Arc-transitive cyclic and dihedral covers of pentavalent symmetric graphs of order twice a prime[*]

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Abstract

A regular cover of a connected graph is called *cyclic* or *dihedral* if its transformation group is cyclic or dihedral respectively, and *arc-transitive* (or *symmetric*) if the fibre-preserving automorphism subgroup acts arc-transitively on the regular cover. In this paper, we give a classification of arc-transitive cyclic and dihedral covers of a connected pentavalent symmetric graph of order twice a prime. All those covers are explicitly constructed as Cayley graphs on some groups, and their full automorphism groups are determined.

Keywords: Symmetric graph, Cayley graph, bi-Cayley graph, regular cover.

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1   Introduction

All groups and graphs considered in this paper are finite, and all graphs are simple, connected and undirected, unless otherwise stated. Let G be a permutation group on a set Ω and let $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$, and *regular* if G is transitive and semiregular. Denote by $\mathbb{Z}_n$, $\mathbb{Z}_n^*$, D_n, A_n and S_n the cyclic group of order n, the multiplicative group of units of $\mathbb{Z}_n$, the dihedral group of order $2n$, the alternating and symmetric group of degree n, respectively. For two groups M and N, we use MN, $M.N$, $M \rtimes N$ and $M \times N$ to denote the product of M and N, an extension of M by N, a split extension of M by N and the direct product of M and N, respectively. For a subgroup H

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of a group G, $C_G(H)$ means the centralizer of H in G and $N_G(H)$ means the normalizer of H in G.

For a graph Γ, we denote its vertex set, edge set and full automorphism group by $V(\Gamma)$, $E(\Gamma)$ and $\mathrm{Aut}(\Gamma)$, respectively. An *s-arc* in Γ is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of Γ such that $\{v_{i-1}, v_i\} \in E(\Gamma)$ for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$. A 1-arc is just an *arc*. A graph Γ is (G, s)-*arc-transitive* for a subgroup G of $\mathrm{Aut}(\Gamma)$ if G acts transitively on the set of s-arcs of Γ, and (G, s)-*transitive* if Γ is (G, s)-arc-transitive but not $(G, s+1)$-arc-transitive. A graph Γ is said to be *s-arc-transitive* or *s-transitive* if it is $(\mathrm{Aut}(\Gamma), s)$-arc-transitive or $(\mathrm{Aut}(\Gamma), s)$-transitive, respectively. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. A graph Γ is *edge-transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on the edge set $E(\Gamma)$.

Let Γ be a graph and $N \leq \mathrm{Aut}(\Gamma)$. The *quotient graph* Γ_N of Γ relative to the orbits of N is defined as the graph with vertices the orbits of N on $V(\Gamma)$ and with two orbits adjacent if there is an edge in Γ between those two orbits. In particular, for a normal subgroup N of $\mathrm{Aut}(\Gamma)$, if Γ and Γ_N have the same valency, then Γ_N is a *normal quotient* of Γ, and if Γ has no proper normal quotient, then Γ is *basic*. To study a symmetric graph Γ, there is an extensive used strategy consisting of two steps: the first one is to investigate normal quotient graph Γ_N for some normal subgroup N of $\mathrm{Aut}(\Gamma)$ and the second one is to reconstruct the original graph Γ from the normal quotient Γ_N by using covering techniques. This strategy was first laid out by Praeger (see [31]), and it is usually done by taking the normal subgroup N as large as possible and then the graph Γ is reduced a basic graph. In the literature, there are many works about basic graphs (see [1, 14, 16] for example), while the works about the second step, that is, covers of graphs, are fewer.

An epimorphism $\pi \colon \widetilde{\Gamma} \to \Gamma$ of graphs is called a *regular covering projection* if $\mathrm{Aut}(\widetilde{\Gamma})$ has a semiregular subgroup K whose orbits in $V(\widetilde{\Gamma})$ coincide with the *vertex fibres* $\pi^{-1}(v)$, $v \in V(\Gamma)$, and whose arc and edge orbits coincide with the *arc fibres* $\pi^{-1}((u, v))$ and the *edge fibres* $\pi^{-1}(\{u, v\})$, $\{u, v\} \in E(\Gamma)$, respectively. In particular, we call the graph $\widetilde{\Gamma}$ a *regular cover* or a *K-cover* of the graph Γ, and the group K the *covering transformation group*. If K is dihedral, cyclic or elementary abelian, then $\widetilde{\Gamma}$ is called a *dihedral*, *cyclic* or *elementary abelian cover* of Γ, respectively. An automorphism of $\widetilde{\Gamma}$ is said to be *fibre-preserving* if it maps a vertex fibre to a vertex fibre, and all such fibre-preserving automorphisms form a group called the *fibre-preserving group*, denoted by F. It is easy to see that $F = N_{\mathrm{Aut}(\widetilde{\Gamma})}(K)$. If $\widetilde{\Gamma}$ is F-arc-transitive, we say that $\widetilde{\Gamma}$ is an *arc-transitive cover* or a *symmetric cover* of Γ. For an extensive treatment of regular cover, one can see [3, 26, 27].

Covering techniques have long been known as a powerful tool in algebraic and topological graph theory. Application of these techniques has resulted in many constructions and classifications of certain families of graphs with particular symmetry properties. For example, by using covering techniques, Djoković [10] constructed the first infinite family of 5-arc-transitive cubic graphs as covers of Tutte's 8-cage, and Biggs [4] constructed some 5-arc-transitive cubic graphs as covers of cubic graphs that are 4-arc-transitive but not 5-arc-transitive. Gross and Tucker [18] proved that every regular cover of a base graph can be reconstructed as a voltage graph on the base graph. Later, Malnič et al. [26] and Du et al. [12] developed these ideas further in a systematic study of regular covering projections of a graph along which a group of automorphisms lifts.

Based on the approaches studied in [12, 26], many arc-transitive covers of symmetric graphs of small orders and small valencies have been classified. For example, Pan et al. [29] studied arc-transitive cyclic covers of some complete graphs of small orders. One

may see [2, 27] for other works. Moreover, a new approach was proposed by Conder and Ma [7, 8] by considering a presentation (quotient group) of a universal group, which can be obtained from Reidemeister-Schreier theory, and representation theory and other methods are applied when determining suitable quotients. As an application, arc-transitive abelian covers of the complete graph K_4, the complete bipartite graph $K_{3,3}$, the 3-dimensional hypercube Q_3, the Petersen graph and the Heawood graph, were classified. Later, arc-transitive dihedral covers of these graphs were determined by Ma [25].

For arc-transitive covers of infinite families of graphs, Du et al. studied 2-arc-transitive elementary abelian and cyclic covers of complete graphs K_n in [11, 13] and $K_{n,n} - nK_2$ in [32, 33]. Recently, Pan et al. [28] determined arc-transitive cyclic covers of the complete bipartite graph $K_{p,p}$ of order $2p$ for a prime p. Compared with symmetric covers of graphs of small orders and valencies, there are only a few contributions on symmetric covers of infinite families of graphs.

Arc-transitive covers of non-simple graphs were also considered in literature. For example, regular covers of the dipole Dip_k (a graph with two vertices and k parallel edges) were extensively studied in [2, 16, 26, 27, 34]. Such covers are called *Haar graphs*, and in particular, cyclic regular covers of dipoles are called *cyclic Haar graphs*, which can be regarded as a generalization of bipartite circulants and were studied in [21] (also see [15]). Construction of Haar graphs have aroused wide concern. Marušič et al. [26] studied elementary abelian covers of the dipole Dip_p for a prime p. In particular, symmetric elementary abelian covers and $\mathbb{Z}_p^2 \times \mathbb{Z}_p$-covers for a prime p of the dipole Dip_5 were classified completely in [16] and [34], respectively.

Let p be a prime. Pentavalent symmetric graphs of order $2p$ were classified by Cheng and Oxley in [6], which are the complete graph K_6 of order 6 and a family of Cayley graphs $\mathcal{CD}_p$ with $p = 5$ or $5 \mid (p-1)$ on dihedral groups (see Proposition 3.4). It has been shown that many pentavalent symmetric graphs are regular covers of them, see [16, 34]. In this paper, we consider arc-transitive cyclic and dihedral covers of these graphs. For K_6, the cyclic covers have been classified in [29], which should be the complete bipartite graph $K_{6,6}$ and the Icosahedron graph $\mathbf{I}_{12}$ (note that $\mathbf{I}_{12}$ is missed in [29]). For $\mathcal{CD}_p$, the cyclic covers consist of six infinite families of graphs, which are Cayley graphs on generalized dihedral groups. In particular, one family of graphs are cyclic Haar graphs and the other five families are non-cyclic Haar graphs. What is more, the full automorphism groups of them are determined. Arc-transitive dihedral covers of K_6 and $\mathcal{CD}_p$ are also classified, and there are only four sporadic graphs of order 24, 48, 60 and 120, respectively. A similar work about cubic graphs was done by Zhou and Feng [37].

Different from regular covers of graphs mentioned above, the method to classify arc-transitive cyclic covers used in this paper is related to the so called bi-Cayley graph. A graph Γ is a *bi-Cayley graph* over some group H if $\mathrm{Aut}(\Gamma)$ has a semiregular subgroup isomorphic to H having exactly two orbits on $V(\Gamma)$. Clearly, a Haar graph is a bipartite bi-Cayley graph. Recently, Zhou and Feng [38] gave a depiction of the automorphisms of bi-Cayley graphs (see Section 4), and based on this work, we classify the cyclic covers. In particular, all these covers are bi-Cayley graphs over some abelian groups. Note that vertex-transitive bi-Cayley graphs of valency 3 over abelian groups were determined in [36], while the case for valency 5 is still elusive. Indeed, even for arc-transitive pentavalent bi-Cayley graphs over abelian groups, it seems to be very difficult to give a classification, and one may see [2, 16, 34] for partial works.

The paper is organized as follows. After this introductory section, in Section 2 we give

some notation and preliminary results. In Section 3, several infinite families of connected pentavalent symmetric graphs are constructed as Cayley graphs on generalized dihedral groups $\mathrm{Dih}(\mathbb{Z}_{mp^e} \times \mathbb{Z}_p)$, where e, m are two positive integers and p is a prime such that $(m, p) = 1$. In Section 4, it is proved that these Cayley graphs include all arc-transitive normal bipartite bi-Cayley graphs over $\mathbb{Z}_{mp^e} \times \mathbb{Z}_p$, and using this result, arc-transitive cyclic and dihedral covers of connected pentavalent symmetric graphs of order $2p$ are classified in Sections 5 and 6, respectively. In Section 7, the full automorphism groups of these covers are determined.

2 Preliminaries

In this section, we describe some preliminary results which will be used later. The following result is important to investigate symmetric pentavalent graphs.

Proposition 2.1 ([19, Theorem 1.1]). *Let Γ be a connected pentavalent (G, s)-transitive graph for some $G \leq \mathrm{Aut}(\Gamma)$ and $s \geq 1$, and let $v \in V(\Gamma)$. Then one of the following holds:*

(1) $s = 1$ and $G_v \cong \mathbb{Z}_5$, D_5 or D_{10};

(2) $s = 2$ and $G_v \cong F_{20}$, $F_{20} \times \mathbb{Z}_2$, A_5 or S_5, where F_{20} is the Frobenius group of order 20;

(3) $s = 3$ and $G_v \cong F_{20} \times \mathbb{Z}_4$, $A_4 \times A_5$, $S_4 \times S_5$ or $(A_4 \times A_5) \rtimes \mathbb{Z}_2$ with $A_4 \rtimes \mathbb{Z}_2 = S_4$ and $A_5 \rtimes \mathbb{Z}_2 = S_5$;

(4) $s = 4$ and $G_v \cong \mathrm{ASL}(2, 4)$, $\mathrm{AGL}(2, 4)$, $\mathrm{A\Sigma L}(2, 4)$ or $\mathrm{A\Gamma L}(2, 4)$;

(5) $s = 5$ and $G_v \cong \mathbb{Z}_2^6 \rtimes \Gamma\mathrm{L}(2, 4)$.

From [24, Theorem 9], we have the following proposition.

Proposition 2.2. *Let Γ be a connected G-arc-transitive graph of prime valency, and let N be a normal subgroup of G. If N has at least three orbits, then it is semiregular on $V(\Gamma)$ and the kernel of G on the quotient graph Γ_N. Furthermore, Γ_N is G/N-arc-transitive, and Γ is a regular cover of Γ_N with N as the covering transformation group.*

Let G and E be two groups. We call an extension E of G by N a *central extension* of G if E has a central subgroup N such that $E/N \cong G$, and if further E is perfect, that is, if it equals its derived group E', we call E a *covering group* of G. Schur proved that for every non-abelian simple group G there is a unique maximal covering group M such that every covering group of G is a factor group of M (see [22, V, § 23]). This group M is called the *full covering group* of G, and the center of M is the *Schur multiplier* of G, denoted by $\mathrm{Mult}(G)$.

Lemma 2.3. *Let G be a group, and let N be an abelian normal subgroup of G such that G/N is a non-abelian simple group. If N is a proper subgroup of $C_G(N)$, then $G = G'N$ and $G' \cap N \lesssim \mathrm{Mult}(G/N)$.*

Proof. Since N is a proper subgroup of $C_G(N)$, we have $1 \neq C_G(N)/N \trianglelefteq G/N$, forcing $C_G(N)/N = G/N$ because G/N is simple. Thus $G = C_G(N)$ and it is a central extension of G/N by N. Since $G/N = (G/N)' = G'N/N \cong G'/(G' \cap N)$, we have $G = G'N$,

and since $G' = (G'N)' = (G')'$, G' is a covering group of G/N. Hence $G' \cap N \lesssim$ Mult(G/N). $\qquad \square$

Denote by soc(G) the *socle* of G, that is, the product of all minimal normal subgroups of G. A list of all proper primitive permutation groups of degree less than 1000 was given by Dixon and Mortimer [9, Appendix B], and based on the list, we have:

Lemma 2.4. *Let G be a primitive permutation group on a set Ω and let $\alpha \in \Omega$, where $|\Omega| \in \{2, 4, 6, 8, 12, 16, 24, 72, 144, 288, 576\}$. If G_α is solvable, then either $G \lesssim$ AGL($n, 2$) and $|\Omega| = 2^n$ with $1 \leq n \leq 4$, or soc(G) $\cong$ PSL($2, p$), PSL($3, 3$) or PSL($2, q$) $\times$ PSL($2, q$) with $|\Omega| = p + 1$, 144 or $(q + 1)^2$ respectively, where $p \in \{5, 7, 11, 23, 71\}$ and $q \in \{11, 23\}$.*

Proof. If $|\Omega| = 2$ or 4, then $G \leq S_2 \cong$ AGL($1, 2$) or $G \leq S_4 \cong$ AGL($2, 2$), respectively. Let $|\Omega| \geq 6$ and write $N := \text{soc}(G)$. Then $N \trianglelefteq G$ and $N_\alpha \trianglelefteq G_\alpha$. Since G_α is solvable, N_α is solvable. By [9, Appendix B, Tables B.2 and B.3], G is an affine group, $N \cong A_{|\Omega|}$, or G is isomorphic to one group listed in [9, Tables B.2 and B.3]. If G is affine, then $|\Omega|$ is a prime power and thus $|\Omega| = 2^n$ with $n = 3$ or 4. By [9, Theorem 4.1A (a)], we have $G \lesssim$ AGL($n, 2$). If $N \cong A_{|\Omega|}$ then $N_\alpha \cong A_{|\Omega|-1}$, which is insolvable because $|\Omega| - 1 \geq 5$, a contradiction. In what follows we assume that G is isomorphic to one group listed in [9, Tables B.2 and B.3]. Note that all groups in the tables are collected into cohorts and all groups in a cohort have the same socle.

Assume that $|\Omega| = 144$. By [9, Table B.4, pp. 324], there are one cohort of type C, two cohorts of type H and four cohorts of type I (see [9, Table B.1, pp. 306] for types of cohorts of primitive groups) of primitive groups of degree 144. For the cohort of type C, by [9, Table B.2, pp. 314], $N \cong$ PSL($3, 3$) and $N_\alpha \cong \mathbb{Z}_{13} \rtimes \mathbb{Z}_3$. For the two cohorts of type H, by [9, Table B.2, pp. 321], they have the same socle $N \cong M_{12}$ and $N_\alpha \cong$ PSL($2, 11$). For the four cohorts of type I, by [9, Table B.3, pp. 323], $N \cong A_{12} \times A_{12}$, PSL($2, 11$) $\times$ PSL($2, 11$), $M_{11} \times M_{11}$ or $M_{12} \times M_{12}$ and $N_\alpha \cong A_{11} \times A_{11}$, $(\mathbb{Z}_{11} \rtimes \mathbb{Z}_5) \times (\mathbb{Z}_{11} \rtimes \mathbb{Z}_5)$, $M_{10} \times M_{10}$ or $M_{11} \times M_{11}$, respectively. Since N_α is solvable, we have $N \cong$ PSL($3, 3$) or PSL($2, 11$) $\times$ PSL($2, 11$).

For $|\Omega| \in \{6, 8, 12, 16, 24, 72, 288, 576\}$, by [9, Tables B.2, B.3 and B.4], a similar argument to the above paragraph implies that either $N \cong$ PSL($2, 23$) $\times$ PSL($2, 23$) with degree $23^2 = 576$ and $N_\alpha \cong (\mathbb{Z}_{23} \rtimes \mathbb{Z}_{11}) \times (\mathbb{Z}_{23} \rtimes \mathbb{Z}_{11})$, or $N \cong$ PSL($2, p$) with degree $p + 1$ and $N_\alpha \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{\frac{p-1}{2}}$ where $p \in \{5, 7, 11, 23, 71\}$. $\qquad \square$

3 Graph constructions as Cayley graphs

Let G be a finite group and S a subset of G with $1 \notin S$ and $S^{-1} = S$. The *Cayley graph* $\Gamma = \text{Cay}(G, S)$ on G with respect to S is defined to have vertex set $V(\Gamma) = G$ and edge set $E(\Gamma) = \{\{g, sg\} \mid g \in G, s \in S\}$. It is well-known that Aut(Γ) contains the right regular representation $R(G)$ of G, the acting group of G by right multiplication, and Γ is connected if and only if $G = \langle S \rangle$, that is, S generates G. By Godsil [17], $N_{\text{Aut}(\Gamma)}(R(G)) = R(G) \rtimes \text{Aut}(G, S)$, where Aut($G, S$) $= \{\alpha \in \text{Aut}(G) \mid S^\alpha = S\}$. A Cayley graph $\Gamma = \text{Cay}(G, S)$ is said to be *normal* if $R(G)$ is normal in Aut(Γ), and in this case, Aut(Γ) $= R(G) \rtimes \text{Aut}(G, S)$.

For an abelian group H, the *generalized dihedral group* Dih(H) is the semidirect product $H \rtimes \mathbb{Z}_2$, where the unique involution in $\mathbb{Z}_2$ maps each element of H to its inverse. In particular, if H is cyclic, then Dih(H) is a dihedral group. In this section, we introduce

several infinite families of connected pentavalent symmetric graphs which are constructed as Cayley graphs on generalized dihedral groups.

Example 3.1. Let $\mathrm{Dih}(\mathbb{Z}_5^3) = \langle a, b, c, h \mid a^5 = b^5 = c^5 = h^2 = [a, b] = [a, c] = [b, c] = 1, a^h = a^{-1}, b^h = b^{-1}, c^h = c^{-1} \rangle$, and define

$$\mathcal{CGD}_{5^3} = \mathrm{Cay}(\mathrm{Dih}(\mathbb{Z}_5^3), \{h, ah, bh, ch, a^{-1}b^{-1}c^{-1}h\}).$$

By [34, Theorem 1.1], $\mathrm{Aut}(\mathcal{CGD}_{5^3}) \cong \mathrm{Dih}(\mathbb{Z}_5^3) \rtimes S_5$ and $\mathcal{CGD}_{5^3}$ is the unique connected pentavalent symmetric graph of order 250 up to isomorphism.

Let m be a positive integer. Consider the following equation in $\mathbb{Z}_m$

$$x^4 + x^3 + x^2 + x + 1 = 0. \tag{3.1}$$

In view of [15, Lemma 3.3], we have the following proposition.

Proposition 3.2. *Equation (3.1) has a solution r in $\mathbb{Z}_m$ if and only if $(r, m) \in \{(0, 1), (1, 5)\}$ or $m = 5^t p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$ and r is an element in $\mathbb{Z}_m^*$ of order 5, where $t \leq 1$, $s \geq 1$, $e_i \geq 1$ and p_i's are distinct primes such that $5 \mid (p_i - 1)$.*

The following infinite family of Cayley graphs was first constructed in [23].

Example 3.3. Let $m > 1$ be an integer such that Equation (3.1) has a solution r in $\mathbb{Z}_m$. Then $m = 5, 11$ or $m \geq 31$. Let

$$\mathcal{CD}_m = \mathrm{Cay}(D_m, \{b, ab, a^{r+1}b, a^{r^2+r+1}b, a^{r^3+r^2+r+1}b\})$$

be a Cayley graph on the dihedral group $D_m = \langle a, b \mid a^n = b^2 = 1, a^b = a^{-1} \rangle$. For $m = 5$ or 11, by [6], $\mathrm{Aut}(\mathcal{CD}_m) \cong (S_5 \times S_5) \rtimes \mathbb{Z}_2$ or $\mathrm{PGL}(2, 11)$, respectively. In particular, $\mathcal{CD}_5 \cong K_{5,5}$. For $m \geq 31$, by [23, Theorem B and Proposition 4.1], $\mathrm{Aut}(\mathcal{CD}_m) \cong D_m \rtimes \mathbb{Z}_5$, and obviously, if m has a prime divisor p with $p < m$, then $\mathrm{Aut}(\mathcal{CD}_m)$ has a normal subgroup $\mathbb{Z}_{m/p}$, and by Proposition 2.2, $\mathcal{CD}_m$ is a symmetric $\mathbb{Z}_{m/p}$-cover of a connected pentavalent symmetric graph of order $2p$.

By [6], we have the following proposition.

Proposition 3.4. *Let Γ be a connected pentavalent symmetric graph of order $2p$ for a prime p. Then $\Gamma \cong K_6$ or $\mathcal{CD}_p$ with $p = 5$ or $5 \mid (p - 1)$.*

In the remaining part of this section, we construct five infinite families of Cayley graphs on some generalized dihedral groups, and for convenience, we always assume that $G = \mathrm{Dih}(\mathbb{Z}_m \times \mathbb{Z}_{p^e} \times \mathbb{Z}_p) = \langle a, b, c, h \mid a^m = b^{p^e} = c^p = h^2 = [a, b] = [a, c] = [b, c] = 1, a^h = a^{-1}, b^h = b^{-1}, c^h = c^{-1} \rangle$ and r is a solution of Equation (3.1) in $\mathbb{Z}_m$, that is, $r^4 + r^3 + r^2 + r + 1 \equiv 0 \pmod{m}$. By Proposition 3.2, m is odd and $5^2 \nmid m$.

Example 3.5. Assume that $e \geq 2$ and p is a prime such that $(m, p) = 1$ and $5 \mid (p - 1)$. Let λ be an element of order 5 in $\mathbb{Z}_{p^e}^*$. Then λ is a solution of Equation (3.1) in $\mathbb{Z}_{p^e}$. Set

$T_1(r, \lambda) =$
  $\{h, hab, ha^{r+1}b^{\lambda+1}c, ha^{r^2+r+1}b^{\lambda^2+\lambda+1}c^{\lambda^4+\lambda+1}, ha^{r^3+r^2+r+1}b^{\lambda^3+\lambda^2+\lambda+1}c\}$,

$T_2(r, \lambda) =$
  $\{h, hab, ha^{r+1}b^{\lambda+1}c, ha^{r^2+r+1}b^{\lambda^2+\lambda+1}c^{\lambda^3+\lambda+1}, ha^{r^3+r^2+r+1}b^{\lambda^3+\lambda^2+\lambda+1}c^{\lambda}\}$,

$T_3(r, \lambda) =$
  $\{h, hab, ha^{r+1}b^{\lambda+1}c, ha^{r^2+r+1}b^{\lambda^2+\lambda+1}c^{\lambda^2+\lambda+1}, ha^{r^3+r^2+r+1}b^{\lambda^3+\lambda^2+\lambda+1}c^{\lambda^2}\}$.

It is easy to see that each of $T_1(r, \lambda)$, $T_2(r, \lambda)$ and $T_3(r, \lambda)$ generates G. Define

$$\mathcal{CGD}^i_{mp^e \times p} = \text{Cay}(G, T_i(r, \lambda)), \ i = 1, 2, 3.$$

The maps

$$a \mapsto a^r, \ b \mapsto b^\lambda c, \ c \mapsto c^{\lambda^4}, \ h \mapsto hab;$$
$$a \mapsto a^r, \ b \mapsto b^\lambda c, \ c \mapsto c^{\lambda^3}, \ h \mapsto hab;$$
$$a \mapsto a^r, \ b \mapsto b^\lambda c, \ c \mapsto c^{\lambda^2}, \ h \mapsto hab$$

induce three automorphisms of order 5 of G, denoted by α_1, α_2 and α_3 respectively, and α_i fixes the set $T_i(r, \lambda)$ and permutes its five elements cyclically. It follows that for each $i = 1, 2, 3$, $\alpha_i \in \text{Aut}(G, T_i(r, \lambda))$ and $\langle R(G), \alpha_i \rangle \cong G \rtimes \mathbb{Z}_5$, which is arc-transitive on $\mathcal{CGD}^i_{mp^e \times p}$.

The graphs $\mathcal{CGD}^1_{mp^e \times p}$, $\mathcal{CGD}^2_{mp^e \times p}$ and $\mathcal{CGD}^3_{mp^e \times p}$ for $(m, e) = (1, 2)$ have been introduced in [34, Example 4.4], and they are not isomorphic to each other by [34, Lemma 4.5]. Indeed, we can also prove that the graphs $\mathcal{CGD}^1_{mp^e \times p}$, $\mathcal{CGD}^2_{mp^e \times p}$ and $\mathcal{CGD}^3_{mp^e \times p}$ for each integers $m \geq 1$ and $e \geq 2$ are not isomorphic to each other. Since the proof is similar to [34, Lemma 4.5], we omit it, and one can see [35] for a detailed proof.

Example 3.6. Let p be a prime such that $p = 5$ or $5 \mid (p \pm 1)$. Assume that $e = 1$ and $(m, p) = 1$. Then $G = \text{Dih}(\mathbb{Z}_m \times \mathbb{Z}_p \times \mathbb{Z}_p)$. For $p = 5$, let $\lambda = 0$, and for $5 \mid (p \pm 1)$, let $\lambda \in \mathbb{Z}_p$ satisfying the equation $\lambda^2 = 5$ in $\mathbb{Z}_p$. Set

$$S(r, \lambda) = \{h, hab, ha^{r+1}c, ha^{r^2+r+1}b^{-2^{-1}(1+\lambda)}c^{2^{-1}(1+\lambda)}, ha^{r^3+r^2+r+1}b^{-2^{-1}(1+\lambda)}c\}.$$

It is easy to see that $S(r, \lambda)$ generates G. Define

$$\mathcal{CGD}^4_{mp \times p} = \text{Cay}(G, S(r, \lambda)).$$

The map $a \mapsto a^r$, $b \mapsto b^{-1}c$, $c \mapsto b^{-2^{-1}(3+\lambda)}c^{2^{-1}(1+\lambda)}$ and $h \mapsto hab$ induces an automorphism of the group G, denoted by α_4, which permutes the elements in $S(r, \lambda)$ cyclically. Then $\alpha_4 \in \text{Aut}(G, S(r, \lambda))$ and $\langle R(G), \alpha_4 \rangle \cong G \rtimes \mathbb{Z}_5$ acts arc-transitive on $\mathcal{CGD}^4_{mp \times p}$. Moreover, for $m = 1$ or 5, we have $r = 0$ or 1 respectively, and the map $a \mapsto a^{-1}$, $b \mapsto b^{-2^{-1}(1+\lambda)}c$, $c \mapsto b^{-2^{-1}(1+\lambda)}c^{2^{-1}(1+\lambda)}$, $h \mapsto h$ induces an automorphism β of G. It is easy to check that $\beta \in \text{Aut}(G, S(r, \lambda))$ and $\langle \alpha_4, \beta \rangle \cong D_5$. In particular, by [16, Theorem 6.1], $\mathcal{CGD}^4_{5 \times 5}$ is the unique connected pentavalent symmetric graph of order 50 up to isomorphism.

Example 3.7. Assume that $e = 1$ and p is a prime such that $(m, p) = 1$ and $5 \mid (p - 1)$. By [34, Case 2, page 14], $x^4 + 10x^2 + 5 = 0$ has a root λ in $\mathbb{Z}_p$. Set

$$S(r, \lambda) = \{h, \ hab, \ ha^{r+1}c, \ ha^{r^2+r+1}b^{8^{-1}(\lambda^3-\lambda^2+7\lambda+1)}c^{2^{-1}(\lambda+1)},$$
$$ha^{r^3+r^2+r+1}b^{-8^{-1}(\lambda^3+\lambda^2+7\lambda-1)}c^{8^{-1}(\lambda^3+\lambda^2+11\lambda+3)}\}.$$

It is easy to check that $S(r, \lambda)$ generates G. Define

$$\mathcal{CGD}^5_{mp \times p} = \text{Cay}(G, S(r, \lambda)).$$

The map $a \mapsto a^r$, $b \mapsto b^{-1}c$, $c \mapsto b^{8^{-1}(\lambda^3 - \lambda^2 + 7\lambda - 7)}c^{2^{-1}(\lambda+1)}$ and $h \mapsto hab$ induces an automorphism of the group G, denoted by α_5, which permutes the elements in $S(r, \lambda)$ cyclicly. Then $\alpha_5 \in \mathrm{Aut}(G, S(r, \lambda))$ and $\langle R(G), \alpha_5 \rangle \cong G \rtimes \mathbb{Z}_5$ acts arc-transitive on $\mathcal{CGD}^5_{mp \times p}$.

Let $\Gamma_i = \mathcal{CGD}^i_{mp^e \times p}$ with $p = 5$ or $5 \mid (p - 1)$, where $1 \leq i \leq 5$. By Examples 3.5 – 3.7, $\mathrm{Aut}(\Gamma_i)$ contains an arc-transitive subgroup $R(G) \rtimes \langle \alpha_i \rangle$ for each $1 \leq i \leq 5$. Let N_i be a subgroup of $R(G) \rtimes \langle \alpha_i \rangle$ as listed in Table 1. In particular, for Γ_4 with $5 \mid (p - 1)$, since $\lambda^2 = 5$ in $\mathbb{Z}_p$, the equation $x^4 + 10x^2 + 5 = 0$ has a root t such that $t^2 = 2\lambda - 5$ (see Example 3.7). It is easy to compute that $N_i \cong \mathbb{Z}_{mp^e}$ and $N_i \trianglelefteq R(G) \rtimes \langle \alpha_i \rangle$ for each $1 \leq i \leq 5$ (see [35] for a detailed computation). By Proposition 2.2, we have the following lemma.

Table 1: Subgroups of $\mathrm{Aut}(\mathcal{CGD}^i_{mp^e \times p})$ for $1 \leq i \leq 5$.

Γ_i	p	N_i
$\mathcal{CGD}^1_{mp^e \times p}$	$5 \mid (p-1)$	$\langle R(a), R(b^5 c^{3\lambda^4 + 2\lambda^2 - \lambda + 1}) \rangle$
$\mathcal{CGD}^2_{mp^e \times p}$	$5 \mid (p-1)$	$\langle R(a), R(b^{-5} c^{2\lambda^3 + 4\lambda^2 + \lambda + 3}) \rangle$
$\mathcal{CGD}^3_{mp^e \times p}$	$5 \mid (p-1)$	$\langle R(a), R(b^{-5} c^{4\lambda^3 + 3\lambda^2 + 2\lambda + 1}) \rangle$
$\mathcal{CGD}^4_{mp \times p}$	$p = 5$	$\langle R(a), R(b^2 c^4) \rangle$
	$5 \mid (p-1)$	$\langle R(a), R(b^{t+1} c^{\lambda - 3}) \rangle$ $(t^2 = 2\lambda - 5)$
$\mathcal{CGD}^5_{mp \times p}$	$5 \mid (p-1)$	$\langle R(a), R(b^{2(\lambda^2 + 5)^{-1}(\lambda^3 + 10\lambda + 5) - (\lambda + 3)} c^4) \rangle$

Lemma 3.8. *Let p be a prime such that $p = 5$ or $5 \mid (p - 1)$. Then for each $1 \leq i \leq 5$, $\mathcal{CGD}^i_{mp^e \times p}$ is a connected symmetric cyclic cover of a connected pentavalent symmetric graph of order $2p$.*

4 Pentavalent symmetric bi-Cayley graphs over abelian groups

Given a group H, let R, L and S be three subsets of H such that $R^{-1} = R$, $L^{-1} = L$, and $1 \notin R \cup L$. The *bi-Cayley graph* over H relative to the triple (R, L, S), denoted by $\mathrm{BiCay}(H, R, L, S)$, is the graph having vertex set $\{h_0 \mid h \in H\} \cup \{h_1 \mid h \in H\}$ and edge set $\{\{h_0, g_0\} \mid gh^{-1} \in R\} \cup \{\{h_1, g_1\} \mid gh^{-1} \in L\} \cup \{\{h_0, g_1\} \mid gh^{-1} \in S\}$. For a bi-Cayley graph $\Gamma = \mathrm{BiCay}(H, R, L, S)$, it is easy to see that $R(H)$ can be regarded as a semiregular subgroup of $\mathrm{Aut}(\Gamma)$ with two orbits, which acts on $V(\Gamma)$ by the rule $h_i^{R(g)} = (hg)_i$, $i = 0, 1$, $h, g \in H$. If $R(H)$ is normal in $\mathrm{Aut}(\Gamma)$, then Γ is a *normal bi-Cayley graph* over H.

Let $\Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, S)$ be a connected bi-Cayley graph over an abelian group H. Then Γ is bipartite. By [38, Lemma 3.1], we may always assume that $1 \in S$. Moreover, $\Gamma \cong \mathrm{BiCay}(H, \emptyset, \emptyset, S^\alpha)$ for $\alpha \in \mathrm{Aut}(H)$, and $H = \langle S \rangle$. Since H is abelian, there is an automorphism of H of order 2, denoted by γ, induced by $g \mapsto g^{-1}$, $\forall g \in H$. For

$\alpha \in \mathrm{Aut}(H)$ and $x \in H$, define

$$\delta_{\gamma,1,1}\colon h_0 \mapsto (h^{-1})_1,\ h_1 \mapsto (h^{-1})_0,\ \forall h \in H;$$
$$\sigma_{\alpha,x}\colon h_0 \mapsto (h^\alpha)_0,\ h_1 \mapsto (xh^\alpha)_1,\ \forall h \in H.$$

Set

$$\mathrm{F} = \{\sigma_{\alpha,x} \mid \alpha \in \mathrm{Aut}(H),\ S^\alpha = x^{-1}S\}.$$

Then $\delta_{\gamma,1,1} \in \mathrm{Aut}(\Gamma)$ and $\mathrm{F} \le \mathrm{Aut}(\Gamma)_{1_0}$ (see [38, Lemma 3.3]). Since Γ is connected, F acts on $N_\Gamma(1_0)$ faithfully. By [38, Theorem 1.1 and Lemma 3.2], we have the following proposition.

Proposition 4.1. *Let* $\Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, S)$ *be a connected bi-Cayley graph over an abelian group H, and let $A = \mathrm{Aut}(\Gamma)$. Then $N_A(R(H)) = R(H)\langle \mathrm{F}, \delta_{\gamma,1,1}\rangle$ with vertex stabilizer $(N_A(R(H)))_{1_0} = \mathrm{F}$, and Γ is isomorphic to the Cayley graph $\mathrm{Cay}(\mathrm{Dih}(H), \gamma S)$, where $\mathrm{Dih}(H) = H \rtimes \langle \gamma \rangle$.*

The following lemma is from [2, Theorem 1.1].

Lemma 4.2. *Let n be a positive integer and p a prime such that $p \ge 5$. Let Γ be a connected pentavalent symmetric bi-Cayley graph over $\mathbb{Z}_{np}$. Then $\Gamma \cong \mathcal{CD}_{np}$, as defined in Example 3.3.*

Let $H = \langle x \rangle \times \langle y \rangle \times \langle z \rangle = \mathbb{Z}_m \times \mathbb{Z}_{p^e} \times \mathbb{Z}_p$, where m and e are two positive integers and p is a prime such that $p \ge 5$ and $(m, p) = 1$. In the remaining of this section, we always let $\Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, S)$ be a connected pentavalent bi-Cayley graph over H such that $N_{\mathrm{Aut}(\Gamma)}(R(H))$ is arc-transitive on Γ. Assume that $S = \{1, a, b, c, d\}$. Then $H = \langle a, b, c, d \rangle$. By Proposition 4.1, there exists a $\sigma_{\alpha,g} \in \mathrm{F}$ of order 5 permuting the neighborhood $\{1_1, a_1, b_1, c_1, d_1\}$ of 1_0 in Γ cyclically. One may assume that $1_1^{\sigma_{\alpha,g}} = a_1$, which implies that $g = a$ because $1_1^{\sigma_{\alpha,g}} = g_1$, and that $b_1 = a_1^{\sigma_{\alpha,a}}$, $c_1 = b_1^{\sigma_{\alpha,a}}$, $d_1 = c_1^{\sigma_{\alpha,a}}$ and $1_1 = d_1^{\sigma_{\alpha,a}}$. It follows that

$$a^\alpha = ba^{-1},\ b^\alpha = ca^{-1},\ c^\alpha = da^{-1},\ d^\alpha = a^{-1}. \tag{4.1}$$

For $h \in H$, denote by $o(h)$ the order of h in H. Since $a^\alpha = ba^{-1}$ by Equation (4.1), $o(ba^{-1}) = o(a^\alpha) = o(a)$, forcing that $o(b) \mid o(a)$. Similarly, since $d^\alpha = a^{-1}$ and $c^\alpha = da^{-1}$, we have $o(d) = o(a)$ and $o(c) \mid o(a)$. Since $H = \langle a, b, c, d \rangle$, we have $o(x) \mid o(a)$ for any $x \in H$, and since $H = \mathbb{Z}_m \times \mathbb{Z}_{p^e} \times \mathbb{Z}_p$, we have $o(a) = mp^e$ and $|H : \langle a \rangle| = p$.

Suppose that $b \in \langle a \rangle$, say $b = a^i$ for some integer i. By Equation (4.1), $a^\alpha = ba^{-1} = a^{i-1} \in \langle a \rangle$ and $ca^{-1} = b^\alpha = (a^i)^\alpha = a^{i(i-1)} \in \langle a \rangle$, implying that $c \in \langle a \rangle$. Similarly, $d \in \langle a \rangle$ because $d = a \cdot c^\alpha$. Since $H = \langle a, b, c, d \rangle$, we have $H = \langle a \rangle \cong \mathbb{Z}_{mp^e}$, a contradiction. Hence $b \notin \langle a \rangle$, and since $|H : \langle a \rangle| = p$, we have $H = \langle a, b \rangle$ and $p \mid o(b)$.

Let $A = \mathrm{Aut}(\Gamma)$. Since Γ is $N_A(R(H))$-arc-transitive, $\mathrm{F} = N_A(R(H))_{1_0}$ acts transitively on $N_\Gamma(1_0)$. Let $\sigma_{\beta,g} \in \mathrm{F}$ for some $\beta \in \mathrm{Aut}(H)$ and $g \in H$ such that $1_1^{\sigma_{\beta,g}} = 1_1$. Then $1_1 = (1^\beta g)_1 = g_1$, forcing that $g = 1$. Hence $\mathrm{F}_{1_1} = \{\sigma_{\beta,1} \mid \beta \in \mathrm{Aut}(H), S^\beta = S\}$, that is, $\mathrm{F}_{1_1} \cong \mathrm{Aut}(H, S)$. By Proposition 4.1,

$$|N_A(R(H))| = 2|H||\mathrm{F}| = 2|H| \cdot |N_\Gamma(1_0)||\mathrm{F}_{1_1}| = 10|H||\mathrm{Aut}(H, S)|.$$

Observation 4.3. $o(a) = mp^e$, $p \mid o(b)$, $H = \langle a, b \rangle$ *and*

$$|N_A(R(H))| = 10|H||\operatorname{Aut}(H, S)|.$$

In the following two lemmas we consider the two cases: $e \geq 2$ and $e = 1$, respectively.

Lemma 4.4. *If $e \geq 2$, then $5 \mid (p - 1)$, $\Gamma \cong \mathcal{CGD}_{mp^e \times p}^i$ for some $1 \leq i \leq 3$ and $|N_A(R(H))| = 10|H|$.*

Proof. By Observation 4.3, $o(a) = mp^e$, $p \mid o(b)$ and $H = \langle a, b \rangle = \langle x, y, z \rangle = \mathbb{Z}_m \times \mathbb{Z}_{p^e} \times \mathbb{Z}_p$, where $(m, p) = 1$. Then H has an automorphism mapping xy to a, and thus we may assume $a = xy$, which implies that $b = x^{r+1}y^{\lambda+1}z^\iota$ for some $r + 1 \in \mathbb{Z}_m$, $\lambda + 1 \in \mathbb{Z}_{p^e}$ and $0 \neq \iota \in \mathbb{Z}_p$ because $H = \langle a, b \rangle$. Furthermore, H has an automorphism fixing x, y and mapping z to z^ι, and so we may assume $b = x^{r+1}y^{\lambda+1}z$. Let $c = x^i y^j z^s$ and $d = x^k y^\ell z^t$, where $i, k \in \mathbb{Z}_m$, $j, \ell \in \mathbb{Z}_{p^e}$ and $s, t \in \mathbb{Z}_p$.

Note that both $\langle x \rangle = \mathbb{Z}_m$ and $\langle y, z \rangle = \mathbb{Z}_{p^e} \times \mathbb{Z}_p$ are characteristic in H. Since $a^\alpha = ba^{-1}$ by Equation (4.1), that is, $(xy)^\alpha = x^r y^\lambda z$, we have $x^\alpha = x^r$ and $y^\alpha = y^\lambda z$. Since $(x^{r+1}y^{\lambda+1}z)^\alpha = b^\alpha = ca^{-1} = x^{i-1}y^{j-1}z^s$, we have $z^\alpha = (x^{-r-1})^\alpha \cdot (y^{-\lambda-1})^\alpha \cdot (x^{i-1}y^{j-1}z^s) = x^{-r^2-r-1+i}y^{-\lambda^2-\lambda-1+j}z^{s-\lambda-1}$, implying that

$$z^\alpha = y^{-\lambda^2-\lambda-1+j}z^{s-\lambda-1}$$

and

$$-r^2 - r - 1 + i \equiv 0 \pmod{m}, \tag{4.2}$$

$$\lambda^2 - \lambda - 1 + j \equiv 0 \pmod{p^{e-1}}. \tag{4.3}$$

Similarly, since $c^\alpha = da^{-1}$ and $d^\alpha = a^{-1}$ by Equation (4.1), we have $x^{k-1}y^{\ell-1}z^t = c^\alpha = (x^i y^j z^s)^\alpha = x^{ir}y^{\lambda j + s(-\lambda^2-\lambda-1+j)}z^{j+s(s-\lambda-1)}$ and $x^{-1}y^{-1} = d^\alpha = (x^k y^\ell z^t)^\alpha = x^{kr}y^{\lambda \ell + t(-\lambda^2-\lambda-1+j)}z^{\ell+t(s-\lambda-1)}$, and by considering the powers of x, y and z, we have the following Equations (4.4)–(4.9).

$$ir \equiv k - 1 \pmod{m}; \tag{4.4}$$

$$kr \equiv -1 \pmod{m}; \tag{4.5}$$

$$\lambda j + s(-\lambda^2 - \lambda - 1 + j) \equiv \ell - 1 \pmod{p^e}; \tag{4.6}$$

$$\lambda \ell + t(-\lambda^2 - \lambda - 1 + j) \equiv -1 \pmod{p^e}; \tag{4.7}$$

$$j + s(s - \lambda - 1) \equiv t \pmod{p}; \tag{4.8}$$

$$\ell + t(s - \lambda - 1) \equiv 0 \pmod{p}. \tag{4.9}$$

By Equation (4.2), $i \equiv r^2 + r + 1 \pmod{m}$, and by Equations (4.4) and (4.5), $k \equiv r^3 + r^2 + r + 1 \pmod{m}$ and $r^4 + r^3 + r^2 + r + 1 \equiv 0 \pmod{m}$. It follows from Proposition 3.2 that either $(r, m) \in \{(0, 1), (1, 5)\}$, or r is an element in $\mathbb{Z}_m^*$ of order 5 and $m = 5^t p_1^{e_1} \cdots p_f^{e_f}$ with $t \leq 1$, $f \geq 1$, $e_\iota \geq 1$ and p_ι's are distinct primes such that $5 \mid (p_\iota - 1)$ for $1 \leq \iota \leq f$.

Note that $e \geq 2$. By Equation (4.3), $j \equiv \lambda^2 + \lambda + 1 \pmod{p^{e-1}}$ and by Equations (4.6) and (4.7), $\ell \equiv \lambda^3 + \lambda^2 + \lambda + 1 \pmod{p^{e-1}}$ and $\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1 \equiv 0 \pmod{p^{e-1}}$, implying $\lambda^5 \equiv 1 \pmod{p^{e-1}}$. It follows from Proposition 3.2 that either $(\lambda, p^{e-1}) =$

$(1, 5)$, or $5 \mid (p - 1)$ and λ is an element in $\mathbb{Z}^*_{p^{e-1}}$ of order 5, forcing that $\lambda \neq 0$ and $\ell^{-1} = (-\lambda^4)^{-1} = -\lambda$. Furthermore, one may assume that $j \equiv \lambda^2 + \lambda + 1 + s_1 p^{e-1}$ (mod p^e), $\ell \equiv \lambda^3 + \lambda^2 + \lambda + 1 + s_2 p^{e-1}$ (mod p^e) and $\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1 \equiv \iota p^{e-1}$ (mod p^e) for some $s_1, s_2, \iota \in \mathbb{Z}_p$.

In what follows all equations are considered in $\mathbb{Z}_p$, unless otherwise stated. As $p \mid p^{e-1}$, the following equations are also true in $\mathbb{Z}_p$:

$$j = \lambda^2 + \lambda + 1,$$
$$\ell = \lambda^3 + \lambda^2 + \lambda + 1,$$
$$\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1 = 0,$$
$$\ell^{-1} = -\lambda.$$

By $s \times (4.9) - t \times (4.8)$, $s = \ell^{-1}(jt - t^2) = -\lambda(jt - t^2)$, and by Equation (4.9), we have $\lambda t^3 - (\lambda^3 + \lambda^2 + \lambda)t^2 - (\lambda + 1)t + (\lambda^3 + \lambda^2 + \lambda + 1) = 0$. Combined with $\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1 = 0$ and $\lambda \neq 0$, we have $(t - 1)(t - \lambda)(t - \lambda^2) = 0$, which implies that $t = 1$, λ or λ^2. Recall that $j = \lambda^2 + \lambda + 1$ and $s = -\lambda(jt - t^2)$. Thus $(t, s) = (1, \lambda^4 + \lambda + 1)$, $(\lambda, \lambda^3 + \lambda + 1)$ or $(\lambda^2, \lambda^2 + \lambda + 1)$.

Since $j \equiv \lambda^2 + \lambda + 1 + s_1 p^{e-1}$ (mod p^e) and $\ell \equiv \lambda^3 + \lambda^2 + \lambda + 1 + s_2 p^{e-1}$ (mod p^e), by Equations (4.6) and (4.7) we have:

$$
\begin{cases}
(\lambda + s)s_1 p^{e-1} \equiv s_2 p^{e-1} \pmod{p^e} \\
t s_1 p^{e-1} + \lambda s_2 p^{e-1} \equiv -(\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1) \pmod{p^e}
\end{cases}
\tag{4.10}
$$

Recall that either $(\lambda, p^{e-1}) = (1, 5)$ or $5 \mid (p - 1)$. Suppose that $p^{e-1} = 5$. Then $p = 5$, $e = 2$ and $(\lambda, s, t) = (1, 3, 1)$. By Equation (4.10), we have $5s_2 = 20s_1$ and $5^2 s_1 + 5 = 0$ in $\mathbb{Z}_{5^2}$, a contradiction. Hence $5 \mid (p - 1)$. Again by Equation (4.10), we have $-(t + \lambda^2 + \lambda s)s_1 p^{e-1} \equiv \iota p^{e-1}$ (mod p^e), where $\iota p^{e-1} = \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1$. Furthermore,

$$
\begin{cases}
(t + \lambda^2 + \lambda s)s_1 = -\iota \\
(t + \lambda^2 + \lambda s)s_2 = -\iota(\lambda + s)
\end{cases}
\tag{4.11}
$$

Since $(t, s) = (1, \lambda^4 + \lambda + 1)$, $(\lambda, \lambda^3 + \lambda + 1)$ or $(\lambda^2, \lambda^2 + \lambda + 1)$, we have $t + \lambda^2 + \lambda s = 2\lambda^2 + \lambda + 2$, $\lambda^4 + 2\lambda^2 + 2\lambda$ or $\lambda^3 + 3\lambda^2 + \lambda$, respectively, and since $(2\lambda^2 + \lambda + 2)(\lambda^4 + 2\lambda^2 + 2\lambda) = 6(\lambda^4 + \lambda^3 + \lambda^2 + \lambda) + 1 = -5$ and $(\lambda^3 + 3\lambda^2 + \lambda)(\lambda^4 - 2\lambda^3 + \lambda^2) = \lambda^4 + \lambda^3 + \lambda^2 + \lambda - 4 = -5$, we have $(t + \lambda^2 + \lambda s)^{-1} = -5^{-1}(\lambda^4 + 2\lambda^2 + 2\lambda)$, $-5^{-1}(2\lambda^2 + \lambda + 2)$ or $-5^{-1}(\lambda^4 - 2\lambda^3 + \lambda^2)$, respectively. By Equation (4.11), $(s_1, s_2) = (5^{-1}\iota(\lambda^4 + 2\lambda^2 + 2\lambda), 5^{-1}\iota(-3\lambda^4 + \lambda^3 + 2\lambda^2))$, $(5^{-1}\iota(2\lambda^2 + \lambda + 2), 5^{-1}\iota(-3\lambda^4 + 2\lambda^3 + \lambda))$ or $(5^{-1}\iota(\lambda^4 - 2\lambda^3 + \lambda^2), 5^{-1}\iota(-2\lambda^4 + \lambda^2 + \lambda))$. Note that $a = xy$, $b = x^{r+1}y^{\lambda+1}z$, and

$$(c, d) = (x^{r^2+r+1}y^{\lambda^2+\lambda+1+s_1 p^{e-1}}z^{\lambda^4+\lambda+1}, x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1+s_2 p^{e-1}}z),$$
$$(x^{r^2+r+1}y^{\lambda^2+\lambda+1+s_1 p^{e-1}}z^{\lambda^3+\lambda+1}, x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1+s_2 p^{e-1}}z^{\lambda}) \text{ or}$$
$$(x^{r^2+r+1}y^{\lambda^2+\lambda+1+s_1 p^{e-1}}z^{\lambda^2+\lambda+1}, x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1+s_2 p^{e-1}}z^{\lambda^2}),$$

we have $S = S_1$, S_2 or S_3, where

$$S_1 = \{1, xy, x^{r+1}y^{\lambda+1}z, x^{r^2+r+1}y^{\lambda^2+\lambda+1+5^{-1}(\lambda^4+2\lambda^2+2\lambda)\iota p^{e-1}}z^{\lambda^4+\lambda+1},$$
$$x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1+5^{-1}(-3\lambda^4+\lambda^3+2\lambda^2)\iota p^{e-1}}z\},$$

$$S_2 = \{1, xy, x^{r+1}y^{\lambda+1}z, x^{r^2+r+1}y^{\lambda^2+\lambda+1+5^{-1}(2\lambda^2+\lambda+2)\iota p^{e-1}}z^{\lambda^3+\lambda+1},$$
$$x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1+5^{-1}(-3\lambda^4+2\lambda^3+\lambda)\iota p^{e-1}}z^{\lambda}\},$$

$$S_3 = \{1, xy, x^{r+1}y^{\lambda+1}z, x^{r^2+r+1}y^{\lambda^2+\lambda+1+5^{-1}(\lambda^4-2\lambda^3+\lambda^2)\iota p^{e-1}}z^{\lambda^2+\lambda+1},$$
$$x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1+5^{-1}(-2\lambda^4+\lambda^2+\lambda)\iota p^{e-1}}z^{\lambda^2}\}.$$

Since $x^5 \equiv 1 \pmod{p^e}$ implies that $x^5 \equiv 1 \pmod{p^{e-1}}$, there exists $f \in \mathbb{Z}_p$ such that $\lambda_1 = \lambda + fp^{e-1}$ is an element of order 5 in $\mathbb{Z}_{p^e}^*$. Then $\lambda = \lambda_1 - fp^{e-1}$, $\lambda_1^5 = 1$ and $\lambda_1^4 + \lambda_1^3 + \lambda_1^2 + \lambda_1 + 1 = 0$ in $\mathbb{Z}_{p^e}$. Hence $\iota p^{e-1} = \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1 = (\lambda_1 - fp^{e-1})^4 + (\lambda_1 - fp^{e-1})^3 + (\lambda_1 - fp^{e-1})^2 + (\lambda_1 - fp^{e-1}) + 1 = -(4\lambda_1^3 + 3\lambda_1^2 + 2\lambda_1 + 1)fp^{e-1}$ in $\mathbb{Z}_{p^e}$, and thus

$$S_1 = \{1, xy, x^{r+1}y^{\lambda+1}z, x^{r^2+r+1}y^{\lambda^2+\lambda+1+5^{-1}(\lambda^4+2\lambda^2+2\lambda)\iota p^{e-1}}z^{\lambda^4+\lambda+1},$$
$$x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1+5^{-1}(-3\lambda^4+\lambda^3+2\lambda^2)\iota p^{e-1}}z\}$$
$$= \{1, xy, x^{r+1}y^{\lambda_1+1}y^{-fp^{e-1}}z, x^{r^2+r+1}y^{\lambda_1^2+\lambda_1+1}y^{-(\lambda_1^4+\lambda_1+1)fp^{e-1}}z^{\lambda_1^4+\lambda_1+1},$$
$$x^{r^3+r^2+r+1}y^{\lambda_1^3+\lambda_1^2+\lambda_1+1}y^{-fp^{e-1}}z\}.$$

Let φ be the automorphism of H induced by $x \mapsto x$, $y \mapsto y$ and $z \mapsto y^{fp^{e-1}}z$. Then $(S_1)^\varphi = \{1, xy, x^{r+1}y^{\lambda_1+1}z, x^{r^2+r+1}y^{\lambda_1^2+\lambda_1+1}z^{\lambda_1^4+\lambda_1+1}, x^{r^3+r^2+r+1}y^{\lambda_1^3+\lambda_1^2+\lambda_1+1}z\}$. Since $\mathrm{BiCay}(H, \emptyset, \emptyset, S_1) \cong \mathrm{BiCay}(H, \emptyset, \emptyset, S_1^\varphi)$, we may assume that $\lambda = \lambda_1$ is an element of order 5 in $\mathbb{Z}_{p^e}^*$, and

$$S_1 = \{1, xy, x^{r+1}y^{\lambda+1}z, x^{r^2+r+1}y^{\lambda^2+\lambda+1}z^{\lambda^4+\lambda+1}, x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1}z\}.$$

Similarly, we can also assume that

$$S_2 = \{1, xy, x^{r+1}y^{\lambda+1}z, x^{r^2+r+1}y^{\lambda^2+\lambda+1}z^{\lambda^3+\lambda+1}, x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1}z^{\lambda}\},$$
$$S_3 = \{1, xy, x^{r+1}y^{\lambda+1}z, x^{r^2+r+1}y^{\lambda^2+\lambda+1}z^{\lambda^2+\lambda+1}, x^{r^3+r^2+r+1}y^{\lambda^3+\lambda^2+\lambda+1}z^{\lambda^2}\}.$$

By Proposition 4.1 and Example 3.5, $\Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, S) \cong \mathcal{CGD}_{mp^e \times p}^i$ with $1 \le i \le 3$.

Note that $|N_A(R(H))| = 10|H||\mathrm{Aut}(H, S)|$ (see Observation 4.3). For S_1, let $\beta \in \mathrm{Aut}(H, S_1)$. Then $S_1^\beta = S_1$. Since $\langle y, z \rangle$ is characteristic in $H = \langle x \rangle \times \langle y \rangle \times \langle z \rangle = \mathbb{Z}_m \times \mathbb{Z}_{p^e} \times \mathbb{Z}_p$, we have

$$\{y, y^{\lambda+1}z, y^{\lambda^2+\lambda+1}z^{\lambda^4+\lambda+1}, y^{\lambda^3+\lambda^2+\lambda+1}z\}^\beta$$
$$= \{y, y^{\lambda+1}z, y^{\lambda^2+\lambda+1}z^{\lambda^4+\lambda+1}, y^{\lambda^3+\lambda^2+\lambda+1}z\}.$$

It follows that $y^\beta = y^s z^t$ with $(s,t) = (1,0)$, $(\lambda+1, 1)$, $(\lambda^2+\lambda+1, \lambda^4+\lambda+1)$, or $(\lambda^3+\lambda^2+\lambda+1, 1)$. Furthermore, we have

$$(y \cdot y^{\lambda+1}z \cdot y^{\lambda^2+\lambda+1}z^{\lambda^4+\lambda+1} \cdot y^{\lambda^3+\lambda^2+\lambda+1}z)^\beta$$
$$= y \cdot y^{\lambda+1}z \cdot y^{\lambda^2+\lambda+1}z^{\lambda^4+\lambda+1} \cdot y^{\lambda^3+\lambda^2+\lambda+1}z$$

and

$$(y^\beta y^{-1})^{\lambda^3 + 2\lambda^2 + 3\lambda + 4} = (z^{-\lambda^4 - \lambda - 3})^\beta z^{\lambda^4 + \lambda + 3}.$$

In particular, $(y^\beta y^{-1})^{(\lambda^3 + 2\lambda^2 + 3\lambda + 4)p} = 1$. Note that $\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1 = 0$ in $\mathbb{Z}_{p^e}$ implies that $\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1 = 0$ in $\mathbb{Z}_p$. If $\lambda^3 + 2\lambda^2 + 3\lambda + 4 = 0$ in $\mathbb{Z}_p$, then $\lambda^3 = -2\lambda^2 - 3\lambda - 4$, $\lambda^4 = \lambda \cdot \lambda^3 = \lambda^2 + 2\lambda + 8$, and thus $0 = \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1 = 5$, contrary to $5 \mid (p-1)$. Hence $\lambda^3 + 2\lambda^2 + 3\lambda + 4 \neq 0$ in $\mathbb{Z}_p$ and $(y^\beta y^{-1})^p = 1$.

Suppose that $(s,t) \neq (1,0)$. Then $y^\beta y^{-1} = y^{s-1} z^t$ with $s - 1 = \lambda, \lambda^2 + \lambda$ or $\lambda^3 + \lambda^2 + \lambda$. Since $\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1 = 0$ in $\mathbb{Z}_p$, we have $\lambda \neq 0, -1$ and thus $(s-1, p) = 1$. This implies that $y^\beta y^{-1} = y^{s-1} z^t$ has order p^e, and since $e \geq 2$, we have $(y^\beta y^{-1})^p \neq 1$, a contradiction. Hence $(s,t) = (1,0)$, that is, $y^\beta = y$. It follows that $(y^{\lambda+1} z)^\beta = y^{\lambda+1} z^\beta \in \{y^{\lambda+1} z, y^{\lambda^2 + \lambda + 1} z^{\lambda^4 + \lambda + 1}, y^{\lambda^3 + \lambda^2 + \lambda + 1} z\}$, and thus $z^\beta \in \{z, y^{\lambda^2} z^{\lambda^4 + \lambda + 1}, y^{\lambda^3 + \lambda^2} z\}$. If $z^\beta = y^{\lambda^2} z^{\lambda^4 + \lambda + 1}$ or $y^{\lambda^3 + \lambda^2} z$, then $(y^{\lambda^2})^p = 1$ or $(y^{\lambda^3 + \lambda^2})^p = 1$. It forces that $\lambda^2 = 0$ or $\lambda^3 + \lambda^2 = 0$ in $\mathbb{Z}_{p^{e-1}}$, and $\lambda = 0, -1$, a contradiction. Hence $z^\beta = z$. Noting that $\langle x \rangle$ is characteristic in H, we have $(xy)^\beta = x^\beta y \in S_1^\beta = S_1$. Then it is easy to check that $(xy)^\beta = xy$ and thus $x^\beta = x$. It implies that β is the identity automorphism. Hence $|\operatorname{Aut}(H, S_1)| = 1$ and $|N_A(R(H))| = 10|H|$. By a similar argument as above, for S_2 and S_3, we also have $|\operatorname{Aut}(H, S_2)| = |\operatorname{Aut}(H, S_3)| = 1$ and $|N_A(R(H))| = 10|H|$. $\qquad \square$

Lemma 4.5. *If $e = 1$, that is, $H \cong \mathbb{Z}_m \times \mathbb{Z}_p \times \mathbb{Z}_p$, then one of the following holds:*

(1) $p = 5$ or $5 \mid (p \pm 1)$ and $\Gamma \cong \mathcal{CGD}_{mp \times p}^4$ as defined in Example 3.6. Furthermore,

    (i) $|N_A(R(H))| = 10|H|$ if $m \neq 1, 5$;

    (ii) $|N_A(R(H))| = 20|H|$ if $m = 5$;

    (iii) $|N_A(R(H))| = 20|H|$ if $m = 1$ and $p \neq 5$; and

    (iv) $|N_A(R(H))| = 40|H|$ if $m = 1$ and $p = 5$;

(2) $5 \mid (p-1)$, $\Gamma \cong \mathcal{CGD}_{mp \times p}^5$ as defined in Example 3.7 and $|N_A(R(H))| = 10|H|$.

Proof. Note that $(m, p) = 1$. By Observation 4.3, we have $o(a) = mp$, $p \mid o(b)$ and $H = \langle a, b \rangle = \langle x, y, z \rangle = \mathbb{Z}_m \times \mathbb{Z}_p \times \mathbb{Z}_p$. Then H has an automorphism mapping xy to a, and we may assume $a = xy$, implying that $b = x^{r+1} y^\lambda z^\iota$ for some $r + 1 \in \mathbb{Z}_m$, $\iota, \lambda \in \mathbb{Z}_p$ and $\iota \neq 0$ because $H = \langle a, b \rangle$. The group H also has an automorphism fixing x, y and mapping z to $y^\lambda z^\iota$, and we may further assume $b = x^{r+1} z$. Let $c = x^i y^j z^s$ and $d = x^k y^\ell z^t$, where $i, k \in \mathbb{Z}_m$, $j, \ell, s, t \in \mathbb{Z}_p$.

By Equation (4.1), $a^\alpha = ba^{-1}$, that is, $(xy)^\alpha = x^r y^{-1} z$. Since both $\langle x \rangle$ and $\langle y, z \rangle$ are characteristic in H, we have $x^\alpha = x^r$ and $y^\alpha = y^{-1} z$. Again by Equation (4.1), since $(x^{r+1} z)^\alpha = b^\alpha = ca^{-1} = x^{i-1} y^{j-1} z^s$, we have $z^\alpha = (x^{-r-1})^\alpha \cdot b^\alpha = x^{-r^2 - r - 1 + i} \cdot y^{j-1} z^s$, implying that $z^\alpha = y^{j-1} z^s$ and

$$-r^2 - r - 1 + i \equiv 0 \pmod{m}. \tag{4.12}$$

Moreover, we have

$$x^{k-1} y^{\ell-1} z^t = da^{-1} = c^\alpha = (x^i y^j z^s)^\alpha$$
$$= (x^r)^i (y^{-1} z)^j (y^{j-1} z^s)^s = x^{ri} y^{-j+s(j-1)} z^{j+s^2},$$

and

$$x^{-1}y^{-1} = a^{-1} = d^\alpha = (x^k y^\ell z^t)^\alpha = (x^r)^k (y^{-1}z)^\ell (y^{j-1}z^s)^t = x^{rk} y^{-\ell+(j-1)t} z^{st+\ell}.$$

Considering the powers of x, y and z, we have Equations (4.13)–(4.18). As shown in these equations, in what follows all equations are considered in $\mathbb{Z}_p$, unless otherwise stated:

$$k - 1 \equiv ri \pmod{m}; \tag{4.13}$$
$$\ell - 1 = -j + s(j-1); \tag{4.14}$$
$$t = j + s^2; \tag{4.15}$$
$$-1 \equiv rk \pmod{m}; \tag{4.16}$$
$$-1 = -\ell + (j-1)t; \tag{4.17}$$
$$0 = st + \ell. \tag{4.18}$$

By Equation (4.12), we have $i \equiv r^2 + r + 1 \pmod{m}$ and by Equations (4.13) and (4.16), $k \equiv r^3 + r^2 + r + 1 \pmod{m}$ and $r^4 + r^3 + r^2 + r + 1 \equiv 0 \pmod{m}$. It follows from Proposition 3.2 that either $(r, m) \in \{(0, 1), (1, 5)\}$ or r is an element of order 5 in $\mathbb{Z}_m^*$ and the prime decomposition of m is $5^t p_1^{e_1} \cdots p_f^{e_f}$ with $t \le 1$, $f \ge 1$, $e_\iota \ge 1$ and $5 \mid (p_\iota - 1)$ for $1 \le \iota \le f$.

By Equation (4.15), $t = j + s^2$, and by Equations (4.14), (4.17) and (4.18), $\ell = 1 - j + s(j-1)$, $\ell = 1 + (j-1)t = 1 + (j-1)(j+s^2)$ and $\ell = -st = -sj - s^3$. It follows

$$j^2 + (s^2 - s)j - (s^2 - s) = 0; \tag{4.19}$$
$$(2s - 1)j + s^3 - s + 1 = 0. \tag{4.20}$$

By Equation (4.19), $(2s-1)^2 j^2 + (2s-1)^2(s^2-s)j - (2s-1)^2(s^2-s) = 0$, and since $(2s-1)j = -(s^3 - s + 1)$, we have $s^6 - 3s^5 + 5s^4 - 5s^3 + 2s - 1 = 0$, that is, $(s^2 - s - 1)(s^4 - 2s^3 + 4s^2 - 3s + 1) = 0$. Hence, either $s^2 - s - 1 = 0$ or $s^4 - 2s^3 + 4s^2 - 3s + 1 = 0$.

Case 1: $s^2 - s - 1 = 0$. Let $\lambda = 2s - 1$. Then $s = 2^{-1}(1 + \lambda)$ and $\lambda^2 = 5$, and thus $(\lambda, p) = (0, 5)$ or $5 \mid (p \pm 1)$ by [34, Example 4.6]. By Equations (4.19) and (4.20), $j^2 + j - 1 = 0$ and $(2s - 1)j + (s + 2) = 0$.

For $(\lambda, p) = (0, 5)$, $j^2 + j - 1 = 0$ implies that $j = 2 = -2^{-1}(1 + \lambda)$. For $5 \mid (p \pm 1)$, we have $\lambda \ne 0$, and since $2s - 1 = \lambda$ and $(2s - 1)j + (s + 2) = 0$, we have $j = -(2s-1)^{-1}(s+2) = -\lambda^{-1} \cdot 2^{-1}(\lambda + 5) = -2^{-1}(1 + \lambda)$ (note that $5 = \lambda^2$). It follows from Equations (4.15) and (4.18) that $t = j + s^2 = 1$ and $\ell = -st = -2^{-1}(1 + \lambda)$. Recall that $i \equiv r^2 + r + 1 \pmod{m}$ and $k \equiv i^3 + i^2 + i + 1 \pmod{m}$. Hence $c = x^{r^2+r+1} y^{-2^{-1}(1+\lambda)} z^{2^{-1}(1+\lambda)}$ and $d = x^{r^3+r^2+r+1} y^{-2^{-1}(1+\lambda)} z$. Now,

$$S = \{1, xy, x^{r+1}z, x^{r^2+r+1} y^{-2^{-1}(1+\lambda)} z^{2^{-1}(1+\lambda)}, x^{r^3+r^2+r+1} y^{-2^{-1}(1+\lambda)} z\}.$$

By Proposition 4.1 and Example 3.6, $\Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, S) \cong \mathcal{CGD}_{mp \times p}^4$.

For $(m, p) = (1, 5)$, we have $\lambda = 0$ and $S = \{1, y, z, y^{-3}z^3, y^{-3}z\}$. By MAGMA [5], $|N_A(R(H))| = 40|H|$. Assume that $(m, p) \ne (1, 5)$, and let $\beta \in \mathrm{Aut}(H, S)$. Then $S^\beta = S$, and since both $\langle x \rangle$ and $\langle y, z \rangle$ are characteristic subgroups of H, we have

$$\{x, x^{r+1}, x^{r^2+r+1}, x^{r^3+r^2+r+1}\}^\beta = \{x, x^{r+1}, x^{r^2+r+1}, x^{r^3+r^2+r+1}\},$$

$$\{y, z, y^{-2^{-1}(1+\lambda)} z^{2^{-1}(1+\lambda)}, y^{-2^{-1}(1+\lambda)} z\}^{\beta}$$
$$= \{y, z, y^{-2^{-1}(1+\lambda)} z^{2^{-1}(1+\lambda)}, y^{-2^{-1}(1+\lambda)} z\}.$$

Similarly to Lemma 4.4, the two equations imply that for $m \neq 1, 5$ ($r \neq 0, \pm 1$), β is the identity automorphism of H, and for $m = 1$ or 5 ($r = 0$ or 1), β has order 2 that are induced by $x \mapsto x^{r^3+r^2+r+1}$, $y \mapsto y^{-2^{-1}(1+\lambda)} z$, and $z \mapsto y^{-2^{-1}(\lambda+1)} z^{2^{-1}(\lambda+1)}$ (one may also see [35] for a detailed computation). It implies that $|\operatorname{Aut}(H, S)| = 1$ for $m \neq 1, 5$ and $|\operatorname{Aut}(H, S)| = 2$ for $m = 1$ or 5. By Observation 4.3, we have $|N_A(R(H))| = 10|H|$ or $20|H|$, respectively.

Case 2: $s^4 - 2s^3 + 4s^2 - 3s + 1 = 0$. By Case 1, we may assume that $s^2 - s - 1 \neq 0$. If $p = 5$, then $s^4 - 2s^3 + 4s^2 - 3s + 1 = 0$ implies that $s = 3$ and thus $s^2 - s - 1 = 0$, a contradiction. Hence $p \neq 5$. By [34, Lemma 5.4, Case 2], we have $5 \mid (p - 1)$ and $s = 2^{-1}(1 + \lambda)$, where $\lambda^4 + 10\lambda^2 + 5 = 0$ and $\lambda \neq 0, \pm 1$.

Since $s^4 - 2s^3 + 4s^2 - 3s + 1 = 0$, we have $(2s - 1)(8s^3 - 12s^2 + 26s - 11) = -5$, and since $p \neq 5$, we have $(2s - 1)^{-1} = -5^{-1}(8s^3 - 12s^2 + 26s - 11)$. Noting that $s^4 = 2s^3 - 4s^2 + 3s - 1$, we have $s^5 = -5s^2 + 5s - 2$ and $s^6 = -5s^3 + 5s^2 - 2s$. By Equation (4.20), $j = -(2s - 1)^{-1}(s^3 - s + 1) = 5^{-1}(8s^3 - 12s^2 + 26s - 11)(s^3 - s + 1) = s^3 - 2s^2 + 3s - 1 = 8^{-1}(\lambda^3 - \lambda^2 + 7\lambda + 1)$ and by Equations (4.15) and (4.18), $t = j + s^2 = s^3 - s^2 + 3s - 1 = 8^{-1}(\lambda^3 + \lambda^2 + 11\lambda + 3)$ and $\ell = -st = -s^3 + s^2 - 2s + 1 = -8^{-1}(\lambda^3 + \lambda^2 + 7\lambda - 1)$. It follows that

$$S = \{1, \ xy, \ x^{r+1} z, \ x^{r^2+r+1} y^{8^{-1}(\lambda^3 - \lambda^2 + 7\lambda + 1)} z^{2^{-1}(1+\lambda)},$$
$$x^{r^3+r^2+r+1} y^{-8^{-1}(\lambda^3 + \lambda^2 + 7\lambda - 1)} z^{8^{-1}(\lambda^3 + \lambda^2 + 11\lambda + 3)}\}.$$

By Proposition 4.1 and Example 3.7, $\Gamma = \operatorname{BiCay}(H, \emptyset, \emptyset, S) \cong \mathcal{CGD}^5_{mp \times p}$.

Let $\beta \in \operatorname{Aut}(H, S)$. Then $S^{\beta} = S$. Since $\langle x \rangle$ and $\langle y, z \rangle$ are characteristic in H, we have

$$\{y, z, y^{8^{-1}(\lambda^3 - \lambda^2 + 7\lambda + 1)} z^{2^{-1}(1+\lambda)}, y^{-8^{-1}(\lambda^3 + \lambda^2 + 7\lambda - 1)} z^{8^{-1}(\lambda^3 + \lambda^2 + 11\lambda + 3)}\}^{\beta}$$
$$= \{y, z, y^{8^{-1}(\lambda^3 - \lambda^2 + 7\lambda + 1)} z^{2^{-1}(1+\lambda)}, y^{-8^{-1}(\lambda^3 + \lambda^2 + 7\lambda - 1)} z^{8^{-1}(\lambda^3 + \lambda^2 + 11\lambda + 3)}\},$$

and since $\lambda \neq 0, \pm 1$, we have $y^{\beta} = y$ and $z^{\beta} = z$ (also see [35] for a detailed computation). Since $(xy)^{\beta} = x^{\beta} y \in S$, it is easy to check that $(xy)^{\beta} = xy$ and thus $x^{\beta} = x$. Hence β is the identity automorphism of H and $|\operatorname{Aut}(H, S)| = 1$. By Observation 4.3, $|N_A(R(H))| = 10|H|$. $\square$

5 Cyclic covers

In this section, we classify connected symmetric cyclic covers of connected pentavalent symmetric graphs of order twice a prime. Denote by $K_{6,6} - 6K_2$ the complete bipartite graph of order 12 minus a one-factor and by $\mathbf{I}_{12}$ the Icosahedron graph. Edge-transitive cyclic covers of K_6 were classified in [29, Theorem 1.1], and by [29, Line 20, pp. 40], such graphs have order 12 and thus isomorphic to $K_{6,6} - 6K_2$ or $\mathbf{I}_{12}$ by [20, Proposition 2.7] (note that the graph $\mathbf{I}_{12}$ is missed in [29, Theorem 1.1]).

Theorem 5.1. *Let Γ be a connected pentavalent symmetric graph of order $2p$ for a prime p, and let $\tilde{\Gamma}$ be a connected symmetric $\mathbb{Z}_n$-cover of Γ with $n \geq 2$. Then $\tilde{\Gamma} \cong K_{6,6} - 6K_2$,*

$\mathbf{I}_{12}$, $\mathcal{CD}_{np}$, or $\mathcal{CGD}^i_{mp^e \times p}$ for $1 \le i \le 5$ with $n = mp^e$, $(m, p) = 1$, $5 \mid (p-1)$ and $e \ge 1$, which are defined in Examples 3.3, 3.5, 3.6 and 3.7.

Proof. By Proposition 3.4, $\Gamma \cong K_6$ for $p = 3$, $K_{5,5}$ for $p = 5$, or $\mathcal{CD}_p$ for $5 \mid (p-1)$. If $\Gamma \cong K_6$ then $\widetilde{\Gamma} \cong K_{6,6} - 6K_2$ or $\mathbf{I}_{12}$ by [29, Theorem 1.1] (also see the proof in [2, Theorem 3.6]). In the following, we assume that $p \ge 5$. Let $A = \mathrm{Aut}(\widetilde{\Gamma})$.

Let $K = \mathbb{Z}_n$ and $F = N_A(K)$. Since $\widetilde{\Gamma}$ is a symmetric K-cover of Γ, F is arc-transitive on $\widetilde{\Gamma}$ and F/K is arc-transitive on $\widetilde{\Gamma}_K = \Gamma$. Let B/K be a minimal arc-transitive subgroup of F/K. By Proposition 3.4, $B/K \cong D_p \rtimes \mathbb{Z}_5$ for $p > 11$; by MAGMA [5], $B/K \cong D_{11} \rtimes \mathbb{Z}_5$ for $p = 11$, and $B/K \cong \mathbb{Z}_5^2 \rtimes \mathbb{Z}_2$, $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_4$ or $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_8$ for $p = 5$. Each minimal normal subgroup of B/K is isomorphic to $\mathbb{Z}_p$ or $\mathbb{Z}_5^2$ with $p = 5$ and $B/K \cong \mathbb{Z}_5^2 \rtimes \mathbb{Z}_8$. Clearly, B is arc-transitive on $\widetilde{\Gamma}$ and B/K is non-abelian.

Set $C = C_B(K)$. Since K is abelian, $K \le Z(C) \le C$, where $Z(C)$ is the center of C. Suppose $K = C$. Then $B/K = B/C \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_n^*$, which forces that B/K is abelian, a contradiction. Hence $K < C$ and $1 \ne C/K \trianglelefteq B/K$. It follows that C/K contains a minimal normal subgroup of B/K, say L/K. Then $L \trianglelefteq B$ and $L \le C \trianglelefteq B$. Furthermore, $L/K \cong \mathbb{Z}_p$, or $L/K \cong \mathbb{Z}_5^2$ with $p = 5$ and $B/K \cong \mathbb{Z}_5^2 \rtimes \mathbb{Z}_8$.

Clearly, L and L/K have two orbits on $V(\widetilde{\Gamma})$ and $V(\widetilde{\Gamma}_K)$, and $\widetilde{\Gamma}$ and $\widetilde{\Gamma}_K$ are bipartite graphs with the two orbits of L and L/K as their bipartite sets, respectively. Since $K \le Z(C)$ and $L \le C$, $K \le Z(L)$.

First, assume $L/K \cong \mathbb{Z}_p$. Since $K \le Z(L)$, L is abelian, and so $L \cong \mathbb{Z}_{np}$ or $\mathbb{Z}_n \times \mathbb{Z}_p$ with $p \mid n$. For the latter, $L \cong \mathbb{Z}_m \times \mathbb{Z}_{p^e} \times \mathbb{Z}_p$ with $n = mp^e$, $(m, p) = 1$ and $e \ge 1$. Since L/K is semiregular on $V(\widetilde{\Gamma}_K)$, L is semiregular on $V(\widetilde{\Gamma})$ and thus $\widetilde{\Gamma}$ is a bi-Cayley graph over L. Noting that $L \triangleleft B$, we have that $N_A(L)$ is arc-transitive on $\widetilde{\Gamma}$, forcing that $\widetilde{\Gamma} \cong \mathrm{BiCay}(L, \emptyset, \emptyset, S)$ for some subset $S \subseteq L$. Recall that $p \ge 5$. By Lemmas 4.2–4.5, $\widetilde{\Gamma} \cong \mathcal{CD}_{np}$ or $\mathcal{CGD}^i_{mp^e \times p}$ ($1 \le i \le 5$), as required.

Now, assume $L/K \cong \mathbb{Z}_5^2$. Then $p = 5$ and $B/K \cong \mathbb{Z}_5^2 \rtimes \mathbb{Z}_8$. Since $K \le Z(L)$ and $K = \mathbb{Z}_n$, $L = P \times H$, where P and H are the Sylow 5-subgroup and the Hall 5'-subgroup of L, respectively. Note that $H \le K$ is abelian, but P may not. Since $(L/K)_{v^K} \cong \mathbb{Z}_5$, we have $L_v = P_v \cong \mathbb{Z}_5$, where $v \in V(\widetilde{\Gamma})$ and v^K is an orbit of K on $V(\widetilde{\Gamma})$ containing v. Note that $P \trianglelefteq B$ as P is characteristic in L and $L \trianglelefteq B$. By Proposition 2.2, P has at most two orbits on $V(\widetilde{\Gamma})$ because $P_v \ne 1$, and since L has exactly two orbits on $V(\widetilde{\Gamma})$, P and L have the same orbits. It follows that $L = PL_v = PP_v = P$, forcing that $H = 1$ and K is a 5-group.

Suppose $|K| = 5^t$ with $t \ge 2$. Since K is cyclic, K has a characteristic subgroup N such that $|K/N| = 25$, and since $K \trianglelefteq B$, $N \trianglelefteq B$. By Proposition 2.2, $\widetilde{\Gamma}_N$ is a connected pentavalent B/N-arc-transitive graph of order $10|K|/|N| = 250$, and by Example 3.1, $\Gamma_N \cong \mathcal{CGD}_{5^3}$. Since $B/K \cong \mathbb{Z}_5^2 \rtimes \mathbb{Z}_8$ and $|K/N| = 5^2$, all Sylow 2-subgroups of B/N are isomorphic to $\mathbb{Z}_8$ and $|B/N| = 8 \cdot 5^4$. However, by MAGMA [5], $\mathrm{Aut}(\mathcal{CGD}_{5^3})$ has no arc-transitive subgroup of order $8 \cdot 5^4$ that has a Sylow 2-subgroup isomorphic to $\mathbb{Z}_8$, a contradiction.

Since $K \ne 1$, we have $|K| = 5$ and $|V(\widetilde{\Gamma})| = 10|K| = 50$. By Example 3.6, $\widetilde{\Gamma} \cong \mathcal{CGD}^4_{5 \times 5}$, as required. $\qquad\square$

6 Dihedral covers

In this section, we aim to classify symmetric dihedral covers of connected pentavalent symmetric graphs of order twice a prime. First, we introduce four graphs which are from [30].

Example 6.1. Let

$$\mathbf{I}_{12}^{(2)} = \text{Cay}(D_{12}, \{b, ba, ba^2, ba^4, ba^9\}),$$
$$\mathcal{G}_{48} = \text{Cay}(D_{24}, \{b, ba, ba^3, ba^{11}, ba^{20}\}).$$

be two Cayley graphs on the dihedral groups $D_{12} = \langle a, b \mid a^{12} = b^2 = 1, a^b = a^{-1} \rangle$ and $D_{24} = \langle a, b \mid a^{24} = b^2 = 1, a^b = a^{-1} \rangle$, respectively. By MAGMA [5], $\text{Aut}(\mathbf{I}_{12}^{(2)}) \cong A_5 \rtimes D_4$ and $\text{Aut}(\mathcal{G}_{48}) \cong \text{SL}(2,5) \rtimes D_4$, and their vertex stabilizers are isomorphic to F_{20}.

Example 6.2. Let

$$\mathcal{G}_{60} = \text{Cay}(A_5, \{(1\,4)(2\,5), (1\,3)(2\,5), (1\,3)(2\,4), (2\,4)(3\,5), (1\,4)(3\,5)\})$$

be a Cayley graph on A_5. By MAGMA [5], it is a connected pentavalent symmetric graph of order 60 and $\text{Aut}(\mathcal{G}_{60}) \cong A_5 \times D_5$ with vertex stabilizer isomorphic to D_5.

Example 6.3. Let G be a subgroup of S_7 generated by the elements $a = (1\,4)(2\,5)(6\,7)$, $b = (1\,3)(2\,5)(6\,7)$, $c = (1\,3)(2\,4)(6\,7)$, $d = (2\,4)(3\,5)(6\,7)$ and $e = (1\,4)(3\,5)(6\,7)$, and define $\mathcal{G}_{120} = \text{Cay}(G, \{a, b, c, d, e\})$. By MAGMA [5], $G \cong A_5 \times \mathbb{Z}_2$ and $\mathcal{G}_{120}$ is a connected pentavalent symmetric graph of order 120. Moreover, $\text{Aut}(\mathcal{G}_{120}) \cong A_5 \times D_{10}$ with vertex stabilizer isomorphic to D_5.

A list of all pentavalent G-arc-transitive graphs on up to 500 vertices with the vertex stabilizer $G_v \cong \mathbb{Z}_5$, D_5 or F_{20} was given in MAGMA code by Potočnik [30]. Based on this list, we have the following lemma.

Lemma 6.4. *Let Γ be a G-arc-transitive graph of order 24, 48, 60, 120 or 240 with vertex stabilizer $G_v \cong \mathbb{Z}_5$, D_5 or F_{20} for some $G \leq \text{Aut}(\Gamma)$ and $v \in V(\Gamma)$. Then Γ is a connected symmetric dihedral cover of K_6 if and only if $\Gamma \cong \mathbf{I}_{12}^{(2)}$, $\mathcal{G}_{48}$, $\mathcal{G}_{60}$ or $\mathcal{G}_{120}$.*

Proof. To show the necessity, let Γ be a connected symmetric dihedral cover of K_6. Then $\text{Aut}(\Gamma)$ has an arc-transitive subgroup having a normal dihedral subgroup of order $|V(\Gamma)|/6$. Since Γ is G-arc-transitive with $G_v \cong \mathbb{Z}_5$, D_5 or F_{20}, by [30] Γ is isomorphic to one of the seven graphs: three graphs of order 24, 48 and 60 respectively, two graphs of order 120 and two graphs of order 240. For the orders 24, 48 and 60, $\Gamma \cong \mathbf{I}_{12}^{(2)}$, $\mathcal{G}_{48}$ or $\mathcal{G}_{60}$ by Examples 6.1 and 6.2. For the order 120, by MAGMA [5] one graph is isomorphic to $\mathcal{G}_{120}$ and the other has no arc-transitive group of automorphisms having a normal dihedral subgroup of order 20; in this case $\Gamma \cong \mathcal{G}_{120}$. For the order 240, again by MAGMA [5] none of the two graphs has an arc-transitive group of automorphisms having a normal dihedral subgroup of order 40.

Now, we show the sufficiency. By MAGMA [5], $\text{Aut}(\mathbf{I}_{12}^{(2)})$ has a normal subgroup $N \cong D_2$. Clearly, N has more than two orbits on $V(\mathbf{I}_{12}^{(2)})$, and by Proposition 2.2, the quotient graph $(\mathbf{I}_{12}^{(2)})_N$ is a connected pentavalent symmetric graph of order 6, that is, the complete graph K_6. Thus $\mathbf{I}_{12}^{(2)}$ is a D_2-cover of K_6. Similarly, one may show that $\mathcal{G}_{48}$, $\mathcal{G}_{60}$ or $\mathcal{G}_{120}$ is a symmetric D_3-, D_5- or D_{10}-cover of K_6, respectively. □

Now, we are ready to classify symmetric dihedral covers of connected pentavalent symmetric graphs of order $2p$ for any prime p. Clearly, we have $p \geq 3$.

Theorem 6.5. *Let Γ be a connected pentavalent symmetric graph of order $2p$ with p a prime, and let $\widetilde{\Gamma}$ be a connected symmetric D_n-cover of Γ with $n \geq 2$. Then $\widetilde{\Gamma} \cong \mathbf{I}_{12}^{(2)}$, $\mathcal{G}_{48}$, $\mathcal{G}_{60}$ or $\mathcal{G}_{120}$.*

Proof. Let $K = D_n$ and let F be the fibre-preserving group. Since $\widetilde{\Gamma}$ is a symmetric K-cover of Γ, F is arc-transitive on $\widetilde{\Gamma}$ and F/K is arc-transitive on $\widetilde{\Gamma}_K = \Gamma$.

Assume $n = 2$. Then $|V(\widetilde{\Gamma})| = 2n \cdot |V(\Gamma)| = 8p$. Recall that $p \geq 3$. By [20, Proposition 2.9], $\widetilde{\Gamma} \cong \mathbf{I}_{12}^{(2)}$ or a graph $\mathcal{G}_{248}$ of order 248 with $\mathrm{Aut}(\mathcal{G}_{248}) = \mathrm{PSL}(2, 31)$. Since $\mathrm{PSL}(2, 31)$ has no proper subgroup of order divisible by 248 by MAGMA [5], $\mathrm{Aut}(\widetilde{\Gamma})$ is the unique arc-transitive group of automorphisms of $\widetilde{\Gamma}$, that is, $F \cong \mathrm{PSL}(2, 31)$. It implies that $\widetilde{\Gamma} \ncong \mathcal{G}_{248}$ because F has no normal subgroup isomorphic to D_n. Hence $\widetilde{\Gamma} \cong \mathbf{I}_{12}^{(2)}$.

Assume $n > 2$. Let $\mathbb{Z}_n$ be the cyclic subgroup of $K = D_n$ of order n. Then $\mathbb{Z}_n$ is characteristic in K and so $\mathbb{Z}_n \trianglelefteq F$ as $K \trianglelefteq F$. By Proposition 2.2, $\widetilde{\Gamma}_{\mathbb{Z}_n}$ is a connected pentavalent $F/\mathbb{Z}_n$-arc-transitive graph of order $4p$, and by [20, Proposition 2.7], $\widetilde{\Gamma}_{\mathbb{Z}_n} \cong \mathbf{I}_{12}$ or $K_{6,6} - 6K_2$. Thus $\widetilde{\Gamma}$ is a symmetric $\mathbb{Z}_n$-cover of $K_{6,6} - 6K_2$ or $\mathbf{I}_{12}$. Note that $|V(\widetilde{\Gamma})| = 12n$.

Let $\widetilde{\Gamma}_{\mathbb{Z}_n} \cong K_{6,6} - 6K_2$. Since each minimal arc-transitive subgroup of $\mathrm{Aut}(K_{6,6} - 6K_2)$ is isomorphic to $A_5 \times \mathbb{Z}_2$ or S_5 by MAGMA [5], $F/\mathbb{Z}_n$ has an arc-transitive subgroup $B/\mathbb{Z}_n = A_5 \times \mathbb{Z}_2$ or S_5. It follows that $|B_v| = 10$ for $v \in V(\widetilde{\Gamma})$, and form Proposition 2.1 that $B_v \cong D_5$. In particular, B is arc-transitive on $\widetilde{\Gamma}$ and $B/\mathbb{Z}_n$ has a normal subgroup $M/\mathbb{Z}_n = A_5$, which is edge-transitive on $\widetilde{\Gamma}_{\mathbb{Z}_n}$ and has exactly two orbits on $V(\widetilde{\Gamma}_{\mathbb{Z}_n})$. Thus $M \trianglelefteq B$ is edge-transitive and has two orbits on $V(\widetilde{\Gamma})$. Since $|B : M| = 2$, we have $M_v \cong D_5$.

Clearly, $\mathbb{Z}_n \leq C_M(\mathbb{Z}_n)$. If $\mathbb{Z}_n = C_M(\mathbb{Z}_n)$, then $A_5 = M/\mathbb{Z}_n = M/C_M(\mathbb{Z}_n) \leq \mathrm{Aut}(\mathbb{Z}_n) = \mathbb{Z}_n^*$, which is impossible. Hence $\mathbb{Z}_n$ is a proper subgroup of $C_M(\mathbb{Z}_n)$, and since $\mathrm{Mult}(A_5) = \mathbb{Z}_2$, Lemma 2.3 implies that either $M = M' \times \mathbb{Z}_n = A_5 \times \mathbb{Z}_n$ or $M = M'\mathbb{Z}_n = \mathrm{SL}(2, 5)\mathbb{Z}_n$ with $M' \cap \mathbb{Z}_n \cong \mathbb{Z}_2$. In particular, M/M' is cyclic. Since M' is characteristic in M and $M \trianglelefteq B$, we have $M' \trianglelefteq B$. If M' has at least three orbits on $V(\widetilde{\Gamma})$, by Proposition 2.2, M' is semiregular on $V(\widetilde{\Gamma})$ and $\widetilde{\Gamma}_{M'}$ is a connected pentavalent B/M'-arc-transitive graph. The stabilizer of $\alpha \in V(\widetilde{\Gamma}_{M'})$ in M/M' is isomorphic to $M_v \cong D_5$, but this is impossible because M/M' is cyclic. Thus M' has at most two orbits on $V(\widetilde{\Gamma})$ and so $|V(\widetilde{\Gamma})| \mid 2|M'|$, that is, $6n \mid |M'|$. If $M = A_5 \times \mathbb{Z}_n$, then $M' = A_5$ and $6n \mid |M'|$ implies that $n = 5$ or 10 as $n > 2$. It follows that $|V(\widetilde{\Gamma})| = 60$ or 120. Since $B_v \cong D_5$, we have $\tilde{\Gamma} \cong \mathcal{G}_{60}$ or $\mathcal{G}_{120}$ by Lemma 6.4. If $M = \mathrm{SL}(2, 5)\mathbb{Z}_n$ with $M' = \mathrm{SL}(2, 5)$ and $\mathrm{SL}(2, 5) \cap \mathbb{Z}_n \cong \mathbb{Z}_2$, then n is even and $6n \mid |M'|$ implies that $n = 4, 10$ or 20. It follows that $|V(\widetilde{\Gamma})| = 48, 120$ or 240, and from Lemma 6.4 that $\tilde{\Gamma} \cong \mathcal{G}_{48}$ or $\mathcal{G}_{120}$.

Let $\widetilde{\Gamma}_{\mathbb{Z}_n} \cong \mathbf{I}_{12}$. By MAGMA [5], under conjugation $\mathrm{Aut}(\mathbf{I}_{12})$ has only one minimal arc-transitive subgroup isomorphic to A_5, and so $F/\mathbb{Z}_n$ has an arc-transitive subgroup $B/\mathbb{Z}_n \cong A_5$. By a similar argument as the previous paragraph, $B = B'\mathbb{Z}_n$ and $B' \cap \mathbb{Z}_n \lesssim \mathrm{Mult}(A_5)$ by Lemma 2.3, forcing that either $B = B' \times \mathbb{Z}_n = A_5 \times \mathbb{Z}_n$ or $B = B'\mathbb{Z}_n = \mathrm{SL}(2, 5)\mathbb{Z}_n$ with $\mathrm{SL}(2, 5) \cap \mathbb{Z}_n \cong \mathbb{Z}_2$. Furthermore, B is arc-transitive on $\widetilde{\Gamma}$ with $B_v \cong \mathbb{Z}_5$ for

$v \in V(\widetilde{\Gamma})$, and B/B' is cyclic. If B' has more than two orbits on $V(\Gamma)$, then $\widetilde{\Gamma}_{B'}$ is a connected pentavalent B/B'-arc-transitive graph by Proposition 2.2, which is impossible because B/B' is abelian. Thus B' has at most two orbits on $V(\widetilde{\Gamma})$ and so $12n \mid 2|B'|$. If $B = A_5 \times \mathbb{Z}_n$, then $B' \cong A_5$, and $12n \mid 2|B'|$ implies that $n = 5$ or 10. It follows that $|V(\widetilde{\Gamma})| = 60$ or 120. Since $B_v \cong \mathbb{Z}_5$, we have $\widetilde{\Gamma} \cong \mathcal{G}_{60}$ or $\mathcal{G}_{120}$ by Lemma 6.4. If $B = \mathrm{SL}(2,5)\mathbb{Z}_n$ with $\mathrm{SL}(2,5) \cap \mathbb{Z}_n \cong \mathbb{Z}_2$, then $B' \cong \mathrm{SL}(2,5)$ and n is even. Since $12n \mid 2|B'|$, we have $n = 4$, 10 or 20, and so $|V(\widetilde{\Gamma})| = 48$, 120 or 240. It follows from Lemma 6.4 that $\widetilde{\Gamma} \cong \mathcal{G}_{48}$ or $\mathcal{G}_{120}$. $\qquad\square$

7 Full automorphism groups of covers

Let Γ be a symmetric D_n- or $\mathbb{Z}_n$-cover of a connected symmetric pentavalent graph of order $2p$, where $n \geq 2$ is an integer and p is a prime. In this section, we aim to determine the full automorphism group of Γ. For D_n, by Theorem 6.5, $\Gamma \cong \mathbf{I}_{12}^{(2)}$, $\mathcal{G}_{48}$, $\mathcal{G}_{60}$ or $\mathcal{G}_{120}$ and by Examples 6.1–6.3, $\mathrm{Aut}(\Gamma)$ is known. For $\mathbb{Z}_n$, by Theorem 5.1, $\Gamma \cong K_{6,6} - 6K_2$, $\mathbf{I}_{12}$, $\mathcal{CD}_{np}$ (see Example 3.3), or $\mathcal{CGD}_{mp^e \times p}^i$ with $1 \leq i \leq 5$ (see Examples 3.5, 3.6 and 3.7). In particular, for the graph $\mathcal{CGD}_{mp^e \times p}^i$, we have $mp^e = n$ and m is given by

$$m = 5^t p_1^{e_1} \cdots p_s^{e_s} \quad \text{s.t.} \quad t \leq 1, \ s \geq 0, \ e_j \geq 1, \ 5 \mid (p_j - 1) \text{ for } 0 \leq j \leq s, \qquad (7.1)$$

where m, p, e satisfy the conditions as listed in the second column in Table 2. Note that m is odd by Equation (7.1). By MAGMA [5], $\mathrm{Aut}(K_{6,6} - 6K_2) = S_6 \times \mathbb{Z}_2$ and $\mathrm{Aut}(\mathbf{I}_{12}) = A_5 \times \mathbb{Z}_2$, and by Example 3.3, $\mathrm{Aut}(\mathcal{CD}_{np}) = D_{np} \rtimes \mathbb{Z}_5$. Hence we only need to determine the full automorphism groups of $\mathcal{CGD}_{mp^e \times p}^i$ for $1 \leq i \leq 5$. All theses graphs are connected symmetric cyclic covers of some pentavalent symmetric graph of order $2p$ except $\mathcal{CGD}_{mp \times p}^4$ with $5 \mid (p + 1)$, which are connected symmetric bi-Cayley graphs over $\mathbb{Z}_{mp} \times \mathbb{Z}_p$.

Theorem 7.1. $\mathrm{Aut}(\mathcal{CGD}_{mp^e \times p}^i)$ *for* $1 \leq i \leq 5$ *is isomorphic to one group listed in Table 2.*

Table 2: Full automorphism groups of $\mathcal{CGD}_{mp^e \times p}^i$ for $1 \leq i \leq 5$.

Γ	Conditions: $(m,p) = 1$, m: Eq. (7.1)	$\mathrm{Aut}(\Gamma)$
$\mathcal{CGD}_{mp^e \times p}^i, i = 1,2$	$5 \mid (p-1)$ and $e \geq 2$	$\mathrm{Dih}(\mathbb{Z}_{mp^e} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_5$
$\mathcal{CGD}_{mp \times p}^4$	$m \neq 1, 5$, and $p = 5$ or $5 \mid (p \pm 1)$	$\mathrm{Dih}(\mathbb{Z}_{mp} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_5$
	$m = 1$ or 5, and $5 \mid (p \pm 1)$	$\mathrm{Dih}(\mathbb{Z}_{mp} \times \mathbb{Z}_p) \rtimes D_5$
	$m = 1$ and $p = 5$	$(\mathrm{Dih}(\mathbb{Z}_5^2) \times F_{20}).\mathbb{Z}_4$
$\mathcal{CGD}_{mp \times p}^5$	$5 \mid (p-1)$	$\mathrm{Dih}(\mathbb{Z}_{mp} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_5$

Proof. Let $\Gamma = \mathcal{CGD}_{mp^e \times p}^i$ for $1 \leq i \leq 5$ and $A = \mathrm{Aut}(\Gamma)$. For $(m,p) = (1,5)$, we have $\Gamma = \mathcal{CGD}_{5 \times 5}^4$ and by [16, Theorem 4.3 (1)], $\mathrm{Aut}(\Gamma) \cong (\mathrm{Dih}(\mathbb{Z}_5^2) \rtimes F_{20}).\mathbb{Z}_4$. In what follows we assume that $(m,p) \neq (1,5)$. By Examples 3.5, 3.6 and 3.7, A has an arc-transitive subgroup F isomorphic to $\mathrm{Dih}(\mathbb{Z}_{mp^e} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_5$ for $\mathcal{CGD}_{mp^e \times p}^i$ ($i = 1, 2, 3$),

$\mathrm{Dih}(\mathbb{Z}_{mp} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_5$ for $\mathcal{CGD}^4_{mp \times p}$ with $m \neq 1, 5$ and $p = 5$ or $5 \mid (p \pm 1)$, $\mathrm{Dih}(\mathbb{Z}_{mp} \times \mathbb{Z}_p) \rtimes D_5$ for $\mathcal{CGD}^4_{mp \times p}$ with $m = 1$ or 5 and $5 \mid (p \pm 1)$, and $\mathrm{Dih}(\mathbb{Z}_{mp} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_5$ for $\mathcal{CGD}^5_{mp \times p}$ with $5 \mid (p - 1)$. Note that $F_v = \mathbb{Z}_5$ or D_5 for $v \in V(\Gamma)$. Furthermore, F has a normal semiregular subgroup $K = \mathbb{Z}_{mp^e} \times \mathbb{Z}_p$ having two orbits on $V(\Gamma)$, and hence Γ is an F-arc-transitive bi-Cayley graph over K. By Lemmas 4.4 and 4.5, $|N_A(K)| = |F|$, implying that $N_A(K) = F$. Note that $|F| = 10|K|$ or $20|K|$, that is, $|F| = 10mp^{e+1}$ or $20mp^{e+1}$ with $p = 5$ or $5 \mid (p \pm 1)$, and by Equation (7.1), both m and $|K|$ are odd. In particular, $|V(\Gamma)| = 2|K| = 2mp^{e+1}$ is twice an odd integer.

Clearly, $K = \mathbb{Z}_{mp^e} \times \mathbb{Z}_p$ has a characteristic Hall $5'$-subgroup, say H. Then $H \trianglelefteq F$ as $K \trianglelefteq F$. If $H \neq K$, then $5 \mid mp^{e+1}$ and H has at least three orbits. For $p \neq 5$, we have $5 \mid m$, and since $5^2 \nmid m$ by Equation (7.1), we have $|K : H| = 5$. For $p = 5$, by Table 2, $\Gamma = \mathcal{CGD}^4_{mp \times p}$ with $(m, 5) = 1$ and $K = \mathbb{Z}_m \times \mathbb{Z}_p \times \mathbb{Z}_p$, implying that $|K : H| = 5^2$. By Proposition 2.2, Γ_H is a connected pentavalent F/H-arc-transitive graph of order $2 \cdot 5$ or $2 \cdot 5^2$. By Proposition 3.4 and Example 3.6, $\Gamma_H \cong K_{5,5}$ or $\Gamma_H \cong \mathcal{CGD}^4_{5 \times 5}$. Since $|F| = 10|K|$ or $20|K|$ and $|K|$ is odd, H is the characteristic Hall $\{2, 5\}'$-subgroup of F. Thus we have the following claim.

Claim 7.2. *H is the characteristic Hall $\{2, 5\}'$-subgroup of F, and we have $H = K$, or $|K : H| = 5$ and $\Gamma_H \cong K_{5,5}$, or $|K : H| = 25$ and $\Gamma_H \cong \mathcal{CGD}^4_{5 \times 5}$.*

To finish the proof, we only need to show that $A = F$. Suppose to the contrary that $A \neq F$. Then A has a subgroup M such that F is a maximal subgroup of M. Since F is arc-transitive on Γ, M is arc-transitive, and since $N_A(K) = F$, we have $K \ntrianglelefteq M$.

By the definitions of the graphs $\mathcal{CGD}^i_{mp^e \times p}$ ($1 \leq i \leq 5$) in Examples 3.5, 3.6 and 3.7, Γ has the 6-cycle $(1, h, a^{-r-1}b^{-\lambda-1}c^{-1}, ha^{-r}b^{-\lambda}c^{-1}, a^{-r}b^{-\lambda}c^{-1}, hab, 1)$ for $1 \leq i \leq 3$, and the 6-cycle $(1, h, a^{-r-1}c^{-1}, ha^{-r}bc^{-1}, a^{-r}bc^{-1}, hab, 1)$ for $4 \leq i \leq 5$. Suppose that Γ is $(M, 4)$-arc-transitive. Then each 4-arc lies in a 6-cycle in Γ and so Γ has diameter at most three. It follows that $|V(\Gamma)| = 2mp^{e+1} \leq 1 + 5 + 5 \cdot 4 + 5 \cdot 4 \cdot 4 = 106$, that is, $mp^{e+1} \leq 53$. Since $p = 5$ or $5 \mid (p \pm 1)$ and $e + 1 \geq 2$ (see the second column of Table 2), we have $p = 5$ and $m \leq 2$. Since m is odd, $(m, p) = (1, 5)$, contrary to assumption. Thus Γ is at most 3-arc-transitive, and by Proposition 2.1, we have $|M_v| \in \{5, 10, 20, 40, 60, 80, 120, 720, 1440, 2880\}$.

Note that $|M : F| = |M_v : F_v| \in \{2, 4, 6, 8, 12, 16, 24, 72, 144, 288, 576\}$ because $M \neq F$ and $|F_v| = 5$ or 10. Let $[M : F]$ be the set of right cosets of F in M. Consider the action of M on $[M : F]$ by right multiplication, and let F_M be the kernel of this action, that is, the largest normal subgroup of M contained in F. Then M/F_M is a primitive permutation group on $[M : F]$ because F/F_M is maximal in M/F_M, and $(M/F_M)_F = F/F_M$, the stabilizer of $F \in [M : F]$ in M/F_M. It follows that $|M/F_M| = |M : F||F/F_M|$ and so $|F/F_M| = |M/F_M|/|M : F|$. Since $|M : F| \in \{2, 4, 6, 8, 12, 16, 24, 72, 144, 288, 576\}$, by Lemma 2.4 we have $M/F_M \leq \mathrm{AGL}(t, 2)$ with $|M : F| = 2^t$ and $1 \leq t \leq 4$, or $\mathrm{soc}(M/F_M) \cong \mathrm{PSL}(2, q)$, $\mathrm{PSL}(3, 3)$ or $\mathrm{PSL}(2, r) \times \mathrm{PSL}(2, r)$ with $|M : F| = q + 1$, 144 or $(r + 1)^2$ respectively, where $q \in \{5, 7, 11, 23, 71\}$ and $r \in \{11, 23\}$.

Suppose $M/F_M \leq \mathrm{AGL}(2, 2)$ and $|M : F| = 4$. Since a 2-group cannot be primitive on $[M : F]$, we have $3 \mid |M/F_M|$ and so $3 \mid |M/F_M|/|M : F| = |F/F_M|$. Since $|F| = 10mp^{e+1}$ or $20mp^{e+1}$ with $p = 5$ or $5 \mid (p \pm 1)$, we have $3 \mid m$, which is impossible by Equation (7.1). Thus $M/F_M \nleq \mathrm{AGL}(2, 2)$. Similarly, since $7 \nmid m$, we have $M/F_M \nleq \mathrm{AGL}(3, 2)$, and if $M/F_M \leq \mathrm{AGL}(4, 2)$, then M/F_M is a $\{2, 5\}$-group. Furthermore, $\mathrm{soc}(M/F_M) \ncong \mathrm{PSL}(2, q)$, $\mathrm{PSL}(3, 3)$ or $\mathrm{PSL}(2, r) \times \mathrm{PSL}(2, r)$ for $q \in$

$\{7, 23, 71\}$ and $r = 23$ because otherwise one of $7, 23, 13, 23$ is a divisor of m. It follows that $M/F_M \cong \mathbb{Z}_2$ with $|M : F| = 2$, $M/F_M \leq \mathrm{AGL}(4, 2)$ with $|M : F| = 2^4$ and M/F_M a $\{2, 5\}$-group, $\mathrm{soc}(M/F_M) \cong \mathrm{PSL}(2, q)$ with $|M : F| = q + 1$ and $q \in \{5, 11\}$, or $\mathrm{soc}(M/F_M) \cong \mathrm{PSL}(2, 11) \times \mathrm{PSL}(2, 11)$ with $|M : F| = 144$.

First assume that $M/F_M \cong \mathbb{Z}_2$ with $|M : F| = 2$. Then $F \trianglelefteq M$ and $H \trianglelefteq M$ as H is characteristic in F by Claim 7.2. Let $C = C_M(H)$. Since K is abelian, $H \leq K \leq C$. Let P be a Sylow 5-subgroup of C containing the unique Sylow 5-subgroup of K. Since H is the Hall $5'$-group of K, $K \leq HP = H \times P$. Clearly, HP/H is a Sylow 5-subgroup of C/H. Recall that $|F/K| \mid 20$ and $|K/H| \mid 25$ (see Claim 7.2). Since $|M| = 2|F|$, we have $|M/H| \mid 2^3 \cdot 5^3$, and by Sylow theorem, M/H has a normal Sylow 5-subgroup. In particular, C/H has a normal Sylow 5-subgroup, that is, $HP/H \trianglelefteq C/H$. This implies $H \times P \trianglelefteq C$, and since $C \trianglelefteq M$ and P is characteristic in C, we have $P \trianglelefteq M$. Since $(m, p) \neq (1, 5)$ and $|V(\Gamma)| = 2mp^{e+1}$, P has at least three orbits on $V(\Gamma)$. By Proposition 2.2, P is semiregular on $V(\Gamma)$. Thus $|P| \mid |V(\Gamma)|$ and $|P| \mid |K|$. It follows that $|HP| = |H||P| \mid |K|$, and since $K \leq HP$, we have $K = HP \trianglelefteq M$, a contradiction.

Assume that $M/F_M \leq \mathrm{AGL}(4, 2)$ with $|M : F| = 2^4$ and M/F_M a $\{2, 5\}$-group. Then M/F_M has a regular normal subgroup of order 2^4, say L/F_M, and hence $L \trianglelefteq M$, $2^4 \mid |L|$ and $5 \mid |M : L|$. If L is semiregular then $2^4 \mid |V(\Gamma)| = 2mp^{e+1}$, which is impossible. Thus L is not semiregular, and so $5 \mid |L_v|$. By Proposition 2.2, L has one or two orbits, yielding that $|L| = |V(\Gamma)||L_v|$ or $|L| = |V(\Gamma)||L_v|/2$. Since $|M| = |V(\Gamma)||M_v|$, we have $|M : L| = |M_v : L_v|$ or $2|M_v : L_v|$, and since $5^2 \nmid |M_v|$, we have $5 \nmid |M : L|$, a contradiction.

Assume that $\mathrm{soc}(M/F_M) \cong \mathrm{PSL}(2, 5)$ with $|M : F| = 6$. Then $M/F_M = \mathrm{PSL}(2, 5)$ or $\mathrm{PGL}(2, 5)$, and $|F/F_M| = |M/F_M|/|M : F| = 10$ or 20. Since H is the unique normal Hall $\{2, 5\}'$-subgroup of F, we have $H \leq F_M$ and so H is characteristic in F_M. This implies $H \trianglelefteq M$ because $F_M \trianglelefteq M$. Since $M/F_M \cong (M/H)/(F_M/H)$, M/H is insolvable, and since $K \ntrianglelefteq M$, we have $H \neq K$. By Claim 7.2, $\Gamma_H \cong K_{5,5}$ or $\mathcal{CGD}_{5\times5}^4$. If $\Gamma_H \cong \mathcal{CGD}_{5\times5}^4$ then $\mathrm{Aut}(\Gamma_H) \cong (\mathrm{Dih}(\mathbb{Z}_5^2) \times F_{20}).\mathbb{Z}_4$ is solvable and so M/H is solvable, a contradiction. If $\Gamma_H \cong K_{5,5}$ then as $\mathrm{Aut}(K_{5,5}) = (S_5 \times S_5) \rtimes \mathbb{Z}_2$, it is easy to show that each insolvable arc-transitive group of $\mathrm{Aut}(K_{5,5})$ contains $A_5 \times A_5$ (this is also easily checked by MAGMA [5]), and so $|M/H| \geq 2 \cdot 60^2$. Noting that F_M is semiregular on $V(\Gamma)$, we have $|F_M| \mid |K|$. By Claim 7.2, $|K : H| \mid 5^2$, and hence $|F_M : H| \mid 5^2$. It follows that $|M/F_M| = |M/H|/|F_M/H| \geq 2 \cdot 60^2/5^2 > |\mathrm{PGL}(2, 5)|$, a contradiction.

Assume that $L/F_M := \mathrm{soc}(M/F_M) \cong \mathrm{PSL}(2, 11)$ with $|M : F| = 12$. Then $M/F_M = \mathrm{PSL}(2, 11)$ or $\mathrm{PGL}(2, 11)$, and $|F/F_M| = |M/F_M|/|M : F| = 55$ or 110. Moreover, $L \trianglelefteq M$ and $K \leq L$ as $|K|$ is odd and $|M : L| \leq 2$. Since $11 \mid |L/F_M|$, F_M has at least three orbits on $V(\Gamma)$, and by Proposition 2.2 F_M is semiregular and Γ_{F_M} is a pentavalent F/F_M-arc-transitive graph. Thus $|F_M| \mid |V(\Gamma)|$ and $|V(\Gamma_{F_M})|$ is even. Since $|V(\Gamma_{F_M})| = |V(\Gamma)|/|F_M| = 2|K|/|F_M|$, $|F_M|$ is odd and $|F_M| \mid |K|$.

Recall that H is the characteristic Hall $\{2, 5\}'$-subgroup of F by Claim 7.2. Set $N = H \cap F_M$. Since F_M has odd order, N is the characteristic Hall $5'$-subgroup of F_M, and since $F_M \trianglelefteq M$, we have $N \trianglelefteq M$. Hence F_M/N is a 5-subgroup. By Claim 7.2, $5^3 \nmid |K|$, and since $|F_M| \mid |K|$, we have $5^3 \nmid |F_M|$, that is, $|F_M/N| \mid 25$. Thus F_M/N is abelian, and $\mathrm{Aut}(F_M/N)$ is cyclic or $\mathrm{Aut}(F_M/N) \cong \mathrm{GL}(2, 5)$. If $F_M/N = C_{L/N}(F_M/N)$, then $\mathrm{PSL}(2, 11) \cong L/F_M \cong (L/N)/(F_M/N) \lesssim \mathrm{Aut}(F_M/N)$, which is impossible. Thus F_M/N is a proper subgroup of $C_{L/N}(F_M/N)$, and since $\mathrm{Mult}(\mathrm{PSL}(2, 11)) \cong \mathbb{Z}_2$, Lemma 2.3 implies that $L/N = (L/N)' \times F_M/N$ with $(L/N)' \cong \mathrm{PSL}(2, 11)$. Since

$|V(\Gamma_N)| = |V(\Gamma)|/|N| = 2|K|/|N|$ with $|K|$ odd, $(L/N)' \cong \mathrm{PSL}(2,11)$ cannot be semiregular on $V(\Gamma_N)$, implying that $5 \mid |(L/N)'_\alpha|$ for $\alpha \in V(\Gamma_N)$. It follows from Proposition 2.2 that $(L/N)'$ has at most two orbits on $V(\Gamma_N)$, and so $|(L/N)|/|(L/N)'| = |V(\Gamma_N)||(L/N)_\alpha|/(|V(\Gamma_N)||(L/N)'_\alpha|) = |(L/N)_\alpha|/|(L/N)'_\alpha|$ or $2|(L/N)_\alpha|/|(L/N)'_\alpha|$, implying that $5 \nmid (|L/N|/|(L/N)'|)$. Since $|(L/N)/(L/N)'| = |F_M/N|$ and F_M/N is a 5-group, we have $|F_M/N| = 1$, that is, $L/N = (L/N)' \cong \mathrm{PSL}(2,11)$.

Since $K \ntrianglelefteq M$ and $N \trianglelefteq M$, we have $N \neq K$, and since $K \leq C_L(N)$ and $|N|$ is odd, Lemma 2.3 implies $L = L' \times N$ with $L' \cong \mathrm{PSL}(2,11)$. Note that $L' \trianglelefteq M$. Since Γ has order twice an odd integer, L' cannot be semiregular on Γ, yielding $5 \mid |L'_v|$. By Proposition 2.2, L' has at most two orbits, and so $|\mathrm{PSL}(2,11)| = |L'| = |V(\Gamma)||L'_v|$ or $|V(\Gamma)||L'_v|/2$. It implies that $|V(\Gamma)| \mid 2|\mathrm{PSL}(2,11)|$, that is, $|V(\Gamma)| \mid 2^3 \cdot 3 \cdot 11$. Since $|V(\Gamma)| = 2mp^{e+1}$ and it is not divided by 3 or 2^2 by Equation (7.1), we have $|V(\Gamma)| = 22$, contrary to the fact that $e+1 \geq 2$.

Assume that $L/F_M := \mathrm{soc}(M/F_M) \cong \mathrm{PSL}(2,11) \times \mathrm{PSL}(2,11)$ with $|M:F| = 144$. Then there exists $L_1/F_M \trianglelefteq L/F_M$ such that $L_1/F_M \cong \mathrm{PSL}(2,11)$ and $11 \mid |L:L_1|$. Since $11 \mid |L:F_M|$, F_M has at least three orbits and so Γ_{F_M} has order twice an odd integer. This implies that L/F_M cannot be semiregular, and by Proposition 2.2, L/F_M has one or two orbits. If L/F_M has one orbit then L_1/F_M is semiregular on Γ_{F_M} as $11 \mid |L:L_1|$ implies that L_1/F_M has at least three orbits, and so $4 \mid |V(\Gamma_{F_M})|$, a contradiction. If L/F_M has two orbits then Γ_{F_M} is bipartite and L/F_M is edge-transitive on Γ_{F_M}. Furthermore, L_1/F_M fixes the bipartite sets setwise. Since $11 \mid |L:L_1|$, L_1/F_M has at least two orbits on each bipartite set, and by [20, Proposition 2.4], L_1/F_M is semiregular on Γ_{F_M}. Since $L_1/F_M \cong \mathrm{PSL}(2,11)$, again we have the contradiction that $4 \mid |V(\Gamma_{F_M})|$. □

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