  is a  $\tau$ -adapted basis in  $H_1(X)$ .

Now let us consider the case (ii). Since  $\tau(\vec{E}) = \vec{E}$ , the cycle  $E$  is invariant under the action of  $\tau$ . The involution  $\tau$  has no fixed points on  $E$  since it preserves the orientation and acts freely on the set of directed edges of  $E$ . Let  $v$  be a fixed point of  $\tau$  on  $X$ , then  $v$  is not a vertex of  $E$ . Consider a shortest path  $\lambda$  from  $v$  to  $E$ . Then  $\lambda$  has no common edges with  $E$ . Denote by  $w$  the terminate point of  $\lambda$ . Set  $\lambda' = \tau(\lambda)$  and  $w' = \tau(w)$ . Then  $\lambda'$  is a path from  $v$  to  $w'$ , and  $\lambda'$  and  $E$  have no edges in common. Denote by  $\gamma$  a path in the cycle  $E$  from  $w$  to  $w'$  and by  $\gamma' = \tau(\gamma)$  the respective path from  $w'$  to  $w$ . Consider two cycles  $F = \lambda\gamma(\lambda')^{-1}$  and  $F' = \lambda'\gamma'\lambda^{-1}$ . Since the cycle  $E$  is not trivial, there exists an edge  $e$  in  $E$  with the source  $s(e) = w$ . Then  $e$  belongs to  $F$  and the respective edge  $e' = \tau(e)$  belongs to  $F'$ .

Note that  $e \in F \setminus F'$  and  $e' \in F' \setminus F$ . Moreover, graphs  $F \setminus \{e\}$  and  $F' \setminus \{e'\}$  share  $v$  as a common vertex. Hence, the graph  $X' = X \setminus \{e, e'\}$  is a connected graph and its genus  $g(X') = g(X) - 2$ . The involution  $\tau$  acts on  $X'$  satisfying conditions of the theorem. By induction, there is a homological basis  $\mathcal{B}'$  in  $H_1(X')$  consisting of  $g(X')$  elements and adapted to  $\tau$ . Consider the set  $\mathcal{B} = \mathcal{B}' \cup \{F, F'\}$  of homological cycles in  $H_1(X)$ . The elements of  $\mathcal{B}$  are linear independent. Indeed, the elements of  $\mathcal{B}'$  are already

linear independent, but  $F$  and  $F'$  are the only elements of  $\mathcal{B}$  containing edges  $e$  and  $e'$  respectively. Hence,  $g(X)$  elements of  $\mathcal{B}$  form a basis in  $H_1(X)$  adapted to  $\tau$ .

In the case (iii) we have  $\tau(\vec{E}) = -\vec{E}$ . It means that  $\tau$  leaves cycle  $E$  invariant reversing the orientation of its edges. By assumption,  $\tau$  acts without edge inversion. Then  $\tau$  has exactly two fixed points  $v$  and  $w$  on  $E$ . Let  $e$  be the edge of  $E$  with the source  $s(e) = v$  and  $e' = \tau(e)$ . Consider the graph  $X' = X \setminus \{e, e'\}$ .

Suppose that graph  $X'$  is connected. Then  $X'$  contains a path  $\gamma$  from  $v$  to  $w$ . The union  $E \cup \gamma$  is a connected graph of genus at least two. So, it contains a cycle  $F$  passing through  $e$  which differs from  $E$ . Then  $\tau(\vec{F}) \neq \vec{F}$  and the proof follows from the case (i).

Now we suppose that graph  $X'$  is disconnected. The  $X'$  consists of two connected components  $X'_1$  and  $X'_2$  containing the vertices  $v$  and  $w$  respectively. Since  $v$  and  $w$  are fixed by  $\tau$ , both  $X'_1$  and  $X'_2$  satisfy the conditions of the theorem. Also we have  $g(X) = g(X'_1) + g(X'_2) + 1$ .

By inductive assumption,  $X'_1$  and  $X'_2$  admit homological bases  $\mathcal{B}'_1$  and  $\mathcal{B}'_2$  adapted to  $\tau$ . Consider the cycle  $E$  as an element of  $H_1(X)$ . Then the set  $\mathcal{B} = \mathcal{B}'_1 \cup \mathcal{B}'_2 \cup \{E\}$  gives a homological basis in  $H_1(X)$  adapted to  $\tau$ . □

**Remark 3.3.** The condition on  $\tau$  to have fixed points in Theorem 3.2 is essential. Indeed, consider a cyclic graph  $X$  on even number of vertices. Let  $\tau$  act on  $X$  by the order two rotation. Then  $\tau$  acts fixed point free on the vertices of  $X$ . The homology group  $H_1(X)$  is generated by a cycle  $A$  and  $\tau_*(A) = A$ . That is,  $\tau_*$  acts trivially on  $H_1(X)$ .

To investigate the action of  $\tau$  without fixed points, we introduce the following definition.

**Definition 3.4.** Let  $X$  be a finite connected graph and  $\tau$  be an involution acting on  $X$  without fixed vertices. A homological basis  $\mathcal{B}$  in  $H_1(X)$  is said to be *adapted to  $\tau$*  if it consists of the elements  $A, B_i, C_i, i = 1, \dots, s$  such that

$$\tau_*(A) = A, \tau_*(B_i) = C_i, \tau_*(C_i) = B_i, i = 1, \dots, s.$$

**Theorem 3.5.** *Let  $X$  be a finite connected graph of genus  $g$ . Let  $\tau$  be an involution acting on  $X$  without edge inversion. Suppose that  $\tau$  has no fixed vertices. Then genus  $g$  is an odd number and  $H_1(X)$  has a basis adapted to  $\tau$ .*

*Proof.* The first statement of the theorem follows from the Riemann-Hurwitz formula. In our case, it has the form  $g - 1 = 2(\gamma - 1)$ , where  $\gamma$  is the genus of the factor graph  $X/\langle\tau\rangle$ . Since  $\gamma \geq 0$  we have  $g \geq 1$ . Hence,  $g = 2(\gamma - 1) + 1$  is a positive odd number.

We prove the second statement of the theorem using induction by genus  $g$ . If  $g = 1$  then  $X$  has only one cycle  $E$ . Provide  $E$  one of its two possible orientations  $\vec{E}$ . Then  $\tau(\vec{E}) = \pm\vec{E}$ . By assumption,  $\tau$  acts without edge inversion. Then in the case  $\tau(\vec{E}) = -\vec{E}$  it has two fixed points. This is impossible, since  $\tau$  acts fixed point free. So,  $\tau(\vec{E}) = \vec{E}$  and  $\mathcal{B} = \{E\}$  is the basis in  $H_1(X)$  adapted to  $\tau$ .

Since  $g = g(X)$  is an odd number, one can assume that  $g \geq 3$ . Consider an arbitrary cycle  $E$  in  $X$ . Choosing an orientation  $\vec{E}$  on  $E$  we have two possibilities: (i)  $\tau(\vec{E}) \neq \pm\vec{E}$  and (ii)  $\tau(\vec{E}) = \vec{E}$ . By the above arguments, the case  $\tau(\vec{E}) = -\vec{E}$  is impossible.

In the case (i) we set  $E' = \tau(E)$ . Since  $E \neq E'$ , there are edges  $e \in E$  and  $e' = \tau(e) \in E'$  such that  $e \in E \setminus E'$  and  $e' \in E' \setminus E$ . Then  $X' = X \setminus \{e, e'\}$  is a connected graph of genus  $g(X') = g(X) - 2 \geq 1$ . Graph  $X'$  is invariant under  $\tau$  and satisfies the conditions of the theorem. By induction,  $H_1(X')$  has a  $\tau$ -adapted basis  $\mathcal{B}'$ . Consider the cycles  $E$  and  $E'$  as elements of  $H_1(X)$ . Then  $\mathcal{B} = \mathcal{B}' \cup \{E, E'\}$  is a basis in  $H_1(X)$  adapted to  $\tau$ .

In the case (ii) we have  $\tau(E) = E$ . Choose an edge  $e \in E$  and set  $e' = \tau(e)$ . Then  $e' \in E$  and  $e \neq e'$ . Consider the graph  $X = X' \setminus \{e, e'\}$ . If  $X'$  is disconnected, it consists of two components  $X'_1$  and  $X'_2$  permuted by  $\tau$ . Thereby  $g(X'_1) = g(X'_2) = g'$  and  $g(X) = 2g' + 1$ . Let the group  $H_1(X'_1)$  be generated by cycles  $B_1, B_2, \dots, B_{g'}$ . Then  $H_1(X'_2)$  is generated by cycles  $C_1, C_2, \dots, C_{g'}$ , where  $C_i = \tau(B_i), i = 1, \dots, g'$  and

$$\mathcal{B} = \{B_1, B_2, \dots, B_{g'}, C_1, C_2, \dots, C_{g'}, E\}$$

is the required  $\tau$ -adapted basis in  $H_1(X)$ .

Now, let the graph  $X = X' \setminus \{e, e'\}$  be connected. Then there is a path  $\gamma$  from  $s(e)$  to  $t(e)$  in  $X'$ . The genus of graph  $E \cup \gamma$  is at least two, so  $E \cup \gamma$  contains a cycle  $F$  such that  $\tau(F) \neq F$ . Then  $\tau(\bar{F}) \neq \pm \bar{F}$  and we are back to the case (i). □

### 4 Main results

Now we apply the Theorems 3.2 and 3.5 to establish the main results of the paper. They are given in Theorems 4.1 and 4.3.

Theorem 4.1 gives two criteria for an involution acting on a graph  $X$  of genus  $g$  with fixed points to be  $\gamma$ -hyperelliptic. Theorem 4.3 provides necessary and sufficient conditions for a  $\gamma$ -hyperelliptic involution to act without fixed points.

**Theorem 4.1.** *Let  $X$  be a finite connected graph of genus  $g$ . Consider an involution  $\tau$  acting freely and without edge inversion on the set of directed edges of  $X$ . Denote by  $\tau_*$  the induced action of  $\tau$  on the first homology group  $H_1(X)$ . Suppose that  $\tau$  has at least one fixed vertex on  $X$ . Then the following conditions are equivalent:*

- (i) *the genus of factor graph  $X/\langle \tau \rangle$  is equal to  $\gamma$ ;*
- (ii) *there is a basis in the homology group  $H_1(X)$  whose  $g$  elements are either invertible or split into  $\gamma$  interchangeable pairs under the action of  $\tau_*$ ;*
- (iii)  $\text{tr}_{H_1(X)}(\tau_*) = 2\gamma - g$ .

*Proof.* Suppose that  $\tau$  has  $r \geq 1$  fixed points on  $\mathcal{V}(X)$ .

We show that (i) implies (ii). By Theorem 3.2 there exists a homological basis  $\mathcal{B}$  in  $H_1(X)$  adapted to  $\tau$ . Without loss of generality, we can assume that  $\mathcal{B}$  consists of  $s$  interchangeable pairs  $B_i, C_i, i = 1, \dots, s$ , and  $D_j, j = 1, \dots, t$  invertible elements, where  $s$  and  $t$  are non-negative integers related by the equation

$$2s + t = g. \tag{4.1}$$

Then the induced action  $\tau_*$  of  $\tau$  on  $H_1(X)$  is given by the formulas:

$$\tau_*(B_i) = C_i, \tau_*(C_i) = B_i, i = 1, \dots, s \text{ and } \tau_*(D_j) = -D_j, j = 1, \dots, t.$$

Consider the matrix representation of  $\tau_*$  in the basis  $\mathcal{B}$ . Then we have:

$$\tau_* = \left( \begin{array}{cccccc} J & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & J & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -1 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & -1 \end{array} \right),$$

$\underbrace{\hspace{10em}}_{2s}$        $\underbrace{\hspace{10em}}_t$

where

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Using the above matrix representation, by direct calculation we obtain

$$\text{tr}_{H_1(X)}(\tau_*) = -t. \tag{4.2}$$

At the same time, by the Hopf-Lefschetz formula [8] we have

$$\text{tr}_{H_1(X)}(\tau_*) = 1 - r, \tag{4.3}$$

where  $r$  is the number of fixed point of  $\tau$  on the graph  $X$ . Now we use the Riemann-Hurwitz formula

$$g - 1 = 2(\gamma - 1) + r \tag{4.4}$$

to find  $s$  and  $t$  through  $g$  and  $\gamma$ . From (4.2), (4.3) and (4.4) we have  $t = r - 1 = g - 2\gamma$ . Hence, taking into account (4.1) we conclude that  $s = \gamma$ .

These are the statements (iii) and (ii) of the theorem.

We note that (iii) follows from (ii) by direct calculations. To finish the proof, we have to show that (iii) implies (i). Again, by the Hopf-Lefschetz formula we have (4.3). Hence  $2\gamma - g = 1 - r$ . Equivalently,  $g - 1 = 2(\gamma - 1) + r$ . By Riemann-Hurwitz formula we conclude that  $\gamma$  is genus of the factor graph  $X/\langle\tau\rangle$ . □

The equivalence of conditions (i) and (ii) for  $\gamma = 0$  was established earlier by the second named author in ([11], Lemma 1).

The following result is an immediate consequence from the proof of Theorem 4.1.

**Corollary 4.2.** *Let  $\tau$  be the same as in Theorem 4.1 and*

$$\{B_i, C_i, i = 1, \dots, \gamma, D_j, j = 1, \dots, t\}, \tau_*(B_i) = C_i, \tau_*(C_i) = B_i, \tau_*(D_j) = D_j,$$

*be a homological basis in  $H_1(X)$  adapted to  $\tau$ . Then  $g(X/\langle\tau\rangle) = \gamma$  and the number of fixed vertices of  $\tau$  is  $t + 1$ .*



**Theorem 4.3.** *Let  $X$  be a finite connected graph of genus  $g \geq 1$ . Consider an involution  $\tau$  acting freely and without edge inversion on the set of directed edges of  $X$ . Denote by  $\tau_*$  the induced action of  $\tau$  on the first homology group  $H_1(X)$  and by  $\gamma$  the genus of factor graph  $X/\langle\tau\rangle$ . Then  $\tau$  acts fixed point free on the set of vertices of  $X$  if and only if one of the following conditions is satisfied:*

- (i) *genera  $g$  and  $\gamma$  are related by the Schreier formula  $g - 1 = 2(\gamma - 1)$ ;*
- (ii) *there is a basis  $\{A, B_i, C_i, i = 1, \dots, \gamma - 1\}$  in the homology group  $H_1(X)$  such that  $\tau_*(A) = A$ ,  $\tau_*(B_i) = C_i$ ,  $\tau_*(C_i) = B_i$ ;*
- (iii)  $\text{tr}_{H_1(X)}(\tau_*) = 1$ .

*Proof.* Let  $r$  be the number of fixed vertices of  $\tau$ . Then (i) immediately follows from the Riemann-Hurwitz formula (4.4).

We show that (ii) and (i) are equivalent. Indeed, by Theorem 3.5 the group  $H_1(X)$  has a basis  $\mathcal{B}$  adapted to  $\tau$  and consisting of  $g$  elements. By (i) we have  $g = 2(\gamma - 1) + 1$ . So,  $\gamma - 1$  elements of  $\mathcal{B}$  are permutable by  $\tau_*$  and one of them is fixed by  $\tau_*$ . This is exactly the statement (ii).

To prove that (ii) implies (i) we note that the basis  $\mathcal{B} = \{A, B_i, C_i, i = 1, \dots, \gamma - 1\}$  of the  $H_1(X)$  consists of  $2(\gamma - 1) + 1$  elements. Hence,  $g = 2(\gamma - 1) + 1$  and the Schreier formula holds. Calculating trace of  $\tau_*$  in the basis  $\mathcal{B}$ , we obtain  $\text{tr}_{H_1(X)}(\tau_*) = 1$ . Therefore, (iii) follows from (ii). Finally, let  $\text{tr}_{H_1(X)}(\tau_*) = 1$ . Then, by the Hopf-Lefschetz formula the number  $r$  of the fixed points of  $\tau$  is equal to 0. Then again, by (4.4) we get (i).  $\square$

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