Lobe, edge, and arc transitivity of graphs of connectivity 1

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Abstract

We give necessary and sufficient conditions for lobe-transitivity of locally finite and locally countable graphs whose connectivity equals 1. We show further that, given any biconnected graph Λ and a "code" assigned to each orbit of $\mathrm{Aut}(\Lambda)$, there exists a unique lobe-transitive graph Γ of connectivity 1 whose lobes are copies of Λ and is consistent with the given code at every vertex of Γ. These results lead to necessary and sufficient conditions for a graph of connectivity 1 to be edge-transitive and to be arc-transitive. Countable graphs of connectivity 1 the action of whose automorphism groups is, respectively, vertex-transitive, primitive, regular, Cayley, and Frobenius had been previously characterized in the literature.

Keywords: Lobe, lobe-transitive, edge-transitive, orbit, connectivity.

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1   Introduction

Throughout this article, Γ denotes a connected simple graph. Consider the equivalence relation $\cong$ on the edge-set $E\Gamma$ of Γ whereby $e_1 \cong e_2$ whenever the edges e_1 and e_2 lie on a common cycle of Γ. A *lobe* is a subgraph of Γ induced by an equivalence class with respect to $\cong$. Equivalently, a *lobe* is a subgraph that either consists of a cut-edge with its two incident vertices or is a maximal biconnected subgraph[1]. A vertex is a *cut-vertex* if it belongs to at least two different lobes. Connected graphs other than K_2 have connectivity 1 if and only if they have a cut-vertex. Clearly no *finite* vertex-transitive graph admits a cut-vertex.

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[1]The term "lobe" is due to O. Ore [6]. We eschew the term "block" for this purpose, as it leads to ambiguity when discussing imprimitivity.

Graphs of connectivity 1 whose automorphism groups have certain given properties have been characterized. Those whose automorphism groups are, respectively, vertex-transitive, primitive, and regular were characterized in [5]. In particular, primitive *planar* graphs of connectivity 1 were characterized in [11][2]. Cayley graphs of connectivity 1 were characterized in [9]. Graphs of connectivity 1 with Frobenius automorphism groups were characterized in [10]. In the present work, we complete this investigation; we characterize graphs of connectivity 1 whose automorphism groups act transitively on their set of lobes. As a consequence, we obtain characterizations of edge-transitive graphs and arc-transitive graphs of connectivity 1.

The conditions for a graph of connectivity 1 to be lobe-transitive or to be vertex-transitive are independent; such a graph may have either property or neither one or both. Such is not the case for edge- and arc-transitivity. In Section 3 we give necessary and sufficient conditions for a graph to be lobe-transitive. We further show that, given any biconnected graph Λ and a "list" of orbit-multiplicities of copies of $\mathrm{Aut}(\Lambda)$, one can construct a lobe-transitive graph of connectivity 1 all of whose lobes are isomorphic to Λ and locally respects the given list. We give necessary and sufficient conditions for a countable graph of connectivity 1 to be edge-transitive in Section 4 and to be arc-transitive in Section 5. As the sets of conditions for these latter two properties are more intertwined with lobe-transitivity than the characterization of vertex-transitivity (for graphs of connectivity 1), scattered throughout are examples that illustrate some algebraic distinctions among these various properties.

2   Preliminaries

Throughout this article, the symbol $\mathbb{N}$ denotes the set of positive integers. The symbols $\mathbb{I}, \mathbb{J}$, and $\mathbb{K}$, often subscripted, denote subsets of $\mathbb{N}$ of the form $\{1, 2, \ldots, n\}$ or the set $\mathbb{N}$ itself; they appear as sets of indices. All graphs (and their valences) in this article are finite or countably infinite. The symbol $\delta_{i,j}$ (the so-called "Kronecker delta") assumes the value 1 if $i = j$ and 0 if $i \neq j$. For a graph Λ and any subgroup $H \leq \mathrm{Aut}(\Lambda)$, the set of orbits of H acting on $V\Lambda$ is denoted by $\mathscr{O}(H)$.

The set of lobes of a graph Γ is denoted by $\mathscr{L}(\Gamma)$. We let $\{\mathscr{L}_k : k \in \mathbb{K}\}$ denote the partition of $\mathscr{L}(\Gamma)$ into isomorphism classes of lobes. For given $k \in \mathbb{K}$ and a lobe $\Lambda \in \mathscr{L}_k$, we let $\mathscr{O}(\mathrm{Aut}(\Lambda)) = \{(V\Lambda)_j : j \in \mathbb{J}_k\}$, and we understand that if $\sigma \colon \Lambda \to \Theta$ is an isomorphism between lobes in $\mathscr{L}_k$, then $\sigma((V\Lambda)_j) = (V\Theta)_j$ for all $j \in \mathbb{J}_k$. Finally, for each $k \in \mathbb{K}$ and $j \in \mathbb{J}_k$, we define the function $\tau_j^{(k)} \colon V\Gamma \to \mathbb{N}$ by

$$\tau_j^{(k)}(v) = |\{\Lambda \in \mathscr{L}_k : v \in (V\Lambda)_j\}|. \tag{2.1}$$

For $\Lambda_0 \in \mathscr{L}(\Gamma)$ and $n \in \mathbb{N}$, we recursively define the subgraphs

$$\Gamma_0(\Lambda_0) = \Lambda_0,$$
$$\Gamma_{n+1}(\Lambda_0) = \bigcup \{\Lambda \in \mathscr{L}(\Gamma) : V\Lambda \cap V\Gamma_n(\Lambda_0) \neq \emptyset\}.$$

Lemma 2.1 ([5, Lemma 3.1])**.** *Let $\Lambda, \Theta \in \mathscr{L}(\Gamma)$ and let $n \in \mathbb{N}$. If for each $k \in \mathbb{K}$ and $j \in \mathbb{J}_k$, the function $\tau_j^{(k)}$ is constant on $V\Gamma$, then any isomorphism $\sigma_n \colon \Gamma_n(\Lambda) \to \Gamma_n(\Theta)$ admits an extension to an automorphism $\sigma \in \mathrm{Aut}(\Gamma)$.*

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[2]For a short algebraic proof that all 1-ended planar graphs with primitive automorphism group are biconnected, see [8].

This lemma was used in [5] to prove the following characterization of vertex-transitive graphs of connectivity 1.

Theorem 2.2 ([5, Theorem 3.2]). *Let Γ be a graph of connectivity 1. A necessary and sufficient condition for Γ to be vertex-transitive is that all the functions $\tau_j^{(k)}$ be constant on $V\Gamma$.*

Notation. When all the lobes of the graph Γ are pairwise isomorphic, that is, the index set $\mathbb{K}$ has but one element, then in Equation (2.1) the index k is suppressed; we simply replace $\mathbb{J}_k$ by $\mathbb{J}$ and $\tau_j^{(k)}$ by τ_j.

3   Lobe-transitivity

Let Γ be a graph of connectivity 1. It is immediate from the above definitions that the edge-sets of the lobes of Γ are blocks of imprimitivity of the group $\mathrm{Aut}(\Gamma)$ acting on $E\Gamma$. Hence any automorphism of Γ must map lobes onto lobes, and therefore, if $\mathrm{Aut}(\Gamma)$ is to act transitively on $\mathscr{L}(\Gamma)$, then all the lobes of Γ must be pairwise isomorphic. However, pairwise-isomorphism of the lobes alone is not sufficient for lobe-transitivity, even when every vertex of Γ lies in the same number of lobes.

Let us first dispense with trees; the proof is elementary and hence omitted.

Proposition 3.1. *A finite or countable tree is lobe-transitive (and simultaneously, edge-transitive) if and only if there exist $n_1, n_2 \in \mathbb{N} \cup \{\aleph_0\}$ such that every edge has one incident vertex of valence n_1 and the other of valence n_2. If $n_1 = n_2$, the tree is also arc-transitive.*

For graphs of connectivity 1 other than trees, we have the following characterization of lobe-transitivity.

Theorem 3.2. *Let Γ be a graph of connectivity 1, and let Λ_0 be an arbitrary lobe of Γ. Let $\{P_i : i \in \mathbb{I}\}$ be the set of orbits of $\mathrm{Aut}(\Gamma)$, and let $\mathscr{Q} = \{Q_j : j \in \mathbb{J}\}$ be the set of those orbits of the stabilizer in $\mathrm{Aut}(\Gamma)$ of Λ_0 that are contained in Λ_0. Then necessary and sufficient conditions for the graph Γ to be lobe-transitive are:*

(1) For each lobe $\Lambda \in \mathscr{L}(\Gamma)$, there exists an isomorphism $\sigma_\Lambda : \Lambda_0 \to \Lambda$.

(2) For each $j \in \mathbb{J}$, there exists a function $\tau_j : V\Gamma \to \mathbb{N} \cup \{0, \aleph_0\}$ such that

   (a) for all $v \in V\Gamma$,

$$\tau_j(v) = |\{\Lambda \in \mathscr{L}(\Gamma) : v \in \sigma_\Lambda(Q_j)\}| \tag{3.1}$$

   and

   (b) for each $i \in \mathbb{I}$, τ_j is constant on P_i and is nonzero if and only if $Q_j \subset P_i$.

Proof. (Necessity) Suppose that Γ is lobe-transitive. For each lobe $\Lambda \in \mathscr{L}(\Gamma)$, there is an automorphism $\overline{\sigma}_\Lambda \in \mathrm{Aut}(\Gamma)$ that maps the fixed lobe Λ_0 onto Λ. The restriction to Λ_0 of $\overline{\sigma}_\Lambda$ is an isomorphism $\sigma_\Lambda : \Lambda_0 \to \Lambda$ that satisfies condition (1).

For any lobe Λ, an automorphism $\alpha \in \mathrm{Aut}(\Gamma)$ is in the stabilizer of Λ if and only if $\overline{\sigma}_\Lambda^{-1} \alpha \overline{\sigma}_\Lambda$ is in the stabilizer of Λ_0. It follows that the partition $\{\sigma_\Lambda(Q_j) : j \in \mathbb{J}\}$ of $V\Lambda$ is the set of orbits of the stabilizer of Λ that are contained in Λ. Furthermore, since the stabilizer of Λ_0 is a subgroup of $\mathrm{Aut}(\Gamma)$, the partition $\{\sigma_\Lambda(Q_j) : j \in \mathbb{J}\}$ of $V\Lambda$

refines the partition $\{P_i \cap V\Lambda : i \in \mathbb{I}\}$. If for some indices i and j, the vertex v satisfies $v \in \sigma_\Lambda(Q_j) \subset P_i$, then for any lobe Θ, the vertex $\sigma_\Theta \sigma_\Lambda^{-1}(v)$ lies in $P_i \cap \sigma_\Theta(Q_j)$. This implies that, for all $j \in \mathbb{J}$, the function τ_j as given in Equation (3.1) is well-defined and constant on P_i.

Suppose that for an arbitrary index $i \in \mathbb{I}$, the vertex v lies in P_i. Since by Equation (3.1), $\tau_j(v)$ counts for each $j \in \mathbb{J}$ the number of lobes Λ such that v lies in $\sigma_\Lambda(Q_j)$, it follows that $\tau_j(v)$ is positive exactly when $\sigma_\Lambda^{-1}(v) \in Q_j \subset P_i$ holds, concluding the proof of condition (2.b).

(Sufficiency) Assume conditions (1) and (2). To prove that Γ is lobe-transitive, it suffices to prove that every isomorphism $\sigma_\Theta \colon \Lambda_0 \to \Theta$ is extendable to an automorphism of Γ. (Note that in this direction of the proof, σ_Θ is not presumed to be the restriction to Λ_0 of an automorphism $\overline{\sigma}_\Theta \in \operatorname{Aut}(\Gamma)$ but, in fact, it is.)

Fix a lobe Θ_0 and a vertex $v \in V\Lambda_0$ and let $w = \sigma_{\Theta_0}(v)$. For some $j \in \mathbb{J}$, the vertex v lies in Q_j, and so $w \in \sigma_{\Theta_0}(Q_j)$. Since both $\tau_j(v)$ and $\tau_j(w)$ are therefore positive, both v and w lie in the same orbit P_i of $\operatorname{Aut}(\Gamma)$ by condition (2.b). Furthermore, since τ_j is constant on P_i for each $j \in \mathbb{J}$, there exists a bijection β_j from the set of lobes Λ such that $v \in \sigma_\Lambda(Q_j)$ onto the set of lobes Θ such that $w \in \sigma_\Theta(Q_j)$. Let Λ_1 be a lobe in the former set, and let $\Theta_1 = \beta_j(\Lambda_1)$.

Although v and w lie in the images of the same orbit Q_j in lobes Λ_1 and Θ_1, respectively, the vertices $\sigma_{\Lambda_1}^{-1}(v)$ and $\sigma_{\Theta_1}^{-1}(w)$ need not be the same vertex of Λ_0. However, since both vertices lie in the same orbit Q_j of Λ_0, there exists an automorphism $\alpha \in \operatorname{Aut}(\Lambda_0)$ such that $\alpha\sigma_{\Lambda_1}^{-1}(v) = \sigma_{\Theta_1}^{-1}(w)$. Then $\sigma_{\Theta_1}\alpha\sigma_{\Lambda_1}^{-1}$ is an isomorphism from Λ_1 onto Θ_1 that maps v onto w and therefore agrees with σ_{Θ_0} at the vertex v common to Λ_0 and Λ_1.

The amalgamation of $\sigma_{\Theta_1}\alpha\sigma_{\Lambda_1}^{-1}$ with σ_{Θ_0} is an isomorphism from $\Lambda_0 \cup \Lambda_1$ to $\Theta_0 \cup \Theta_1$. By repeating this same technique, we can extend σ_{Θ_0} to all lobes adjacent to Λ_0 and then to all of their adjacent lobes and inductively to all of Γ. $\qquad\square$

Example 3.3. Suppose that the lobes of Γ are copies of some biconnected, vertex-transitive graph and that every vertex of Γ is incident with exactly m lobes where $m \geq 2$. By Theorem 2.2, Γ is vertex-transitive. By Theorem 3.2, Γ is lobe-transitive, with $\mathscr{O}(\operatorname{Aut}(\Gamma))$ being the trivial partition (with just one big cell P_1). Also $|\mathbb{J}| = 1$ and $\tau_1(v) = m$ for all $v \in V\Gamma$.

Remark 3.4. There exists a "degenerate" family of lobe-transitive graphs Γ of connectivity 1 that have but a single cut-vertex. For some cardinal $\mathfrak{K} \geq 2$, consider a collection of $\mathfrak{K}$ copies of a biconnected graph Λ_0, and let $v_0 \in V\Lambda_0$. Let $\sigma_\Lambda \colon \Lambda_0 \to \Lambda$ be an isomorphism as in Theorem 3.2, and let $\sigma_\Lambda(v_0) = v_\Lambda$ for each copy Λ of Λ_0 in the collection. We obtain Γ by identifying v_0 and all the vertices v_Λ and naming the new amalgamated vertex w, which forms a singleton orbit $\{w\}$ of $\operatorname{Aut}(\Gamma)$. Clearly Γ is lobe-transitive and w is its unique cut-vertex. If Λ_0 has finite diameter, then Γ has zero ends (see [3]) when Λ_0 is finite and has $\mathfrak{K}$ ends when Λ_0 is infinite; if Λ_0 has infinite diameter, then Γ has at least $\mathfrak{K}$ ends. Other than the graphs just described, all *countable* lobe-transitive graphs of connectivity 1 are "tree-like" with $\aleph_0$ cut vertices and either 2 or $2^{\aleph_0}$ ends.

Theorem 3.5. *Let Λ_0 be any biconnected graph. Let $H \leq \operatorname{Aut}(\Lambda_0)$, let $\mathscr{Q} = \mathscr{O}(H) = \{Q_j : j \in \mathbb{J}\}$, and let $\mathscr{R} = \{R_k : k \in \mathbb{K}\}$ be a partition of $V\Lambda_0$ refined by $\mathscr{Q}$. For each $k \in \mathbb{K}$, let the function $\mu_k \colon \mathbb{J} \to \mathbb{N} \cup \{0, \aleph_0\}$ satisfy $\mu_k(j) > 0$ if and only if $Q_j \subseteq R_k$ and (to avoid the triviality of a single lobe) $\sum_{j \in \mathbb{J}} \mu_k(j) \geq 2$ for at least one $k \in \mathbb{K}$. Then there exists (up to isomorphism) a unique lobe-transitive graph Γ of connectivity 1 such that*

(1) for each lobe $\Lambda \in \mathscr{L}(\Gamma)$, there exists an isomorphism $\sigma_\Lambda \colon \Lambda_0 \to \Lambda$;

(2) for each vertex $v \in V\Gamma$ and each $j \in \mathbb{J}$, we have

$$\mu_k(j) = |\{\Lambda \in \mathscr{L}(\Gamma) : \sigma_\Lambda^{-1}(v) \in Q_j \subseteq R_k\}|.$$

Proof. Let Λ_0, H, $\mathscr{Q}$, and μ_k be as postulated. Let $\Gamma_0 = \Lambda_0$ from which we construct Γ_1 as follows.

Let v be any vertex of Λ_0. For some j, k, it must hold that $v \in Q_j \subseteq R_k$, and so $\mu_k(j) > 0$. For each ℓ such that $Q_\ell \subseteq R_k$, we postulate the existence of $\mu_k(\ell)$ copies Λ of Λ_0 (including Λ_0 itself when $\ell = j$) such that, if $\sigma_\Lambda \colon \Lambda_0 \to \Lambda$ is an isomorphism, then some vertex in $\sigma_\Lambda(Q_\ell)$ is identified with the vertex v. The graph Γ_1 is produced by repeating this process for each vertex of Λ_0. We repeat this process starting at each vertex $w \in V\Gamma_1 \setminus V\Gamma_0$, the only notational change being that, if specifically $w \in \sigma_\Lambda(Q_{j'})$ for some $j' \in \mathbb{J}$, then we consider the subset $\sigma_\Lambda(Q_{j'})$ of $V\Lambda$ (instead of Q_j in Λ_0) to which w belongs. Thus we construct Γ_2.

Inductively, suppose that Γ_n has been constructed for some $n \geq 2$. Let $w \in V\Gamma_n \setminus V\Gamma_{n-1}$, and so $w \in \sigma_\Lambda(Q_m)$ holds for some $m \in \mathbb{J}$ and a unique lobe $\Lambda \in \mathscr{L}(\Gamma_n) \setminus \mathscr{L}(\Gamma_{n-1})$. Supposing that $Q_m \subseteq R_k$, we postulate the existence of $\mu_k(m)$ new copies of Λ_0 that share only the vertex w with Γ_n according to the above identification. In this way we construct Γ_{n+1}. Finally, let $\Gamma = \bigcup_{n=0}^\infty \Gamma_n$.

It remains only to prove that Γ so-constructed is lobe-transitive. Let Θ be any lobe of Γ. By the above construction, all lobes of Γ are pairwise isomorphic, and so there exists an isomorphism $\sigma_\Theta \colon \Lambda_0 \to \Theta$. Starting with $\Gamma_0' = \Theta$ and by using the technique in the proof of Sufficiency in Theorem 3.2, one constructs a sequence $\Gamma_0', \Gamma_1', \ldots$ so that for all $n \in \mathbb{N}$, we have $\Gamma_n' \cong \Gamma_n$, and σ_Θ is extendable to an isomorphism from Γ_n to Γ_n'. Thus $\Gamma \cong \bigcup_{n=0}^\infty \Gamma_n'$, and σ_Θ can be extended to an automorphism of Γ. □

Example 3.6. In the notation of Theorem 3.5, let Λ_0 be the 5-cycle with one chord as shown in Figure 1(a), and let $H = \mathrm{Aut}(\Lambda)$, yielding the orbit partition $\{Q_1, Q_2, Q_3\}$ as indicated. Let $R_1 = Q_1 \cup Q_3$ and $R_2 = Q_2$, giving $\mathbb{J} = \{1, 2, 3\}$ and $\mathbb{K} = \{1, 2\}$. Define $\mu_1(1) = 3$, $\mu_1(3) = 1$, and $\mu_2(2) = 2$. Note that all other values of μ_1 and μ_2 must equal 0. Then Γ_1 is as seen in Figure 1(b).

(a) Λ_0 (b) Γ_1

Figure 1: Λ_0 and Γ_1 from Example 3.6.
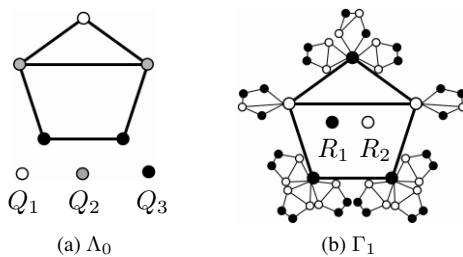
The pairs of conditions in Theorems 3.2 and 3.5 may appear alike, but there is a notable difference between them. This occurs when the arbitrarily chosen subgroup $H \leq \mathrm{Aut}(\Lambda_0)$

of Theorem 3.5 is a *proper* subgroup of the stabilizer of Λ_0 in $\mathrm{Aut}(\Gamma)$, where Γ is the graph constructed from Λ_0 and the functions μ_k of Theorem 3.5. We illustrate this distinction with following example.

Example 3.7. Our initial lobe Λ_0 is a copy of K_4, with vertices labeled as in Figure 2(a), and so $\mathrm{Aut}(\Lambda_0) \cong \mathrm{Sym}(4)$ of order 24. We use Λ_0 to "build" the lobe-transitive graph Γ shown four times in Figure 2(b). The action on Λ_0 by the stabilizer of Λ_0 in $\mathrm{Aut}(\Gamma)$ is the 4-element group $\langle g_1 \rangle \times \langle g_2 \rangle$ whose generators have cycle representation $g_1 = (v_1, v_2)$ and $g_2 = (v_3, v_4)$. The shadings of the vertices in the four depictions of Γ in Figure 2(b) correspond respectively to the four different subgroups of $\langle g_1 \rangle \times \langle g_2 \rangle$ described below. For the sake of simplicity, we assume $\mathscr{R} = \mathscr{Q}$.
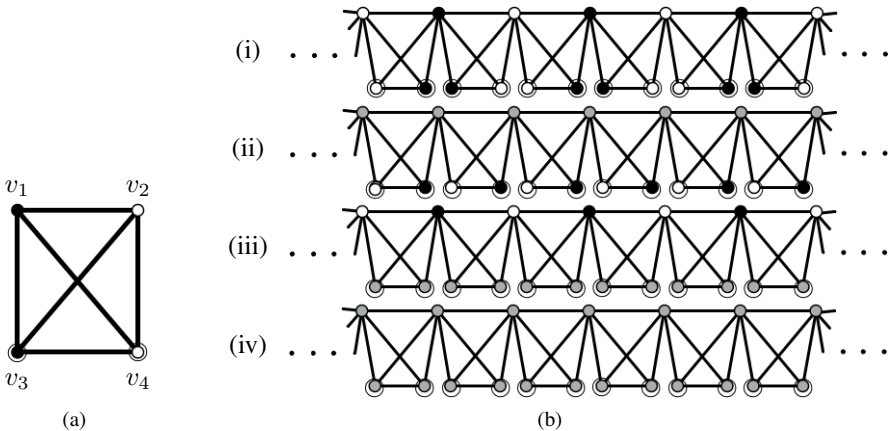

Figure 2: The clothesline graph.

(i) H is the trivial group $\{\iota\}$. Thus H induces four orbits $Q_j = \{v_j\}$ for $j \in \{1, 2, 3, 4\}$. The functions μ_k are then given by $\mu_k(j) = 2\delta_{j,k}$ for $k \in \{1, 2\}$ and $\mu_k(j) = \delta_{j,k}$ for $k \in \{3, 4\}$.

(ii) $H = \langle g_1 \rangle$. There are three orbits of H: $Q_1 = \{v_1, v_2\}$, $Q_2 = \{v_3\}$, and $Q_3 = \{v_4\}$, which give $\mu_1(1) = 2, \mu_2(2) = \mu_3(3) = 1$. All other functional values are zero.

(iii) $H = \langle g_2 \rangle$. Again there are three orbits of H but not the same ones: $Q_1 = \{v_1\}$, $Q_2 = \{v_2\}$, and $Q_3 = \{v_3, v_4\}$. This gives $\mu_k(j) = 2\delta_{j,k}$ for $k \in \{1, 2\}$ and $\mu_3(j) = \delta_{j,3}$.

(iv) $H = \langle g_1 \rangle \times \langle g_2 \rangle$. Now there are just two orbits: $Q_1 = \{v_1, v_2\}$ and $Q_2 = \{v_3, v_4\}$. Finally, $\mu_1(1) = 2, \mu_2(2) = 1$, and all other functional values are zero.

All four choices for H, the partition $\mathscr{Q}$, and the functions μ_k clearly yield the same lobe-transitive graph Γ of connectivity 1 by the construction of Theorem 3.5

4   Edge-transitivity

Lemma 4.1. *If Γ is an edge-transitive (respectively, arc-transitive) graph, then Γ is lobe-transitive and its lobes are also edge-transitive (respectively, arc-transitive).*

Proof. For $i = 1, 2$, let $\Theta_i \in \mathcal{L}(\Gamma)$, and let e_i be an edge (respectively, arc) of Θ_i. There exists an an automorphism $\varphi \in \mathrm{Aut}(\Gamma)$ such that $\varphi(e_1) = e_2$. Since φ maps cycles through e_1 onto cycles through e_2, φ must map Θ_1 onto Θ_2. If e_1 and e_2 lie in the same lobe Θ, then φ leaves Θ invariant, and so its restriction to Θ is an automorphism of Θ.   □

Theorem 4.2. *Let Γ be a graph of connectivity 1 with more than one lobe, and let $\Lambda \in \mathcal{L}(\Gamma)$. Necessary and sufficient conditions for Γ to be edge-transitive are the following:*

(1) *The lobes of Γ are edge-transitive.*

(2) *For each lobe $\Theta \in \mathcal{L}(\Gamma)$, there exists an isomorphism $\sigma_\Theta \colon \Lambda \to \Theta$.*

(3) *Exactly one of the following descriptions of Γ holds:*

 (a) *Both Γ and Λ are vertex-transitive, in which case every vertex is incident with the same number ≥ 2 of lobes.*

 (b) *The graph Γ is vertex-transitive but Λ is not vertex-transitive, in which case Λ is bipartite with bipartition $\{Q_1, Q_2\}$, and there exist constants $m_1, m_2 \in \mathbb{N} \cup \{\aleph_0\}$ such that for $j = 1, 2$ and all $v \in V\Gamma$, it holds that*

 $$m_j = |\{\Theta \in \mathcal{L}(\Gamma) : v \in \sigma_\Theta(Q_j)\}|.$$

 (c) *The graph Γ is not vertex-transitive, in which case Γ is bipartite with bipartition $\{P_1, P_2\}$ and there exist constants $m_1, m_2 \in \mathbb{N} \cup \{\aleph_0\}$, at least one of which is at least 2, such that for $i = 1, 2$, if $v \in P_i$, then*

 $$|\{\Theta \in \mathcal{L}(\Gamma) : v \in \sigma_\Theta(P_j \cap V\Lambda)\}| = m_j \delta_{i,j}.$$

Proof. *(Sufficiency)* Assume all the conditions in the hypothesis and let $e_1, e_2 \in E\Gamma$ be arbitrary edges in lobes Λ_1 and Λ_2, respectively. By condition (2), there exists an isomorphism $\sigma \colon \Lambda_1 \to \Lambda_2$. By condition (1), there exists an automorphism $\alpha \in \mathrm{Aut}(\Lambda_2)$ such that $e_2 = \alpha\sigma(e_1)$. Each of the three cases of condition (3) is seen to satisfy the hypothesis of Lemma 2.1, implying that $\alpha\sigma \colon \Lambda_1 \to \Lambda_2$ is extendable to an isomorphism of Γ mapping e_1 to e_2.

(Necessity) Suppose that Γ is an edge-transitive graph of connectivity 1. By Lemma 4.1, Γ is also lobe-transitive and its lobes are edge-transitive, proving condition (1). Condition (2), which establishes notation for the remainder of this proof, also follows from Lemma 4.1.

To prove (3), we continue the notation of Theorem 3.2 with $\mathbb{I}$ being the index set for the set of orbits of $\mathrm{Aut}(\Gamma)$ and $\mathbb{J}$ being the index set for the orbits of the stabilizer of Λ that are contained in Λ. Since both Γ and all of its lobes are edge-transitive, $|\mathbb{I}|$ and $|\mathbb{J}|$ equal either 1 or 2.

If both Γ and Λ are vertex-transitive, then $|\mathbb{I}| = |\mathbb{J}| = 1$, and for every vertex $v \in V\Gamma$, $\tau_1(v) = m$ holds for some $m \geq 2$. This is case (3.a).

Since any odd cycle in Γ would be contained in a lobe of Γ, it holds that Γ is bipartite if and only if every lobe is bipartite. If either Γ or Λ is not vertex-transitive, then each – and hence both – are bipartite, and the sides of the bipartitions (whether or not they are entire orbits of the appropriate automorphism group) are blocks of imprimitivity systems. Let $\{P_1, P_2\}$ be the bipartition of $V\Gamma$, and so $\{P_1 \cap V\Theta, P_2 \cap V\Theta\}$ is the bipartition of

any lobe Θ. Equivalently, letting $\{Q_1, Q_2\}$ denote the bipartition of Λ, we have $P_i = \bigcup_{\Theta \in \mathscr{L}(\Gamma)} \sigma_\Theta(Q_i)$ for $i = 1, 2$.

Suppose that Γ is vertex-transitive but Λ is not, and so $|\mathbb{I}| = 1$ and $|\mathbb{J}| = 2$. By Theorem 2.2, there exist constants $m_1, m_2 \in \mathbb{N} \cup \{\aleph_0\}$ such that for all $v \in V\Gamma$ and $j \in \mathbb{J}$, we have $m_j = \tau_j(v) = |\{\Theta \in \mathscr{L}(\Gamma) : v \in \sigma_\Theta(Q_j)\}|$.

Finally, suppose that Γ is not vertex-transitive, and so P_1 and P_2 are the orbits of $\mathrm{Aut}(\Gamma)$. Also, Λ is bipartite with bipartition $\{Q_1, Q_2\}$, where $Q_i = P_i \cap V\Lambda$. As no automorphism of Γ swaps P_1 with P_2, no automorphism of Γ swaps Q_1 with Q_2 (even though Λ may be vertex-transitive!). Hence $|\mathbb{I}| = |\mathbb{J}| = 2$. Since Γ is lobe-transitive, it follows now from the "necessity" argument of Theorem 3.2 that, for $j = 1, 2$, the function τ_j satisfies the condition $\tau_j(v) > 0$ if and only if $v \in P_j$. That means that there exist constants $m_1, m_2 \in \mathbb{N} \cup \{\aleph_0\}$, at least one of which is greater than 1, such that, if $v \in P_i$, then $\tau_j(v) = m_j \delta_{i,j}$. $\qquad\square$

Example 4.3. Suppose in the notation of Theorem 4.2 that Γ is edge-transitive and Λ is the complete bipartite graph $K_{s,t}$ with $|Q_1| = s$ and $|Q_2| = t$. Suppose that every vertex of Γ is incident with exactly two lobes isomorphic to Λ. If $s = t$, then both Γ and Λ are vertex-transitive, and we have case (3.a) of the theorem. If $s \neq t$ and every vertex lies in one image of Q_1 and one image of Q_2, then we have the situation of case (3.b). If again $s \neq t$ but each vertex lies in either two images of Q_1 or two images of Q_2, then we have the situation described in case (3.c).

Remark 4.4. With regard to Example 4.3, we note that having $s = t$ does not assure vertex-transitivity of edge-transitive bipartite graphs. There exist edge-transitive, non-vertex-transitive, finite bipartite graphs where the two sides of the bipartition have the same size. Such graphs are called *semisymmetric*. The smallest such graph, on 20 vertices with valence 4, was found by J. Folkman [2], who also found several infinite families of semisymmetric graphs. Many more such families as well as forbidden values for s were determined by A. V. Ivanov [4].

Example 4.5. This simple example illustrates how the converse of Lemma 4.1 is false, even though the lobes themselves may be highly symmetric. Let Γ be a graph of connectivity 1 whose lobes are copies of the Petersen graph (which is 3-arc-transitive!). For each lobe Λ, let $(V\Lambda)_1$ and $(V\Lambda)_2$ denote the vertex sets of disjoint 5-cycles indexed "consistently," i.e., if Λ and Θ share a vertex v, then $v \in (V\Lambda)_i \cap (V\Theta)_i$ for $i = 1$ or $i = 2$. (Observe that Γ is not bipartite.) For $i = 1, 2$ define $P_i = \bigcup_{\Theta \in \mathscr{L}(\Gamma)} (V\Theta)_i$, and suppose that each vertex in P_i belongs to exactly m_i lobes. The graph Γ is lobe-transitive by Theorem 3.2, and Γ is both vertex- and edge-transitive when $m_1 = m_2$, but Γ is neither vertex- nor edge-transitive when $m_1 \neq m_2$.

5   Arc-transitivity

Theorem 5.1. *Let Γ be a graph of connectivity 1. Necessary and sufficient conditions for Γ to be arc-transitive are the following:*

(1) The lobes of Γ are arc-transitive.

(2) The lobes of Γ are pairwise isomorphic.

(3) All vertices of Γ are incident with the same number of lobes.

Proof. (Necessity) Suppose that Γ is arc-transitive. Conditions (1) and (2) follow from Lemma 4.1. Since arc-transitivity implies vertex-transitivity, condition (3) holds.

(Sufficiency) Assume that the three conditions hold. For $k = 1, 2$, let a_k be an arc of Γ, and let Θ_k be the lobe containing a_k. By condition (2), there exists an isomorphism $\sigma \colon \Theta_1 \to \Theta_2$. By condition (1), there exists an automorphism $\alpha \in \mathrm{Aut}(\Theta_2)$ such that $\alpha\sigma(a_1) = a_2$. By condition (3), the functions τ_j of Equation (2.1) are constant on $V\Gamma$. (In fact, since the lobes are vertex-transitive, there is only one such function.) It now follows from Lemma 2.1 that $\alpha\sigma$ is extendable to all of Γ. $\qquad\square$

Remark 5.2. If conditions (1) and (3) of Theorem 5.1 were replaced by *the lobes are edge-transitive and Γ is vertex-transitive*, the *sufficiency* argument would fail. There exist finite graphs [1] and countably infinite graphs with polynomial growth rate [7] that are vertex- and edge-transitive but not arc-transitive. Let Λ denote such a graph, and consider a graph Γ whose lobes are isomorphic to Λ with the same number of lobes incident with every vertex. Then Γ itself is vertex- and edge-transitive but not arc-transitive.

The following proposition is elementary.

Proposition 5.3. *For all $k \geq 2$, the only k-arc-transitive graphs of connectivity* 1 *are trees of constant valence.*

References

[1] I. Z. Bouwer, Vertex and edge transitive, but not 1-transitive, graphs, *Canad. Math. Bull.* **13** (1970), 231–237, doi:10.4153/cmb-1970-047-8.

[2] J. Folkman, Regular line-symmetric graphs, *J. Comb. Theory* **3** (1967), 215–232, doi:10.1016/s0021-9800(67)80069-3.

[3] R. Halin, Automorphisms and endomorphisms of infinite locally finite graphs, *Abh. Math. Sem. Univ. Hamburg* **39** (1973), 251–283, doi:10.1007/bf02992834.

[4] A. V. Ivanov, On edge but not vertex transitive regular graphs, in: C. J. Colbourn and R. Mathon (eds.), *Combinatorial Design Theory*, North-Holland, Amsterdam, volume 149 of *North-Holland Mathematics Studies*, pp. 273–285, 1987, doi:10.1016/s0304-0208(08)72893-7.

[5] H. A. Jung and M. E. Watkins, On the structure of infinite vertex-transitive graphs, *Discrete Math.* **18** (1977), 45–53, doi:10.1016/0012-365x(77)90005-x.

[6] O. Ore, *Theory of Graphs*, volume XXXVIII of *American Mathematical Society Colloquium Publications*, American Mathematical Society, Providence, Rhode Island, 1962.

[7] C. Thomassen and M. E. Watkins, Infinite vertex-transitive, edge-transitive non-1-transitive graphs, *Proc. Amer. Math. Soc.* **105** (1989), 258–261, doi:10.2307/2046766.

[8] J. Širáň and M. E. Watkins, Imprimitivity of locally finite, 1-ended, planar graphs, *Ars Math. Contemp.* **5** (2012), 213–217, doi:10.26493/1855-3974.213.fcd.

[9] M. E. Watkins, Les graphes de Cayley de connectivité un, in: *Problèmes combinatoires et théorie des graphes*, CNRS, Paris, volume 260 of *Colloques Internationaux du Centre National de la Recherche Scientifique*, pp. 419–422, 1978, Colloque International CNRS held at the Université d'Orsay, Orsay, July 9 – 13, 1976.

[10] M. E. Watkins, Infinite graphical Frobenius representations, *Electron. J. Combin.* **25** (2018), #P4.22 (23 pages), https://www.combinatorics.org/ojs/index.php/eljc/article/view/v25i4p22.

[11] M. E. Watkins and J. E. Graver, A characterization of infinite planar primitive graphs, *J. Comb. Theory Ser. B* **91** (2004), 87–104, doi:10.1016/j.jctb.2003.10.005.