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Where Do We Stand?

This volume completes the 11^{th} year of publication of *AMC*. A very natural question for our readers, authors, reviewers and editors is this: *Where do we stand?* Although we do not believe in one-dimensional ranking, we would like to present some figures that tell how SCImago sees our journal.

SCImago uses Scopus data to rank a total of 31971 scientific journals worldwide, according to their SJR index. Among these, *AMC* ranks 3612^{th}, which puts *AMC* in the top 12%. Similarly, among the 2011 mathematics journals in the Scopus database, *AMC* ranks 246^{th}, putting it in the top 13%. At first sight this might not seem all that impressive. But among the 926 scientific journals published in Eastern Europe, *AMC* ranks 7^{th}, and among the 152 of those that are in mathematics, *AMC* ranks first!

Moreover, according to SCImago, Ars Mathematica Contemporanea covers papers from Algebra and Number Theory, Discrete Mathematics and Combinatorics, Geometry and Topology, and Theoretical Computer Science, and in each of these categories, *AMC* is in Q1 (the first quartile). It is also in the Scopus first quartile in the field of Mathematics for the year 2018.

Still, these are just numbers, and are not the most important for our community. What matters is the quality of the papers that we publish.

Also we are aware of the fact that some authors are not happy with our large backlog, and for their career, other rankings may be more important. For *AMC* itself, however, these numbers matter quite a lot. In particular, it is notable that among the 78 scientific journals covered by Scopus that are published in Slovenia, our journal also ranks first. This is of crucial importance when we seek financial support from our government.

Klavdija Kutnar, Dragan Marušič and Tomaž Pisanski
Editors in Chief

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Girth-regular graphs*

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Abstract

We introduce a notion of a *girth-regular* graph as a k-regular graph for which there exists a non-descending sequence $(a_1, a_2, \ldots, a_k)$ (called the *signature*) giving, for every vertex u of the graph, the number of girth cycles the edges with end-vertex u lie on. Girth-regularity generalises two very different aspects of symmetry in graph theory: that of vertex transitivity and that of distance-regularity. For general girth-regular graphs, we give some results on the extremal cases of signatures. We then focus on the cubic case and provide a characterisation of cubic girth-regular graphs of girth up to 5.

Keywords: Graph, girth-regular, cubic, girth.

Math. Subj. Class.: 05C38

1 Introduction

This paper stems from our research of finite connected vertex-transitive graphs of small girth. The girth (the length of a shortest cycle in the graph) is an important graph theoretical invariant that is often studied in connection with the symmetry properties of graphs. For example, cubic arc-transitive graphs (a graph is called arc-transitive if its automorphism group acts transitively on its *arcs*, where an arc is an ordered pair of adjacent vertices) and cubic semisymmetric (regular, edge-transitive but not vertex-transitive) graphs of girth up

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to 9 and 10 have been studied in [5, 11] and [6], respectively, and tetravalent edge-transitive graphs of girths 3 and 4 have been considered in [14]. Recently, a classification of all cubic vertex-transitive graphs of girth up to 5 was obtained in [8].

In our investigation of vertex-transitive graphs of small girth, it became apparent to us that the condition of vertex-transitivity is almost never used in its full strength. What was needed in most of the arguments was only a particular form of uniformity of the distribution of girth cycles throughout the graph. Let us make this more precise.

For an edge e of a graph Γ, let $\epsilon(e)$ denote the number of girth cycles containing the edge e. Let v be a vertex of Γ and let $\{e_1, \ldots, e_k\}$ be the set of edges incident to v ordered in such a way that $\epsilon(e_1) \leq \epsilon(e_2) \leq \cdots \leq \epsilon(e_k)$. Then the k-tuple $(\epsilon(e_1), \epsilon(e_2), \ldots, \epsilon(e_k))$ is called the *signature* of v. A graph Γ is called *girth-regular* provided all of its vertices have the same signature. The signature of a vertex is then called the signature of the graph.

We should like to point out that girth-regular graphs of signature $(a, a, \ldots, a)$ for some a have been introduced under the name *edge-girth-regular graphs* in [10], where the authors focused on the families of cubic and tetravalent edge-girth-regular graphs.

By definition, every girth-regular graph is regular (in the sense that all its vertices have the same valence). Further, it is clear that every vertex-transitive as well as every semisymmetric graph is girth-regular. Slightly less obvious is the fact that every distance-regular graph is also girth-regular. The notion of girth-regularity is thus a natural generalisation of all these notions. On the other hand, examples of girth-regular graphs exist that are neither vertex-transitive, nor semisymmetric nor distance-regular (for example, the truncation of a 3-prism is such a graph; see Section 3.2).

The central question we would like to propose and address in this paper is the following:

Question 1.1. Given integers k and g, for which tuples $\sigma = (a_1, a_2, \ldots, a_k) \in \mathbb{Z}^k$ does a girth-regular graph of girth g and signature σ exist?

The above question seems to be very difficult if considered in its full generality. We begin by stating three theorems proved in Section 2, which give an upper bound on the entries a_i of the signature in terms of the valence k and the girth g, and consider the case where this upper bound is attained.

Theorem 1.2. *If Γ is a girth-regular graph of valence k, girth g, and signature $(a_1, \ldots, a_k)$, then $a_k \leq (k-1)^d$, where $d = \lfloor g/2 \rfloor$.*

Theorem 1.3. *If Γ is a connected girth-regular graph of valence k, girth $2d$ for some integer d, and signature $(a_1, \ldots, a_k)$ such that $a_k = (k-1)^d$, then $a_1 = a_2 = \cdots = a_k$ and Γ is the incidence graph of a generalised d-gon of order $(k-1, k-1)$.*

In particular, if $k = 3$, then $g \in \{4, 6, 8, 12\}$ and Γ is isomorphic to $K_{3,3}$ (if $g = 4$), the Heawood graph (if $g = 6$), the Tutte-Coxeter graph (if $g = 8$) or to the Tutte 12-cage (if $g = 12$).

For a description of the graphs mentioned in the above theorem, see Note 2.2.

Theorem 1.4. *If Γ is a connected 3-valent girth-regular graph of girth $2d+1$ for some integer d and signature (a_1, a_2, a_3) such that $a_3 = 2^d$, then Γ is isomorphic to K_4 or the Petersen graph.*

In the second part of the paper, we focus on 3-valent graphs (also called *cubic* graphs) and obtain a complete classification of cubic girth-regular graphs of girth at most 5 (see

Theorem 1.5 below). Prisms and Möbius ladders are defined in Section 4, the notion of a dihedral scheme and truncation is defined in Section 3.2, and graphs arising from maps are discussed in Section 3.3.

Theorem 1.5. *Let Γ be a connected cubic girth-regular of girth g with $g \leq 5$. Then either the signature of Γ is $(0,1,1)$ and Γ is a truncation of a dihedral scheme on some g-regular graph (possibly with parallel edges), or one of the following occurs:*

1. *$g = 3$ and $\Gamma \cong K_4$ with signature $(2,2,2)$;*
2. *$g = 4$ and Γ is isomorphic to a prism or to a Möbius ladder, with signature $(4,4,4)$ if $\Gamma \cong K_{3,3}$, signature $(2,2,2)$ if Γ is isomorphic to the cube Q_3, and signature $(1,1,2)$ otherwise;*
3. *$g = 5$ and Γ is isomorphic to the Petersen graph with signature $(4,4,4)$, or to the dodecahedron with signature $(2,2,2)$.*

Since every vertex-transitive graph is girth-regular, the above result can be viewed as a partial generalisation of the classification [5] of arc-transitive cubic graphs of girth at most 9 and also a recent classification [8] of vertex-transitive cubic graphs of girth at most 5.

Unless explicitly stated otherwise, by a *graph*, we will always mean a finite simple graph, defined as a pair $(V, \sim)$ where V is the *vertex-set* and $\sim$ an irreflexive symmetric *adjacency relation* on V.

However, in Section 3.2 it will be convenient to allow graphs possessing parallel edges; details will be explained there. Finally, in Section 3.3, when considering embeddings of graphs onto surfaces, we will intuitively think of a graph in a topological context as a 1-dimensional CW complex. See that section for details.

2 An upper bound on the signature

This section is devoted to the proof of Theorems 1.2, 1.3 and 1.4 that give an upper bound on the number of girth cycles through an edge in a girth-regular graph and in some cases characterise the graphs attaining this bound.

2.1 Moore graphs and generalised n-gons

We begin by a well-known result that sets a lower bound on the number of vertices for a k-regular graph of finite girth g.

Proposition 2.1 (Tutte [16, 8.39], cf. Brouwer, Cohen & Neumaier [2, §6.7])**.** *Let Γ be a k-regular graph with n vertices and finite girth $g \geq 2$. Let $d = \lfloor g/2 \rfloor$. Then*

$$n \geq \begin{cases} 1 + k\sum_{j=0}^{(g-3)/2}(k-1)^j = \frac{k(k-1)^d - 2}{k-2} & \text{if } g \text{ is odd,} \\ 2\sum_{j=0}^{(g-2)/2}(k-1)^j = 2\frac{(k-1)^d - 1}{k-2} & \text{if } g \text{ is even.} \end{cases} \tag{2.1}$$

□

Note 2.2. Let Γ be a k-regular graph of girth g for which equality holds in (2.1). If g is odd, then such an extremal graph is called a *Moore graph*. It is well known (see [7] or [1], for example) that a Moore graph is either a complete graph, an odd cycle, or has girth 5 and valence $k \in \{3, 7, 57\}$. Of the latter, the first two cases uniquely determine the Petersen graph and the Hoffman-Singleton graph, respectively, while no example is known for $k = 57$.

If the girth g is even, then Γ is an incidence graph of a generalised $(g/2)$-gon of order $(k-1, k-1)$ (see [15] or [2, §6.5], for example). For $k = 2$, we have ordinary polygons, and their incidence graphs are even cycles. For $k \geq 3$, such generalised $(g/2)$-gons only exist if $g/2 \in \{2, 3, 4, 6\}$ (see [9, Theorem 1]). In particular, if $k = 3$, then Γ is the incidence graph of a generalised d-gon of order $(2, 2)$, where $d \in \{2, 3, 4, 6\}$. For $d = 2$, this is a geometry with three points incident to three lines, so its incidence graph is $K_{3,3}$. For $d = 3$, we get the Fano plane, and its incidence graph is the Heawood graph, which is the unique cubic arc-transitive graph on 14 vertices. For $d = 4$, there is a unique generalised quadrangle of order $(2, 2)$, cf. Payne & Thas [12, 5.2.3], and its incidence graph is the Tutte-Coxeter graph, also known as the Tutte 8-cage, which is the unique connected cubic arc-transitive graph on 30 vertices. For $d = 6$, there is a unique dual pair of generalised hexagons of order $(2, 2)$, cf. Cohen & Tits [3], and their incidence graph (on 126 vertices), also known as the Tutte 12-cage, is not vertex-transitive. However, the latter graph is edge-transitive, making it *semisymmetric* – in fact, it is the unique cubic semisymmetric graph on 126 vertices (see [4], where this graph is denoted by S126).

2.2 Proof of Theorems 1.2, 1.3 and 1.4

Equipped with these facts, we are now ready to prove Theorems 1.2, 1.3 and 1.4. Let us thus assume that Γ is a simple connected girth-regular graph of valence $k \geq 3$, let g be its girth and let $(a_1, a_2, \ldots, a_k)$ be its signature. Set $d = \lfloor g/2 \rfloor$.

In order to prove Theorem 1.2, we need to show that $a_k \leq (k-1)^d$, or equivalently, that $\epsilon(e) \leq (k-1)^d$ for every edge e of Γ.

For an integer i and a vertex v of Γ, let $S_i(v)$ denote the set of vertices of Γ that are at distance i from v, and for an edge uv of Γ, let $D^i_j(u,v) = S_i(u) \cap S_j(v)$. If i and j are integers such that $|i-j| \geq 2$, then clearly $D^i_j(u,v) = \emptyset$.

Now let uv be an arbitrary edge of Γ and let $i \in \{2, \ldots, d\}$. For simplicity, let $D^i_j = D^i_j(u,v)$. If A and B are two sets of vertices of Γ, let $E(A, B)$ be the set of edges with one end-vertex in A and the other in B. Since $g \geq 2d$, the following facts can be easily deduced:

(1) $D^i_i = \emptyset$ if $i \leq d-1$;

(2) each of D^{i-1}_i and D^i_{i-1} is an independent set;

(3) each vertex in D^{i-1}_i has precisely one neighbour in D^{i-2}_{i-1}, and if $i \leq d-1$, precisely $k-1$ neighbours in D^i_{i+1};

(3') each vertex in D^i_{i-1} has precisely one neighbour in D^{i-1}_{i-2}, and if $i \leq d-1$, precisely $k-1$ neighbours in D^{i+1}_i;

(4) $|D^i_{i-1}| = |D^{i-1}_i| = (k-1)^{i-1}$;

(5) if g is even, then $\epsilon(uv) = |E(D^{d-1}_d, D^d_{d-1})|$;

(6) if g is odd, then every vertex in D^d_d has precisely one neighbour in each of the sets D^{d-1}_d, D^d_{d-1} and $\epsilon(uv) = |D^d_d|$.

Henceforth, let uv be an arbitrary edge of Γ such that $\epsilon(uv) = a_k$ and let $D^j_i = D^j_i(u,v)$. The structure of Γ with respect to the sets D^j_i is then depicted in Figure 1.

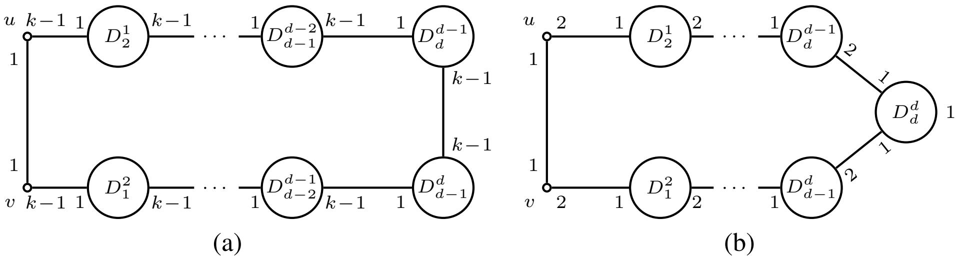

Figure 1: The partitions of the vertices of Γ of girth g corresponding to an edge uv lying on $(k-1)^d$ girth cycles, where $d = \lfloor g/2 \rfloor$. (a) shows the case when g is even, while (b) shows the case when Γ is cubic and g is odd. The sets D^i_j with $i + j < 2d$ are independent sets, while the set D^d_d in the odd case induces a perfect matching.

Suppose first that g is even. Let

$$D = \bigcup_{i=1}^{d} (D^i_{i-1} \cup D^{i-1}_i)$$

and observe that all of the vertices in D, except possibly those in D^{d-1}_d and D^d_{d-1}, have all of their neighbours contained in D. By (4), we see that

$$|D| = 2(1 + (k-1) + \cdots + (k-1)^{d-1}) = 2\frac{(k-1)^d - 1}{k-2}.$$

Moreover, it follows from (1) – (5) that

$$a_k = \epsilon(uv) = |E(D^{d-1}_d, D^d_{d-1})| \leq (k-1)|D^{d-1}_d| = (k-1)^d.$$

This proves Theorem 1.2 in the case when g is even. (The case when g is odd will be considered later.)

To prove Theorem 1.3, assume that $a_k = (k-1)^d$. Then equality holds in the above equation, implying that $|E(D^{d-1}_d, D^d_{d-1})| = (k-1)|D^{d-1}_d|$, which means that each vertex in D^{d-1}_d has $k-1$ neighbours within D^d_{d-1}. This implies that every vertex from the set D has all of its neighbours contained in D, and by connectivity of Γ, we see that $V(\Gamma) = D$. But then by Proposition 2.1 and Note 2.2, the graph Γ is the incidence graph of a generalised $g/2$-gon of order $(k-1, k-1)$. If, in addition, $k = 3$ holds, then Γ is one of the graphs mentioned in the statement of Theorem 1.3. This proves Theorem 1.3.

Let us now move to the case where g is odd, prove Theorem 1.4 and finish the proof of Theorem 1.2. Suppose henceforth that g is odd. Even though Theorem 1.4 is only about cubic graphs, we will try to continue the proof without this assumption for as long as we can. Let

$$D = D^d_d \cup \bigcup_{i=1}^{d} (D^i_{i-1} \cup D^{i-1}_i)$$

and observe that

$$|D| \leq (k-1)^d + 2(1 + (k-1) + \cdots + (k-1)^{d-1}) = \frac{k(k-1)^d - 2}{k-2}.$$

If we prove that every vertex in D has all of its neighbours contained in D, the connectivity of Γ will imply that $V(\Gamma) = D$. But then Proposition 2.1 will imply that Γ is a Moore graph. Since the only cubic Moore graphs are K_4 and the Petersen graph, this will then imply Theorem 1.4.

Note that by (2), (3) and (3'), it follows that the neighbourhoods of all vertices, except possibly those contained in D_d^{d-1}, D_{d-1}^d or D_d^d, are contained in D. By (3), (3') and (6), it follows that $|D_d^d| \le (k-1)|D_d^{d-1}| = (k-1)^d$, and by (6) we see that

$$a_k = \epsilon(uv) = |D_d^d| \le (k-1)^d,$$

thus proving Theorem 1.2 also for the case when g is odd.

Assume now that $a_k = (k-1)^d$. Then, by (6), $|D_d^d| = (k-1)^d$, implying that every vertex in D_d^{d-1} (as well as in D_{d-1}^d) has $k-1$ neighbours in D_d^d, and thus none outside the set D. To prove Theorem 1.4, it thus suffices to show that every vertex from D_d^d has all of its neighbours in D.

Since every vertex in D_{d+1}^d or D_d^{d+1} has to have at least one neighbour in D_d^{d-1} or D_{d-1}^d, respectively, and since all of the neighbours of vertices in the latter two sets lie in D_d^d, D_{d-1}^{d-2} and D_{d-2}^{d-1}, it follows that the sets D_{d+1}^d and D_d^{d+1} are empty. By consequence, the sets $D_i^{i+1}(u,v)$ and $D_{i+1}^i(u,v)$ for $i \ge d$ are also empty. Let us summarise that in Lemma 2.3.

Lemma 2.3. *Let Γ be a girth-regular graph of girth $2d+1$ and signature $(a_1,\dots,a_k)$ such that $a_k = (k-1)^d$. If uv is an edge of Γ such that $\epsilon(uv) = a_k$, then for $i \ge d$ the sets $D_i^{i+1}(u,v)$ and $D_{i+1}^i(u,v)$ are empty.*

Suppose now that $V(\Gamma) \ne D$. Then a vertex $y \in D_d^d$ has a neighbour w outside D. Since the girth of Γ is $2d+1$, there exists a unique path of length d from y to u. Let v' be the neighbour of u through which this path passes, and let u' be a neighbour of v' other than u such that $\epsilon(v'u') = \epsilon(uv)$. Let $E_j^i = D_j^i(u',v')$ and observe that by Lemma 2.3, the sets E_d^{d+1} and E_{d+1}^d are empty. Furthermore, since w is not in D but has a neighbour y in D, we see that $d(w,u) = d+1$, implying that $w \in D_{d+1}^{d+1}$.

We shall now partition the set D_d^{d-1} with respect to the distance to the vertices v' and u'. In particular, we will show that D_d^{d-1} is a disjoint union of the sets

$$X = D_d^{d-1} \cap E_{d-2}^{d-3},$$
$$Y = D_d^{d-1} \cap E_{d-2}^{d-1},$$
$$Z = D_d^{d-1} \cap E_d^d.$$

To prove this, note first that a vertex in D_d^{d-1} is at distance $d-1$ from u and thus by (1), it is either at distance $d-2$ or d from v'. Furthermore, those vertices that are at distance $d-2$ from v' are either at distance $d-3$ or $d-1$ from u', and therefore belong to X or Y. Now let x be an element of D_d^{d-1} that is at distance d from v'. Since $E_d^{d+1} = \emptyset$, this implies that x is either in E_d^{d-1} or in E_d^d. If $x \in E_d^{d-1}$, then there exist two distinct paths of length d from x to v', one passing through u and one passing through u', yielding a cycle of length at most $2d$, which is a contradiction. Hence $x \in E_d^d$, and therefore $x \in Z$.

We will now determine the sizes of X, Y and Z. In particular, we will show that:

$$|X| = (k-1)^{d-3},$$
$$|Y| = (k-2)(k-1)^{d-3},$$
$$|Z| = (k-2)(k-1)^{d-2}.$$

To prove the first equality, observe that X consists of all the ends of paths of length $d-2$ that start with $v'u'$. The equality for $|X|$ then follows from the fact that there are $(k-1)^{d-3}$ such paths. Further, note that Y consists of all the ends of paths of length $d-2$ that start in v' but do not pass through u' or u. There are $(k-2)(k-1)^{d-3}$ such paths, proving the equality for $|Y|$. Finally, to prove the equality for $|Z|$, observe that Z consists of all the ends of paths of length $d-1$ that start in u but do not pass through v or v'; there are clearly $(k-2)(k-1)^{d-2}$ such paths.

We will now partition the set D_d^d into sets X', Y' and Z' defined as follows. Let x be a vertex of D_d^d and observe that there is a unique path from x to u of length d. If this path passes through X, then we let $x \in X'$, if it passes through Y, then we let $x \in Y'$, and if it passes through Z, we let $x \in Z'$.

Since each vertex in D_d^{d-1} has $k-1$ neighbours in D_d^d and each vertex in D_d^d has precisely one neighbour in D_d^{d-1}, we see that

$$|X'| = (k-1)|X| = (k-1)^{d-2},$$
$$|Y'| = (k-1)|Y| = (k-2)(k-1)^{d-2},$$
$$|Z'| = (k-1)|Z| = (k-2)(k-1)^{d-1}.$$

Observe furthermore that a vertex x in X', having a neighbour in X, is at distance at most $d-2$ from u', but since it is at distance d from u, it is at distance exactly $d-2$ from u'. Similarly, $d(x, v') \leq d-1$ and since $d(x, u) = d$, we see that $d(x, v') = d-1$. In particular, $x \in E_{d-1}^{d-2}$ and thus

$$X' = D_d^d \cap E_{d-1}^{d-2} = E_{d-1}^{d-2}.$$

A similar argument shows that

$$Y' = D_d^d \cap E_{d-1}^d.$$

Let us now consider the set Z', and in particular the intersection $A = Z' \cap E_d^{d-1}$. Note that each vertex in Z must have at least one neighbour in A, for otherwise it could not be at distance d from u'. This implies that $|A| \geq |Z| = (k-2)(k-1)^{d-2}$. On the other hand, for a similar reason, each vertex in A must have a neighbour in X'. By comparing the sizes of A and X', we may thus conclude that every vertex in X' has $k-2$ neighbours in A and each vertex in A has precisely one neighbour in X'. In particular, every vertex in X' has all of its neighbours in D, and consequently, the vertex w has no neighbours in X'. Therefore, we have $y \in Y'$. Now recall that $w \in D_{d+1}^{d+1}$, implying that $d(w, v') \geq d$. On the other hand, w has a neighbour in Y', which is a subset of E_{d-1}^d, implying that $d(w, v') = d$. Since $E_d^{d+1} = \emptyset$, it follows that $w \in E_d^d$, and hence there exists a path $wz_1z_2 \ldots z_{d-1}u'$ of length d from w to u'. By considering possibilities for such a path, one can now easily see that $z_1 \in Z'$ and $z_2 \in X'$. But then z_1 has at least four neighbours: z_2, w, a neighbour in Z, and a neighbour in D_{d-1}^d, see Figure 2. This contradicts our assumption that the valence k is 3. This contradiction shows that $V(\Gamma) = D$, and thus completes the proof of Theorem 1.4.

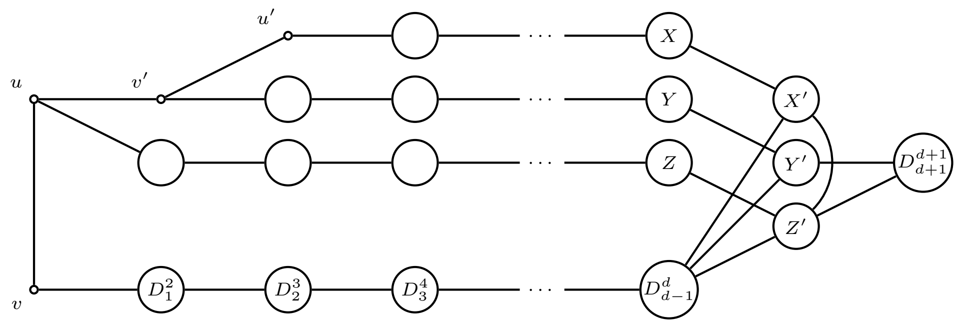

Figure 2: The partitions of the vertices of Γ of girth g, where g is odd, corresponding to the edges uv and $u'v'$, both lying on 2^d girth cycles, where $d = \lfloor g/2 \rfloor$. Assuming there is a vertex $w \in D_{d+1}^{d+1}$, we show that $w \in E_d^d$ has a neighbour in Z', which in turn must have at least four neighbours.

3 Cubic girth-regular graphs

Let us now turn our attention to cubic girth-regular graphs. After proving a few auxiliary lemmas, we will characterise cubic girth-regular graphs of some specific signatures. As an application of our analysis, we provide a characterisation of all cubic girth-regular graphs of girth at most 5 in Sections 4 and 5.

3.1 Auxiliary results

Lemma 3.1. *If (a, b, c) is the signature of a cubic girth-regular graph Γ of girth g, then:*

1. *$a + b + c$ is even,*
2. *$a + b \geq c$, and*
3. *if $a \geq 1$ and $c = a + b$, then g is even.*

Proof. Let u be a vertex of Γ and let and e_1, e_2 and e_3 be the three edges incident to u, lying on a, b and c g-cycles, respectively. Further, let x, y, z be the number of g-cycles the 2-paths e_1e_2, e_2e_3 and e_3e_1 lie on, respectively. Clearly, we have $a = x + z$, $b = x + y$ and $c = y + z$. Then $a + b + c = 2(x + y + z)$, showing that this sum is even.

Further we may express $x = (a+b-c)/2$, $y = (-a+b+c)/2$ and $z = (a-b+c)/2$. Since these numbers are nonnegative, it follows that $a + b \geq c$.

Now suppose that $a \geq 1$ and $c = a + b$. Let us call an edge e with $\epsilon(e) = c$ *saturated* and others *unsaturated*. Note that $c > b$, implying that e_1 and e_2 are unsaturated while e_3 is saturated. Since $y + z = c = a + b = 2x + y + z$, we see that $x = 0$. Since u was an arbitrary vertex of Γ, this shows that a 2-path in Γ consisting of two unsaturated edges belongs to no g-cycles. In particular, when traversing a g-cycle in Γ, saturated and unsaturated edges must alternate, implying that g is even. □

Lemma 3.2. *If the signature of a cubic girth-regular graph is $(0, b, c)$, then $b = c = 1$.*

Proof. Let Γ be a cubic girth-regular graph with signature $(0, b, c)$ and let g be its girth. By part (2) of Lemma 3.1, it follows that $b = c$. Suppose that $b > 1$. Let e be an edge of

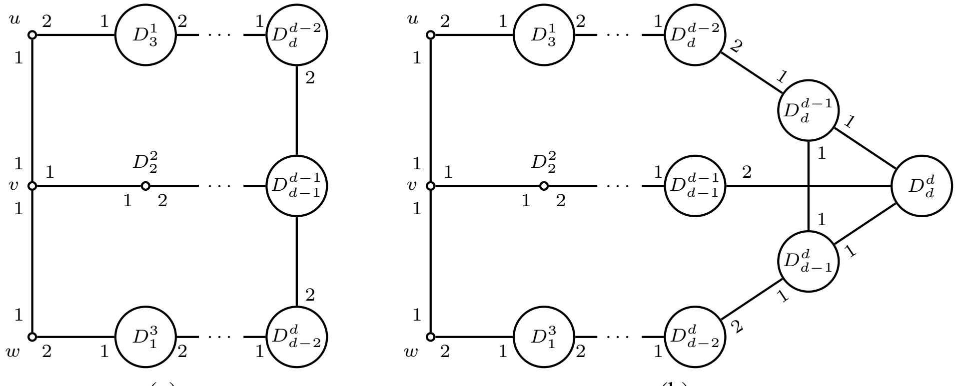

Figure 3: The partitions of the vertices of Γ of girth g corresponding to a 2-path uvw lying on 2^{d-1} girth cycles, where $d = \lfloor g/2 \rfloor$. (a) shows the case when g is even, while (b) shows the case when g is odd. The sets D^i_j with $i + j < 2d$ are independent sets, while the set D^d_d may contain edges. Note that no vertex of D^i_i ($i \in \{d-1, d\}$) with a neighbour in D^{i-1}_{i-1} can have a neighbour in D^{i-1}_i or D^i_{i-1}.

Γ lying on b g-cycles, and let C, C' be two distinct g-cycles containing e. Since $C \neq C'$, there exists a vertex u such that one of the edges incident to u lies on both C and C', while each of the remaining two edges incident to u belongs to exactly one of C and C'. However, this contradicts $a = 0$. □

Corollary 3.3. *If Γ is a cubic girth-regular graph with signature (a, b, c) and girth g, where g is odd, then $a \neq 1$.*

Proof. Suppose that $a = 1$. By part (2) of Lemma 3.1, $c = b$ or $c = b + 1$. If $b = c$, then $a + b + c$ is odd, contradicting part (1) of Lemma 3.1. Hence $c = b + 1 = a + b$, and by part (3) of Lemma 3.1, g is even, contradicting our assumptions. □

Lemma 3.4. *Let Γ be a cubic girth-regular graph of girth g with signature (a, b, c). Let $m = 2^{\lfloor g/2 \rfloor - 1}$. Then $a \geq c - m$ and $b \leq a - c + 2m$.*

Proof. Let us first show that any 2-path in Γ lies on at most m girth cycles. Let uvw be a 2-path in Γ, and let D^i_j be the set of vertices at distance i from u and at distance j from w. Set $d = \lfloor g/2 \rfloor$. Similarly as in the proof of Theorem 1.2, we can see that the number of girth cycles containing the 2-path uvw equals the number of common neighbours of vertices in the sets D^{d-2}_d and D^d_{d-2} if g is even, and the number of edges between the vertices in the sets D^{d-1}_d and D^d_{d-1} if g is odd, see Figure 3. In the even case, $|D^{d-2}_d| = |D^d_{d-2}| = 2^{d-2}$, and each of the vertices from D^{d-2}_d or D^d_{d-2} may have at most two common neighbours with vertices of the other set, so uvw can lie on at most $2^{d-1} = m$ girth cycles. In the odd case, we have $|D^{d-1}_d|, |D^d_{d-1}| \leq 2^{d-1}$, and each vertex from D^{d-1}_d or D^d_{d-1} may have at most one neighbour in the other set, as otherwise we would have a cycle of length $2d < g$. Therefore, uvw can lie on at most m girth cycles also in this case.

As each of a, b, c is the sum of the number of girth cycles two distinct 2-paths sharing the central vertex lie on, the quantity $c - a$ equals the difference between the numbers of girth cycles two such 2-paths lie on, and is therefore at most m, from which $a \geq c - m$ follows. Also, the quantity $-a + b + c$ equals twice the number of girth cycles a 2-path in Γ lies on, and is therefore at most $2m$. From this, $b \leq a - c + 2m$ follows. □

3.2 Dihedral schemes, truncations and signature $(0, 1, 1)$

In this section we will allow graphs to have parallel edges and loops. A graph *with parallel edges and loops* is defined as a triple (V, E, ∂) where V and E are the *vertex-set* and the *edge-set* of the graph and $\partial \colon E \to \{X : X \subseteq V, |X| \leq 2\}$ is a mapping that maps an edge to the set of its end-vertices. If $|\partial(e)| = 1$, then e is a *loop*. Further, we let each edge consist of two mutually inverse *arcs*, each of the two arcs having one of the end-vertices as its *tail*. If the graph has no loops, we may identify an arc with tail v underlying edge e with the pair (v, e). The set of arcs of a graph Γ is denoted by $A(\Gamma)$ and the set of the arcs with their tail being a specific vertex u by $\mathrm{out}_\Gamma(u)$. The valence of a vertex u is defined as the cardinality of $\mathrm{out}_\Gamma(u)$.

A *dihedral scheme* on a graph Γ (possibly with parallel edges and loops) is an irreflexive symmetric relation $\leftrightarrow$ on the arc-set $A(\Gamma)$ such that the simple graph $(A(\Gamma), \leftrightarrow)$ is a 2-regular graph each of whose connected components is the set $\mathrm{out}_\Gamma(u)$ for some $u \in V(\Gamma)$. (Intuitively, we may think of a dihedral scheme as a collection of circles drawn around each vertex u of Γ intersecting each of the arcs in $\mathrm{out}_\Gamma(u)$ once.) Note that, according to this definition, the minimum valence of a graph admitting a dihedral scheme is at least 3.

The group of all automorphisms of Γ that preserve the relation $\leftrightarrow$ will be denoted by $\mathrm{Aut}(\Gamma, \leftrightarrow)$ and the dihedral scheme $\leftrightarrow$ is said to be *arc-transitive* if $\mathrm{Aut}(\Gamma, \leftrightarrow)$ acts transitively on $A(\Gamma)$.

Given a dihedral scheme $\leftrightarrow$ on a graph Γ, let $\mathrm{Tr}(\Gamma, \leftrightarrow)$ be the simple graph whose vertices are the arcs of Γ and two arcs $s, t \in \Gamma$ are adjacent in Γ if either $t \leftrightarrow s$ or t and s are inverse to each other. The graph $\mathrm{Tr}(\Gamma, \leftrightarrow)$ is then called the *truncation of* Γ *with respect to the dihedral scheme* $\leftrightarrow$. Note that $\mathrm{Tr}(\Gamma, \leftrightarrow)$ is a cubic graph which is connected whenever Γ is connected.

As we shall see in Section 3.3, a natural source of arc-transitive dihedral schemes are arc-transitive maps (either orientable or non-orientable). However, not all dihedral schemes arise in this way.

Clearly, the automorphism group $\mathrm{Aut}(\Gamma, \leftrightarrow)$ acts naturally as a group of automorphisms of $\mathrm{Tr}(\Gamma, \leftrightarrow)$, implying that $\mathrm{Tr}(\Gamma, \leftrightarrow)$ is vertex-transitive whenever the dihedral scheme $\leftrightarrow$ is arc-transitive. The following result gives a characterisation of arc-transitive dihedral schemes in group theoretical terms. Here, the symbol $\mathbb{D}_d$ denotes the dihedral group of order $2d$ acting naturally on d points, while $\mathbb{Z}_d$ is the cyclic group acting transitively on d points.

Lemma 3.5. *Let Γ be an arc-transitive graph (possibly with parallel edges) of valence d for some $d \geq 3$. Then Γ admits an arc-transitive dihedral scheme if and only if there exists an arc-transitive subgroup $G \leq \mathrm{Aut}(\Gamma)$ such that the group $G_u^{\mathrm{out}_\Gamma(u)}$ induced by the action of the vertex stabiliser G_u on the set $\mathrm{out}_\Gamma(u)$ is permutation isomorphic to the transitive action of $\mathbb{D}_d$, $\mathbb{Z}_d$ or (when d is even) $\mathbb{D}_{\frac{d}{2}}$ on d vertices.*

Proof. Suppose that $\leftrightarrow$ is a dihedral scheme on Γ and that $G = \mathrm{Aut}(\Gamma, \leftrightarrow)$. Then $G_u^{\mathrm{out}_\Gamma(u)}$

preserves the restriction $\leftrightarrow_u$ of the relation $\leftrightarrow$ onto $\mathrm{out}_\Gamma(u)$, and thus acts as a vertex-transitive group of automorphisms on the simple graph $(\mathrm{out}_\Gamma(u), \leftrightarrow_u)$. Since the latter graph is a cycle of length d, we thus see that $G_u^{\mathrm{out}_\Gamma(u)}$ is a transitive subgroup of $\mathbb{D}_d$ and thus permutation isomorphic to one of the transitive actions mentioned in the statement of the lemma.

Conversely, suppose that for some vertex u, the group $G_u^{\mathrm{out}_\Gamma(u)}$ is permutation isomorphic to the transitive action of $\mathbb{D}_d$, $\mathbb{Z}_d$, or (if d is even) $\mathbb{D}_{d/2}$ on d vertices. In all three cases, we may choose an adjacency relation $\leftrightarrow_u$ on $\mathrm{out}_\Gamma(u)$ preserved by $G_u^{\mathrm{out}_\Gamma(u)}$ in such a way that $(\mathrm{out}_\Gamma(u), \leftrightarrow_u)$ is a cycle. For every $v \in V(\Gamma)$, choose an element $g_v \in G$ such that $v^{g_v} = u$, and let $\leftrightarrow_v$ be the relation on $\mathrm{out}_\Gamma(v)$ defined by $s \leftrightarrow_v t$ if and only if $s^{g_v} \leftrightarrow_u t^{g_v}$. Then clearly $(\mathrm{out}_\Gamma(v), \leftrightarrow_v)$ is a cycle, implying that the union $\leftrightarrow$ of all $\leftrightarrow_u$ for $u \in V(\Gamma)$ is a dihedral scheme. Moreover, it is a matter of straightforward computation to show that $\leftrightarrow$ is invariant under G. □

We are now ready to prove the following characterisation of cubic girth-regular graphs of signature $(0, 1, 1)$.

Theorem 3.6. *If Γ is a simple cubic girth-regular graph of girth g with signature $(0, 1, 1)$, then $\Gamma \cong \mathrm{Tr}(\Lambda, \leftrightarrow)$, where $\leftrightarrow$ is a dihedral scheme on a g-regular graph Λ (possibly with parallel edges). Moreover, if Γ is vertex-transitive, then the dihedral scheme is arc-transitive.*

Proof. Let V be the vertex-set of Γ, let $\mathcal{T}$ be the set of girth cycles in Γ, let $\mathcal{M}$ be the set of edges that belong to no girth cycle in Γ, and let $G = \mathrm{Aut}(\Gamma)$. Note that since the signature of Γ is $(0, 1, 1)$, each vertex $v \in V$ is incident to exactly one edge in $\mathcal{M}$ and to exactly one girth cycle in $\mathcal{T}$.

For an edge $v'v \in \mathcal{M}$, let C and C' be the girth cycles that pass through v and v', respectively, and let $\partial(v'v) = \{C, C'\}$. This allows us to define a graph $\Lambda = (\mathcal{T}, \mathcal{M}, \partial)$.

Note that since $C, C' \in V(\Lambda)$ are girth cycles of Γ, we have $C \neq C'$, and so Λ has no loops. This allows us to view an arc of Λ as a pair (C, e) where $e \in \mathcal{M}$ and C is a girth cycle of Γ passing through one of the two end-vertices of e. For two such pairs (C_1, e_1) and (C_2, e_2) we write $(C_1, e_1) \leftrightarrow (C_2, e_2)$ if and only if $C_1 = C_2$ and the end-vertices of e_1 and e_2 that belong to C_1 are two consecutive vertices of C_1. Then $\leftrightarrow$ is a dihedral scheme on Λ. Let $\Gamma' = \mathrm{Tr}(\Lambda, \leftrightarrow)$.

We will now show that $\Gamma' \cong \Gamma$. By the definition of truncation, the vertex-set of Γ' equals the arc-set of Λ. For an arc (C, e) of Λ let $\varphi(C, e)$ be the unique end-vertex of e that belongs to C. Since each vertex of Γ is incident to exactly one edge in $\mathcal{M}$ and exactly one cycle in $\mathcal{T}$, it follows that φ is a bijection between $V(\Gamma')$ and $V(\Gamma)$. If (C_1, e_1) and (C_2, e_2) are adjacent in Γ', then either $(C_1, e_1) \leftrightarrow (C_2, e_2)$ or (C_1, e_1) and (C_2, e_2) are inverse arcs in Γ'. In the first case, $C_1 = C_2$ and the vertices $\varphi(C_1, e_1)$ and $\varphi(C_2, e_2)$ are adjacent on C_1. In the second case, $e_1 = e_2$ and the vertices $\varphi(C_1, e_1)$ and $\varphi(C_2, e_2)$ are the two end-vertices of e_1. In both cases $\varphi(C_1, e_1)$ and $\varphi(C_2, e_2)$ are adjacent in Γ. By a similar argument we see that whenever $\varphi(C_1, e_1)$ and $\varphi(C_2, e_2)$ are adjacent in Γ, (C_1, e_1) and (C_2, e_2) are adjacent in Γ'. Since both Γ and Γ' are simple graphs (one by assumption, the other by definition), this shows that φ is a graph isomorphism.

Suppose now that G is transitive on the vertices of Γ. Since both sets $\mathcal{T}$ and $\mathcal{M}$ are invariant under the action of G, there exists a natural action of G on Λ that preserves the dihedral scheme $\leftrightarrow$; that is, $G \leq \mathrm{Aut}(\Lambda, \leftrightarrow)$. Now let (C_1, e_1) and (C_2, e_2) be two arcs of Λ, and for $i \in \{1, 2\}$, let v_i be the unique end-vertex of e_i that lies on C_i. Since G is vertex-transitive on Γ, there exists $g \in G$ mapping v_1 to v_2. Since C_i is the unique girth-cycle through v_i for $i \in \{1, 2\}$, it follows that $C_1^g = C_2$. Similarly, since e_i is the unique edge in $\mathcal{M}$ incident with v_i for $i \in \{1, 2\}$, it follows that $e_1^g = e_2$. This shows that G acts transitively on the arcs of Λ. □

Note 3.7. Parallel edges occur in the graph Λ as in Theorem 3.6 whenever there exist two girth cycles in Γ such that there are at least two edges with an end-vertex in each of the two girth cycles. In fact, it can be easily seen that in a girth-regular graph Γ with signature $(0, 1, 1)$, there are at most two such edges between any two girth cycles, leading to at most two parallel edges between each two vertices, with the exception of the case when Γ is the 3-prism (see Section 4) and Λ is the graph with two vertices and three parallel edges between them.

Note 3.8. No nontrivial bound on the girth of the graph Λ as in Theorem 3.6 can be given. In fact, we can construct a family of graphs of constant girth such that their truncations with respect to appropriate dihedral schemes are cubic girth-regular graphs with signature $(0, 1, 1)$ and unbounded girth. Let Λ be a graph obtained by doubling all edges in a k-regular graph of girth at least $\frac{k+1}{2}$ – the girth of Λ is then 2. Equip Λ with a dihedral scheme $\leftrightarrow$ such that each two arcs with a common tail belonging to two parallel edges are antipodal in the connected component of the graph defined by $\leftrightarrow$ they belong to. Then $\mathrm{Tr}(\Lambda, \leftrightarrow)$ is a cubic girth-regular graph of girth k and signature $(0, 1, 1)$.

3.3 Maps and signatures $(2, 2, 2)$ and $(1, 1, 2)$

In this section, it will be convenient to think of a graph (possibly with parallel edges) as a topological space having the structure of a regular 1-dimensional CW complex with the vertices of the graph corresponding to the 0-cells of the complex and the edges corresponding to the 1-cells. A simple closed walk (that is, a closed walk that traverses each edge at most once) in the graph then corresponds to a closed curve in the corresponding topological space which may intersect itself only in the points that correspond to the vertices of the graph.

Given a graph Γ (viewed as a CW complex) and a set of simple closed walks $\mathcal{T}$ in Γ, one can construct a 2-dimensional CW complex in the following way. First, take a collection $\mathcal{D}$ of topological disks, one for each walk in $\mathcal{T}$. Then choose a surjective continuous mapping from the boundary of each disk to the closed curve in Γ representing the corresponding walk in $\mathcal{T}$, such that the preimage of each point that is not a vertex of the graph is a singleton. Finally, identify each point of the boundary of the disk with its image under that continuous mapping. Note that the resulting topological space is independent of the choice of the homeomorphisms $\mathcal{D}$ and thus depends only on the choice of the graph and the set of closed walks $\mathcal{T}$.

When Γ is connected and the resulting topological space is a closed surface (either orientable or non-orientable), the CW complex is also called a *map*. Its open 2-cells are then called the *faces* of the map, the closed walks in $\mathcal{T}$ are called the *face-cycles* and the graph Γ is the *skeleton* of the map. A map whose skeleton is a k-regular graph and all of whose face cycles are of length m is called an $\{m, k\}$-map. The following lemma provides a sufficient condition on the set of cycles $\mathcal{T}$ under which the resulting 2-dimensional CW complex is indeed a map.

Lemma 3.9. *Let* Γ *be a graph and* $\mathcal{T}$ *a set of simple closed walks in* Γ *such that every edge of* Γ *belongs to precisely two walks in* $\mathcal{T}$*. For two arcs* s *and* t *with a common tail, write* $s \leftrightarrow t$ *if and only if the underlying edges of* s *and* t *are two consecutive edges on a walk in* $\mathcal{T}$*. If* $\leftrightarrow$ *is a dihedral scheme, then* Γ *is the skeleton of a map whose face cycles are precisely the walks in* $\mathcal{T}$*.*

Proof. Let us think of Γ as a 1-dimensional CW complex and let us turn it into a 2-dimensional CW complex by adding to it one 2-cell for each walk in $\mathcal{T}$ as described above.

Let us now prove that the resulting topological space $\mathcal{M}$ is a closed surface. It is clear that the internal vertices of the 2-cells have a regular neighbourhood. Further, since each edge of Γ lies on precisely two walks in $\mathcal{T}$, the internal points of edges also have a regular neighbourhood, made up from two half-disks, each contained in the 2-cell glued to one of the walks in $\mathcal{T}$ passing through that edge. Finally, let u be a vertex of Γ, let k be the valence of u, and let $\{s_i : i \in \mathbb{Z}_k\}$ be the set of arcs with the initial vertex u such that $s_0 \leftrightarrow s_1 \leftrightarrow \cdots \leftrightarrow s_{k-1} \leftrightarrow s_0$. By the definition of $\leftrightarrow$, each pair of arcs (s_i, s_{i+1}) $(i \in \mathbb{Z}_k)$ lies on a unique walk C_i in $\mathcal{T}$. Note that $C_i \neq C_{i+1}$, for otherwise the edge underlying s_{i+1} would lie on only one walk in $\mathcal{T}$. This implies that a regular neighbourhood of u in $\mathcal{M}$ can be built by taking appropriate half-disks from the 2-cells corresponding to the cycles C_i $(i \in \mathbb{Z}_k)$, and gluing them together in the order suggested by the relation $\leftrightarrow$. This shows that $\mathcal{M}$ is a 2-manifold without a boundary. Finally, since Γ is finite, $\mathcal{M}$ is compact, and thus a closed surface. Hence, $\mathcal{M}$ is a map with Γ as its skeleton. □

Each face of a map can be decomposed further into *flags*, that is, triangles with one vertex in the centre of a face, one vertex in the centre of an edge on the boundary of that face and one in a vertex incident with that edge. In most cases, a flag can be viewed as a triple consisting of a vertex, an edge incident to that vertex, and a face incident to both the vertex and the edge.

An automorphism of a map is then defined as a permutation of the flags induced by a homeomorphism of the surface that preserves the embedded graph. A map is said to be *vertex-transitive* or *arc-transitive* provided that its automorphism group induces a vertex-transitive or arc-transitive group on the skeleton of the map, respectively.

Note 3.10. If a map is built from a graph Γ and a set of simple closed walks $\mathcal{T}$ as in Lemma 3.9, then each automorphism of Γ that preserves the set of walks $\mathcal{T}$ clearly extends to an automorphism of the map.

If $\mathcal{M}$ is a map on a surface $\mathcal{S}$, then the sets V, E and F of the vertices, edges and faces, respectively, satisfy the *Euler formula*

$$|V| - |E| + |F| = \chi(\mathcal{S})$$

where $\chi(\mathcal{S})$ is the *Euler characteristic* of the surface $\mathcal{S}$. It is well known that $\chi(\mathcal{S}) \leq 2$ with equality holding if and only if $\mathcal{S}$ is homeomorphic to a sphere. Moreover, if $\chi(\mathcal{S})$ is odd, then $\mathcal{S}$ is non-orientable.

As the following two results show, skeletons of maps arise naturally when analysing cubic vertex-transitive graphs of signature $(2, 2, 2)$ or $(1, 1, 2)$.

Theorem 3.11. *Let* Γ *be a simple connected cubic girth-regular graph of girth* g *and order* n *with signature* $(2, 2, 2)$*. Then* g *divides* $3n$ *and* Γ *is the skeleton of a* $\{g, 3\}$*-map* *embedded on a surface with Euler characteristic*

$$\chi = n\left(\frac{3}{g} - \frac{1}{2}\right).$$

Moreover, every automorphism of Γ *extends to an automorphism of the map. In particular, if* Γ *is vertex-transitive, so is the map.*

Proof. Let $\mathcal{T}$ be the set of girth cycles of Γ. Since the valence of Γ is 3, it follows easily that the relation $\leftrightarrow$ from Lemma 3.9 satisfies the conditions stated in the lemma; that is, $\leftrightarrow$ is a dihedral scheme. Lemma 3.9 thus yields a map $\mathcal{M}$ whose skeleton is Γ and whose face-cycles are precisely the walks in $\mathcal{T}$; in particular, $\mathcal{M}$ is a $\{g, 3\}$-map, as claimed.

Since Γ is a cubic graph with n vertices, it has $3n/2$ edges, and since each vertex lies on three face-cycles and since each face-cycle contains g vertices, the map $\mathcal{M}$ has $3n/g$ faces (showing that g must divide $3n$). The Euler characteristic of $\mathcal{M}$ thus equals $n - \frac{3n}{2} + \frac{3n}{g} = n(\frac{3}{g} - \frac{1}{2})$.

Since every automorphism of Γ preserves $\mathcal{T}$, it extends to an automorphism of $\mathcal{M}$ (see Note 3.10). □

Theorem 3.11 has the following interesting consequence.

Corollary 3.12. *There exists only finitely many connected cubic girth-regular graphs with signature* $(2, 2, 2)$ *of girth at most* 5.

Proof. Suppose that Γ is a connected cubic girth-regular graph with signature $(2, 2, 2)$ of girth g and order n. By Theorem 3.11, Γ is a skeleton of a map on a surface of Euler characteristic $\chi = n(3/g - 1/2)$. Hence, if $g \leq 5$, then $\chi \geq n/10$, and since $\chi \leq 2$, it follows that $n \leq 20$. □

Note 3.13. For each $g \geq 6$, there are infinitely many girth-regular graphs of girth g with signature $(2, 2, 2)$.

If $\mathcal{M}$ is a map and Γ is its skeleton, then one can define a dihedral scheme $\leftrightarrow$ on Γ by letting $s \leftrightarrow t$ whenever the arcs s and t have a common tail and the underlying edges of s and t are two consecutive edges on some face-cycle of $\mathcal{M}$. The truncation $\mathrm{Tr}(\Gamma, \leftrightarrow)$ is then simply referred to as the *truncation of the map* $\mathcal{M}$ and denoted $\mathrm{Tr}(\mathcal{M})$. Note that this construction in some sense complements Lemma 3.9. We are now equipped for a characterisation of cubic girth-regular graphs with signature $(1, 1, 2)$.

Theorem 3.14. *Let* Γ *be a simple connected cubic girth-regular graph of girth* g *with* n *vertices and signature* $(1, 1, 2)$*. Then* g *is even and* Γ *is the truncation of some map* $\mathcal{M}$ *with face cycles of length* $g/2$*. In particular,* $g/2$ *divides* n*. Moreover, if* Γ *is vertex-transitive,* $\mathcal{M}$ *is an arc-transitive* $\{g/2, \ell\}$*-map for some* $\ell > g$*.*

Proof. By part (3) of Lemma 3.1 we know that g is even and in particular, $g \geq 4$. Let $\mathcal{X}$ be the set of edges of Γ that belong to exactly one girth cycle and let $\mathcal{Y}$ be the set of edges that belong to two girth cycles. Since the signature of Γ is $(1, 1, 2)$, every vertex of Γ is incident to two edges in $\mathcal{X}$ and one edge in $\mathcal{Y}$. Consequently, the edges in $\mathcal{Y}$ form a perfect matching of Γ and the subgraph induced by the edges in $\mathcal{X}$ is a union of vertex-disjoint cycles of Γ that cover all the vertices of Γ. Let us denote the set of these cycles by $\mathcal{C}$.

Observe also that two edges in $\mathcal{X}$ sharing a common end-vertex, say v, cannot be two consecutive edges on the same girth cycle, for otherwise that would be a unique girth cycle through v, contradicting the fact that the third edge incident with v belongs to two girth cycles. Since the edges in $\mathcal{Y}$ form a complete matching of Γ, the same holds for the edges in $\mathcal{Y}$, implying that the edges on any girth cycle alternate between the sets $\mathcal{X}$ and $\mathcal{Y}$.

For an edge e in $\mathcal{Y}$ with end-vertices u and v, let C_u and C_v be the unique cycles in $\mathcal{C}$ that pass through u and v, respectively, and define $\partial(e)$ to be the pair $\{C_u, C_v\}$. Let $\Lambda = (\mathcal{C}, \mathcal{Y}, \partial)$. Note that since the edges of Λ are precisely those edges of Γ that belong to $\mathcal{Y}$, we may think of the arc-set $A(\Lambda)$ as being the set of arcs of Γ that underlie edges in $\mathcal{Y}$. Note also that it may happen that for some $e \in \mathcal{Y}$, we may have $C_u = C_v$ and then the graph Λ has loops. If D is a girth cycle of Γ, then the edges of D that belong to $\mathcal{Y}$ induce a simple closed walk in the graph Λ of length $g/2$, which we denote $\hat{D}$.

Let $\mathcal{T}$ be the set of walks $\hat{D}$ where D runs through the set of girth cycles of Γ. Since edges of Λ correspond to the edges of Γ that pass through two girth cycles of Γ, each edge of Λ belongs to two walks in $\mathcal{T}$. As $|\mathcal{Y}| = n/2$, it follows that $g/2$ divides n.

Let $\leftrightarrow$ be the relation on the arcs of Λ defined by $\mathcal{T}$ as explained in Lemma 3.9. It is easy to see that $\leftrightarrow$ is a dihedral scheme. Indeed, let $C \in \mathcal{C}$ be a vertex of Λ viewed as a cycle in Γ and let $v_0, v_1, \ldots, v_{k-1} \in V(\Gamma)$ be its vertices listed in a cyclical order as they appear on C. Further, for each $i \in \mathbb{Z}_k$, let s_i be the arc of Γ with tail v_i that underlies an edge contained in $\mathcal{Y}$. The arc s_i can thus also be viewed as an arc of Λ. Observe that $\mathrm{out}_\Lambda(C) = \{s_0, s_1, \ldots, s_{k-1}\}$ and that $s_0 \leftrightarrow s_1 \leftrightarrow \cdots \leftrightarrow s_{k-1} \leftrightarrow s_0$. In particular, $\leftrightarrow$ is a dihedral scheme.

By Lemma 3.9, there exists a map $\mathcal{M}$ with skeleton Λ in which $\mathcal{T}$ is the set of face-cycles. Moreover, $\leftrightarrow$ equals the dihedral scheme arising from that map.

Let $\Gamma' = \mathrm{Tr}(\mathcal{M})$ and let s be a vertex of Γ'. Then s is an arc of Λ and thus also an arc of Γ underlying an edge in $\mathcal{Y}$. By letting $\varphi(s)$ be the tail of s (viewed as an arc of Γ), we define a mapping $\varphi\colon V(\Gamma') \to V(\Gamma)$. Note that the mapping which assigns to a vertex $v \in V(\Gamma)$ the unique arc of Γ with tail v that underlies an edge in $\mathcal{Y}$ is the inverse of φ, showing that φ is a bijection. Furthermore, note that two vertices s and t of Γ' are adjacent in Γ' if and only if one of the following happens: (1) they are inverse to each other as arcs of Λ; or (2) they have a common tail and $s \leftrightarrow t$. In case (1), $\varphi(s)$ and $\varphi(t)$ are adjacent in Γ via an edge in $\mathcal{Y}$, while in case (2), $\varphi(s)$ and $\varphi(t)$ are adjacent in Γ via an edge in $\mathcal{X}$. Conversely, if for some $s, t \in V(\Gamma')$, the images $\varphi(s)$ and $\varphi(t)$ are adjacent in Γ, then either s and t are inverse to each other as arcs of Λ (this happens if $\varphi(s)$ and $\varphi(t)$ form an edge in $\mathcal{Y}$), or s and t have a common tail and $s \leftrightarrow t$ (this happens if $\varphi(s)$ and $\varphi(t)$ form an edge in $\mathcal{X}$). In both cases, s and t are adjacent in Γ'. This implies that φ is an isomorphism of graphs and thus $\Gamma \cong \mathrm{Tr}(\mathcal{M})$, as claimed.

Since every automorphism of Γ preserves each of the sets $\mathcal{Y}$ and $\mathcal{X}$ (and thus also $\mathcal{C}$), it clearly induces an automorphism of the graph Λ which preserves the set $\mathcal{T}$. In particular, every automorphism of Γ induces an automorphism of the map $\mathcal{M}$.

Finally, suppose that Γ is vertex-transitive. Then all cycles of $\mathcal{C}$ have the same length $\ell > g$. As each vertex of a cycle of $\mathcal{C}$ is incident to precisely one edge of $\mathcal{Y}$, it follows that Λ is an ℓ-regular graph, and $\mathcal{M}$ is then a $\{g/2, \ell\}$-map. Let G be a group of automorphisms of Γ acting transitively on $V(\Gamma)$. Note that every vertex of Γ is the tail of precisely one arc of Γ that underlies an edge of $\mathcal{Y}$. In view of our identification of the arcs of Γ' with the arcs of Γ that underlie an edge in $\mathcal{Y}$, we thus see that the transitivity of the action of G on $V(\Gamma)$ implies the transitivity of the action of G on the arcs of $\mathcal{M}$. □

4 Cubic girth-regular graphs of girths 3 and 4

Before stating the theorem about girth-regular cubic graphs of girth 3, let us point out that every cubic graph admits a unique dihedral scheme, which is preserved by every automorphism of the graph. This allows us to talk about truncations of cubic graphs without specifying the dihedral scheme.

Theorem 4.1. *Let* Γ *be a connected cubic girth-regular graph of girth* 3*. Then one of the following holds:*

(a) Γ *is isomorphic to the complete graph* K_4*;*

(b) Γ *has signature* $(0, 1, 1)$ *and is isomorphic to the truncation of a cubic graph.*

Proof. Let (a, b, c) be the signature of Γ. By Theorem 1.2 it follows that $c \leq 2$. If $c = 2$, then Theorem 1.4 implies that Γ is isomorphic to K_4. On the other hand, if $c = 1$, then Lemmas 3.1 and 3.2 imply that the signature of Γ is $(0, 1, 1)$, and by Theorem 3.6, it follows that Γ is the truncation of a cubic graph. □

Let us now move our attention to graphs of girth 4. Before stating the classification theorem, let us define two families of cubic vertex-transitive graphs.

For $n \geq 3$, let the *n-Möbius ladder* M_n be the Cayley graph $\mathrm{Cay}(\mathbb{Z}_{2n}, \{-1, 1, n\})$. Note that such a graph has girth 4. The graph M_n has signature $(4, 4, 4)$ if $n = 3$ (in this case it is isomorphic to the complete bipartite graph $K_{3,3}$), and $(1, 1, 2)$ if $n \geq 4$. An n-Möbius ladder can also be seen as the skeleton of the truncation of the $\{2, 2n\}$-map with a single vertex embedded on a projective plane.

For $n \geq 3$, the *n-prism* Y_n is defined as the Cartesian product $C_n \,\square\, K_2$ or, alternatively, as the Cayley graph $\mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, \{(-1, 0), (1, 0), (0, 1)\})$. The girth of Y_3 is 3, while the girth of Y_n for $n \geq 4$ is 4. The graph Y_n has signature $(2, 2, 2)$ if $n = 4$ (in this case it is isomorphic to the cube Q_3), and $(1, 1, 2)$ if $n \geq 5$. An n-prism can also be seen as the skeleton of the truncation of the $\{2, n\}$-map with two vertices embedded on a sphere, i.e., an n-gonal hosohedron.

Theorem 4.2. *Let* Γ *be a connected cubic girth-regular graph of girth* 4*. Then* Γ *is isomorphic to one of the following graphs:*

(a) the n*-Möbius ladder* M_n *for some* $n \geq 3$*;*

(b) the n*-prism* Y_n *for some* $n \geq 4$*;*

(c) $\mathrm{Tr}(\Lambda, \leftrightarrow)$ *for some tetravalent graph* Λ *and a dihedral scheme* $\leftrightarrow$ *on* Λ*.*

Proof. Let (a, b, c) be the signature of Γ. By Theorem 1.2, we see that $c \leq 4$, and by Theorem 1.3, if $c = 4$, then the signature of Γ is $(4, 4, 4)$ and $\Gamma \cong K_{3,3} \cong M_3$.

Suppose now that $c = 3$. Then, by Lemma 3.1, $a + b$ is odd, and by Lemma 3.2, $a \geq 1$. Hence either $a = 1$ and then $b = 2$, or $a = 2$ and then $b = 3$. The possible signatures in this case are thus $(1, 2, 3)$ and $(2, 3, 3)$. Let us show that neither can occur.

Let uv be an edge of Γ lying on three 4-cycles, and u_0, u_1 and v_0, v_1 be the remaining neighbours of u and v, respectively. There must be three edges with one end-vertex in $\{u_0, u_1\}$ and the other in $\{v_0, v_1\}$; without loss of generality, these edges are u_0v_0, u_0v_1 and u_1v_1 (see Figure 4(c)). Then the edges uu_0 and vv_1 already lie on three 4-cycles,

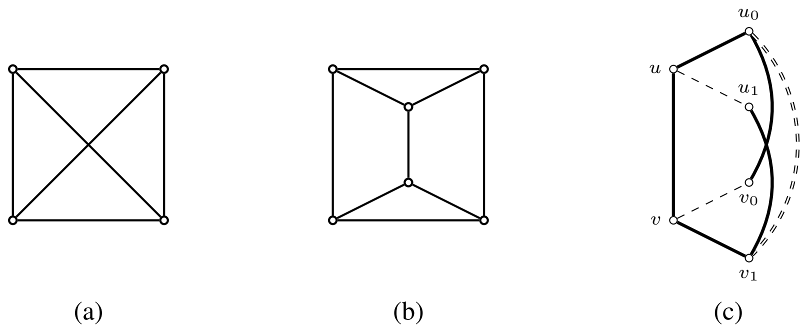

Figure 4: (a) The graph K_4 of girth 3 with signature $(2,2,2)$. (b) The graph Y_3 of girth 3 with signature $(1,1,2)$. (c) Constructing a graph of girth 4 with $c = 3$. The dashed edges should lie on two 4-cycles, however the doubled edge already lies on three 4-cycles.

so we have $b = 3$, and thus $a = 2$. In particular, $\epsilon(uu_1) = \epsilon(vv_0) = 2$. Then the edge u_1v_1, being incident to both u_1 and v_1, belongs to precisely three 4-cycles, that is $\epsilon(u_1v_1) = 3$. Similarly, $\epsilon(u_0v_0) = 3$. It follows that the edge u_0v_1 lies on precisely two 4-cycles. However, we have already determined three 4-cycles on which u_0v_1 lies; these are uu_0v_1v, $v_0u_0v_1v$, and $uu_0v_1u_1$. This contradiction shows that the case $c = 3$ is not possible.

Suppose now that $c = 2$. By Lemma 3.1, $a + b$ is even, and by Lemma 3.2, $a \geq 1$. Hence the signature of Γ is either $(1,1,2)$ or $(2,2,2)$.

If $(a,b,c) = (1,1,2)$, then, by Theorem 3.14, Γ is the skeleton of the truncation of a connected map $\mathcal{M}$ with face cycles of length 2. Since every edge belongs to two faces and every face is surrounded by two edges, the number of faces equals the number of edges. The Euler characteristics $\chi(\mathcal{S})$ of the underlying surface $\mathcal{S}$ thus equals $|V(\mathcal{M})|$. Since $\chi(\mathcal{S}) \leq 2$, it follows that $\mathcal{M}$ has one or two vertices, depending on whether $\mathcal{S}$ is the projective plane or the sphere – in particular, the skeleton of $\mathcal{M}$ is an ℓ-regular graph for some $\ell > 4$. If $\mathcal{M}$ has one vertex only, then it consists of $\ell/2$ loops embedded onto the projective plane in such a way that its truncation is the Möbius ladder M_n with $n = \ell/2 \geq 4$, see Figure 5(a) (note that $M_3 \cong K_{3,3}$ has signature $(4,4,4)$). On the other hand, if $\mathcal{M}$ has two vertices, then $\mathcal{M}$ is the map with two vertices and ℓ parallel edges embedded onto the sphere. The graph Γ is then isomorphic to the n-prism Y_n with $n = \ell \geq 5$.

If $(a,b,c) = (2,2,2)$, then, by Theorem 3.11, Γ is the skeleton of a $\{4,3\}$-map embedded on a surface of Euler characteristic $\chi = n/4 > 0$. As above, $\chi \leq 2$ and thus $\chi = 1$ or 2. For $\chi = 1$, we get the hemicube on the projective plane (see Figure 5(b)), and its skeleton is isomorphic to K_4 of girth 3. For $\chi = 2$, we get the cube on a sphere, and its skeleton is isomorphic to Y_4 with signature $(2,2,2)$. This completes the case $c = 2$.

If $c = 1$, then since $a + b + c$ is even, we see that $a = 0$ and $b = 1$, and then by Theorem 3.6, Γ is the truncation of a 4-regular graph with respect to some dihedral scheme. □

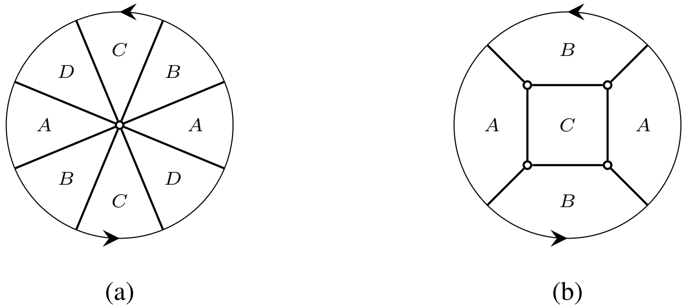

Figure 5: (a) A $\{2,8\}$-map with a single vertex, four edges and four labelled faces embedded on the projective plane. Its truncation has the graph M_4 with signature $(1,1,2)$ as its skeleton. (b) The hemicube on the projective plane with labelled faces. Its skeleton is the graph K_4.

5 Cubic girth-regular graphs of girth 5

Theorem 5.1. *Let* Γ *be a connected cubic girth-regular graph of girth* 5*. Then either the signature of* Γ *is* $(0,1,1)$ *and* Γ *is the truncation of a* 5*-regular graph with respect to some dihedral scheme, or* Γ *is isomorphic to the Petersen graph or to the dodecahedron graph.*

Proof. Let (a,b,c) be the signature of Γ. By Theorem 1.2, we see that $c \leq 4$. If $c = 4$, then by Theorem 1.3 and Theorem 1.4, the signature of Γ is $(4,4,4)$ and Γ is isomorphic to the Petersen graph. We may thus assume that $c \leq 3$.

If $a = 0$, then by Lemma 3.2 the signature of Γ is $(0,1,1)$, and then by Theorem 3.6, Γ is the truncation of a 5-regular graph with respect to some dihedral scheme. Moreover, by Corollary 3.3, $a \neq 1$. We may thus assume that $a \geq 2$.

If $c = 2$, then the signature of Γ is $(2,2,2)$ and by Theorem 3.11, Γ is the skeleton of a $\{5,3\}$-map embedded on a surface of Euler characteristic $\chi = n/10$, where n is the order of the graph Γ. In particular, $\chi \in \{1,2\}$. If $\chi = 1$, then $n = 10$ and since the girth of Γ is 5, Proposition 2.1 and Note 2.2 imply that Γ is the Petersen graph (whose signature is in fact $(4,4,4)$). If $\chi = 2$, then $n = 20$ and Γ is the skeleton of a $\{5,3\}$-map on the sphere. It is well known that there is only one such map, namely the dodecahedron.

Finally, suppose that $c = 3$. Then, by Lemma 3.1, $a + b$ is odd, and since $a \geq 2$, the signature of Γ is $(2,3,3)$. We will now show that this possibility does not occur.

Let uv be an edge of Γ lying on three 5-cycles, and u_0, u_1, v_0, v_1 be vertices of Γ with adjacencies $u_0 \sim u \sim u_1$ and $v_0 \sim v \sim v_1$. Then there should be three vertices adjacent to one of u_0, u_1 and one of v_0, v_1. Without loss of generality, let w_{00}, w_{10}, w_{11} be vertices such that $u_0 \sim w_{00} \sim v_0 \sim w_{10} \sim u_1 \sim w_{11} \sim v_1$. Further, let x be the neighbour of u_0 other than u and w_{00}, and let y be the neighbour of v_1 other than v and w_{11}, see Figure 6(a). Observe that $x \neq y$, for otherwise the edge uv would belong to four 5-cycles. Note also that x is not adjacent to any of the three neighbours of v, for otherwise the girth of Γ would be at most 4.

The signature implies that for each vertex, two edges incident to it lie on three 5-cycles. Suppose that uu_0 lies on three 5-cycles. As x and v have no common neighbours, w_{00} and x must have a common neighbour with u_1, so we have $w_{00} \sim w_{11}$ and $x \sim w_{10}$, see Figure 6(b). But then the edge uu_1 lies on four 5-cycles, contradiction. Therefore, the edge

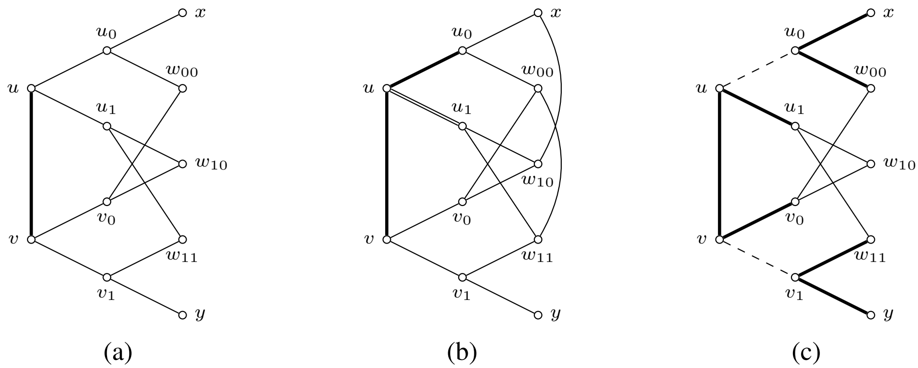

Figure 6: Constructing a graph of girth 5 with $c = 3$. The bold edges lie on three 5-cycles, and the dashed edges lie on two 5-cycles. The general setting is shown in (a). In (b), the arc (u, u_0) is assumed to lie on three 5-cycles, but the doubled edge then lies in four 5-cycles. In (c), the obtained distribution of edges among cycles is shown, which, however, cannot be completed.

uu_0 must lie on two 5-cycles, and a similar argument shows the same for vv_1. Thus, the arcs uu_1, vv_0, u_0x, u_0w_{00}, v_1w_{11} and v_1y must lie on three 5-cycles, see Figure 6(c).

Since the edge u_0w_{00} lies on three 5-cycles, there should be three vertices adjacent to one of u, x and one of v_0 and the remaining neighbour of w_{00}. Similarly, v_1w_{11} lying on three 5-cycles implies that there should be three vertices adjacent to one of v, y and one of u_1 and the remaining neighbour of w_{11}. As w_{10} is the only potential common neighbour for x, v_0 and for y, u_1, it follows that at least one of these pairs does not have a common neighbour. Without loss of generality, we may assume that x and v_0 do not have a common neighbour. The vertex u already has a common neighbour with v_0, and it must also have a common neighbour with the remaining neighbour of w_{00}. Then the remaining neighbour of w_{00} must be w_{11}, which however has no common neighbour with x, contradiction. Therefore, $(a, b, c) = (2, 3, 3)$ is not possible. □

6 Concluding remarks

Theorem 1.5 gives a complete classification of simple connected cubic girth-regular graphs of girths up to 5. While extending the classification to non-simple graphs (i.e., girths 1 and 2) is straightforward, increasing the girth leads to exponentially many more possible signatures. For example, the census of connected cubic vertex-transitive graphs on at most 1280 vertices by Potočnik, Spiga and Verret [13] shows that 9 distinct signatures appear among graphs of girth 6, while many more signatures are allowed by the results in Sections 1, 2 and Subsection 3.1. A classification of connected cubic vertex-transitive graphs of girth 6 will thus be given in a follow-up paper.

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The existence of square non-integer Heffter arrays

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Abstract

A Heffter array $H(n;k)$ is an $n \times n$ matrix such that each row and column contains k filled cells, each row and column sum is divisible by $2nk+1$ and either x or $-x$ appears in the array for each integer $1 \leqslant x \leqslant nk$. Heffter arrays are useful for embedding the graph K_{2nk+1} on an orientable surface. An integer Heffter array is one in which each row and column sum is 0. Necessary and sufficient conditions (on n and k) for the existence of an integer Heffter array $H(n;k)$ were verified by Archdeacon, Dinitz, Donovan and Yazıcı (2015) and Dinitz and Wanless (2017). In this paper we consider square Heffter arrays that are not necessarily integer. We show that such Heffter arrays exist whenever $3 \leqslant k < n$.

Keywords: Heffter arrays, biembedding cycle systems.

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1 Introduction

A Heffter array $H(m,n;s,t)$ is an $m \times n$ matrix of integers such that:

(1) each row contains s filled cells and each column contains t filled cells;

(2) the elements in every row and column sum to 0 in $\mathbb{Z}_{2ms+1}$; and

(3) for each integer $1 \leqslant x \leqslant ms$, either x or $-x$ appears in the array.

If the Heffter array is square, then $m = n$ and necessarily $s = t$. We denote such Heffter arrays by $H(n;k)$, where each row and each column contains k filled cells. A Heffter array is called an *integer* Heffter array if Condition (2) in the definition of a Heffter array above is strengthened so that the elements in every row and every column sum to zero in $\mathbb{Z}$.

Archdeacon, in [1], was the first to define and study a Heffter array $H(m,n;s,t)$. He showed that a Heffter array with a pair of special orderings can be used to construct an embedding of the complete graph K_{2ms+1} on a surface. This connection is formalised in the following theorem. For definitions of simple and compatible orderings refer to [1].

Theorem 1.1 ([1])**.** *Given a Heffter array $H(m,n;s,t)$ with compatible orderings ω_r of the symbols in the rows of the array and ω_c on the symbols in the columns of the array, then there exists an embedding of K_{2ms+1} such that every edge is on a face of size s and a face of size t. Moreover, if ω_r and ω_c are both simple, then all faces are simple cycles.*

The embedding of K_{2ms+1} given in Theorem 1.1 provides a connection with the embedding of cycle systems. A t-cycle system on n points is a decomposition of the edges of K_n into t-cycles. A t-cycle system C on K_n is cyclic if there is a labeling of the vertex set of K_n with the elements of $\mathbb{Z}_n$ such that the permutation $x \to x+1$ preserves the cycles of C. A biembedding of an s-cycle system and a t-cycle system is a face 2-colorable topological embedding of the complete graph K_{2ms+1} in which one color class is comprised of the cycles in the s-cycle system and the other class contains the cycles in the t-cycle system, see for instance [4, 5, 6, 8, 9, 10, 11] for further details.

A number of papers have appeared on the construction of Heffter arrays, $H(m,n;s,t)$. The case where the array contained no empty cells was studied in [2], with results summarised in Theorem 1.2.

Theorem 1.2 ([2])**.** *There is an $H(m,n;n,m)$ for all $m,n \geqslant 3$ and an integer Heffter array $H(m,n;n,m)$ exists if and only if $m,n \geqslant 3$ and $mn \equiv 0,3 \pmod 4$.*

The papers [3, 7] focused on square integer Heffter arrays $H(n;k)$ and verified their existence for all admissible orders. This result is summarized in the following theorem.

Theorem 1.3 ([3, 7])**.** *There exists an integer $H(n;k)$ if and only if $3 \leqslant k \leqslant n$ and $nk \equiv 0,3 \pmod 4$.*

Table 1 lists the possible cases and cites the article which verifies existence of square integer Heffter arrays, where DNE represents a value that does not exist. In these cases we will verify existence for the non-integer Heffter arrays $H(n;k)$. The main result of this paper is the following.

Theorem 1.4. *There exists an $H(n;k)$ if and only if $3 \leqslant k \leqslant n$.*

Table 1: Existence results for square integer Heffter arrays $H(n;k)$.

$n \setminus k$	0	1	2	3
0	[3]	[3, 7]	[3]	[3]
1	[3]	DNE	DNE	[3]
2	[3]	DNE	[3]	DNE
3	[3]	[3, 7]	DNE	DNE

Table 2: Cases for non-integer Heffter arrays $H(n;k)$.

	Case A	Case B	Case C	Case D	Case E
k	$2 \pmod 4$	$3 \pmod 4$	$3 \pmod 4$	$1 \pmod 4$	$1 \pmod 4$
n	$1,3 \pmod 4$	$3 \pmod 4$	$2 \pmod 4$	$1 \pmod 4$	$2 \pmod 4$

From Theorem 1.2 above, the case $n = k$ has been solved, so we henceforth assume that $n > k$. The cases that need to be addressed are set out in Table 2. Cases A, B, C, D and E are solved by Theorems 3.2, 4.2, 5.2, 6.2 and 7.2, respectively, thus proving Theorem 1.4.

In this paper the rows and columns of a square $n \times n$ array are always indexed by the elements of $\{1, 2, \ldots, n\}$. Unless otherwise stated, when working modulo n, replace 0 by n, so we use the symbols $1, 2, \ldots, n$ instead of $0, 1, \ldots, n-1$. While rows and columns are calculated modulo an integer, entries are always expressed as non-zero integers. Throughout this paper $A[r, c] = x$ denotes the occurrence of symbol x in cell (r, c) of array A.

By $A \pm z$ we refer to the array obtained by replacing $A[r, c]$ by $A[r, c]+z$ (if $A[r, c] > 0$) and $A[r, c]-z$ (if $A[r, c] < 0$). If each row and each column of A contains the same number of positive and negative numbers, then $A \pm z$ has the same row and column sums as A. In this case we say A is *shiftable*. The *support* of an array A is defined to be the set containing the absolute value of the elements contained in A. If A is an array with support S and z a nonnegative integer, then $A \pm z$ has support $S + z$.

2 Increasing *k* from base cases

For each of the cases set out in Table 2 our overall strategy is to generate a base case $H(n; k)$ where k takes the smallest possible value and then increase k by multiples of 4, adjoining 4 additional entries to each row and column. In this section we outline various tools to enable this process. To this end, we introduce the following definitions.

We associate the cells of an $n \times n$ array with the complete bipartite graph $K_{n,n}$ where partite sets are denoted $\{a_i \mid i = 1, 2, \ldots, n\}$ and $\{b_j \mid j = 1, 2, \ldots, n\}$ and the edge $\{a_i, b_j\}$ corresponds to the cell (i, j). We say that in an $n \times n$ array a set of cells S forms a 2-*factor* if the corresponding set of edges in the graph $K_{n,n}$ forms a spanning 2-regular graph and forms a *Hamilton cycle* if the corresponding set of edges forms a single cycle of length $2n$.

For each $d \in \{0, 1, \ldots, n-1\}$, we define the *diagonal* D_d to be the set of cells of the form $(r + d, r)$, $1 \leqslant r \leqslant n$ (evaluated modulo n). Observe that the cells $D_i \cup D_j$ form a Hamilton cycle whenever $j - i$ is coprime to n.

Lemma 2.1. *Let S_1 and S_2 be two disjoint sets of cells in an $n \times n$ array which each form*

Hamilton cycles. The cells of $S_1 \cup S_2$ *can be filled with the elements of* $\{1, 2, \dots, 4n\}$ *so that each row and column sum is equal to* $8n + 2$.

Proof. Let the cells of S_1 and S_2 be $\{e_i \mid 1 \leqslant i \leqslant 2n\}$ and $\{f_i \mid 1 \leqslant i \leqslant 2n\}$, respectively, where:

- Cells e_i and e_{i+1} are in the same row (column) whenever i is odd (respectively, even);
- Cells f_i and f_{i+1} are in the same row (column) whenever i is odd (respectively, even);
- Cells e_1 and f_1 are in the same row.

Place 1 in cell e_1, $4n$ in cell f_1 and:

- $2n - 2i + 1$ in cell e_{2i+1}, where $1 \leqslant i \leqslant n - 1$; $2n + 2i - 1$ in cell e_{2i} where $1 \leqslant i \leqslant n$;
- $2n + 2i$ in cell f_{2i+1} where $1 \leqslant i \leqslant n - 1$; $2n - 2i + 2$ in cell f_{2i} where $1 \leqslant i \leqslant n$.

The entries in cells e_1, e_2, f_1 and f_2 add to $1 + (2n + 1) + 4n + 2n = 8n + 2$. For every other row, there are two cells from S_1 with entries adding to $4n + 2$ and two cells from S_2 with entries adding to $4n$. For every column, there are two cells from S_1 adding to $4n$ and two cells from S_2 adding to $4n + 2$. See the example below. ☐

We demonstrate Lemma 2.1 below when $n = 9$. The elements of S_1 are underlined.

$S_1 \cup S_2$

$\underline{1}$	$\underline{19}$	18				36		
34	$\underline{17}$	$\underline{21}$				2		
		$\underline{15}$	$\underline{23}$	10	26			
4			$\underline{13}$	$\underline{25}$				32
	14			$\underline{11}$	$\underline{27}$		22	
		20			$\underline{9}$	$\underline{29}$	16	
			30			$\underline{7}$	$\underline{31}$	6
	24				12		$\underline{5}$	$\underline{33}$
$\underline{35}$			8	28				$\underline{3}$

The following theorem will be crucial in Cases A and D.

Theorem 2.2. *Let* $H(n; k)$ *be a Heffter array such that each row and column sums to* $2nk + 1$. *Suppose there exist Hamilton cycles* H_1 *and* H_2 *disjoint to each other and to the filled cells of* $H(n; k)$. *Then there exists an* $H(n; k + 4)$ *Heffter array with row and column sums equal to* $2n(k + 4) + 1$, *where the filled cells are precisely the filled cells of* $H(n; k)$, H_1 *and* H_2.

Proof. Let A_0 represent the $H(n; k)$ and negate each element so that each row and column has sum equal to $-(2nk + 1)$. From Lemma 2.1, there exists an array A_1' on the cells of H_1 and H_2 such that each row and column sum is equal to $8n + 2$; add $n(k + 4) - (4n) = nk$ to each element of A_1' to create a new array A_1 that has support $\{nk{+}1, nk{+}2, \dots, n(k{+}4)\}$. Note that in A_1 each row and column sum is equal to $8n + 2 + 4nk$. Let A be the union of A_0 with A_1. The resulting array A has support $\{1, 2, \dots, n(k + 4)\}$, with $k + 4$ filled cells in each row and column. Finally, each row and column sum of A is

$$-(2nk + 1) + (8n + 2) + 4(nk) = 2n(k + 4) + 1,$$

as desired. □

The following lemma generalizes Theorem 2.2 from [7], and is used in Cases B and C.

Lemma 2.3. *Let S_1 and S_2 be two disjoint sets of cells in an $n \times n$ array which each form Hamilton cycles. Then for any positive integers t and $s > t + 2n$, the cells of $S_1 \cup S_2$ can be filled with elements to make a shiftable array with support $\{s + i, t + i \mid 1 \leqslant i \leqslant 2n\}$ so that the four elements in each row and each column sum to 0.*

Proof. Let the sets of cells of S_1 and S_2 be $\{e_i \mid 1 \leqslant i \leqslant 2n\}$ and $\{f_i \mid 1 \leqslant i \leqslant 2n\}$, respectively, where:

- Cells e_i and e_{i+1} are in the same row (column) whenever i is odd (respectively, even);
- Cells f_i and f_{i+1} are in the same row (column) whenever i is odd (respectively, even);
- Cells e_1 and f_1 are in the same row.

Place:

- $s + 2n$ in cell e_1 and $-(t + 2n)$ in cell f_1, with sum $s - t$;
- $s + 2i$ in cell e_{2i+1} and $-(t + 2i)$ in cell f_{2i+1}, with sum $s - t$, for $1 \leqslant i \leqslant n - 1$,
- $-(s + 2i - 1)$ in cell e_{2i} and $t + 2i - 1$ in cell f_{2i}, with sum $t - s$, for $1 \leqslant i \leqslant n$.

It now follows that the row sums are 0. Using similar arguments it can be seen that the columns also sum to 0. Observe that there are two positive and two negative integers in each row and column; thus the array is shiftable. □

The proof of the following lemma is similar to the proof of Lemma 2.3; we use this in Case E.

Lemma 2.4. *Let n be even. Let S_1 and S_2 be two disjoint sets of cells in an $n \times n$ array which each form 2-factors that are the union of two n-cycles. Then for any positive integers s, t, u and v where $s > t + n$, $t > u + n$ and $u > v + n$, the cells of $S_1 \cup S_2$ can be filled with elements to make a shiftable array with support $\{s + i, t + i, u + i, v + i \mid 1 \leqslant i \leqslant n\}$ so that the four elements in each row and column sum to 0.*

Proof. Let C_i, C_i' be the cycles of the 2-factor S_i, $i \in \{1, 2\}$, where C_1 and C_1' share a row and C_2 and C_2' share a row. Let the sets of cells of C_1, C_1', C_2 and C_2' be $\{e_i \mid 1 \leqslant i \leqslant n\}$, $\{f_i \mid 1 \leqslant i \leqslant n\}$, $\{g_i \mid 1 \leqslant i \leqslant n\}$ and $\{h_i \mid 1 \leqslant i \leqslant n\}$, respectively, where:

- Cells e_i and e_{i+1} are in the same row (column) whenever i is odd (respectively, even);

- Cells f_i and f_{i+1} are in the same row (column) whenever i is odd (respectively, even);
- Cells g_i and g_{i+1} are in the same row (column) whenever i is odd (respectively, even);
- Cells h_i and h_{i+1} are in the same row (column) whenever i is odd (respectively, even);
- Cells e_1 and g_1 are in the same row; cells f_1 and h_1 are in the same row.

Place:

- $s+n$ in cell e_1, $-(t+n)$ in cell g_1 and $u+n$ in cell f_1, $-(v+n)$ in cell h_1;
- $s+2i$ in cell e_{2i+1} and $-(t+2i)$ in cell g_{2i+1}, for $1 \leqslant i \leqslant n/2-1$;
- $-(s+2i-1)$ in cell e_{2i} and $t+2i-1$ in cell g_{2i}, for $1 \leqslant i \leqslant n/2$;
- $u+2i$ in cell f_{2i+1} and $-(v+2i)$ in cell h_{2i+1}, for $1 \leqslant i \leqslant n/2-1$;
- $-(u+2i-1)$ in cell f_{2i} and $v+2i-1$ in cell h_{2i}, for $1 \leqslant i \leqslant n/2$.

It now follows that the row sums are 0. Using similar arguments it can be seen that the columns also sum to 0. □

3 Case A: $k \equiv 2 \pmod 4$

In this section we construct a Heffter array $H(n;k)$, for $n \equiv 1,3 \pmod 4$ and $k \equiv 2 \pmod 4$, where $k < n$. Row and column sums will always equal $2nk+1$. We start with an example of our construction.

$H(15;6)$

6									−4	89	81	1	8	
	12									−88	83	87	85	2
86		18									−82	77	3	79
73	80		24									−76	71	9
15	67	74		30									−70	65
59	21	61	68		36									−64
−58	53	27	55	62		42								
	−52	47	33	49	56		48							
		−46	41	39	43	50		54						
			−40	35	45	37	44		60					
				−34	29	51	31	38		66				
					−28	23	57	25	32		72			
						−22	17	63	19	26		78		
							−16	11	69	13	20		84	
								−10	5	75	7	14		90

Lemma 3.1. *For $n \equiv 1, 3 \pmod 4$, $n \geqslant 7$ and $k = 6$ there exists a Heffter array $H(n; 6)$.*

Proof. We remind the reader that rows and columns are calculated modulo n but the array entries are not. The array $A = A[r, c]$ is defined as follows, where $1 \leqslant i \leqslant n$:

$$A[i,i] = 6i, \qquad A[i+2,i] = 6n+2-6i,$$
$$A[i+1,n-2+i] = 6n+1-6i, \qquad A[i+2,n-2+i] = 6i-3,$$
$$A[i,n-5+i] = 6n+5-6i, \qquad A[i+1,n-5+i] = -6n-4+6i.$$

Then the support of A is $\{1, 2, \dots, 6n\}$. The sets of elements in rows 1, 2 and i, $3 \leqslant i \leqslant n$, are, respectively:

$$\{6, 8, 1, 6n-9, 6n-1, -4\},$$
$$\{12, 2, 6n-5, 6n-3, 6n-7, -(6n-2)\},$$
$$\{6i, 6n+2-6(i-2), 6n+1-6(i-1), 6(i-2)-3,$$
$$6n+5-6i, -6n-4+6(i-1)\}.$$

Thus in each case the sum of elements in a row is $12n + 1$.

The set of elements in column i, $1 \leqslant i \leqslant n-5$ is:

$$\{6i, 6n+2-6i, 6n+1-6(i+2), 6(i+2)-3,$$
$$6n+5-6(i+5), -6n-4+6(i+5)\}.$$

The set of elements in columns $n-4$, $n-3$, $n-2$, $n-1$ and n are, respectively:

$$\{6n-24, 26, 13, 6n-15, 6n-1, -(6n-2)\},$$
$$\{6n-18, 20, 7, 6n-9, 6n-7, -(6n-8)\},$$
$$\{6n-12, 14, 1, 6n-3, 6n-13, -(6n-14)\},$$
$$\{6n-6, 8, 6n-5, 3, 6n-19, -(6n-20)\},$$
$$\{6n, 2, 6n-11, 9, 6n-25, -(6n-26)\}.$$

Thus in each case the sum of elements in a column is $12n + 1$. □

Theorem 3.2. *There exists a Heffter array $H(n; k)$ for all $n \equiv 1, 3 \pmod 4$ and $k \equiv 2 \pmod 4$, where $n > k \geqslant 6$.*

Proof. Let $k = 4p + 6$. Then $4p + 6 \leqslant n - 1$ so $p \leqslant (n-7)/4$. We have solved the case $p = 0$ in Lemma 3.1 so we may assume $p \geqslant 1$. Observe that the Heffter array given in the proof of that lemma uses only elements in diagonals D_0, D_2, D_3, D_4, D_5 and D_6 and so does not intersect the diagonals $D_7, D_8, \dots, D_{n-1}$. We can apply Theorem 2.2 recursively, where the diagonals $D_7, D_8, \dots, D_{6+2p-1}, D_{6+2p}$ can be paired to give sets of cells S_1 and $D_{6+2p+1}, D_{6+2p+2}, \dots, D_{6+4p-1}, D_{6+4p}$ paired to give sets of cells S_2. The result is a Heffter array $H(n; k)$ with constant row and column sum $2nk + 1$ whenever $k \equiv 2 \pmod 4$, $n \equiv 1, 3 \pmod 4$ and $n > k \geqslant 6$. □

4 Case B: $k \equiv 3 \pmod 4$ and $n \equiv 3 \pmod 4$

In this section we construct a Heffter array $H(n; k)$ where $n = 4m + 3$, $k = 4p + 3$ and $k < n$.

We will begin with $k = 3$. We first assume that $m \geqslant 4$ and construct an $n \times n$ array which is the concatenation of three smaller arrays, $A_0 = A_0[r,c]$, of dimension $(4m-7) \times (4m-7)$, $A_1 = A_1[r,c]$ of dimension 7×7 and C of dimension 3×3, each containing 3 filled cells per row and column. So we see that $n = 4m + 3$. The sum of the rows and columns in A_0 and A_1 will be 0, while the sum of the rows and columns in C will be $2nk + 1$.

We begin with an example of the main construction of this section.

$H(19;3)$

16	−48	32																
17	27		−44															
−33		−14		47														
	21		15		−36													
		−18		−13		31												
			29		−9		−20											
				−34		−12		46										
					45		−10	−35										
						−19	30	−11										
									−25	24	1							
									22	−50		28						
									3		37		−40					
										26		23		−49				
											−38		42		−4			
												−51		8	43			
													−2	41	−39			
																5	57	53
																54	6	55
																56	52	7

Lemma 4.1. *For* $n \equiv 3 \pmod 4$ *and* $n \geqslant 7$ *there exists a Heffter array* $H(n;3)$.

Proof. Let $n = 4m + 3$. We first assume that $m \geqslant 4$ and so $n \geqslant 19$. The small cases will be dealt with at the end of the proof. Let A_0' be:

$$\begin{aligned}
A_0'[2i-1, 2i] &= 8m + 1 - i, && 1 \leqslant i \leqslant m, \\
A_0'[2i, 2i-1] &= -(8m + i), && 1 \leqslant i \leqslant m, \\
A_0'[2i, 2i+1] &= 12m - i, && 1 \leqslant i \leqslant m-1, \\
A_0'[2i+1, 2i] &= -(4m + 1 + i), && 1 \leqslant i \leqslant m-1, \\
A_0'[2m-2+2i, 2m-1+2i] &= 5m + i, && 1 \leqslant i \leqslant m-3, \\
A_0'[2m-1+2i, 2m-2+2i] &= -(11m + 1 - i), && 1 \leqslant i \leqslant m-3,
\end{aligned}$$

$$\begin{aligned}
A_0'[2m-1+2i, 2m+2i] &= 9m+i, && 1 \leqslant i \leqslant m-4,\\
A_0'[2m+2i, 2m-1+2i] &= -(7m+1-i), && 1 \leqslant i \leqslant m-4,\\
A_0'[i+1, i+1] &= -(4m-1-i), && 1 \leqslant i \leqslant 2m-2,\\
A_0'[2m+i, 2m+i] &= 2m-i, && 1 \leqslant i \leqslant 2m-8,
\end{aligned}$$

$$A_0'[2m,2m] = 4m-1, \quad A_0'[1,4m-7] = -12m,$$
$$A_0'[1,1] = 4m, \qquad A_0'[4m-7,1] = 4m+1,$$
$$A_0'[4m-7,4m-7] = 6m+3.$$

We illustrate A_0' in the case $m = 4$:

$A_0'(m = 4)$

16	32							−48
−33	−14	47						
	−18	−13	31					
		−34	−12	46				
			−19	−11	30			
				−35	−10	45		
					−20	−9	29	
						−36	15	21
17							−44	27

First observe that the array A_0' is a $(4m-7) \times (4m-7)$ array that has 3 filled cells in each row and column.

To confirm that the row and columns sums are 0, note that this array was constructed by taking the first $4m-8$ rows and columns of the integer Heffter array $H(4m;3)$ given in [3], then placing entry $-12m$ in cell $(1, 4m-7)$, entry $4m+1$ in cell $(4m-7, 1)$ and entry $6m+3$ in cell $(4m-7, 4m-7)$. Thus we need only check the sum of row $4m-7$ which is $(4m+1)+(6m+3)-(10m+4) = 0$ and the sum of column $4m-7$ which is $-12m+(6m-3)+(6m+3) = 0$. Hence each row and column in the array A_0' sums to zero.

Although not necessary for the case $k = 3$, for larger values of k (see the following theorem) we map the rows and columns of A_0' so that the filled cells are a subset of the union of diagonals $D_0 \cup D_1 \cup D_2 \cup D_{n-2} \cup D_{n-1}$. This can be done by applying the mapping

$$i \mapsto \begin{cases} 2i-1, & \text{when } 1 \leqslant i \leqslant 2m-3,\\ 8m-2i-12, & \text{when } 2m-2 \leqslant i \leqslant 4m-7,\end{cases}$$

to the rows and column of A_0'. This does not change the row and column sum and the support is still

$$\{1,2,\dots,4m+3\} \setminus \{1,2,\dots,7,2m,6m-2,\dots,6m+2,6m+4,\\ 10m-3,\dots,10m+3,12m+1,12m+2,\dots,12m+9\}.$$

We call this rearranged array A_0; see $H(19;3)$ above for A_0 when $m = 4$.

Next, let A_1 be:

A_1

$-(6m+1)$	$6m$	1				
$6m-2$	$-(12m+2)$		$6m+4$			
3		$10m-3$		$-10m$		
	$6m+2$		$6m-1$		$-(12m+1)$	
		$-(10m-2)$		$10m+2$		-4
			$-(12m+3)$		$2m$	$10m+3$
				-2	$10m+1$	$-(10m-1)$

It is easy to check that this array has row and column sum 0 and support

$$\{1,2,3,4,2m,6m-2,\dots,6m+2,6m+4,\\ 10m-3,\dots,10m+3,12m+1,12m+2,12m+3\}.$$

We place A_1 on the intersection of row and column sets $\{4m-6,4m-5,\dots,4m\}$.

Finally, place the block C on the intersection of the row and column sets $\{4m+1, 4m+2, 4m+3\}$. Observe that the rows and columns sum to $2nk+1$. It is convenient to express C in terms of n and k, as it will be part of more general constructions in the next theorems. However, if $k=3$, observe that $\{12m+4,12m+5,\dots,12m+9\} = \{nk-5,nk-4,\dots,nk\}$.

C

5	nk	$nk-4$
$nk-3$	6	$nk-2$
$nk-1$	$nk-5$	7

Let A be the concatenation of A_0, A_1 and C to obtain an $H(4m+3;3)$ Heffter array for all $m \geqslant 4$. An $H(7;3)$ is given in the Appendix. When $n = 11$ or 15, concatenating the array $B(n)$ given in the Appendix for these size with C will produce an $H(n;3)$ with similar properties and support. □

We now consider the case when $k>3$; we apply the techniques developed in Section 2.

Theorem 4.2. *There exists a Heffter array $H(n;k)$ for all $n \equiv 3 \pmod 4$ and $k \equiv 3 \pmod 4$, where $n > k \geqslant 3$.*

Proof. Given Lemma 4.1 we only need to address the case $k = 4p+3$ where $1 \leqslant p < m$. Since $k \leqslant n-4$, $p \leqslant (n-7)/4$. First observe that the filled cells of $H(n;3)$ given in the previous lemma are a subset the union of diagonals $D_0, D_1, D_2, D_{n-2}, D_{n-1}$ with support $\{1,2,\dots,12m+3\} \cup \{nk-5,nk-4,\dots,nk\}$. We next identify p disjoint Hamilton cycles, that are also disjoint from diagonals $D_0, D_1, D_2, D_{n-2}, D_{n-1}$, by pairing the remaining $(n-5)$ diagonals. Then we apply Lemma 2.3, contributing $\{12m+4,12m+5,\dots,nk-6\}$ to the support. The result is a Heffter array $H(n;k)$ with each row and column sum equal to 0 except for the final three rows and columns which sum to $2nk+1$. □

5 Case C: $k \equiv 3 \pmod 4$ and $n \equiv 2 \pmod 4$

We construct a Heffter array $H(n; k)$ where $n = 4m + 6$ and $k = 4p + 3$ for $k < n$.

As in the previous section, we will begin with $k = 3$. Firstly we will assume that $m \geqslant 7$ and construct an $n \times n$ array which is the concatenation of several smaller arrays. Here we use four smaller arrays, $A_0 = A_0[r, c]$ of size $(4m - 13) \times (4m - 13)$, $A_1 = A_1[r, c]$ of size 13×13, $A_2 = A_2[r, c]$ of size 3×3 and $A_3 = A_3[r, c]$ of size 3×3, each containing 3 filled cells per row and column. The sum of the rows and columns in A_0, A_1 and A_2 is 0, while the sum of the rows and columns in A_3 is $2nk + 1$.

Lemma 5.1. *For* $n \equiv 2 \pmod 4$ *and* $n \geqslant 34$ *there exists a Heffter array* $H(n; 3)$.

Proof. Let $n = 4m + 6$ and $m \geqslant 7$. Let A_0' be:

$$\begin{aligned}
A_0'[2i-1, 2i] &= 8m + 1 - i, && 1 \leqslant i \leqslant m,\\
A_0'[2i, 2i-1] &= -(8m + i), && 1 \leqslant i \leqslant m,\\
A_0'[2i, 2i+1] &= 12m - i, && 1 \leqslant i \leqslant m-1,\\
A_0'[2i+1, 2i] &= -(4m + 1 + i), && 1 \leqslant i \leqslant m-1,\\
A_0'[2m-2+2i, 2m-1+2i] &= 5m + i, && 1 \leqslant i \leqslant m-6,\\
A_0'[2m-1+2i, 2m-2+2i] &= -(11m + 1 - i), && 1 \leqslant i \leqslant m-6,\\
A_0'[2m-1+2i, 2m+2i] &= 9m + i, && 1 \leqslant i \leqslant m-7,\\
A_0'[2m+2i, 2m-1+2i] &= -(7m + 1 - i), && 1 \leqslant i \leqslant m-7,\\
A_0'[i+1, i+1] &= -(4m - 1 - i), && 1 \leqslant i \leqslant 2m-2,\\
A_0'[2m+i, 2m+i] &= 2m - i, && 1 \leqslant i \leqslant 2m-14,\\
A_0'[2m, 2m] &= 4m - 1, \quad A_0'[1, 4m-13] = -(12m), &&\\
A_0'[1, 1] &= 4m, \quad A_0'[4m-13, 1] = 4m + 1, &&\\
A_0'[4m-13, 4m-13] &= 6m + 6. &&
\end{aligned}$$

We illustrate A_0' when $m = 7$:

$A_0'(m = 7)$

28	56													−84
−57	−26	83												
	−30	−25	55											
		−58	−24	82										
			−31	−23	54									
				−59	−22	81								
					−32	−21	53							
						−60	−20	80						
							−33	−19	52					
								−61	−18	79				
									−34	−17	51			
										−62	−16	78		
											−35	−15	50	
												−63	27	36
29													−77	48

Observe that the array A_0' is a $(4m-13)\times(4m-13)$ array that has 3 filled cells in each row and column. Similarly to the previous case, this array was constructed by taking the first $4m-14$ rows and columns of the integer Heffter array $H(4m;3)$ given in [3], then placing entry $-12m$ in cell $(1,4m-13)$, entry $4m+1$ in cell $(4m-13,1)$ and entry $6m+6$ in cell $(4m-13,4m-13)$. Thus we need only check the sum of row $4m-13$ which is $(4m+1)+(6m+6)-(10m+7)=0$ and the sum of column $4m-13$ which is $-12m+(6m-6)+(6m+6)=0$. Hence all rows and columns in the array A_0' sum to zero. We apply the same mapping as in the proof of Lemma 4.1 to the rows and columns so that all non-empty cells are a subset of the diagonals $D_0, D_1, D_2, D_{n-2}, D_{n-1}$. Let A_0 be the resultant array. The support for A_0 is

$$\{1,2,\dots,12m+8\}\setminus\{1,2,\dots,13,2m,6m-5,6m-4,\dots,6m+5,6m+7,\\ 10m-6,10m-5,\dots,10m+6,12m+1,12m+2,\dots,12m+18\}.$$

We now arrange the 57 missing symbols into a 13×13 array A_1 and two 3×3 arrays A_2 and A_3, where A_2 and A_3 ares shown below and A_1 is given at the end of the Appendix.

A_2

-8	$nk-2$	$-(nk-10)$
$-(nk-9)$	-9	nk
$nk-1$	$-(nk-11)$	-10

A_3

11	$nk-3$	$nk-7$
$nk-6$	12	$nk-5$
$nk-4$	$nk-8$	13

The support for A_1 is

$$\{1,2,\dots,7,2m,6m-5,6m-4,\dots,6m+5,6m+7,\\ 10m-6,10m-5,\dots,10m+6,12m+1,12m+2,\dots,12m+6\}.$$

Finally we give A_2 and A_3 with support

$$\{8,9,10,11,12,13\}\cup\{nk-11,nk-10,\dots,nk\}.$$

In the case $k=3$, observe that $\{nk-11,nk-10,\dots,nk\}=\{12m+7,12m+8,\dots,12m+18\}$.

The concatenation of A_0, A_1, A_2 and A_3 gives a $H(4m+6;3)$ Heffter array for $m\geqslant 7$. □

Theorem 5.2. *There exists a Heffter array $H(n;k)$ for all $n\equiv 2 \pmod 4$ and $k\equiv 3 \pmod 4$, where $n>k\geqslant 3$.*

Proof. An $H(6;3)$ is given in the Appendix. Otherwise let $n=4m+6$ where $m\geqslant 1$. When $1\leqslant m\leqslant 5$ concatenate the $B(4m+6)$ given in the Appendix with the array C given in Case B to get an $H(4m+6;3)$. Observe that when $1\leqslant m\leqslant 4$ the entries are only on diagonals $D_0, D_1, D_2, D_{n-2}, D_{n-1}$ as before. When $n=26$ the entries are on diagonals $D_0, D_1, D_2, D_{24}, D_{25}$ and D_9. We pair up the rest of the diagonals as $\{D_{2i}, D_{2i+1}\}$ for $5\leqslant i\leqslant 11$, and $\{D_3, D_4\}$, $\{D_5, D_6\}$, $\{D_7, D_8\}$ to get the required Hamilton cycles. When $m=6$, $n=30$; an $H(30;3)$ is given in the Appendix. Two Hamilton cycles H and K are also given as a reference on the array. These Hamilton cycles together with $H(30;3)$ only have entries on diagonals $D_0, D_1, D_2, D_3, D_{27}, D_{28}, D_{29}, D_{18}$ and D_{11}. For the rest of the diagonals, pair them up as $\{D_4, D_5\}$, $\{D_6, D_7\}$, $\{D_8, D_9\}$, $\{D_{12}, D_{13}\}$,

$\{D_{14}, D_{15}\}$, $\{D_{16}, D_{17}\}$ and $\{D_{2i-1}, D_{2i}\}$ for $10 \leqslant i \leqslant 13$ to get the necessary Hamilton cycles.

When $m \geqslant 7$, a $H(n; 3)$ exists by Lemma 5.1, then the proof follows as in Lemma 4.1. □

6 Case D: $k \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 4$

In this section we construct a Heffter array $H(n; k)$ where $n = 4m + 1$ and $k = 4p + 1$, with $k < n$ and hence $m \geqslant 2$. The case $H(9; 5)$ is given in the Appendix and so henceforth we assume $m \geqslant 3$.

We begin with $k = 5$ construct an $n \times n$ array for which the sum of each row and column is $2nk + 1 = 40m + 11$. First we give an example $H(17; 5)$ of our construction. (Note a Hamilton cycle H has been included for the case $k > 5$.)

$H(17; 5)$

85	-27		11				50		H	H						52
68	84	-28		12				35		H	H					
	67	83	-29		13						H	H			37	
		66	82	-30		14				39		H	H			
			65	81	-31		15				41		H	H		
				64	80	-32						43		H	16	H
					63	79	-33		17				45		H	H
H	H					62	78		-34	18				47		
-20	H	H					61	60			21				49	
		H	H					77	59	-19		3				51
36			H	H					76	58	-22		23			
	38			H	H					75	57	-4		5		
		40			H	H		25			74	56	-24			
			42			H	H					73	55	-6		7
2				44			H	H					72	54	-1	
	9				46			H	53					71	H	-8
H		10				48		-26	H						70	69

Lemma 6.1. *For $n \equiv 1 \pmod 4$, $n \geqslant 13$ and $k = 5$ there exists a Heffter array $H(n; 5)$.*

Proof. Let $n = 4m + 1$ for $m \geqslant 3$. In every case here row and column sums will equal $40m + 11$.

We give the general construction of A below.

$$A[4m-1, 4m] = -1, \qquad A[4m-1, 1] = 2,$$
$$A[4m, 2] = 2m + 1, \qquad A[4m+1, 3] = 2m + 2,$$
$$A[2m-2, 4m] = 4m, \qquad A[2m-1, 2m+2] = 4m + 1,$$
$$A[2m, 2m+3] = 4m + 2, \qquad A[2m+2, 2m+3] = -(4m+3),$$
$$A[2m+1, 1] = -(4m+4), \qquad A[1, 2m] = 12m + 2,$$

$$A[4m+1,2m+1]=-(6m+2),\quad A[4m-3,2m+1]=6m+1,$$
$$A[2m,2m+2]=-(8m+2),\quad A[2,2m+1]=8m+3,$$
$$A[3,4m]=8m+5,\quad A[1,4m+1]=12m+4,$$
$$A[4m,2m+2]=12m+5,\quad A[4m+1,4m+1]=16m+5.$$

$$A[i,i+3]=2m+i+2,\qquad 1\leqslant i\leqslant 2m-3, \tag{6.1}$$
$$A[i+1,i]=16m+5-i,\qquad 1\leqslant i\leqslant 2m, \tag{6.2}$$
$$A[i,i]=20m+6-i,\qquad 1\leqslant i\leqslant 2m, \tag{6.3}$$
$$A[2m-i,2m+1-i]=-(8m+2-i),\qquad 1\leqslant i\leqslant 2m-1, \tag{6.4}$$
$$A[2m+3-i,4m+2-i]=12m+5-2i,\qquad 1\leqslant i\leqslant 2m-1, \tag{6.5}$$
$$A[2m+i,2m+i]=14m+5-i,\qquad 1\leqslant i\leqslant 2m-1, \tag{6.6}$$
$$A[2m+1+i,2m+i]=18m+6-i,\qquad 1\leqslant i\leqslant 2m, \tag{6.7}$$
$$A[4m+2-i,2m-i]=12m+2-2i,\qquad 1\leqslant i\leqslant 2m-1, \tag{6.8}$$
$$A[2m+2i,2m+2i+3]=2i+1,\qquad 1\leqslant i\leqslant m-1, \tag{6.9}$$
$$A[2m+2i+2,2m+2i+3]=-(2i+2),\qquad 1\leqslant i\leqslant m-1, \tag{6.10}$$
$$A[2m+2i-1,2m+2i+2]=4m+2i+3,\qquad 1\leqslant i\leqslant m-2, \tag{6.11}$$
$$A[2m+2i+1,2m+2i+2]=-(4m+2i+4),\qquad 1\leqslant i\leqslant m-2. \tag{6.12}$$

We note that the support of A contains:

- $3,4,\dots,2m$ by (6.9) and (6.10),
- $2m+3,2m+4,\dots,4m-1$ by (6.1),
- $4m+5,4m+6,\dots,6m$ by (6.11) and (6.12),
- $6m+3,6m+4,\dots,8m+1$ by (6.4),
- $8m+4,8m+6,8m+7,\dots,12m,12m+1,12m+3$ by (6.5) and (6.8),
- $12m+6,12m+7,\dots,16m+4$ by (6.6) and (6.2),
- $16m+6,16m+7,\dots,20m+5$ by (6.7) and (6.3).

It follows that the support of A is $\{1,2,\dots,20m+5\}$ as required.

To verify the sum of each row and column is $40m+11$ we begin by noting that, respectively, (6.1), (6.2), (6.3), (6.4) and (6.5) give the sum for row r (where $4\leqslant r\leqslant 2m-3$) and (6.1), (6.2), (6.3), (6.4) and (6.8) give the sum for column c (where $4\leqslant c\leqslant 2m-1$) as:

$$(2m+r+2)+(16m+6-r)+(20m+6-r)-(6m+r+2)+(8m-1+2r)=40m+11$$

and

$$(2m+c-1)+(16m+5-c)+(20m+6-c)-(6m+1+c)+(8m+2+2c)=40m+11.$$

For row $2m+r$, where $3\leqslant r\leqslant 2m-4$ (or $r=2m-2$) and column $2m+c$, where $4\leqslant c\leqslant 2m-1$, we argue as follows.

Respectively (6.6), (6.7) and (6.8) give a partial sum of $40m + 10$ for row r and (6.5), (6.6) and (6.7) give a partial sum of $40m + 12$ for column c:

$$(14m + 5 - r) + (18m + 7 - r) + (8m - 2 + 2r) = 40m + 10, \qquad (6.13)$$

$$(8m + 1 + 2c) + (14m + 5 - c) + (18m + 6 - c) = 40m + 12. \qquad (6.14)$$

Next (6.9) and (6.10) imply that if r is even, row $2m + r$ contain the entries $r + 1$ and $-r$ giving a partial sum of 1. If r is odd, (6.11) and (6.12) imply that row $2m + r$ contains the entries $4m+r+4$ and $-(4m+r+3)$ also giving a partial sum of 1. Adding this to the partial sum in (6.13) we get an overall row sum of $40m + 11$. Then (6.9) and (6.10) imply that if c is odd, column $2m + c$ contains the entries $c - 2$ and $-(c - 1)$ giving a partial sum of -1. If c is even, (6.11) and (6.12) imply that column $2m + c$ contains the entries $4m + c + 1$ and $-(4m + c + 2)$ also giving a partial sum of -1. Adding this to the partial sum in (6.14) we get a column sum of $40m + 11$.

The remaining rows and columns can be checked individually to complete the proof that all rows and columns sum to $40m+11$. Thus we have the required $H(4m+1; 5)$. □

Next, with care, we add up to $2(m-2)$ Hamilton cycles to obtain an $H(4m+1; 4p+5)$ where $p \leqslant m - 2$.

Theorem 6.2. *For $n \equiv 1 \pmod 4$ and $k \equiv 1 \pmod 4$ there exists a Heffter array $H(n; k)$, where $k < n$.*

Proof. Let $n = 4m + 1$. The Heffter array $H(9; 5)$ is given in the Appendix. Otherwise, an $H(n; 5)$ exists by Lemma 6.1. This was labeled A in the proof of that lemma. Observe that the occupied cells of A are a subset of the union of diagonals

$$\mathcal{D} := D_{n-3} \cup D_{n-2} \cup D_{n-1} \cup D_0 \cup D_1 \cup D_4 \cup D_{2m-4} \cup D_{2m-2} \cup D_{2m-1} \cup D_{2m} \cup D_{2m+2}.$$

For each $m \geqslant 3$, the following set of cells is a subset of $\mathcal{D}$, that does *not* intersect A and forms a Hamilton cycle H:

$$\begin{aligned}\{(i, 2m + 1 + i), (i, 2m + 2 + i) \mid 1 \leqslant i \leqslant 2m - 3\}&\\ \cup \{(2m - 2, 4m - 1), (2m - 2, 4m + 1)\}&\\ \cup \{(2m - 1, 4m + 1), (2m - 1, 4m), (4m, 4m), (4m, 2m + 1)\}&\\ \cup \{(i, i - 2m + 1), (i, i - 2m + 2) \mid 2m \leqslant i \leqslant 4m - 1\}&\\ \cup \{(4m + 1, 1), (4m + 1, 2m + 2)\}.&\end{aligned}$$

(See the array $H(17; 5)$ above for an example.) Thus there exists $4m+1-11 = 4(m-2)-2$ diagonals that do not intersect $A \cup H$ and so it is possible to construct $2m - 5$ disjoint Hamilton cycles by pairing empty diagonals that are either distance 1 or 2 apart.

For $m \geqslant 6$ a possible pairing of diagonals is:

$$\begin{aligned}&\{D_2, D_3\};\\ &\{D_{2m-5}, D_{2m-3}\};\\ &\{D_{5+2i}, D_{6+2i}\},\ 0 \leqslant i \leqslant m - 6;\\ &\{D_{2m+1}, D_{2m+3}\};\\ &\{D_{2m+4+2i}, D_{2m+5+2i}\},\ 0 \leqslant i \leqslant m - 4.\end{aligned}$$

When $m = 4$ we pair the diagonals as $\{D_2, D_3\}; \{D_9, D_{11}\}; \{D_{12}, D_{13}\}$ and when $m = 5$ we pair the diagonals as $\{D_2, D_3\}; \{D_5, D_7\}; \{D_{11}, D_{13}\}; \{D_{14}, D_{15}\}; \{D_{16}, D_{17}\}$.

Together with H this gives a total of $2m - 4$ Hamilton cycles. Thus applying Theorem 2.2 recursively, we can form a Heffter array for each k such that $k \equiv 1 \pmod 4$ and $k \leqslant n - 4$. □

7 Case E: $k \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$

In this section we construct a Heffter array $H(n; k)$ where $n = 4m + 2$ and $k = 4p + 1$, where $k < n$.

We demonstrate the following construction in the case $m = 4$ with an example of an $H(18; 5)$; the cycles H and K will be needed later for the case $k > 5$.

$H(18; 5)$

2					51	20	−38	−35									
−40	4					53	19	−36									
−42	−28	6					37	27									
39	−44	−29	8					26									
41	25	−46	−30	10													
	43	24	−48	−31	12												
		45	23	−50	−32	14											
			47	22	−52	−33	16										
				49	21	−54	−34	18									
									1	81	K	H	69	−60	K	90	H
									H	3	80	K	H	70	−61	K	89
									88	H	5	79	K	H	71	−62	K
									72	87	H	7	78	K	H	K	−63
									−55	K	86	H	9	77	K	H	64
									K	−56	K	85	H	11	76	65	H
									75	66	−57	K	84	H	13	H	K
									K	H	67	−58	K	83	H	15	74
									H	K	H	68	−59	K	82	73	17

Lemma 7.1. *For $n \equiv 2 \pmod 4$, $n \geqslant 6$ there exists a Heffter array $H(n; 5)$.*

Proof. Let $n = 4m + 2$. The case $H(n; k) = H(6; 5)$ is given in the Appendix and so henceforth we assume $m \geqslant 2$. Our Heffter array will be the concatenation of an array A_0 in the first $2m + 1$ rows and columns and an array A_1 in the last $2m + 1$ rows and columns. The row and column sums of A_0 will be 0 and the row and column sums of A_1 will be $2nk + 1$.

In our definition of A_0, rows and columns are calculated modulo $2m + 1$ rather than n. We begin by defining a $(2m + 1) \times (2m + 1)$ array A_0' which has support

$$\{2, 4, 5, \dots, 4m + 2\} \cup \{4m + 3, 4m + 4, \dots, 12m + 6\}$$

and for which each row sums to 0 and each column sums to 0 except for columns 1 and $2m+1$, which sum to $-(2m+1)$ and $2m+1$, respectively. Then we swap the entry $-(8m+4)$ in cell $(2,1)$ with the entry $-(8m+8)$ in cell $(2,2m+1)$ and swap the entry $6m+2$ in cell $(4,1)$ with the entry $8m+7$ in cell $(4,2m+1)$. The result will be an array A_0 defined on row and column set $\{1,2,\ldots,2m+1\}$ with row and column sums equal to 0.

To this end, for $1 \leqslant i \leqslant 2m+1$ let

$$A'_0[i,i] = 2i; \qquad A'_0[3-i, 2m+1-i] = 4m+2+i;$$
$$A'_0[2+i,1+i] = -(6m+3+i); \qquad A'_0[2+i,i-2] = 8m+3+2i;$$
$$A'_0[i,i-2] = -(8m+4+2i).$$

Now let

$$A_0[r,c] = \begin{cases} -(8m+8), & (r,c)=(2,1), \\ -(8m+4), & (r,c)=(2,2m+1), \\ 8m+7, & (r,c)=(4,1), \\ 6m+2, & (r,c)=(4,2m+1), \\ A'_0[r,c], & \text{otherwise.} \end{cases}$$

The case $m=4$ is illustrated in the example $H(18;5)$ given above. It will be useful to note that the non-empty cells of this $(2m+1)\times(2m+1)$ array A_0 are a subset of the union of diagonals

$$\mathcal{D}_0 := D_0 \cup D_1 \cup D_2 \cup D_3 \cup D_4.$$

Next, we define a $(2m+1)\times(2m+1)$ array A_1, on row and column set $\{2m+2, 2m+3,\ldots,4m+2\}$ with support $\{1,3,4,\ldots,4m+1\} \cup \{12m+7,12m+8,\ldots, 16m+8\} \cup \{kn-4m-1, kn-4m,\ldots,kn\}$. For the case $k=5$, observe that $\{kn-4m-1,kn-4m,\ldots,kn\} = \{16m+9,16m+10,\ldots,20m+10\}$. Thus when $k=5$ the support of $A_0 \cup A_1$ is $\{1,2,\ldots,20m+10\}$ as required. Each row and column of A_1 will sum to $2nk+1$. We first give A_1 for the cases $m=2$ and $m=3$ separately; then we present the general formula.

$A_1(m=2, n=10)$

1	$10k-4$	-31	$10k-2$	37
39	3	$10k$	-34	$10k-7$
-33	40	5	$10k-8$	$10k-3$
$10k-1$	$10k-6$	36	7	-35
$10k-5$	-32	$10k-9$	38	9

$A_1\ (m=3, n=14)$

1	$14k$			-43	$14k-7$	50
$14k-1$	3	51	$14k-8$		-44	
	-45	5	$14k-2$	55		$14k-12$
		$14k-3$	7	$14k-11$	54	-46
56		-47		9	$14k-13$	$14k-4$
$14k-6$	53		-48	$14k-9$	11	
-49	$14k-10$	$14k-5$	52			13

Otherwise $m \geqslant 4$ and we define A_1 as follows. The rows and columns are defined modulo $2m+1$ rather than modulo n. To construct the overall Heffter array, the array A_1 is then shifted by adding $2m+1$ (as an integer) to each row and column.

$$A_1[4,1] = 14m+8; \qquad A_1[5,2m+1] = 14m+9; \tag{7.1}$$
$$A_1[6,2m] = 16m+8; \qquad A_1[2m-1,1] = kn-4m+1; \tag{7.2}$$
$$A_1[2m,2m+1] = kn-4m; \qquad A_1[2m+1,2m] = kn-4m-1; \tag{7.3}$$
$$A_1[i,i] = 2i-1, \qquad 1 \leqslant i \leqslant 2m+1; \tag{7.4}$$
$$A_1[i,2m-1+i] = kn+1-i, \qquad 1 \leqslant i \leqslant 2m+1; \tag{7.5}$$
$$A_1[i,i+1] = kn-2m-i, \qquad 1 \leqslant i \leqslant 2m-2; \tag{7.6}$$
$$A_1[4+i,i] = -(12m+6+i), \qquad 1 \leqslant i \leqslant 2m+1; \tag{7.7}$$
$$A_1[6+i,1+i] = 14m+9+i, \qquad 1 \leqslant i \leqslant 2m-2. \tag{7.8}$$

We note that the support for A_1 is the union of the sets:

- $\{1,3,5,6,\dots,4m+1\}$ (by (7.4)),
- $\{12m+7,12m+8,\dots,14m+7\}$ (by (7.7)),
- $\{14m+8,14m+9\}$ (by (7.1)),
- $\{14m+10,\dots,16m+7\}$ (by (7.8)),
- $\{16m+8\}$ (by (7.2)),
- $\{kn-4m-1,kn-4m,kn-4m+1\}$ (by (7.2), (7.3)),
- $\{kn-4m+2,\dots,kn-2m-1\}$ (by (7.6)) and
- $\{kn-2m,\dots,kn\}$ (by (7.5)).

We next check the row and column sums. For row r in the range 7 to $2m+1$, (7.5) and (7.6) give a partial sum of

$$(kn+1-r)+(kn-2m-r) = 2kn-2m+1-2r,$$

while (7.7) and (7.8) give a partial sum of

$$(-12m-6-(r-4))+(14m+9+(r-6)) = 2m+1.$$

Now combined with (7.4) the sum of these rows is

$$(2nk-2m+1-2r)+(2m+1)+(2r-1) = 2nk+1,$$

as required.

For column c in the range 2 to $2m-1$, (7.5) and (7.6) give a partial sum of

$$(kn+1-(c+2))+(kn-2m-(c-1)) = 2kn-2m-2c,$$

while (7.7) and (7.8) give a partial sum of

$$(-12m-6-c)+(14m+9+(c-1)) = 2m+2.$$

Now combined with (7.4) the sum of these columns is

$$(2nk - 2m - 2c) + (2m + 2) + (2c - 1) = 2nk + 1.$$

The sum of the remaining rows and columns can be calculated individually and overall the rows and columns of A_1 sum to $2nk + 1$ as required. Thus the concatenation of A_0 with A_1 gives an $H(4m + 2; 5)$. □

Theorem 7.2. *For $n \equiv 2 \pmod 4$, $n \geqslant 6$ and $k \equiv 1 \pmod 4$ there exists a Heffter array $H(n; k)$, where $k < n$.*

Proof. Let $n = 4m + 2$ and $k = 4p + 1$. A $H(n; 5)$ exists by Lemma 7.1. Otherwise, $k \geqslant 9$ and $m \geqslant 2$. We take the array $A = A_0 \cup A_1$ from the previous lemma.

We will construct $m - 2$ cycles of length n (that is, on $2(2m + 1) = n$ cells) in the upper left-hand (A_0) and lower right-hand (A_1) quadrants, and m further cycles of length n in each of the remaining quadrants. Together these form $2m - 2$ disjoint 2-factors.

From the proof of Lemma 7.1, within A_0 there are $2m + 1 - 5 = 2(m - 2)$ empty diagonals, which we take in pairs to obtain $m - 2$ cycles of length n. Next take the intersection of the last $2m + 1$ rows and columns, this is the $(2m + 1) \times (2m + 1)$ subarray that contains A_1. We will refer to diagonals within that subarray only, recalling that the rows and columns are calculated modulo $2m + 1$. We aim to find $m - 2$ cycles of length n from this subarray. The case $m = 2$ is trivial and for the case $m = 3$, observe that the empty cells of A_1 in the previous lemma form a cycle of length 14. Otherwise $m \geqslant 4$ and the array A_1 occupies diagonals $D_0, D_1, D_2, D_3, D_4, D_5, D_7, D_{2m-2}$ and D_{2m}.

Next take the following cells

$$\begin{aligned} H &= \big(\{(i + 1, i), (2m - 2 + i, i) \mid 1 \leqslant i \leqslant 2m + 1\} \setminus \\ &\qquad \{(2m - 1, 1), (2m + 1, 2m)\}\big) \cup \{(2m - 1, 2m), (2m + 1, 1)\} \\ K &= \big(\{(3 + i, i), (7 + i, i) \mid 1 \leqslant i \leqslant 2m + 1\} \setminus \\ &\qquad \{(4, 1), (6, 2m)\}\big) \cup \{(4, 2m), (6, 1)\}. \end{aligned}$$

In the example $H(18; 5)$ above these cycles are shown in cells marked by H and K, respectively. Observe that H and K form two cycles of length $4m+2$ disjoint from A_1 but are a subset of

$$D_1 \cup D_{2m-2} \cup D_{2m} \cup D_3 \cup D_7 \cup D_5.$$

Thus there exists $2m + 1 - 9 = 2(m - 4)$ diagonals that do not intersect $A_1 \cup H \cup K$. For $m \geqslant 6$, we can thus form $m - 4$ cycles of length n by taking pairs of diagonals:

$$\begin{aligned} &\{D_6, D_8\}; \\ &\{D_{2m-3}, D_{2m-1}\}; \\ &\{D_{9+2i}, D_{10+2i}\},\ 0 \leqslant i \leqslant m - 7, \end{aligned}$$

and when $m = 5$, $2m + 1 = 11$ so we get one cycle of length n by taking the diagonals D_6 and D_9. Thus with H and K we have $m - 2$ cycles of length n that are disjoint from A_1 and each other; together these form $m - 2$ 2-factors, each consisting of two cycles of length n.

We can create a further m cycles of length n in each of the remaining quadrants, as these cells are all empty. Altogether we have $2m - 2$ disjoint 2-factors. Thus by Lemma 2.4, we can fill $4(p-1)$ cells in each row and column with support $\{16m+9, 16m+10, \ldots, kn-4m-2\}$ without changing the row and column sums, where $k = 4p+1$. Thus there exists an $H(4m+2; 4p+1)$ Heffter array for each $m \geqslant 2$ and $p \leqslant m$. □

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Appendix

Case B

$H(7;3)$

15	−13	−2				
−11	14		−3			
−4		−8	12			
	−1	10	−9			
				5	21	17
				18	6	19
				20	16	7

$B(11)$

−1	18	−17					
24	−2		−22				
−23		−3	26				
	−16	20	−4				
				19	−8	−11	
				−9	21		−12
				−10		25	−15
					−13	−14	27

$B(15)$

1	−36	35									
−34	−3		37								
33			−22	−11							
	39	−21			−18						
		−14			−12	26					
			−15	−17			32				
				28			−19	−9			
					30	−10			−20		
						−16			24	−8	
							−13	38			−25
								−29		31	−2
									−4	−23	27

Case C

$H(6;3)$

-1	-16				17
-11		-4			15
12		-9	-3		
	-2		10	-8	
		13	-7	-6	
	18			14	5

$B(10)$

1	22	-23				
17	2		-19			
-18			15	3		
	-24	14			10	
		9			11	-20
			4	-16		12
				13	-21	8

$B(14)$

-34	-1	35								
-2	24		-22							
36		-32	-4							
	-23	-3	26							
				-20	28	-8				
				30	-9	-21				
				-10	-19	29				
							-11	27	-16	
							25	-12		-13
							-14		-17	31
								-15	33	-18

$B(18)$

1	21	-22												
-36	-4		40											
35			-12	-23										
	-17	33			-16									
		-11			42	-31								
			-28	41			-13							
				-18			43	-25						
					-26	45			-19					
						-14			34	-20				
							-30	-2			32			
								27		-24	-3			
									-15	44	-29			
												-8	46	-38
												-39	-9	48
												47	-37	-10

$B(22)$

1	−36	35																
−30	−4		34															
29			−3	−26														
	40	−2			−38													
		−33			54	−21												
			−31	48			−17											
				−22			42	−20										
					−16	53			−37									
						−32			−15	47								
							−25	39			−14							
								−19			−27	46						
									52	−24		−28						
										−23	41	−18						
													−11	55	−44			
													−45	−12	57			
													56	−43	−13			
																−8	58	−50
																−51	−9	60
																59	−49	−10

$B(26)$

−64	1	63																				
	65	−16	−49																			
−3			53	−50																		
	−66		−4		70																	
		−47			−22	69																
				68	−48		−20															
				−18			59	−41														
						−52		−10	62													
						−17		51	−34													
67							−39		−28													
										−29	61	−32										
										−42	−30	72										
										71	−31	−40										
													−2	60	−58							
													23		15	−38						
													−21	−33			54					
														−27			−8	35				
															43	14			−57			
																24			13	−37		
																	−46	−9			55	
																		−26			−19	45
																			44	12		−56
																				25	−36	11

$H(30,3)$

1	−62	61	K																								K	H	H
−77	K	H	−3	80																								K	H
	83	−12	H	H	−71									K															K
K	−21	K	−60	H	H	81																							
	H	−49	K	K	75	H	−26																						
		H	63	K	−4	K	H	−59																					
			H	−22	K	−57	K	79	H																				
				−58	H	K	K	−20	78	H																			
					K	−24	74	K	H	−50	H																		
						H	−48	H	K	−34	82	K																	
							H	K	−38	H	−35	73	K																
76								H	−40	K	K	−36	H																
									K	84	−47	−37	H	H	K														
										K	H	H	2	70	−72	K													
											K	H	−27	−19	H	46	K												
												K	25	H	39	H	−64	K											
													K	−51	H	−18	H	69	K										
														K	33	H	8	H	−41	K									
															K	−28	H	−17	H	45	K								
																K	56	H	9	H	−65	K							
																	K	−52	H	−16	H	68	K						
																		K	32	H	10	H	−42	K					
																			K	−29	H	−15	H	44	K				
																				K	55	H	11	H	−66	K			
																					K	−53	H	−14	H	67	K		
																						K	31	H	23	−54	H	K	
																							K	−30	43	−13	H	H	K
K																								K	H	H	5	kn	$kn-4$
H	K																								K	H	$kn-3$	6	$kn-2$
H	H	K																								K	$kn-1$	$kn-5$	7

A_1

1	$10m-4$	$-(10m-3)$										
$6m-5$	2		$-(6m-3)$									
$-(6m-4)$			$-(6m+7)$	$12m+3$								
	$-(10m-2)$	$10m+4$			-6							
		-7			$10m+1$	$-(10m-6)$						
			$12m+4$	$-(6m+5)$			$-(6m-1)$					
				$-(6m-2)$			$-(6m+3)$	$12m+1$				
					$-(10m-5)$	-5			$10m$			
						$10m-1$			$-(10m+3)$	4		
							$12m+2$	$-6m$			$-(6m+2)$	
								$-(6m+1)$			$-(6m+4)$	$12m+5$
									3	$10m+2$		$-(10m+5)$
										$-(10m+6)$	$12m+6$	$-2m$

Cases D and E

$H(9;5)$

45	36	20					−18	8
−16	24	43	34					6
	44	35	22	7		−17		
		5	42	−15	33		26	
9			−10	32	41	19		
	1			40	−2	21	31	
		−12			23	30	39	11
25			3		−4	38		29
28	−14			27			13	37

$H(6;5)$

1	2	3		−25	19
5	6	16	4		30
23	7	9	8	14	
11		15	12	10	13
	24	18	17	29	−27
21	22		20	−28	26

$H(13;9)$

65	−21	H	9		38		K	K			H	40
52	64	−22	H	10		27		K	K			H
H	51	63	−23	H	11				K	K	29	
		50	62	−24	H	H		31		K	12	K
	H	H	49	61	−25		13		33		K	K
K	K		H	48	60	H	−26	14		35		
−16	K	K			47	46	H	H	17		37	
		K	K		H	59	45	−15	H	3		39
28	H		K	K		19	58	44	−18	H		
	30			K	K		H	57	43	−4	H	5
2		32		H	K	K			56	42	−1	H
H	7		34			K	41	H		55	K	−6
K		8		36		−20	K		H	H	54	53

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Semigroups with fixed multiplicity and embedding dimension*

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Abstract

Given $m \in \mathbb{N}$, a numerical semigroup with multiplicity m is called a packed numerical semigroup if its minimal generating set is included in $\{m, m+1, \ldots, 2m-1\}$. In this work, packed numerical semigroups are used to build the set of numerical semigroups with a given multiplicity and embedding dimension, and to create a partition of this set. Wilf's conjecture is verified in the tree associated to some packed numerical semigroups. Furthermore, given two positive integers m and e, some algorithms for computing the minimal Frobenius number and minimal genus of the set of numerical semigroups with

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multiplicity m and embedding dimension e are provided. We also compute the semigroups where these minimal values are achieved.

Keywords: Embedding dimension, Frobenius number, genus, multiplicity, numerical semigroup.

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1 Introduction

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of non-negative integers. A numerical semigroup is a subset S of $\mathbb{N}$ which is closed under addition, such that $0 \in S$ and $\mathbb{N} \setminus S$ is finite. If S is a numerical semigroup, we define the multiplicity of S, denoted by $m(S)$, to be the least positive integer in S, the Frobenius number ($F(S)$) to be the greatest integer that is not in S, and the genus, $g(S)$, to be the cardinality of $\mathbb{N} \setminus S$.

Given a non-empty subset A of $\mathbb{N}$ we denote by $\langle A \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by A, that is,

$$\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N},\ a_1, \ldots, a_n \in A,\ \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}.$$

It is well known (for example, see Lemma 2.1 from [11]) that $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$. If S is a numerical semigroup and $S = \langle A \rangle$, we say that A is a system of generators of S. Moreover, A is a minimal system of generators of S if $S \neq \langle B \rangle$ for every $B \subsetneq A$. In Theorem 2.7 from [11] it is shown that every numerical semigroup has a unique minimal system of generator and this system is finite. We denote by $\operatorname{msg}(S)$ and $e(S)$ the minimal system of generators of S and its cardinality, also called the embedding dimension of S.

If m and e are positive integers we use the following notation:

$$\mathcal{L}(m, e) = \{S \mid S \text{ is a numerical semigroup},\ m(S) = m,\ e(S) = e\}.$$

In this work, one of our aims is to present a procedure that allows us to recursively construct the set $\mathcal{L}(m, e)$.

We say that a numerical semigroup S is a packed numerical semigroup if

$$\operatorname{msg}(S) \subseteq \{m(S), m(S) + 1, \ldots, 2m(S) - 1\}.$$

The set of all packed numerical semigroups with multiplicity m and embedding dimension e is denoted by $\mathcal{C}(m, e)$.

In Section 2, an equivalence relation $\mathcal{R}$ in the set $\mathcal{L}(m, e)$ is defined. For each $S \in \mathcal{L}(m, e)$ we denote by $[S]$, the equivalence class of S. We show that if $S \in \mathcal{L}(m, e)$ then $[S] \cap \mathcal{C}(m, e)$ has cardinality 1, so $\{[S] \mid S \in \mathcal{C}(m, e)\}$ is a partition of $\mathcal{L}(m, e)$. Hence, for computing all the elements of the set $\mathcal{L}(m, e)$ it is only necessary to perform the following steps:

1. Compute $\mathcal{C}(m, e)$.
2. For every $S \in \mathcal{C}(m, e)$ compute $[S]$.

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We see that it is possible to compute $\mathcal{C}(m,e)$, since this problem is equivalent to computing all the subsets A of $\{1,2,\ldots,m-1\}$ such that A has cardinality $e-1$ and $\gcd(A\cup\{m\})=1$. For computing $[S]$, we order its elements by making a tree whose root is S, and describing the children of each of the vertices. In this way, we can recursively build the elements of $[S]$ by adding in at each step the children of the vertices that were obtained in the previous step. This procedure is not algorithmic because $[S]$ is infinite so we can not build it in a finite number of steps.

The Frobenius number and genus have been widely studied (see [7]) and they, together with the embedding dimension, are the background of one the most important problems in this theory: Wilf's conjecture which asserts that if S is a numerical semigroup then $e(S)g(S)\leq(e(S)-1)(F(S)+1)$ (see [15]). At present, it is still open.

In this work, we show that if we go along through a branch of the tree associated to $[S]$, the numerical semigroups have a greater Frobenius number and genus. These facts enable us to give an algorithm for building all the elements of $\mathcal{L}(m,e)$ with a fixed Frobenius number and/or genus. Finally, in order to compute the Frobenius number and the genus of the numerical semigroups of $[S]$, we give an algorithm based on [3]. We would like to note that these new algorithms enable us to study the tree of numerical semigroups with a given multiplicity and embedding dimension and with Frobenius number and/or genus up to any given bound. In some works, there appear algorithms for the computation of the tree of numerical semigroups up to a certain genus (see for example [4]). In our work, there can also be a bound on the Frobenius number. In addition, the computation of the complete tree of numerical semigroups up to a certain genus is not a practical method to obtain the numerical semigroups with a fixed multiplicity and embedding dimension, since it performs unnecessary calculations and does not obtain as large a genus as we can get with our algorithms. We are also interested in giving algorithms for computing

$$g(m,e)=\min\{g(S)\mid S\in\mathcal{L}(m,e)\},$$
$$F(m,e)=\min\{F(S)\mid S\in\mathcal{L}(m,e)\},$$
$$\{S\in\mathcal{L}(m,e)\mid g(S)=g(m,e)\},$$

and

$$\{S\in\mathcal{L}(m,e)\mid F(S)=F(m,e)\}.$$

These methods are illustrated with several examples. To accomplish this, we have used the library `FrobeniusNumberAndGenus` developed by the authors in Mathematica ([16]). This library is freely available online at [5].

The content of this work is organized as follows. In Section 2, a partition of the set $\mathcal{L}(m,e)$ is studied and we construct a map $\phi\colon\mathcal{L}(m,e)\to\mathcal{C}(m,e)$ such that $[S]\cap\mathcal{C}(m,e)$ is equal to $\{\phi(S)\}$ for every $S\in\mathcal{L}(m,e)$. Theorem 3.3, in Section 3, is used to recursively compute the elements of $[S]$. In Section 4, we give some algorithms for computing the elements of $[S]$ with Frobenius number and/or genus less than fixed integer bounds. In Section 5, we show how the Apéry set of the elements of $[S]$ is used to compute their Frobenius number and genus. We also check that Wilf's conjecture is satisfied for some elements of $[S]$. Section 6 illustrates the preceding section and Section 7 contains some known results on Frobenius pseudo-varieties which allow us to construct the tree of all numerical semigroups with any given multiplicity. In Section 8 and Section 9, the minimal genus and minimal Frobenius number of the set of numerical semigroups with fixed multiplicity and embedding dimension are studied, giving some algorithms for computing them and obtaining the semigroups with these minimal values.

2 A partition of $\mathcal{L}(m, e)$

If A and B are subsets of $\mathbb{N}$ we denote by $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$. It is well known (for example see Proposition 2.10 from [11]) that if S is a numerical semigroup then $e(S) \leq m(S)$. Note that if $e(S) = 1$ then $S = \mathbb{N}$. Therefore, in the sequel, we assume that e and m are integers such that $2 \leq e \leq m$.

Given $S \in \mathcal{L}(m, e)$ we denote by $\phi(S)$ the numerical semigroup generated by $\{m\} + \{x \bmod m \mid x \in \operatorname{msg}(S)\}$. Clearly, $\phi(S)$ is a packed numerical semigroup and therefore we have the following result.

Lemma 2.1. *With the previous assumptions, ϕ defines a surjective map from $\mathcal{L}(m, e)$ to $\mathcal{C}(m, e)$.*

We define in $\mathcal{L}(m, e)$ the following equivalence relation: $S \mathrel{\mathcal{R}} T$ if $\phi(S) = \phi(T)$. Given $S \in \mathcal{L}(m, e)$, $[S]$ denotes the set $\{T \in \mathcal{L}(m, e) \mid S \mathrel{\mathcal{R}} T\}$. Therefore, the quotient set $\mathcal{L}(m, e)/\mathcal{R} = \{[S] \mid S \in \mathcal{L}(m, e)\}$ is a partition of $\mathcal{L}(m, e)$.

Lemma 2.2. *If $S \in \mathcal{L}(m, e)$, then $[S] \cap \mathcal{C}(m, e) = \{\phi(S)\}$.*

Proof. By Lemma 2.1, we know that $\phi(S) \in \mathcal{C}(m, e)$. Moreover, it is clear that $\phi(\phi(S)) = \phi(S)$. Therefore, $S \mathrel{\mathcal{R}} \phi(S)$ and $\phi(S) \in [S] \cap \mathcal{C}(m, e)$.

If $T \in [S] \cap \mathcal{C}(m, e)$, then $\phi(T) = \phi(S)$ and $\phi(T) = T$, so $T = \phi(S)$. □

The following result is a consequence of the previous lemmas.

Theorem 2.3. *Let m and e be integers such that $2 \leq e \leq m$. Then $\{[S] \mid S \in \mathcal{C}(m, e)\}$ is a partition of $\mathcal{L}(m, e)$. Moreover, if $\{S, T\} \subseteq \mathcal{C}(m, e)$ and $S \neq T$ then $[S] \cap [T] = \emptyset$.*

Therefore, as a consequence of Theorem 2.3, for computing all the elements of the set $\mathcal{L}(m, e)$ it is only necessary to do the following steps:

1. Compute $\mathcal{C}(m, e)$.
2. For every $S \in \mathcal{C}(m, e)$ compute $[S]$.

$\mathcal{C}(m, e)$ is easy to compute using the following result.

Proposition 2.4. *Let m and e be integers such that $2 \leq e \leq m$, and let A be a subset of $\{1, \ldots, m-1\}$ with cardinality $e - 1$ such that $\gcd(A \cup \{m\}) = 1$. Then*

$$S = \langle \{m\} + (A \cup \{0\}) \rangle \in \mathcal{C}(m, e).$$

Moreover, every element of $\mathcal{C}(m, e)$ has this form.

Proof. The set S is a numerical semigroup because

$$\gcd(\{m\} + (A \cup \{0\})) = \gcd(A \cup \{m\}) = 1.$$

It is straightforward to prove that $\operatorname{msg}(S) = \{m\} + (A \cup \{0\})$, so $S \in \mathcal{C}(m, e)$.

If $S \in \mathcal{C}(m, e)$ then $\operatorname{msg}(S) = \{m, m + r_1, \ldots, m + r_{e-1}\}$ with $\{r_1, \ldots, r_{e-1}\} \subseteq \{1 \ldots, m-1\}$. Moreover, since $\gcd\{m, m + r_1, \ldots, m + r_{e-1}\} = 1$,

$$\gcd\{m, r_1, \ldots, r_{e-1}\} = 1.$$ □

We illustrate the content of the previous proposition with an example.

Example 2.5. We are going to compute the set $\mathcal{C}(6,3)$ formed by all the packed numerical semigroups of multiplicity 6 and embedding dimension 3. For this purpose, and using Proposition 2.4, it is enough computing the subsets A of $\{1,2,3,4,5\}$ of cardinality 2 such that $\gcd(A \cup \{6\}) = 1$. This set is equal to

$$\{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}.$$

Therefore,

$$\begin{aligned}\mathcal{C}(6,3) = \{&\langle 6,7,8\rangle, \langle 6,7,9\rangle, \langle 6,7,10\rangle, \langle 6,7,11\rangle,\\ &\langle 6,8,9\rangle, \langle 6,8,11\rangle, \langle 6,9,10\rangle, \langle 6,9,11\rangle, \langle 6,10,11\rangle\}.\end{aligned}$$

Note that if m is a prime number then every subset A of $\{1,\dots,m-1\}$ with cardinality $e-1$ verifies that $\gcd(A \cup \{m\}) = 1$. Therefore, we have the following result.

Proposition 2.6. *If m is a prime number and e is an integer number such that $2 \le e \le m$ then $\mathcal{C}(m,e)$ has cardinality $\binom{m-1}{e-1}$.*

Our next goal in this work is to show a recursive procedure to compute $[S]$ for every $S \in \mathcal{C}(m,e)$. In order to achieve it, in the next section, we set the elements of $[S]$ in a tree.

3 The tree associated to $[S]$

A graph G is pair (V,E) where V is a set (with elements called vertices) and E is a subset of $\{(v,w) \in V \times V \mid v \neq w\}$ (with elements called edges). A path which connects the vertices x and y of G is a sequence of different edges of the form (v_0,v_1), $(v_1,v_2),\dots,(v_{n-1},v_n)$ such that $v_0 = x$ and $v_n = y$. A graph G is a tree if there exists a vertex r (known as the root of G) such that for any other vertex x of G there exists a unique path connecting x and r. If (x,y) is an edge of a tree, we say that x is a child of y.

Lemma 3.1. *If $\{n_1 < n_2 < \cdots < n_e\}$ is a minimal system of generators of a numerical semigroup and $n_e - n_1 > n_1$ then $\{n_1,\dots,n_{e-1},n_e-n_1\}$ is also a minimal system of generators of a numerical semigroup.*

Proof. In other case, there exists $k \in \{1,\dots,e-1\}$ such that

$$n_k \in \{n_e - n_1\} + \langle n_1,\dots,n_{k-1},n_{k+1},\dots,,n_{e-1},n_e-n_1\rangle.$$

But it is not possible because $n_e - n_1 + n_1 = n_e > n_k$. □

Let S be a numerical semigroup. We denote by $M(S)$ the maximum of $\mathrm{msg}(S)$. If $S \in \mathcal{L}(m,e)$, we define the following sequence of elements of $\mathcal{L}(m,e)$:

- $S_0 = S$,
- $S_{n+1} = \langle(\mathrm{msg}(S_n) \setminus \{M(S_n)\}) \cup \{M(S_n) - m\}\rangle$ if $M(S_n) - m > m$.

Because of Lemma 3.1, there exists a sequence:

$$S = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k = \phi(S) \in \mathcal{C}(m,e).$$

Example 3.2. Let $S \in \mathcal{L}(5,3)$ be the semigroup minimally generated by $\{5,13,21\}$. Then, we have the following sequence of elements of $\mathcal{L}(5,3)$:

$$S_0 = \langle 5,13,21\rangle \subsetneq S_1 = \langle 5,13,16\rangle \subsetneq S_2 = \langle 5,11,13\rangle \\ \subsetneq S_3 = \langle 5,8,11\rangle \subsetneq S_4 = \langle 5,6,8\rangle = \phi(S) \in \mathcal{C}(5,3).$$

Let S be in $\mathcal{C}(m,e)$. We define the graph $G([S])$ as follows: $[S]$ is the set of vertices and $(A,B) \in [S]\times[S]$ is an edge if $\operatorname{msg}(B) = (\operatorname{msg}(A)\setminus\{M(A)\})\cup\{M(A)-m\}$.

Theorem 3.3. *If $S \in \mathcal{C}(m,e)$ then $G([S])$ is a tree with root S. Moreover, if $P \in [S]$ and $\operatorname{msg}(P) = \{n_1 < n_2 < \cdots < n_e\}$ then the children of P in $G([S])$ are the numerical semigroups of the form $\langle(\{n_1,\dots,n_e\}\setminus\{n_k\})\cup\{n_k+n_1\}\rangle$ such that $k \in \{2,\dots,e\}$, $n_k+n_1 > n_e$ and $n_k+n_1 \notin \langle\{n_1,\dots,n_e\}\setminus\{n_k\}\rangle$.*

Proof. From the definition and the comment after Lemma 3.1, we have that $G([S])$ is a tree with root S.

Let k be in $\{2,\dots,e\}$ such that $n_k+n_1 > n_e$ and $n_k+n_1 \notin \langle\{n_1,\dots,n_e\}\setminus\{n_k\}\rangle$. If $H = \langle(\{n_1,\dots,n_e\}\setminus\{n_k\})\cup\{n_k+n_1\}\rangle$ is clear that

$$\operatorname{msg}(H) = (\{n_1,\dots,n_e\}\setminus\{n_k\})\cup\{n_k+n_1\} \quad \text{and} \\ \operatorname{msg}(P) = (\operatorname{msg}(H)\setminus\{M(H)\})\cup\{M(H)-m\}.$$

Therefore H is a child of P.

Conversely, if H is a child of P then (H,P) is an edge of $G([S])$ and we obtain that H is as the theorem describes. □

The previous theorem provides us with a method to recursively build the elements of $[S]$ as it is shown in the next example.

Example 3.4. Figure 1 shows some levels of the tree $G([\langle 5,6,8\rangle])$.

Note that the cardinality of $[S]$ is infinity, so it is impossible to compute all the elements of $[S]$. However, in the next section, we show that it is possible to compute all the elements of $[S]$ with a fixed Frobenius number or genus.

4 Frobenius number and genus

Let P be a numerical semigroup with minimal generating set $\{n_1 < n_2 < \cdots < n_e\}$, $k \in \{2,\dots,e\}$ and H be the numerical semigroup generated by $(\{n_1,\dots,n_e\}\setminus\{n_k\})\cup\{n_k+n_1\}$. Then $H \subset P$, $F(P) \le F(H)$ and $g(P) < g(H)$. We can formulate the following result.

Proposition 4.1. *If $S \in \mathcal{C}(m,e)$, $P \in [S]$ and (H,P) is an edge of $G([S])$ then $F(P) \le F(H)$ and $g(P) < g(H)$.*

As a consequence of the previous proposition, we have that if we go along through the branches of the tree $G([S])$, the numerical semigroups that we are finding have greater or equal Frobenius number, and also a greater genus. This fact enables us to formulate the Algorithms 1 and 3 in order to compute all the elements in $[S]$ with Frobenius number less than or equal to a given integer, and genus less than or equal to another given integer, respectively.

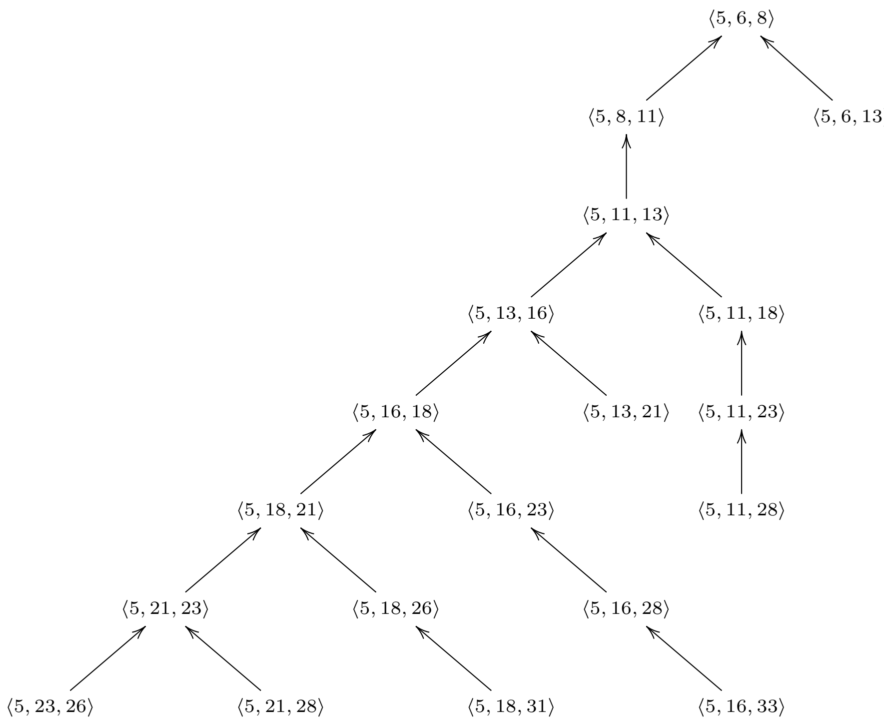


Figure 1: Seven levels of the tree of the packed numerical semigroup $\langle 5, 6, 8\rangle$.

Algorithm 1 An algorithm to determinate the elements $T \in [S]$ such that $F(T) \leq F$ for a fixed integer F.

INPUT: (S, F) where S is a packed numerical semigroup and F is a positive integer.
OUTPUT: $\{T \in [S] \mid F(T) \leq F\}$.

if $F(S) > F$ **then**
  return $\emptyset$
while true **do**
  $A = \{S\}$ and $B = \{S\}$.
  $C = \{H \mid H$ is a child of an element of $B,\ F(H) \leq F\}$.
  if $C = \emptyset$ **then**
    return A
  $A = A \cup C$, $B = C$.

The following example illustrates how the previous algorithm works.

Example 4.2. We compute all the elements of $[\langle 5, 6, 8\rangle]$ with Frobenius number less than or equal to 25.

- $A = \{\langle 5, 6, 8\rangle\}$, $B = \{\langle 5, 6, 8\rangle\}$ and $C = \{\langle 5, 8, 11\rangle, \langle 5, 6, 13\rangle\}$.
- $A = \{\langle 5, 6, 8\rangle, \langle 5, 8, 11\rangle, \langle 5, 6, 13\rangle\}$, $B = \{\langle 5, 8, 11\rangle, \langle 5, 6, 13\rangle\}$ and $C = \{\langle 5, 11, 13\rangle\}$.

- $A = \{\langle 5,6,8\rangle, \langle 5,8,11\rangle, \langle 5,6,13\rangle, \langle 5,11,13\rangle\}$, $B = \{\langle 5,11,13\rangle\}$ and $C = \{\langle 5,11,18\rangle\}$.
- $A = \{\langle 5,6,8\rangle, \langle 5,8,11\rangle, \langle 5,6,13\rangle, \langle 5,11,13\rangle, \langle 5,11,18\rangle\}$, $B = \{\langle 5,11,18\rangle\}$ and $C = \emptyset$.

Therefore, the set $\{T \in [\langle 5,6,8\rangle] \mid F(T) \leq 25\}$ is equal to $\{\langle 5,6,8\rangle, \langle 5,8,11\rangle, \langle 5,6,13\rangle, \langle 5,11,13\rangle, \langle 5,11,18\rangle\}$.

The next algorithm allows us to compute all the numerical semigroups with multiplicity m, embedding dimension e and Frobenius number less than or equal to F. Note that if S is a numerical semigroup, such that $S \neq \mathbb{N}$ then $m(S) - 1 \notin S$ and then $m(S) - 1 \leq F(S)$.

Algorithm 2 An algorithm to determinate the numerical semigroups with a fixed embedding dimension and multiplicity, and bounded Frobenius number.

INPUT: m, e, and F positive integers such that $2 \leq e \leq m \leq F+1$.
OUTPUT: $\{S \mid S \text{ numerical semigroup, } m(S) = m,\ e(S) = e \text{ and } F(S) \leq F\}$.

compute $\mathcal{C}(m,e)$, using Proposition 2.4.
for all $S \in \mathcal{C}(m,e)$ **do**
    compute $A(S) = \{T \in [S] \mid F(T) \leq F\}$, using Algorithm 1.
return $\cup_{S \in \mathcal{C}(m,e)} A(S)$.

Now, we change Frobenius number by the genus in Algorithm 1 and Algorithm 2.

Algorithm 3 An algorithm to determinate the elements $T \in [S]$ such that $g(T) \leq g$ for a fixed integer g.

INPUT: (S, g) where S is a packed numerical semigroup and g is a positive integer.
OUTPUT: $\{T \in [S] \mid g(T) \leq g\}$.

if $g(S) > g$ **then**
    return $\emptyset$
$A = \{S\}$ and $B = \{S\}$.
while true **do**
    $C = \{H \mid H \text{ is a child of an element of } B,\ g(H) \leq g\}$.
    if $C = \emptyset$ **then**
        return A
    $A = A \cup C$, $B = C$.

We illustrate now the above algorithm.

Example 4.3. We compute all the elements of $[\langle 5,6,8\rangle]$ with genus less than or equal to 15.

- $A = \{\langle 5,6,8\rangle\}$, $B = \{\langle 5,6,8\rangle\}$ and $C = \{\langle 5,8,11\rangle, \langle 5,6,13\rangle\}$.
- $A = \{\langle 5,6,8\rangle, \langle 5,8,11\rangle, \langle 5,6,13\rangle\}$, $B = \{\langle 5,8,11\rangle, \langle 5,6,13\rangle\}$ and $C = \{\langle 5,11,13\rangle\}$.
- $A = \{\langle 5,6,8\rangle, \langle 5,8,11\rangle, \langle 5,6,13\rangle, \langle 5,11,13\rangle\}$, $B = \{\langle 5,11,13\rangle\}$ and $C = \{\langle 5,11,18\rangle\}$.

- $A = \{\langle 5,6,8\rangle, \langle 5,8,11\rangle, \langle 5,6,13\rangle, \langle 5,11,13\rangle, \langle 5,11,18\rangle\}$, $B = \{\langle 5,11,18\rangle\}$ and $C = \emptyset$.

Algorithm 3 returns $\{\langle 5,6,8\rangle, \langle 5,8,11\rangle, \langle 5,6,13\rangle, \langle 5,11,13\rangle, \langle 5,11,18\rangle\}$.

Note that if S is a numerical semigroup such that $S \neq \mathbb{N}$ then $\{1,\ldots,m(S)-1\} \subseteq \mathbb{N} \setminus S$ and then $m(S)-1 \leq g(S)$.

Combining the above results, we obtain Algorithm 4.

Algorithm 4 An algorithm to compute numerical semigroups with fixed multiplicity, embedding dimension and bounded genus.

INPUT: m, e, and g positive integers such that $2 \leq e \leq m \leq g+1$.
OUTPUT: $\{S \mid S$ numerical semigroup, $m(S)=m,\ e(S)=e$ and $g(S) \leq g\}$.

1: compute $\mathcal{C}(m,e)$, using Proposition 2.4.
2: **for all** $S \in \mathcal{C}(m,e)$ **do**
3:     compute $A(S) = \{T \in [S] \mid g(T) \leq g\}$, using Algorithm 3.
4: **return** $\cup_{S\in\mathcal{C}(m,e)} A(S)$.

Note that applying Algorithms 1 and 2 we have to compute the Frobenius number and the genus, respectively, of the numerical semigroups we recursively obtain when we build $[S]$. Results of the next section enable us to easily compute the Frobenius number and the genus of every semigroup of $[S]$.

5 The Apéry set of the elements of $[S]$

Let S be a numerical semigroup and $n \in S \setminus \{0\}$. The Apéry set (named by [1]) of n in S is $\mathrm{Ap}(S,n) = \{s \in S \mid s-n \notin S\}$. The next result is a consequence of Lemma 2.4 from [11].

Lemma 5.1. *Let S be a numerical semigroup and $n \in S \setminus \{0\}$. Then $\mathrm{Ap}(S,n)$ has cardinality n. Moreover, $\mathrm{Ap}(S,n) = \{w(0)=0, w(1),\ldots,w(n-1)\}$ where $w(i)$ is the least element in S congruent with i modulo n.*

The set $\mathrm{Ap}(S,n)$ gives us a lot of information of S. The following result is found in [13].

Lemma 5.2. *Let S be a numerical semigroup and $n \in S \setminus \{0\}$. Then:*

- $F(S) = \max(\mathrm{Ap}(S,n)) - n$.
- $g(S) = \frac{1}{n}(\sum_{w\in\mathrm{Ap}(S,n)} w) - \frac{n-1}{2}$.

The following result is a consequence of Lemma 5.1.

Lemma 5.3. *Let S be a numerical semigroup with minimal system of generators $\{n_1, n_2, \ldots, n_e\}$ and $\mathrm{Ap}(S,n_1) = \{0, w(1),\ldots,w(n_1-1)\}$. Then*

$$w(i) = \min\{a_2n_2 + \cdots + a_en_e \mid (a_2,\ldots,a_e) \in \mathbb{N}^{e-1} \text{ and } a_2n_2+\cdots+a_en_e \equiv i \pmod{n_1}\}.$$

Note that the set $\{(a_2,\ldots,a_e)\in\mathbb{N}^{e-1} \mid a_2n_2+\cdots+a_en_e\equiv i \pmod{n_1}\}$ has a finite number of minimal elements (using the usual ordering in $\mathbb{N}^{e-1}$) by Dickson's Lemma (Theorem 5.1 from [10]). We denote the set of these minimal elements by $\mathcal{M}((n_1,\ldots,n_e),i)$. The following result is obtained from Lemma 5.3.

Proposition 5.4. *Let S be a numerical semigroup with minimal system of generators $\{n_1,n_2,\ldots,n_e\}$ and* $\mathrm{Ap}(S,n_1)=\{0,w(1),\ldots,w(n_1-1)\}$. *Then*

$$w(i)=\min\{a_2n_2+\cdots+a_en_e \mid (a_2,\ldots,a_e)\in\mathcal{M}((n_1,\ldots,n_e),i)\}.$$

We illustrate the above proposition with an example.

Example 5.5. In this example we try to compute the Apéry set of the numerical semigroups of $[\langle 5,6,8\rangle]$ that we obtained in Example 3.4.

For every $i\in\{1,2,3,4\}$ let $A(i)$ be the set

$$\{(a_2,a_3)\in\mathbb{N}^2 \mid a_2\cdot 1+a_3\cdot 3\equiv i \pmod 5\},$$

and let $\mathcal{M}(i)$ be the set of the minimal elements of $A(i)$. Then,

$$\begin{aligned}\mathcal{M}(1)&=\{(1,0),(0,2)\},\\ \mathcal{M}(2)&=\{(2,0),(0,4),(1,2)\},\\ \mathcal{M}(3)&=\{(3,0),(0,1)\}\quad\text{and}\\ \mathcal{M}(4)&=\{(4,0),(0,3),(1,1)\}.\end{aligned}$$

Now, if we take an element from $[\langle 5,6,8\rangle]$, for example $S=\langle 5,21,13\rangle$, and we want to compute $\mathrm{Ap}(S,5)=\{0,w(1),w(2),w(3),w(4)\}$, by applying Proposition 5.4 we have that $w(1)=\min\{21,26\}=21$, $w(2)=\min\{42,52,47\}=42$, $w(3)=\min\{63,13\}=13$ and $w(4)=\min\{84,39,34\}=34$.

Note that in the previous example it was easy to compute $\mathcal{M}(i)$ for every $i\in\{1,2,3,4\}$. Now, our next goal is to give an algorithm for computing $\mathcal{M}((n_1,\ldots,n_e),i)$. In order to do it, we introduce the following sets:

$$\begin{aligned}C(1)&=\{(x_2,\ldots,x_e)\in\mathbb{N}^{e-1} \mid n_2x_2+\cdots+n_ex_e\equiv i \pmod{n_1}\},\\ C(2)&=\{(x_1,x_2,\ldots,x_e)\in\mathbb{N}^{e} \mid (-n_1)x_1+n_2x_2+\cdots+n_ex_e=i\},\\ C(3)&=\{(x_1,x_2,\ldots,x_e,x_{e+1})\in\mathbb{N}^{e+1} \mid \\ &\qquad (-n_1)x_1+n_2x_2+\cdots+n_ex_e+(-i)x_{e+1}=0\}.\end{aligned}$$

Lemma 5.6. *If $(a_2,\ldots,a_e)\in C(1)$ then there exists $a_1\in\mathbb{N}$ such that $(a_1,a_2,\ldots,a_e)\in C(2)$.*

Proof. It is enough to note that if $n_2a_2+\cdots+n_ea_e\equiv i \pmod{n_1}$ then, there exist $a_1\in\mathbb{N}$ such that $n_2a_2+\cdots+n_ea_e=i+a_1n_1$. □

Thanks to [12] we know that $C(3)$ is a finitely generated submonoid of $\mathbb{N}^{e+1}$. The next result can be deduced from Lemma 2 of [12].

Lemma 5.7. *Let A be the set $\{\alpha_1,\ldots,\alpha_t\}$ with $\alpha_i=(\alpha_{i1},\alpha_{i2},\ldots,\alpha_{ie},\alpha_{i\,e+1})$ a system of generators of $C(3)$. If we suppose that $\alpha_1,\ldots,\alpha_d$ are the elements in A with the last coordinate equal to zero and $\alpha_{d+1},\ldots,\alpha_q$ are the elements of S with the last coordinate equal to* 1*, then $C(2)=\{\bar\alpha_{d+1},\ldots\bar\alpha_q\}+\langle\bar\alpha_1,\ldots,\bar\alpha_d\rangle$ where $\bar\alpha_i=(\alpha_{i1},\alpha_{i2},\ldots,\alpha_{ie})$.*

Note that $\mathcal{M}((n_1,\ldots,n_e),i)$ are the minimal elements of $C(1)$. Hence, the following result allows us to compute it.

Proposition 5.8. *The minimal elements of* $C(1)$ *are the same that the minimal elements of the set* $\{(\alpha_{d+1,2},\ldots,\alpha_{d+1,e}),\ldots,(\alpha_{q2},\ldots,\alpha_{qe})\}$.

Proof. Let k be in $\{d+1,\ldots,q\}$. We check if $(\alpha_{k2},\ldots,\alpha_{ke})\in C(1)$. Since $(\alpha_{k1},\ldots,\alpha_{ke},1)\in C(3)$, then $(-n_1)\alpha_{k1}+n_2\alpha_{k2}+\cdots+n_e\alpha_{ke}-i=0$. Therefore $n_2\alpha_{k2}+\cdots+n_e\alpha_{ke}\equiv i \pmod{n_1}$ so $(\alpha_{k2},\ldots,\alpha_{ke})\in C(1)$.

We finish the proof checking that if $(a_2,\ldots,a_e)\in C(1)$ then there exists $k\in\{d+1,\ldots,q\}$ such that $(\alpha_{k2},\ldots,\alpha_{ke})\le(a_2,\ldots,a_e)$. By Lemma 5.6, there exists $a_1\in\mathbb{N}$ such that $(a_1,a_2,\ldots,a_e)\in C(2)$. Hence by Lemma 5.7, there exists $k\in\{d+1,\ldots,q\}$ such that $(\alpha_{k1},\alpha_{k2},\ldots,\alpha_{ke})\le(a_1,a_2,\ldots,a_e)$. Therefore, we have that $(\alpha_{k2},\ldots,\alpha_{ke})\le(a_2,\ldots,a_e)$. □

An efficient algorithm for computing a finite system of generators of $C(3)$ is given in [3]. So, applying the previous result we have an algorithm which allows us to compute the minimal elements of $C(1)$. Therefore, using Proposition 5.4 and the idea exposed in Example 5.5, we have an algorithm for computing easily $\mathrm{Ap}(T,m)$ for every $T\in[S]$. Finally, thanks to Lemma 5.2 we can compute $F(T)$ and $g(T)$ for every $T\in[S]$.

6 Examples

We devote this section to illustrate the previous results with several examples. They show all the semigroups with a fixed multiplicity, embedding dimension, and Frobenius number or genus. Besides, we check Wilf's conjecture for many semigroups in the tree associated to $[S]$ for several packed numerical semigroups. The computations have been done in an Intel i7 with 32 GB of RAM, and using `Mathematica` ([16]).

Example 6.1. In this example we compute all the numerical semigroups with multiplicity 6, embedding dimension 3, and Frobenius number equal to 23.

With these fixed conditions, the set $\mathcal{C}(m,e)$ is

$$\{\langle 6,7,8\rangle,\langle 6,7,9\rangle,\langle 6,7,10\rangle,\langle 6,7,11\rangle,\langle 6,8,9\rangle,\\ \langle 6,8,11\rangle,\langle 6,9,10\rangle,\langle 6,9,11\rangle,\langle 6,10,11\rangle\}.$$

The Frobenius number of these semigroups are $17,17,15,16,19,21,23,25$ and 25, respectively. So, by Proposition 4.1, for computing the semigroups with Frobenius number 23, we only consider the packed numerical semigroups $L=\{\langle 6,7,8\rangle,\langle 6,7,9\rangle,\langle 6,7,10\rangle,\langle 6,7,11\rangle,\langle 6,8,9\rangle,\langle 6,8,11\rangle,\langle 6,9,10\rangle\}$. Applying Algorithm 2, we compute the elements in $G([S])$ with the fixed Frobenius number. For example, from the first packed numerical semigroups in L only one numerical semigroup with Frobenius number equal to 23 is obtained (see Figure 2), but there is no numerical semigroups with Frobenius number equal to 23 in $G[\langle 6,8,9\rangle]$ (see Figure 3). Hence, the set of numerical semigroups with multiplicity 6, embedding dimension 3, and Frobenius number equal to 23 is

$$\{\langle 6,8,13\rangle,\langle 6,7,15\rangle,\langle 6,7,22\rangle,\langle 6,7,29\rangle,\langle 6,9,10\rangle\}.$$

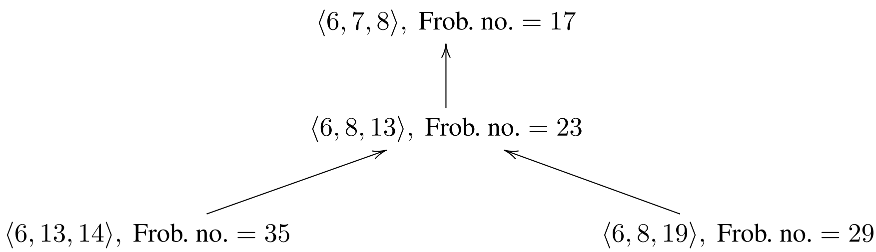


Figure 2: Two levels of the tree associated to the semigroup $\langle 6,7,8\rangle$.

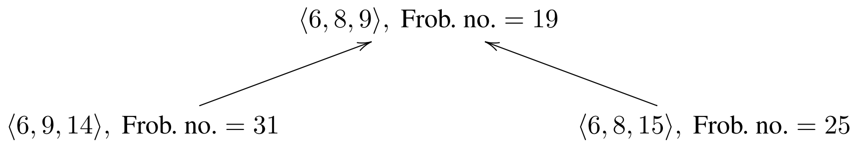


Figure 3: One level of the tree associated to the semigroup $\langle 6,8,9\rangle$.

Example 6.2. In this example, all the numerical semigroups with multiplicity 6, embedding dimension 3, and genus equal to 16 are computed. From Example 6.1, the set $\mathcal{C}(6,3)$ is

$$\{\langle 6,7,8\rangle,\langle 6,7,9\rangle,\langle 6,7,10\rangle,\langle 6,7,11\rangle,\\ \langle 6,8,9\rangle,\langle 6,8,11\rangle,\langle 6,9,10\rangle,\langle 6,9,11\rangle,\langle 6,10,11\rangle\}.$$

The genus of these semigroups are $9,9,9,10,10,11,12,13$ and 13, respectively. So, by Proposition 4.1, for computing the semigroups with genus 16, we apply Algorithm 3 to all elements in $\mathcal{C}(6,3)$. For example, for the semigroups $\langle 6,7,8\rangle$ and $\langle 6,8,9\rangle$ we obtain the trees showed in Figures 4 and 5, respectively. Thus, the set of numerical semigroups with multiplicity 6, embedding dimension 3, and genus 16 is

$$\{\langle 6,14,9\rangle,\langle 6,8,21\rangle,\langle 6,15,11\rangle,\langle 6,10,17\rangle\}.$$

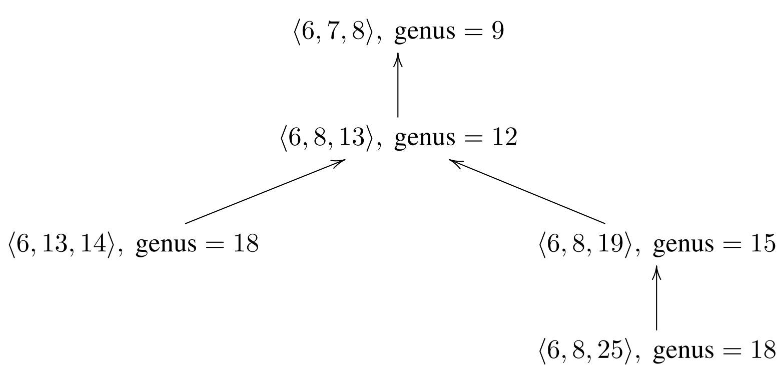


Figure 4: Three levels of the tree associated to the semigroup $\langle 6,7,8\rangle$.

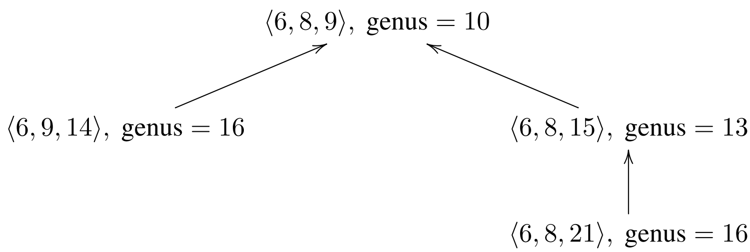


Figure 5: Two levels of the tree associated to the semigroup $\langle 6, 8, 9\rangle$.

Example 6.3. Now, we check Wilf's conjecture for several elements in the tree associated to some packed numerical semigroups. In this example, the elements are showed as a set with three entries $\{A, f, g\}$ where A is the minimal generating set of a numerical semigroup, and f and g are its Frobenius number and genus, respectively. Figure 6 illustrates two levels of the tree associated to the semigroup $S = \langle 110, 216, 217, 218, 219\rangle$. Note that for all its elements the inequality $\frac{e(S)}{e(S)-1} = \frac{5}{4} \leq \frac{F(S)+1}{g(S)}$ is held, and therefore they all satisfy Wilf's conjecture.

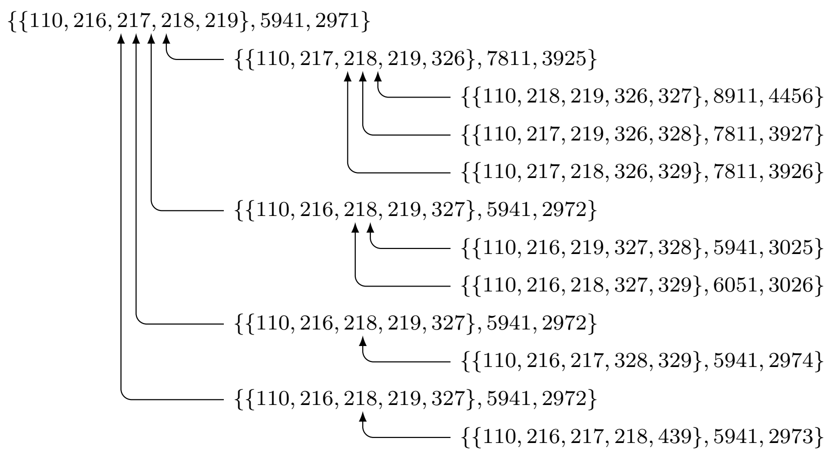


Figure 6: Tree for checking Wilf's conjecture.

In Table 1 we show some packed numerical semigroups and the minimum and maximum of the quotients $(F(T) + 1)/g(T)$ of the semigroups T in their associated trees until a fixed level. Note that all tested semigroups (more than 71 000) satisfy Wilf's conjecture.

7 Frobenius pseudo-variety of numerical semigroups with a fixed multiplicity

According to the notation of [8], a Frobenius pseudo-variety is a non-empty family $\mathcal{P}$ of numerical semigroups which verifies the following conditions:

1. $\mathcal{P}$ has a maximum (according to the inclusion order).

Table 1: Checking Wilf's conjecture (up to level 15).

Semigroup	Number	$\min\left\{\frac{F(\bullet)+1}{g(\bullet)}\right\}$	$\max\left\{\frac{F(\bullet)+1}{g(\bullet)}\right\}$
$\{\{97, 111, 142, 159, 171\}, 958, 525\}$	3 694	1496/981	2705/1357
$\{\{110, 216, 217, 218, 219\}, 5941, 2971\}$	425	2055/1081	2
$\{\{115, 151, 172, 189, 201\}, 1282, 724\}$	2 656	1937/1224	670/339
$\{\{111, 115, 122, 171, 181, 200, 201\}, 702, 445\}$	35 735	1488/1027	2012/1041
$\{\{117, 125, 142, 173, 191, 203, 213\}, 794, 476\}$	28 688	382/261	899/458

2. If $\{S, T\} \subseteq \mathcal{P}$ then $S \cap T \in \mathcal{P}$.
3. If $S \in \mathcal{P}$ and $S \neq \max(\mathcal{P})$ then $S \cup \{F(S)\} \in \mathcal{P}$.

If $\mathcal{P}$ is a Frobenius pseudo-variety we define the graph $G(\mathcal{P})$ as follows: $\mathcal{P}$ is its set of vertices and $(S, T) \in \mathcal{P} \times \mathcal{P}$ is an edge if $T = S \cup \{F(S)\}$.

The following result is a direct consequence from Lemma 12 and Theorem 3 of [8].

Proposition 7.1. *If $\mathcal{P}$ is a Frobenius pseudo-variety, then $G(\mathcal{P})$ is a tree with root $\max(\mathcal{P})$. Moreover, the set of children of a vertex $S \in \mathcal{P}$ is*

$$\{S \setminus \{x\} \in \mathcal{P} \mid x \in \operatorname{msg}(S),\ x > F(S)\}.$$

Let m be a positive integer. We denote by $\mathcal{L}(m)$ the set

$$\{S \mid S \text{ is a numerical semigroup with } m(S) = m\}.$$

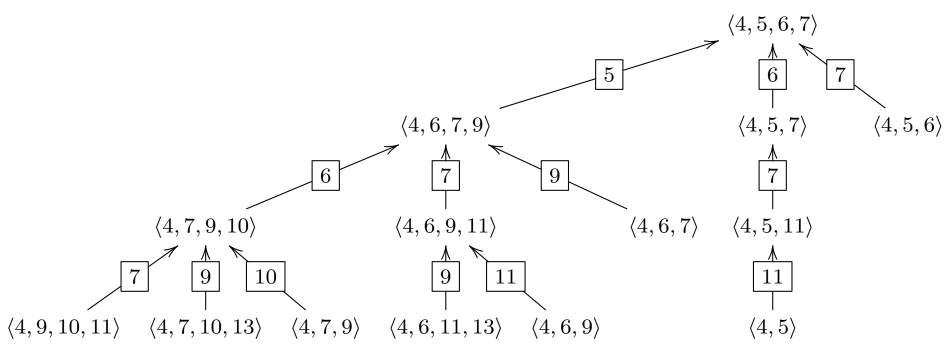
Clearly $\mathcal{L}(m)$ is a Frobenius pseudo-variety and $\max(\mathcal{L}(m)) = \{0, m, \rightarrow\} = \langle m, m+1, \ldots, 2m-1\rangle$. So, as a consequence of Proposition 7.1, we have the following result which is fundamental in this work.

Theorem 7.2. *The graph $G(\mathcal{L}(m))$ is a tree rooted in $\langle m, m+1, \ldots, 2m-1\rangle$. Moreover, the set formed by the children of a vertex $S \in \mathcal{L}(m)$ is*

$$\{S \setminus \{x\} \mid x \in \operatorname{msg}(S),\ x > F(S) \text{ and } x \neq m\}.$$

The previous theorem allows us to recursively construct $\mathcal{L}(m)$ from its root by recursively adding its children to the computed vertices. We illustrate this with an example.

Example 7.3. We show some levels of the tree $G(\mathcal{L}(4))$ giving its vertices and edges, and the minimal removed generators for obtaining the children.



If G is a tree with root r, the level of a vertex x is the length of the only path which connect x and r. The height of a tree is the value of its maximum level. If $k \in \mathbb{N}$, we denote by $N(k, G) = \{v \in G \mid v \text{ has level } k\}$. So in Example 7.3 we have:

$N(0, \mathcal{L}(4)) = \{\langle 4, 5, 6, 7\rangle\}$,
$N(1, \mathcal{L}(4)) = \{\langle 4, 6, 7, 9\rangle, \langle 4, 5, 7\rangle, \langle 4, 5, 6\rangle\}$,
$N(2, \mathcal{L}(4)) = \{\langle 4, 7, 9, 10\rangle, \langle 4, 6, 9, 11\rangle, \langle 4, 6, 7\rangle, \langle 4, 5, 11\rangle\}$,
$N(3, \mathcal{L}(4)) = \{\langle 4, 9, 10, 11\rangle, \langle 4, 7, 10, 13\rangle, \langle 4, 7, 9\rangle, \langle 4, 6, 11, 13\rangle, \langle 4, 6, 9\rangle, \langle 4, 5\rangle\}$.

8 Elements of $\mathcal{L}(m, e)$ with minimum genus

Our aim in this section is to give an algorithm that allows us to compute $g(m, e)$ and $\{S \mid S \in \mathcal{L}(m, e) \text{ and } g(S) = g(m, e)\}$. The following result is a consequence of Theorem 7.2.

Proposition 8.1. *If m is a positive integer and (S, T) an edge of $G(\mathcal{L}(m))$, then*

$$g(S) = g(T) + 1.$$

As a direct consequence of the previous proposition we have the following result.

Corollary 8.2. *Let us fix $m, e \in \mathbb{N}$. If*

$$P = \min\{k \in \mathbb{N} \mid N(k, G(\mathcal{L}(m))) \cap \mathcal{L}(m, e) \neq \emptyset\}$$

then

$$\{S \in \mathcal{L}(m, e) \mid g(S) = g(m, e)\} = N(P, G(\mathcal{L}(m))) \cap \mathcal{L}(m, e).$$

Moreover, $g(m, e) = m - 1 + P$.

It is clear that if $m \geq e \geq 2$ then $\langle m, m+1, \ldots, m+e-1\rangle \in \mathcal{L}(m, e)$. In this way, we have the following result.

Proposition 8.3. *Let m and e be positive integers.*

1. *If $m < e$ then $\mathcal{L}(m, e) = \emptyset$.*
2. *If $e = 1$ and $\mathcal{L}(m, e) \neq \emptyset$ then $m = 1$ and $\mathcal{L}(m, e) = \{\mathbb{N}\}$.*
3. *If $m \geq e \geq 2$ then $\mathcal{L}(m, e) \neq \emptyset$.*

We now give an algorithm to compute $g(m, e)$ and $\{S \in \mathcal{L}(m, e) \mid g(S) = g(m, e)\}$.

Algorithm 5 An algorithm to compute $g(m, e)$ and the set of semigroups with a fixed multiplicity and embedding dimension such that its genus is $g(m, e)$.

INPUT: m and e positive integers such that $m \geq e \geq 2$.
OUTPUT: $g(m, e)$ and $\{S \mid S \in \mathcal{L}(m, e) \text{ and } g(S) = g(m, e)\}$.

```
Set k = 0 and A = {⟨m, m+1, ..., 2m−1⟩}.
while True do
    if A ∩ L(m, e) ≠ ∅ then
        return m − 1 + k and A ∩ L(m, e)
    for S ∈ A do
        C(S) = {T | T is a child of S}.
    A = ⋃_{S∈A} C(S), k = k + 1.
```

We illustrate the above algorithm in the following example.

Example 8.4. We compute $g(5,3)$ and $\{S \in \mathcal{L}(5,3) \mid g(S) = g(5,3)\}$ using Algorithm 5.

- $k = 0$ and $A = \{\langle 5,6,7,8,9\rangle\}$.
- $k = 1$ and $A = \{\langle 5,7,8,9,11\rangle, \langle 5,6,8,9\rangle, \langle 5,6,7,9\rangle, \langle 5,6,7,8\rangle\}$.
- $k = 2$ and

$$A = \{\langle 5,8,9,11,12\rangle, \langle 5,7,9,11,13\rangle, \langle 5,7,8,11\rangle, \langle 5,7,8,9\rangle, \langle 5,6,9,13\rangle, \langle 5,6,8\rangle, \langle 5,6,7\rangle\}.$$

It returns $g(5,3) = 6$ and $\{S \in \mathcal{L}(5,3) \mid g(S) = 6\} = \{\langle 5,6,8\rangle, \langle 5,6,7\rangle\}$.

In the package `FrobeniusNumberAndGenus` ([5]), we can run the command `ComputeMinimumGenusLme[5,3]` to obtain this result.

If S is a numerical semigroup, $n \in S \setminus \{0\}$ and $\mathrm{Ap}(S,n) = \{w(0) = 0, w(1), \ldots, w(n-1)\}$ (see [11, Lemma 2.4, Lemma 2.6]), then $w(i) = k_i n + i$ for some $k_i \in \mathbb{N}$ and $kn + i \in S$ if and only if $k \geq k_i$. Therefore, using Lemma 5.2, we have the following (see the proof of [11, Proposition 2.12]).

Lemma 8.5. *Let S be a numerical semigroup, $n \in S \setminus \{0\}$ and $\mathrm{Ap}(S,n) = \{0, k_1 n + 1, \ldots, k_{n-1}n + n - 1\}$. Then $g(S) = k_1 + \cdots + k_{n-1}$.*

The next result is easily deduced from Corollary 4 of [6].

Lemma 8.6. *Let m, e, q, r be integers such that $m \geq e \geq 2$, $S = \langle m, m+1, \ldots, m+e-1\rangle$, and $m - 1 = q(e-1) + r$, with $q, r \in \mathbb{N}$ and $r \leq e - 2$. Then*

$$\begin{aligned}\mathrm{Ap}(S,m) = \{&0, m+1, \ldots, m+e-1, 2m+(e-1)+1, \ldots, \\ &2m + 2(e-1), \ldots, qm + (q-1)(e-1) + 1, \ldots, qm + q(e-1), \\ &(q+1)m + q(e-1) + 1, \ldots, (q+1)m + q(e-1) + r\}.\end{aligned}$$

If $a, b \in \mathbb{N}$ and $b \neq 0$ we denote by $a \bmod b$ the remainder of dividing a by b. If q is a rational number we denote by $\lfloor q \rfloor = \max\{z \in \mathbb{Z} \mid z \leq q\}$. Note that $a = \lfloor \frac{a}{b} \rfloor b + (a \bmod b)$. From Lemma 8.5 and Lemma 8.6 we have the following result.

Proposition 8.7. *Let m and e be integers such that $m \geq e \geq 2$ and $S = \langle m, m+1, \ldots, m+e-1\rangle$. Then,*

$$g(S) = \left(\left\lfloor \frac{m-1}{e-1} + 1 \right\rfloor\right) \left(\frac{\left\lfloor \frac{m-1}{e-1} \right\rfloor (e-1)}{2} + (m-1) \bmod (e-1)\right).$$

Clearly $\langle m, m+1, \ldots, m+e-1\rangle \in \mathcal{L}(m,e)$ and therefore we have the following result.

Corollary 8.8. *If m and e are integers such that $m \geq e \geq 2$ then*

$$g(m,e) \leq \left(\left\lfloor \frac{m-1}{e-1} + 1 \right\rfloor\right) \left(\frac{\left\lfloor \frac{m-1}{e-1} \right\rfloor (e-1)}{2} + (m-1) \bmod (e-1)\right).$$

For many examples the equality holds. However, there are some cases where the semigroup $\langle m, m+1, \ldots, m+e-1\rangle$ does not have minimum genus in the set $\mathcal{L}(m,e)$ as we show in the next example.

Example 8.9. $S = \langle 8, 9, 10\rangle$ is a numerical semigroup and $g(S) = 16$. $\bar{S} = \langle 8, 9, 11\rangle$ is a numerical semigroup and $g(\bar{S}) = 14$. Therefore, in this case $g(\langle 8, 9, 10\rangle) \neq g(8, 3)$.

The following result is a consequence from Proposition 2.4.

Proposition 8.10. *If* $S \in \mathcal{L}(m,e)$ *then*

$$\bar{S} = \langle \{m\} + \{x \bmod m \mid x \in \mathrm{msg}(S)\}\rangle \in \mathcal{C}(m,e) \quad \textit{and} \quad g(\bar{S}) \leq g(S).$$

Moreover, if $S \notin \mathcal{C}(m,e)$ *then* $g(\bar{S}) < g(S)$.

We illustrate the previous proposition with an example.

Example 8.11. If $S = \langle 5, 11, 17\rangle \in \mathcal{L}(5,3)$ then $\bar{S} = \langle \{5\} + \{0,1,2\}\rangle = \langle 5,6,7\rangle \in \mathcal{C}(5,3)$. Therefore, $g(\bar{S}) \leq g(S)$. Moreover, $S \notin \mathcal{C}(5,3)$, so $g(\bar{S}) < g(S)$.

The next result is a consequence of Proposition 8.10.

Corollary 8.12. *Let* m *and* e *be integers such that* $m \geq e \geq 2$. *Then*

1. $g(m,e) = \min\{g(S) \mid S \in \mathcal{C}(m,e)\}$.
2. $\{S \in \mathcal{L}(m,e) \mid g(S) = g(m,e)\} = \{S \in \mathcal{C}(m,e) \mid g(S) = g(m,e)\}$.

Note that $\mathcal{C}(m,e)$ is finite and therefore the previous corollary gives us another algorithm for computing $g(m,e)$ and $\{S \in \mathcal{L} \mid g(S) = g(m,e)\}$. We give more details about this method using Proposition 2.4 and the calculations which appear in Example 2.5.

Example 8.13. From the calculations of Example 2.5, we have

$$\mathcal{C}(6,3) = \{\langle 6,7,8\rangle, \langle 6,7,9\rangle, \langle 6,7,10\rangle, \langle 6,7,11\rangle, \\ \langle 6,8,9\rangle, \langle 6,8,11\rangle, \langle 6,9,10\rangle, \langle 6,9,11\rangle, \langle 6,10,11\rangle\}.$$

A simple computation shows us

$$\begin{array}{lll} g(\langle 6,7,8\rangle) = 9, & g(\langle 6,7,9\rangle) = 9, & g(\langle 6,7,10\rangle) = 9, \\ g(\langle 6,7,11\rangle) = 10, & g(\langle 6,8,9\rangle) = 10, & g(\langle 6,8,11\rangle) = 11, \\ g(\langle 6,9,10\rangle) = 12, & g(\langle 6,9,11\rangle) = 13 \quad \text{and} & g(\langle 6,10,11\rangle) = 13. \end{array}$$

Therefore, $g(6,3) = 9$ and the set $\{S \in \mathcal{L}(6,3) \mid g(S) = 9\}$ is equal to $\{\langle 6,7,8\rangle, \langle 6,7,9\rangle, \langle 6,7,10\rangle\}$.

9 Elements of $\mathcal{L}(m,e)$ with minimum Frobenius number

Our aim in this section is to obtain algorithmic methods for computing $F(m,e)$ and $\{S \in \mathcal{L}(m,e) \mid F(S) = F(m,e)\}$. The next result is a consequence of Theorem 7.2.

Proposition 9.1. *If* m *is a positive integer and* (S,T) *is an edge of* $G(\mathcal{L}(m))$, *then* $F(T) < F(S)$.

The following result can be deduced from [9].

Proposition 9.2. *If m is a positive integer and (S,T) is an edge of $G(\mathcal{L}(m))$, then $e(S) \leq e(T)$.*

Clearly $F(m,m) = m-1$ and

$$\{S \in \mathcal{L}(m,m) \mid F(S) = m-1\} = \{\langle m, m+1, \ldots, 2m-1\rangle\}.$$

It is well known (see [14] for example) that if $S = \langle n_1, n_2\rangle$ is a numerical semigroup, then $F(S) = n_1 n_2 - n_1 - n_2$. Therefore, we obtain the following result.

Proposition 9.3. *Let m be an integer such that $m \geq 2$.*

1. *$F(m,m) = m-1$ and $\{S \in \mathcal{L}(m,m) \mid F(S) = m-1\} = \{\langle m, m+1, \ldots, 2m-1\rangle\}$.*
2. *$F(m,2) = m^2-m-1$ and $\{S \in \mathcal{L}(m,2) \mid F(S) = m^2-m-1\} = \{\langle m, m+1\rangle\}$.*

If q is a rational number we denote by $\lceil q \rceil = \min\{z \in \mathbb{Z} \mid q \leq z\}$. The next result is deduced from Corollary 5 of [6].

Proposition 9.4. *If m and e are integers such that $m \geq e \geq 2$, then*

$$F(\langle m, m+1, \ldots, m+e-1\rangle) = \left\lceil \frac{m-1}{e-1} \right\rceil m - 1.$$

As a consequence of the previous proposition we get the following result.

Corollary 9.5. *If m and e are integers such that $m \geq e \geq 2$, then*

$$F(m,e) \leq \left\lceil \frac{m-1}{e-1} \right\rceil m - 1.$$

In the previous corollary, equality often holds, but in some cases

$$F(\langle m, m+1, \ldots, m+e-1\rangle) \neq \min\{F(S) \mid S \in \mathcal{L}(m,e)\}.$$

For example, $F(\langle 4,5,6\rangle) = 7$ and $F(\langle 4,5,7\rangle) = 6$.

From the above results, we obtain the following algorithm where the projections from the cartesian product $\mathcal{L}(m) \times \mathbb{N}$ are denoted by π_1 and π_2.

Algorithm 6 An algorithm to compute $F(m,e)$ and the set of semigroups with a fixed multiplicity and embedding dimension such that its Frobenius number is $F(m,e)$.

INPUT: m and e integers such that $m \geq e \geq 2$.
OUTPUT: $F(m,e)$ and $\{S \in \mathcal{L}(m,e) \mid F(S) = F(m,e)\}$.

```
1: A = {⟨m, m+1, ..., 2m−1⟩}, I = ∅ and α = ⌈(m−1)/(e−1)⌉m − 1.
2: while True do
3:     C = {(S, F(S)) | S is child of some element of A and F(S) ≤ α}.
4:     K = {S ∈ π1(C) | e(S) ≥ e}.
5:     if K = ∅ then
6:         return F(m,e) = π2(I) and {S ∈ L(m,e) | F(S) = F(m,e)} = π1(I)
7:     A = K, B = {(S, F(S)) | S ∈ K and e(S) = e}.
8:     α = min(π2(B) ∪ {α}), I = {(S, F(S)) ∈ I ∪ B | F(S) = α}.
```

We illustrate how this algorithm works with an example.

Example 9.6. We compute $F(4,3)$ and $\{S \in \mathcal{L}(4,3) \mid F(S) = F(4,3)\}$ using Algorithm 6.

- $A = \{\langle 4,5,6,7\rangle\}$, $I = \emptyset$ and $\alpha = \lceil \frac{3}{2} \rceil 4 - 1 = 7$.
- $C = \{(\langle 4,6,7,9\rangle, 5), (\langle 4,5,7\rangle, 6), (\langle 4,5,6\rangle, 7)\}$ and $K = \{\langle 4,6,7,9\rangle, \langle 4,5,7\rangle, \langle 4,5,6\rangle\}$.
- $A = \{\langle 4,6,7,9\rangle, \langle 4,5,7\rangle, \langle 4,5,6\rangle\}$, $B = \{(\langle 4,5,7\rangle, 6), (\langle 4,5,6\rangle, 7)\}$, $\alpha = \min\{6,7\} = 6$ and $I = \{(\langle 4,5,7\rangle, 6)\}$.
- $C = \{(\langle 4,7,9,10\rangle, 6)\}$ and $K = \{\langle 4,7,9,10\rangle\}$.
- $A = \{\langle 4,7,9,10\rangle\}$, $B = \emptyset$, $\alpha = 6$ and $I = \{(\langle 4,5,7\rangle, 6)\}$.
- $C = \emptyset$ and $K = \emptyset$.

Therefore, $F(4,3) = 6$ and $\{S \in \mathcal{L}(4,3) \mid F(S) = 6\} = \{\langle 4,5,7\rangle\}$. Using the Mathematica package [5], we obtain 6 and $\langle 4,5,7\rangle$, running the commands `MinFrob[4,3]` and `FrobeniusEmbeddingDimensionMultiplicity[6,3,4]`, respectively.

Our next goal is to give an alternative algorithm for computing $F(m,e)$ and $\{S \in \mathcal{L}(m,e) \mid F(S) = F(m,e)\}$. The next result is deduced from Proposition 2.4.

Proposition 9.7. *If $S \in \mathcal{L}(m,e)$ then $\bar{S} = \langle \{m\} + \{x \bmod m \mid x \in \operatorname{msg}(S)\}\rangle \in \mathcal{C}(m,e)$ and $F(\bar{S}) \leq F(S)$.*

As a consequence of the previous proposition we get the following result.

Corollary 9.8. *If m and e are integers such that $m \geq e \geq 2$, then $F(m,e) = \min\{F(S) \mid S \in \mathcal{C}(m,e)\}$.*

The set $\mathcal{C}(m,e)$ is finite, so previous corollary give us an algorithmic method for computing $F(m,e)$.

Example 9.9. We compute $F(6,5)$. First, we calculate $\mathcal{C}(6,5)$ by using Proposition 2.4 and then we apply Corollary 9.8. So,

$$\mathcal{C}(6,5) = \{\langle 6,7,8,9,10\rangle, \langle 6,7,8,9,11\rangle, \langle 6,7,8,10,11\rangle, \langle 6,7,9,10,11\rangle, \langle 6,8,9,10,11\rangle\}$$

and therefore

$$F(6,5) = \min\{F(\langle 6,7,8,9,10\rangle) = 11, \quad F(\langle 6,7,8,9,11\rangle) = 10, \quad F(\langle 6,7,8,10,11\rangle) = 9, \quad F(\langle 6,7,9,10,11\rangle) = 8, \quad F(\langle 6,8,9,10,11\rangle) = 13\} = 8.$$

We are now interested in giving a method for computing $\{S \in \mathcal{L}(m,e) \mid F(S) = F(m,e)\}$. The next example shows us that there exist semigroups $S \in \mathcal{L}(m,e)$ such that $S \notin \mathcal{C}(m,e)$ and $F(S) = F(m,e)$.

Example 9.10. The numerical semigroups $S_1 = \langle 7,9,10,15\rangle$ and $S_2 = \langle 7,8,10,19\rangle$ verify that $S_1, S_2 \in \mathcal{L}(7,4) \setminus \mathcal{C}(7,4)$ and $F(S_1) = F(S_2) = 13 = F(7,4)$.

If $S \in \mathcal{L}(m, e)$ we denote by $\theta(S)$ the numerical semigroup generated by $\{m\} + \{x \bmod m \mid x \in \operatorname{msg}(S)\}$. Clearly, $\theta(S) \in \mathcal{C}(m, e)$.

Using the partition given in Section 2 and Theorem 2.3, the following two steps are sufficient for computing $\{S \in \mathcal{L}(m, e) \mid F(S) = F(m, e)\}$.

1. Compute $A = \{S \in \mathcal{C}(m, e) \mid F(S) = F(m, e)\}$.
2. For every $S \in A$, compute $\{T \in [S] \mid F(T) = F(S)\}$.

We already know how to compute 1. We now focus on giving an algorithm that allows us to compute 2.

Using Algorithm 1, for $S \in \mathcal{C}(m, e)$ and $F \in \mathbb{N}$ we get the set $\{T \in [S] \mid F(T) \leq F\}$. Clearly if $S \in \mathcal{C}(m, e)$ then $\{T \in [S] \mid F(T) = F(S)\} = \{T \in [S] \mid F(T) \leq F(S)\}$. We are going to adapt Algorithm 1 to our needs for computing 2.

We now recall some definitions of Section 3. If S is a numerical semigroup, $M(S) = \max(\operatorname{msg}(S))$. If $S \in \mathcal{C}(m, e)$ the graph $G([S])$ was defined as follows: $[S]$ is its set of vertices and $(A, B) \in [S] \times [S]$ is an edge if

$$\operatorname{msg}(B) = (\operatorname{msg}(A) \setminus \{M(A)\}) \cup \{M(A) - m\}.$$

Now, using the Theorem 3.3, we give an algorithm which for a semigroup $S \in \mathcal{C}(m, e)$ computes the set $\{T \in [S] \mid F(T) = F(S)\}$.

Algorithm 7 An algorithm to compute the semigroups of each equivalence class such that their Frobenius number is minimum.

INPUT: $S \in \mathcal{C}(m, e)$.
OUTPUT: $\{T \in [S] \mid F(T) = F(S)\}$.

```
A = {S} and B = {S}.
while True do
    C = {H | H is child of an element of B and F(H) = F(S)}.
    if C = ∅ then
        return A
    A = A ∪ C, B = C.
```

We finish this section with an example to illustrate the above algorithm.

Example 9.11. We use now Algorithm 7 for computing $\{T \in [S] \mid F(T) = F(S) = 10\}$ where $S = \langle 6, 7, 8, 9, 11\rangle \in \mathcal{C}(6, 5)$.

- $A = \{\langle 6, 7, 8, 9, 11\rangle\}$ and $B = \{\langle 6, 7, 8, 9, 11\rangle\}$.
- $C = \{\langle 6, 8, 9, 11, 13\rangle, \langle 6, 8, 11, 13, 15\rangle\}$.
- $A = \{\langle 6, 7, 8, 9, 11\rangle, \langle 6, 8, 9, 11, 13\rangle, \langle 6, 8, 11, 13, 15\rangle\}$ and $B = \{\langle 6, 8, 9, 11, 13\rangle, \langle 6, 8, 11, 13, 15\rangle\}$.
- $C = \emptyset$.

Thus, $\{T \in [S] \mid F(T) = 10\} = \{\langle 6, 7, 8, 9, 11\rangle, \langle 6, 8, 9, 11, 13\rangle, \langle 6, 8, 11, 13, 15\rangle\}$. This result is also obtained with the package `FrobeniusNumberAndGenus` ([5]) by executing the command `ComputeEquivalenceClass[{6,7,8,9,11}]`.

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Grundy domination and zero forcing in Kneser graphs*

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Abstract

In this paper, we continue the investigation of different types of (Grundy) dominating sequences. We consider four different types of Grundy domination numbers and the related zero forcing numbers, focusing on these numbers in the well-known class of Kneser graphs $K_{n,r}$. In particular, we establish that the Grundy total domination number $\gamma_{gr}^t(K_{n,r})$ equals $\binom{2r}{r}$ for any $r \geq 2$ and $n \geq 2r+1$. For the Grundy domination number of Kneser graphs we get $\gamma_{gr}(K_{n,r}) = \alpha(K_{n,r})$ whenever n is sufficiently larger than r. On the other hand, the zero forcing number $Z(K_{n,r})$ is proved to be $\binom{n}{r} - \binom{2r}{r}$ when $n \geq 3r+1$ and $r \geq 2$, while lower and upper bounds are provided for $Z(K_{n,r})$ when $2r+1 \leq n \leq 3r$. Some lower bounds for different types of minimum ranks of Kneser graphs are also obtained along the way.

Keywords: Grundy domination number, Grundy total domination number, Kneser graph, zero forcing number, minimum rank.

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1 Introduction

The *Kneser graph*, $K_{n,r}$, where n, r are positive integers such that $n \geq 2r$, has the r-subsets of the n-set as its vertices, and two r-subsets are adjacent in $K_{n,r}$ if they are disjoint. The class of graphs became well-known by the celebrated Erdős-Ko-Rado theorem [11], which determined the independence number $\alpha(K_{n,r})$ of the Kneser graph $K_{n,r}$ to be equal to $\binom{n-1}{r-1}$. Another famous result is Lovász's proof of Kneser's conjecture, which determines the chromatic number of Kneser graphs [22], see also Matoušek for a combinatorial proof of this result [23]. Many other invariants were later considered in Kneser graphs by a number of authors. In particular, the domination number of Kneser graphs was studied in several papers [16, 19, 24], but there is no such complete solution for domination number of Kneser graphs as is the case with the chromatic and the independence number.

Various Grundy domination invariants have been introduced in recent years [5, 7, 8], arising from two standard domination invariants, the domination number and the total domination number. In the context of domination, we say that a vertex x *totally dominates* vertices in its (open) neighborhood, $N(x) = \{y \mid xy \in E(G)\}$, and that x *dominates* vertices in its closed neigborhood, $N[x] = N(x) \cup \{x\}$. A set D in a graph G is a *dominating set* (resp., a *total dominating set*) of G if every vertex in $V(G)$ is dominated (resp., totally dominated) by a vertex in D. (Clearly, G must not have an isolated vertex to have a total dominating set.) The minimum cardinality of a dominating set (resp., total dominating set) is the *domination number* (resp., the *total domination number*) of G, and is denoted by $\gamma(G)$ (resp., $\gamma_t(G)$). Now, Grundy (total) domination number is introduced while applying a greedy algorithm to obtain a (total) dominating set D as follows. Vertices are added to the set D one by one, requiring that a vertex x that was added to D (totally) dominates at least one vertex that was not (totally) dominated before this vertex was added. The longest length of such a sequence in a graph G is the *Grundy (total) domination number*, $\gamma_{gr}(G)$ (resp., $\gamma_{gr}^t(G)$). By imposing an additional condition on such dominating sequences, one gets the so-called Z-dominating sequences and Z-Grundy domination number [4], denoted by $\gamma_{gr}^Z(G)$. See Section 2 for formal definitions, and [6, 9] for more results on Grundy domination and Grundy total domination numbers.

In [5] a strong connection between the Z-Grundy domination number and the zero forcing number of a graph was established. The latter graph invariant has been intensively studied in recent years; it is closely related to another well-known domination concept called power domination, cf. [13, 14]. Moreover, the zero forcing number is very useful in determining the minimum rank of a graph; see the seminal paper about this connection [1] and some further studies [2, 3, 10]. Minimum rank $\mathrm{mr}(G)$ of a graph G is defined as the minimum rank over all symmetric matrices that have non-zero (real) values in the non-zero entries of the adjacency matrix $A(G)$ of G, arbitrary real values in the diagonal, and zero values in all other entries. (See the survey on minimum rank, where many applications and other interesting results on this parameter can be found [12].) In addition, the skew zero forcing number (denoted by $Z_-(G)$) was introduced in [18], and studied in the context of the invariant mr_0, which is a version of the minimum rank in which matrices are in addition required to have empty diagonals. Motivated by the results of [5], Lin [20] noticed a similar connection between the Grundy total domination number and the skew zero forcing number of graphs, and also between the Grundy domination number and another version of the minimum rank parameter, denoted by $\mathrm{mr}_{\not\ell}$. As shown by Lin [20], the following bounds hold:

$$\gamma_{gr}(G) \leq \mathrm{mr}_{\not\ell}(G), \qquad \gamma_{gr}^t(G) \leq \mathrm{mr}_0(G), \qquad \gamma_{gr}^Z(G) \leq \mathrm{mr}(G).$$

Consequently, any lower bound on a Grundy domination parameter gives a lower bound on the corresponding minimum rank parameter.

In this paper, we study four types of Grundy domination parameters (beside the mentioned ones also the L-Grundy domination number) in Kneser graphs, and apply the obtained results to give some bounds or exact results about zero forcing parameters and minimum rank parameters in Kneser graphs. In the next section, we give all the necessary definitions, establish the notation and present some preliminary results. In Section 3 we prove that $\gamma_{gr}(K_{n,2}) = \alpha(K_{n,2}) = n-1$ for $n \geq 6$, while $\gamma_{gr}(K_{5,2}) = 5$. More generally, for Kneser graphs $K_{n,r}$ when $r > 2$ we establish an asymptotic result, which states that there exists an $n_0 \in \mathbb{N}$ dependent on r such that for all n, $n \geq n_0$, we have $\gamma_{gr}(K_{n,r}) = \alpha(K_{n,r}) = \binom{n-1}{r-1}$. The most complete result is obtained for the Grundy total domination number, where we show that for $r \geq 2$ and $n \geq 2r+1$, $\gamma_{gr}^t(K_{n,r}) = \binom{2r}{r}$. This result is proven in Section 4, using a set theoretic result due to Gyárfás and Hubenko [15], which is the ordered version of Lovász's result for set-pair collections [21]. Section 5 is about Z-Grundy domination number of Kneser graphs. We prove that for any $r \geq 2$ and $n \geq 3r+1$, we have $\gamma_{gr}^Z(K_{n,r}) = \binom{2r}{r}$. This immediately implies the result for Kneser graphs with $r \geq 2$ and $n \geq 3r+1$ about their zero forcing number, notably $Z(K_{n,r}) = \binom{n}{r} - \binom{2r}{r}$; and a lower bound for the minimum rank, which reads as $\mathrm{mr}(K_{n,r}) \geq \binom{2r}{r}$. On the other hand, when $2r+1 \leq n \leq 3r$ and $r \geq 2$ we only prove the lower bound for Z-Grundy domination number: $\gamma_{gr}^Z(K_{n,r}) \geq \binom{2r}{r} - \binom{4r-1-n}{3r-n}$, which is dependent on both r and n. In Section 6, we investigate the L-Grundy domination number of Kneser graphs $K_{n,r}$, provide the exact result when $r = 2$, and when r is bigger than 2 we prove a lower and an upper bound, which are not far apart. Finally, in the concluding section, we rephrase the Z-Grundy domination number and the Grundy domination number as set-theoretic problems.

2 Preliminaries and notation

Let $[n] = \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$. In [7] the first type of Grundy dominating sequences was introduced.

Let $S = (v_1, \ldots, v_k)$ be a sequence of distinct vertices of a graph G. The corresponding set $\{v_1, \ldots, v_k\}$ of vertices from the sequence S will be denoted by $\widehat{S}$. The initial segment $(v_1, \ldots, v_i)$ of S will be denoted by S_i. Given a sequence $S' = (u_1, \ldots, u_m)$ of vertices in G such that $\widehat{S} \cap \widehat{S'} = \emptyset$, $S \oplus S'$ is the *concatenation* of S and S', i.e., $S \oplus S' = (v_1, \ldots, v_k, u_1, \ldots, u_m)$. A sequence $S = (v_1, \ldots, v_k)$, where $v_i \in V(G)$, is called a *closed neighborhood sequence* if, for each i

$$N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset. \tag{2.1}$$

If for a closed neighborhood sequence S, the set $\widehat{S}$ is a dominating set of G, then S is called a *dominating sequence* of G. We will say that v_i *footprints* the vertices from $N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j]$, and that v_i is the *footprinter* of any $u \in N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j]$. For a dominating sequence S, any vertex in $V(G)$ has a unique footprinter in $\widehat{S}$. Clearly, the length k of a dominating sequence $S = (v_1, \ldots, v_k)$ is bounded from below by the domination number $\gamma(G)$ of a graph G. We call the maximum length of a dominating sequence in G the *Grundy* *domination number* of a graph G and denote it by $\gamma_{gr}(G)$. The corresponding sequence is called a *Grundy dominating sequence* of G or γ_{gr}-sequence of G.

In a similar way total dominating sequences were introduced in [8], when G is a graph without isolated vertices. Using the same notation as in the previous paragraph, we say that a sequence $S = (v_1, \ldots, v_k)$, where $v_i \in V(G)$, is called an *open neighborhood sequence* if, for each i

$$N(v_i) \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset. \tag{2.2}$$

We also speak of *total footprinters* of which meaning should be clear. It is easy to see that an open neighborhood sequence S in G of maximum length yields $\widehat{S}$ to be a total dominating set; the sequence S is then called a *Grundy total dominating sequence* or γ_{gr}^t*-sequence*, and the corresponding invariant the *Grundy total domination number* of G, denoted $\gamma_{gr}^t(G)$. Any open neighborhood sequence S, where $\widehat{S}$ is a total dominating set is called a *total dominating sequence*.

Two additional variants of the Grundy (total) domination number were introduced in [5]. Let G be a graph without isolated vertices. A sequence $S = (v_1, \ldots, v_k)$, where $v_i \in V(G)$, is called a *Z-sequence* if, for each i

$$N(v_i) \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset. \tag{2.3}$$

Then the *Z-Grundy domination number* $\gamma_{gr}^Z(G)$ of the graph G is the length of a longest Z-sequence. Note that such a sequence is also a closed neighborhood sequence and hence $\gamma_{gr}^Z(G) \le \gamma_{gr}(G)$. Given a Z-sequence S, the corresponding set $\widehat{S}$ of vertices will be called a *Z-set*. Note that $\gamma_{gr}^Z(G) = \gamma_{gr}(G)$ if and only if there exists a Grundy dominating sequence for G each vertex of which footprints some of its neighbors.

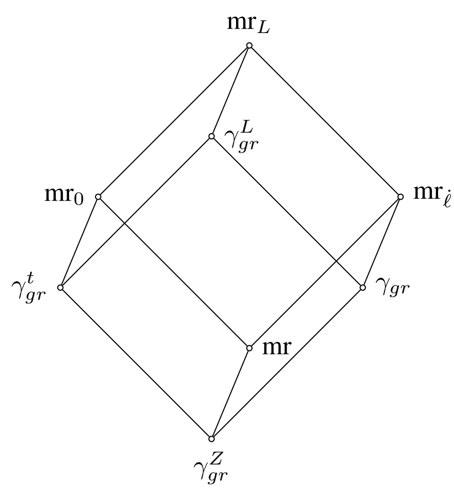

Figure 1: Connections between different Grundy domination and minimum rank parameters.

While observing the defining conditions of the three concepts in (2.1), (2.2) and (2.3), it is natural to consider also the fourth concept. It gives the longest sequences among all four versions, and we call it L-Grundy domination. Given a graph G, a sequence $S = (v_1, \ldots, v_k)$, where v_i are distinct vertices from G, is called an *L-sequence* if, for each i

$$N[v_i] \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset. \tag{2.4}$$

Then the *L-Grundy domination number* $\gamma_{gr}^L(G)$ of the graph G is the length of a longest L-sequence. Given an L-sequence S, the corresponding set $\widehat{S}$ of vertices will be called an *L-set* (the requirement that all vertices in S are distinct, indeed makes $\widehat{S}$ to be a set, and prevents the creation of an infinite sequence by repetition of one and the same vertex). Note that it is possible that v_i L-footprints only itself. We will make use of the following result.

Proposition 2.1 ([5, Proposition 3.1]). *If G is a graph with no isolated vertices, then*

(i) $\gamma_{gr}^Z(G) \leq \gamma_{gr}(G) \leq \gamma_{gr}^L(G) - 1$,

(ii) $\gamma_{gr}^Z(G) \leq \gamma_{gr}^t(G) \leq \gamma_{gr}^L(G)$

and all the bounds are sharp.

As mentioned earlier, Lin established the connections between different Grundy domination numbers, zero forcing numbers and minimum rank invariants. (For definitions of different zero forcing and minimum rank parameters we refer to [20].)

Theorem 2.2 ([20]). *If G is a graph without isolated vertices, then*

(1) $|V(G)| - Z_{\mathring{\ell}}(G) = \gamma_{gr}(G) \leq \mathrm{mr}_{\mathring{\ell}}(G)$,

(2) $|V(G)| - Z_{-}(G) = \gamma_{gr}^t(G) \leq \mathrm{mr}_0(G)$

(3) $|V(G)| - Z(G) = \gamma_{gr}^Z(G) \leq \mathrm{mr}(G)$.

(4) $|V(G)| - Z_L(G) = \gamma_{gr}^L(G) \leq \mathrm{mr}_{\mathrm{L}}(G)$.

The connections between different Grundy domination and minimum rank parameters that follow from Proposition 2.1 and Theorem 2.2 are presented in the Hasse diagram in Figure 1.

3 Grundy domination number

It is easy to check (we did this by computer) that the Grundy domination number of the Petersen graph, $K_{5,2}$, equals 5. Since this is bigger (by 1) than the independence number of the Petersen graph, it is somewhat surprising that for all n bigger than 5, the Grundy domination number equals the independence number of $K_{n,2}$.

Proposition 3.1. *For $n \geq 6$, $\gamma_{gr}(K_{n,2}) = \alpha(K_{n,2}) = n - 1$.*

Proof. Clearly, as a sequence of vertices forming a maximum independent set is a Grundy dominating sequence, we have $\gamma_{gr}(K_{n,2}) \geq \alpha(K_{n,2}) = n-1$. In addition, we have checked by computer small cases, establishing $\gamma_{gr}(K_{n,2}) = n-1$, for $6 \leq n \leq 8$.

For the proof of the reversed inequality for $n > 8$, let $S = (v_1, \ldots, v_k)$ be a Grundy dominating sequence of $K_{n,2}$. Suppose $k > n-1$. Hence, $\widehat{S}$ is not a stable set. We may assume that $\widehat{S_i}$ is a stable set for some $i \geq 1$, while v_{i+1} is adjacent to some $v_j \in \widehat{S_i}$. Note that, since $n \geq 6$, we may assume without loss of generality that $v_j = \{1, j+1\}$ for $1 \leq j \leq i$ (if $i = 3$ and $\widehat{S_i} = \{\{1,2\},\{1,3\},\{2,3\}\}$, then all vertices are already dominated by $\widehat{S_i}$, a contradiction).

First, if $i \leq 2$, then after $S_3 = (v_1, v_2, v_3)$, or $S_2 = (v_1, v_2)$ if $i = 1$, at most four vertices remain undominated, notably vertices from $\{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$. Hence, counting the largest possible number of steps, we get $k \leq 7$. As $n \geq 8$, this is only possible when $n = 8$, in which case we have $\gamma_{gr}(K_{8,2}) = 7$, already noticed above.

Hence, we may assume that $i > 2$. Therefore, after the segment S_i is included, only the vertices in $\{\{1, i+2\}, \ldots, \{1, n\}\}$ remain undominated. Since each vertex in the sequence S has to dominate some undominated vertex $k \leq i + n - (i+1) = n-1$, a contradiction. □

While we could not find exact values of Grundy domination number for all Kneser graphs $K_{n,r}$, we prove in some sense similar result as we have for $K_{n,2}$. Namely, as soon as n is large enough with respect to a given r (e.g., when $r = 2$, this was $n \geq 6$), we can prove that $\gamma_{gr}(K_{n,r}) = \alpha(K_{n,r})$.

Theorem 3.2. *For any $r \geq 2$, there exists $n_0 \in \mathbb{N}$ such that for all n, $n \geq n_0$, we have*

$$\gamma_{gr}(K_{n,r}) = \alpha(K_{n,r}) = \binom{n-1}{r-1}.$$

Proof. The result is proven for $r = 2$ by Proposition 3.1. Fix $r \geq 3$, and consider a dominating sequence $S = (v_1, \ldots, v_m)$ of $K_{n,r}$, assuming that S is not a stable set.

In the first case, suppose that already v_1 and v_2 are adjacent, and without loss of generality we may assume that $v_1 = \{1, \ldots, r\}$ and $v_2 = \{r+1, \ldots, 2r\}$. Let V' be the set of vertices, not dominated by $\{v_1, v_2\}$. Note that every element in V' is an r-set that contains at least one element from v_1 and at least one element from v_2. Denoting by $c = \{2r+1, \ldots, n\}$, any element in V' consists of i elements from c, where $0 \leq i \leq r-2$, j elements from v_1, where $1 \leq j \leq r-i-1$, and consequently, $r-i-j$ elements from v_2 (note that $r-i-j \geq 1$).

Hence

$$|V'| = \sum_{i=0}^{r-2} \binom{n-2r}{i} \sum_{j=1}^{r-i-1} \binom{r}{j} \binom{r}{r-j-i}.$$

Note that $\sum_{j=1}^{r-i-1} \binom{r}{j}\binom{r}{r-j-i}$ is not dependent on n, hence for fixed r this is a constant, while $\sum_{i=0}^{r-2} \binom{n-2r}{i}$ is a polynomial in n of degree $r-2$. Hence $|V'| = O(n^{r-2})$. On the other hand, $\alpha(K_{n,r}) = \binom{n-1}{r-1}$, hence the resulting dominating sequence is of size $\Omega(n^{r-1})$. Note that the length of the sequence S is at most $2 + |V'|$. Hence, if n is big enough, S is not a Grundy dominating sequence, because its length is less than $\binom{n-1}{r-1}$.

In the second case, the initial segment of S, S_k, is a stable set, for some $k > 1$, while v_{k+1} is adjacent to some vertex in S_k. Without loss of generality, we may assume that

$v_k = \{1, \dots, r\}$ and $v_{k+1} = \{r+1, \dots, 2r\}$. Note as above that the size of the set V' of vertices in $K_{n,r}$, which are not in $N(v_k) \cup N(v_{k+1})$ is $O(n^{r-2})$. To complete the proof of this case, consider the subgraph $G' = K_{n,r} - (\cup_{v \in S_k} N[v])$. Note that G' must have an edge, for otherwise the proof is already done (indeed, no edges in this graph means, that an optimal sequence S consists of the stable set $\widehat{S_k} \cup V(G')$). Let ab be an edge in $V(G')$. Clearly, for all vertices v in S_k, v is not adjacent to the endvertices of ab, i.e., we have $v \cap a \neq \emptyset$ and $v \cap b \neq \emptyset$. In the same way as above we get that the longest such sequence S_k has at most $\sum_{i=0}^{r-2} \binom{n-2r}{i} \sum_{j=1}^{r-i-1} \binom{r}{j}\binom{r}{r-j-i}$ vertices, and so $|S_k| = k$ is of the order $O(n^{r-2})$. Note that the set of vertices not footprinted by vertices in S_{k+1} is contained in V'. Then $|S_{k+1}| + |V'| = k + 1 + |V'|$ is an upper bound for $|S|$ and remains of the order $O(n^{r-2})$. Therefore, for n big enough, $|S|$ will not be greater that $\binom{n-1}{r-1}$, which is of order $\Omega(n^{r-1})$. □

For the zero forcing parameter $Z_{\dot{\ell}}$ and minimum rank parameter $\mathrm{mr}_{\dot{\ell}}$ Theorem 3.2 gives the following observation.

Corollary 3.3. *For any $r \geq 2$, there exists $n_0 \in \mathbb{N}$ such that for all n, $n \geq n_0$, we have*

$$Z_{\dot{\ell}}(K_{n,r}) = \binom{n-1}{r} \quad \textit{and}$$
$$\mathrm{mr}_{\dot{\ell}}(K_{n,r}) \geq \alpha(K_{n,r}) = \binom{n-1}{r-1}.$$

Note that for $2r + 1 \leq n < n_0$ a lower bound could be improved by improving the values of $\gamma_{gr}(K_{n,r})$. It would be even more interesting to find an upper bound for $\mathrm{mr}_{\dot{\ell}}(K_{n,r})$, which might require tools from linear algebra.

Unfortunately, we do not know how large must be n_0 in Theorem 3.2 and Corollary 3.3 when $r \geq 3$. This is an interesting issue yet to be investigated. We could only check by computer that $\gamma_{gr}(K_{7,3}) = 20$, while clearly $\alpha(K_{7,3}) = 15$.

4 Grundy total domination number

Unlike the Grundy domination number, we prove that the Grundy total domination number of $K_{n,r}$ does not depend on n. Moreover, we provide the exact value of $\gamma_{gr}^t(K_{n,r})$ for all cases. To this end, we use an ordered version of Lovász's result for set-pair collections [21] provided by Gyárfás and Hubenko [15].

Lemma 4.1 ([15, Lemma 1]). *Let $T = \{(A_i, B_i) \mid 1 \leq i \leq k\}$ be a set-pair collection with $|A_i| = a$, $|B_i| = b$ satisfying the following conditions:*

1. *$A_i \cap B_i = \emptyset$ for $1 \leq i \leq k$;*
2. *$A_i \cap B_j \neq \emptyset$ for $1 \leq i < j \leq k$.*

Then $k \leq \binom{a+b}{a}$.

Proposition 4.2. *For $r \geq 2$ and $n \geq 2r+1$, $\gamma_{gr}^t(K_{n,r}) = \binom{2r}{r}$.*

Proof. In order to obtain the lower bound, it is enough to note that any sequence S such that $\widehat{S} = \{A \subset [2r] \mid |A| = r\}$ is a total dominating sequence of $K_{n,r}$.

To prove the upper bound, let $S = (v_1, \dots, v_k)$ be a total dominating sequence of $K_{n,r}$. For $1 \leq i \leq k$, let u_i be a vertex totally footprinted by v_i. It is not hard to see that the set-pair collection $T = \{(v_i, u_i) \mid 1 \leq i \leq k\}$ satisfies both conditions in Lemma 4.1. Since $|v_i| = |u_i| = r$ for $1 \leq i \leq k$, then $k \leq \binom{2r}{r}$ and the result follows. □

For the skew zero forcing number $Z_-(G)$ and the minimum rank parameter $\mathrm{mr}_0(G)$ we get the following consequences.

Corollary 4.3. *If* $r \geq 2, n \geq 2r+1$, *then* $Z_-(K_{n,r}) = \binom{n}{r} - \binom{2r}{r}$ *and* $\mathrm{mr}_0(K_{n,r}) \geq \binom{2r}{r}$.

5 Z-Grundy domination number

It is easy to check that the Z-Grundy domination number of the Petersen graph $K_{5,2}$ and of $K_{6,2}$ equal to 5. Note that Proposition 4.2 provided a general upper bound, i.e., for $r \geq 2$ and $n \geq 2r+1$, $\gamma_{gr}^Z(K_{n,r}) \leq \binom{2r}{r}$. We can prove that this bound is reached in the following cases.

Proposition 5.1. *For* $r \geq 2$ *and* $n \geq 3r+1$, $\gamma_{gr}^Z(K_{n,r}) = \binom{2r}{r}$.

Proof. Let us consider the following sets

$$\widehat{S_1} = \{A \mid A \subset [2r], 1 \in A, |A| = r\},$$
$$\widehat{S_2} = \{A \mid A \subset \{r+2, r+3, \dots, 3r\}, |A| = r\} - \{2r+1, \dots, 3r\}.$$

Let S_1 be any sequence of $\widehat{S_1}$ and let S_2 be any sequence of $\widehat{S_2}$. We claim that $S = S_1 \oplus (\{2r+1, \dots, 3r\}) \oplus S_2$ is a Z-dominating sequence. Indeed, each $u \in \widehat{S_1}$ Z-footprints at least $f = [2r] - u$, $\{2r+1, \dots, 3r\}$ Z-footprints $[r-1] \cup \{3r+1\}$ and each $v \in \widehat{S_2}$ Z-footprints at least $f = \{1\} \cup (\{r+2, r+3, \dots, 3r\} - v)$. Hence, $\gamma_{gr}^Z(K_{n,r}) \geq |S| = |S_1| + 1 + |S_2| = 2\binom{2r-1}{r} = \binom{2r}{r}$.

As we have mentioned, the equality follows directly from Proposition 2.1 and 4.2. □

Proposition 5.2. *For* $r \geq 2$ *and* $2r+1 \leq n \leq 3r$, $\gamma_{gr}^Z(K_{n,r}) \geq \binom{2r}{r} - \binom{4r-1-n}{3r-n}$.

Proof. Let us consider the following sets

$$\widehat{S_1} = \{A \mid A \subset [2r], 1 \in A, |A| = r\},$$
$$\widehat{S_2} = \{A \mid A \subset \{n-2r+2, n-2r+3, \dots, n\}, \{2r+1, \dots, n\} \not\subset A, |A| = r\}.$$

Note, that

$$|\widehat{S_2}| = \binom{2r-1}{r} - \binom{2r-1-(n-2r)}{r-(n-2r)} = \binom{2r-1}{r} - \binom{4r-1-n}{3r-n}.$$

Let S_1 be any sequence of $\widehat{S_1}$ and let S_2 be any sequence of $\widehat{S_2}$. We claim that $S = S_1 \oplus S_2$ is a Z-dominating sequence. Indeed, each $u \in \widehat{S_1}$ Z-footprints at least $f = [2r] - u$ and each $v \in \widehat{S_2}$ Z-footprints at least $f = \{1\} \cup (\{n-2r+2, n-2r+3, \dots, n\} - v)$. Hence, $\gamma_{gr}^Z(K_{n,r}) \geq |S| = |S_1| + |S_2| = 2\binom{2r-1}{r} - \binom{4r-1-n}{3r-n} = \binom{2r}{r} - \binom{4r-1-n}{3r-n}$. □

Perhaps the most interesting case is that of odd graphs $K(2r+1, r)$:

Corollary 5.3. *For* $r \geq 2$, $\gamma_{gr}^{Z}(K_{2r+1,r}) \geq \binom{2r}{r} - \binom{2r-2}{r-1} = \frac{3r-2}{r}\binom{2r-2}{r-1}$.

For the zero forcing numbers of Kneser graphs we get the following.

Corollary 5.4.

(i) For $r \geq 2$ *and* $2r+1 \leq n \leq 3r$, $Z(K_{n,r}) \leq \binom{n}{r} - \binom{2r}{r} + \binom{4r-1-n}{3r-n}$.

(ii) For $r \geq 2$ *and* $n \geq 3r+1$, $Z(K_{n,r}) = \binom{n}{r} - \binom{2r}{r}$.

Propositions 5.1 and 5.2 yield the following consequences for the minimum ranks, $\mathrm{mr}(K_{n,r})$.

Corollary 5.5.

(i) For $r \geq 2$ *and* $2r+1 \leq n \leq 3r$, $\mathrm{mr}(K_{n,r}) \geq \binom{2r}{r} - \binom{4r-1-n}{3r-n}$.

(ii) For $r \geq 2$ *and* $n \geq 3r+1$, $\mathrm{mr}(K_{n,r}) \geq \binom{2r}{r}$.

6 L-Grundy domination number

Proposition 6.1. *For* $n \geq 5$, $\gamma_{gr}^{L}(K_{n,2}) = n+2$.

Proof. First, we prove that $S = (\{1,2\},\{1,3\},\{1,4\},\ldots,\{1,n\},\{2,3\},\{2,4\},\{3,4\})$ is a L-sequence of $K_{n,2}$, where $n \geq 5$. Note that S is an L-sequence, since each $\{1,i\}$, where $i \in [n]-\{1\}$, L-footprints itself, $\{2,3\}$ L-footprints $\{1,4\}$, $\{2,4\}$ L-footprints $\{1,3\}$, and $\{3,4\}$ L-footprints $\{1,2\}$. Hence, $\gamma_{gr}^{L}(K_{n,2}) \geq n+2$.

For the proof of the reversed inequality for $n \geq 5$, let $S = (v_1,\ldots,v_k)$ be a maximal L-sequence of $K_{n,2}$. We may assume without loss of generality that $v_1 = \{1,2\}$. Clearly, v_1 L-footprints itself and vertices of the form $\{i,j\}$, where $i,j \in [n]-\{1,2\}$ and $i \neq j$. Next, assume v_2 is a neighbor of v_1. Without loss of generality we may assume $v_2 = \{3,4\}$. Thus, v_2 L-footprints itself and vertices of the form $\{i,j\}$, where $i \in \{1,2\}, j \in [n]-\{3,4\}$ and $i \neq j$. Hence, just the vertices $\{1,3\},\{1,4\},\{2,3\}$ and $\{2,4\}$ remain not L-dominated by S_2. The rest can be L-dominated with at most 4 vertices. Follows, $|S| \leq 6 < n+2$.

Next, suppose v_2 is not a neighbor of v_1. Without loss of generality let $v_2 = \{1,3\}$. In this case, the vertex $\{2,3\}$ and the vertices of the form $\{1,i\}$, where $i \in [n]-\{1\}$, remain not L-dominated by S_2. Next, we distinguish again 4 cases:

(i) v_3 is a neighbor of v_1 and v_2 (v_3 is of the form $\{i,j\}$, where $i,j \in [n]-\{1,2,3\}$ and $i \neq j$),

(ii) v_3 is a neighbor of v_1 and not of v_2 (v_3 is of the form $\{3,i\}$, where $i \in [n]-\{1,2,3\}$),

(iii) v_3 is a neighbor of v_2 and not of v_1 (v_3 is of the form $\{2,i\}$, where $i \in [n]-\{1,2,3\}$), and

(iv) v_3 is not a neighbor of v_1 or v_2 (v_3 is $\{2,3\}$ or is of the form $\{1,i\}$, where $i \in [n]-\{1,2,3\}$).

In the cases (i), (ii), (iii) or if $v_3 = \{2,3\}$ in the case (iv), there are at most 3 vertices left, that are not L-dominated by $\widehat{S_3}$. In all cases we can L-dominate the rest with at most 4 vertices. Hence, $|S| \leq 7 \leq n+2$.

In the case (iv), where $v_3 = \{1, i\}$ ($i \in [n] - \{1, 2, 3\}$), the vertex v_3 L-footprints just itself and $\{2, 3\}$. Hence, just the vertices of the form $\{1, i\}$, where $i \in [n] - \{1\}$, remain not L-dominated by $\widehat{S_3}$. Note, that it is possible that v_i is first L-footprinted by itself and then later again by another vertex from S. Next, if $v_4 = \{1, 4\}$, then v_4 just L-footprints itself and the same vertices stay not L-dominated. Hence, to make the sequence S as large as possible, next in the sequence can be vertices $v_{i-1} = \{1, i\}$ for $i = 4, \dots, n$.

Without loss of generality let $v_n = \{2, 3\}$ (until now in the sequence are all vertices that contain 1). Hence, just the vertices $\{1, 2\}$ and $\{1, 3\}$ remain not L-dominated by $\widehat{S_n}$. Follows, $|S| \leq n + 2$. □

Proposition 6.2. *For $r \geq 2$ and $n \geq 2r + 1$, $\gamma_{gr}^L(K_{n,r}) \geq \binom{n-1}{r-1} + \binom{2r-1}{r}$.*

Proof. Let $\widehat{S_1} = \{A \mid A \subset [n], 1 \in A, |A| = r\}$ and let $\widehat{S_2} = \{A \mid A \subset [2r] - \{1\}, |A| = r\}$. Let S_1 be any sequence of $\widehat{S_1}$ and let S_2 be any sequence of $\widehat{S_2}$. We claim that $S = S_1 \oplus S_2$ is an L-dominating sequence.

Indeed, each $u \in \widehat{S_1}$ L-footprints at least itself and each $v \in \widehat{S_2}$ L-footprints at least $f = [2r] - v$. Hence, $\gamma_{gr}^L(K_{n,r}) \geq |S| = |S_1| + |S_2| = \binom{n-1}{r-1} + \binom{2r-1}{r}$. □

For the upper bound, we first present a general result bounding the L-Grundy domination number of a graph G with no isolated vertices by using the independence number $\alpha(G)$ and the Grundy total domination number $\gamma_{gr}^t(G)$.

Proposition 6.3. *For a graph G with no isolated vertices, $\gamma_{gr}^L(G) \leq \alpha(G) + \gamma_{gr}^t(G) - 1$.*

Proof. Let $S = (v_1, \dots, v_k)$ be an L-sequence of G. Let A, B and C be the sets of vertices of $\widehat{S}$ such that every vertex in A only L-footprints itself, every vertex in B L-footprints itself and (at least) one more vertex and every vertex in C does not L-footprint itself. Note that $\{A, B, C\}$ is a partition of $\widehat{S}$. Besides, $A \cup B$ is a stable set of G. Hence, $|A| + |B| \leq \alpha(G)$. Let $S' = (w_1, \dots, w_m)$ be the subsequence of S (respecting the order in S) such that $\widehat{S'} = B \cup C$. Clearly, S' is an open neighborhood sequence in G, thus $m \leq \gamma_{gr}^t(G)$.

Therefore, $|S| = |A| + |B| + |C| \leq \alpha(G) + \gamma_{gr}^t(G) - |B| \leq \alpha(G) + \gamma_{gr}^t(G) - 1$, since $v_1 \in B$. □

Corollary 6.4. *For $r \geq 2$ and $n \geq 2r + 1$, $\gamma_{gr}^L(K_{n,r}) \leq \binom{n-1}{r-1} + \binom{2r}{r} - 1$.*

Note that the gap between the lower and the upper bound in Proposition 6.2 and Corollary 6.4 is $\binom{2r-1}{r} - 1$, which is fixed with respect to n.

7 Set-theoretic connections

Following the set-theoretic connections as in the case of Grundy total domination number (see Lemma 4.1), we ask the following.

Problem 7.1. Let $T = \{(A_i, B_i) \mid (A_i \cup B_i) \subseteq [n], |A_i| = |B_i| = r,$ for all $i \in [k]\}$ be a set-pair collection satisfying the following conditions:

1. $A_i \cap B_i = \emptyset$ or $A_i = B_i$ for $1 \leq i \leq k$;
2. $A_i \cap B_j \neq \emptyset$ and $A_i \neq B_j$ for $1 \leq i < j \leq k$.

Note that $|T| = k$.

Determine the smallest value $f(n,r)$ for which $k \leq f(n,r)$ for all such set-pair collections T.

From Theorem 3.2, for any $r \geq 2$ there exists $n_0 \in \mathbb{N}$ such that for all n, $n \geq n_0$, we have $f(n,r) = \gamma_{gr}(K_{n,r}) = \alpha(K_{n,r}) = \binom{n-1}{r-1}$. Note that in this case, the cardinality n of the universal set in which A_i and B_i are contained plays an important role in determining $f(n,r)$. As we see in the next case, allowing condition $A_i = B_i$ is also relevant to the problem.

In a similar way, we propose the corresponding question for the Z-Grundy domination number.

Problem 7.2. Let $T = \{(A_i, B_i) \mid (A_i \cup B_i) \subseteq [n], |A_i| = |B_i| = r, \text{ for all } i \in [k]\}$ be a set-pair collection satisfying the following conditions:

1. $A_i \cap B_i = \emptyset$ for $1 \leq i \leq k$;
2. $A_i \cap B_j \neq \emptyset$ and $A_i \neq B_j$ for $1 \leq i < j \leq k$.

Note that $|T| = k$.

Determine the smallest value $f_Z(n,r)$ for which $k \leq f_Z(n,r)$ for all such set-pair collections T.

Note that $f_Z(n,r) = \gamma_{gr}^Z(K_{n,r}) = \binom{2r}{r}$ for $n \geq 3r+1$. In this case $f_Z(n,r)$ is independent of n, but it is unclear whether this function is dependent on n when $2r+1 \leq n \leq 3r$.

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S_{12} and P_{12}-colorings of cubic graphs*

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Abstract

If G and H are two cubic graphs, then an H-coloring of G is a proper edge-coloring f with the edges of H, such that for each vertex x of G, there is a vertex y of H with $f(\partial_G(x)) = \partial_H(y)$. If G admits an H-coloring, then we will write $H \prec G$. The Petersen coloring conjecture of Jaeger (P_{10}-conjecture) states that for any bridgeless cubic graph G, one has: $P_{10} \prec G$. The S_{10}-conjecture states that for any cubic graph G, $S_{10} \prec G$. In this paper, we introduce two new conjectures that are related to these conjectures. The first of them states that any cubic graph with a perfect matching admits an S_{12}-coloring. The second one states that any cubic graph G whose edge-set can be covered with four perfect matchings, admits a P_{12}-coloring. We call these new conjectures S_{12}-conjecture and P_{12}-conjecture, respectively. Our first results justify the choice of graphs in S_{12}-conjecture and P_{12}-conjecture. Next, we characterize the edges of P_{12} that may be fictive in a P_{12}-coloring of a cubic graph G. Finally, we relate the new conjectures to the already known conjectures by proving that S_{12}-conjecture implies S_{10}-conjecture, and P_{12}-conjecture and $(5, 2)$-Cycle cover conjecture together imply P_{10}-conjecture. Our main tool for proving the latter statement is a new reformulation of $(5, 2)$-Cycle cover conjecture, which states that the edge-set of any claw-free bridgeless cubic graph can be covered with four perfect matchings.

Keywords: Cubic graph, Petersen graph, Petersen coloring conjecture, S_{10}-conjecture.

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1 Introduction

In this paper, we consider finite, undirected graphs. They do not contain loops, though they may contain parallel edges. We also consider pseudo-graphs, which may contain both loops and parallel edges, and simple graphs, which contain neither loops nor parallel edges. As usual, a loop contributes to the degree of a vertex by two.

Within the frames of this paper, we assume that graphs, pseudo-graphs and simple graphs are considered up to isomorphisms. This implies that the equality $G = G'$ means that G and G' are isomorphic.

For a graph G, let $V(G)$ and $E(G)$ be the set of vertices and edges of G, respectively. Moreover, let $\partial_G(x)$ be the set of edges of G that are incident to the vertex x of G. A matching of G is a set of edges of G such that any two of them do not share a vertex. A matching of G is perfect, if it contains $\frac{|V(G)|}{2}$ edges. A block of G is a maximal 2-connected subgraph of G. An end-block is a block of G containing at most one vertex that is a cut-vertex of G. A subgraph H of G is even, if every vertex of H has even degree in H. A subgraph H is odd, if every vertex of G has odd degree in H. Sometime, we will refer to odd subgraphs as joins. Observe that a perfect matching is a join of a cubic graph. A subgraph H is a parity subgraph if for every vertex v of G $d_G(v)$ and $d_H(v)$ have the same parity. Observe that H is a parity subgraph of G if $G - E(H)$ is an even subgraph of G.

Let G is a cubic graph, and let K be a triangle in G such that each of K is of multiplicity one. For an edge e of K, let f be the edge of G that is incident to a vertex of K and is not adjacent to e. Edges e and f will be called opposite edges.

Let G and H be two cubic graphs. An H-coloring of G is a mapping $f\colon E(G) \to E(H)$, such that for each vertex x of G there is a vertex y of H, such that $f(\partial_G(x)) = \partial_H(y)$. If G admits an H-coloring, then we will write $H \prec G$. It can be easily seen that if $H \prec G$ and $K \prec H$, then $K \prec G$. In other words, $\prec$ is a transitive relation defined on the set of cubic graphs.

If $H \prec G$ and f is an H-coloring of G, then for any adjacent edges e, e' of G, the edges $f(e)$, $f(e')$ of H are adjacent. Moreover, if the graph H contains no triangle, then the converse is also true, that is, if a mapping $f\colon E(G) \to E(H)$ has a property that for any two adjacent edges e and e' of G, the edges $f(e)$ and $f(e')$ of H are adjacent, then f is an H-coloring of G (see Lemma 2.1).

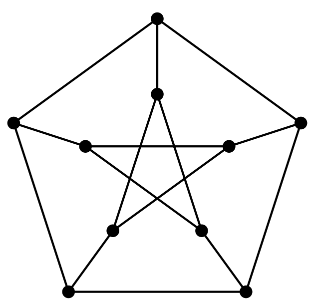

Figure 1: The graph P_{10}.

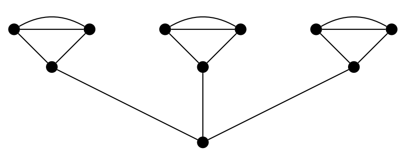

Figure 2: The graph S_{10}.

Let P_{10} be the well-known Petersen graph (Figure 1) and let S_{10} be the graph from Figure 2. The Petersen coloring conjecture of Jaeger states:

Conjecture 1.1 (Jaeger, 1988 [9]). *For any bridgeless cubic graph G, $P_{10} \prec G$.*

Sometimes, we will call this conjecture as P_{10}-conjecture. The conjecture is difficult to prove, since it can be seen that it implies the following classical conjectures:

Conjecture 1.2 (Berge-Fulkerson, 1972 [4, 15]). *Any bridgeless cubic graph G contains six (not necessarily distinct) perfect matchings $F_1, \ldots, F_6$ such that any edge of G belongs to exactly two of them.*

This list of six perfect matchings usually is called a Berge-Fulkerson cover of G. If $k(G)$ is the smallest number of perfect matchings that are needed to cover the edge-set of G, then observe that this conjecture implies that $k(G) \leq 5$ for any bridgeless cubic graph. This weaker statement is known as Berge conjecture.

Conjecture 1.3 ($(5, 2)$-even-subgraph-cover conjecture [1, 13]). *Any bridgeless graph G (not necessarily cubic) contains five even subgraphs such that any edge of G belongs to exactly two of them.*

Let us note that some of the even subgraphs stated in this conjecture might be empty.

Related with the Jaeger conjecture, the following conjecture has been introduced in [11]:

Conjecture 1.4 (V. V. Mkrtchyan, 2012 [11]). *For any cubic graph G, $S_{10} \prec G$.*

We will call this the S_{10}-conjecture.

A k-edge-coloring is an assignment of colors to edges of a graph from a set of k colors such that adjacent edges receive different colors. The smallest k for which a graph G admits a k-edge-coloring is called a chromatic index of G and is denoted by $\chi'(G)$. If α is a k-edge-coloring of a cubic graph G, then an edge $e = uv$ is called poor (rich) in α, if the five edges of G incident to u or v are colored with three (five) colors. α is called a normal k-edge-coloring of G if any edge of G is either poor or rich in α. Observe that not all cubic graphs admit a normal k-edge-coloring for some k. An example of such a graph is the graph from Figure 2. On the positive side, all simple cubic graphs admit a normal 7-edge-coloring [10]. The smallest k (if it exists) for which a cubic graph G admits a normal k-edge-coloring is called a normal chromatic index of G and is denoted by $\chi'_N(G)$.

Normal colorings were introduced by Jaeger in [8], where among other results, he showed that for a cubic graph G, $\chi'_N(G) \leq 5$ if and only if G admits a P_{10}-coloring. This allowed him to obtain a reformulation of Conjecture 1.1, which states that for any bridgeless cubic graph G, $\chi'_N(G) \leq 5$.

In this paper, we introduce two new conjectures that are related to Conjectures 1.1 and 1.4. In Section 2, we discuss some auxiliary results that will be useful later in the paper. In Section 3, we briefly discuss so-called coloring-hereditary classes of cubic graphs that allowed us to come up with these two new conjectures. Then in Section 4, we present our main results. Finally, in Section 5, we discuss some open problems. Terms and concepts that we do not define in the paper can be found in [17, 18].

2 Auxiliary results

In this section, we present some auxiliary results that will be useful later. Our first two results are lemmas about some properties of H-colorings of cubic graphs. Though all these properties are known before, for the sake of completeness we give complete proofs.

Lemma 2.1. *Assume that G and H are two cubic graphs. Moreover, let H be triangle-free. If a mapping $f\colon E(G) \to E(H)$ has a property that for any two adjacent edges e and e′ of G, the edges $f(e)$ and $f(e')$ of H are adjacent, then f is an H-coloring of G.*

Proof. In order to see this, assume that f is not an H-coloring of G. Then G contains a vertex w where the definition of an H-coloring is violated. Let e_1, e_2 and e_3 be the three edges incident to w. Assume that the colors of e_1 and e_2 in f are the edges xy and yz of H. Observe that $z \neq x$, as otherwise we will have $f(\partial_G(w)) = \partial_H(x)$ or $f(\partial_G(w)) = \partial_H(y)$ violating the choice of w. Now, the edge $f(e_3)$ of H cannot be incident to y. On the other hand, it must be adjacent to xy and yz. Hence $f(e_3)$ connects x and z. Observe that the edges x, y and z form a triangle in H. This contradicts our condition on H. □

Note that the condition H is triangle-free is important in the previous lemma. If G is any 3-edge-colorable cubic graph and H contains a triangle with edges h_1, h_2 and h_3, then consider the 3-edge-coloring of G with colors h_1, h_2 and h_3. Observe that for any two adjacent edges of G, their colors are adjacent edges in H. However, the coloring is not an H-coloring as in every vertex of G its definition is violated.

Lemma 2.2. *Suppose that G and H are cubic graphs with $H \prec G$, and let f be an H-coloring of G. Then:*

(a) If M is any matching of H, then $f^{-1}(M)$ is a matching of G;

(b) $\chi'(G) \leq \chi'(H)$;

(c) If M is a perfect matching of H, then $f^{-1}(M)$ is a perfect matching of G;

(d) $k(G) \leq k(H)$;

(e) If H admits a Berge-Fulkerson cover, then G also admits a Berge-Fulkerson cover;

(f) For every even subgraph C of H, $f^{-1}(C)$ is an even subgraph of G;

(g) For every bridge e of G, the edge $f(e)$ is a bridge of H;

(h) If H is bridgeless, then G is bridgeless as well;

(i) $\chi'_N(G) \leq \chi'_N(H)$.

Proof. (a) and (c): The proof of (a) follows from the definition of H-coloring: as adjacent edges of G must be colored with adjacent edges of H, then clearly the pre-image of a matching in H must be a matching in G. For the proof of (c) let M be a perfect matching of H. Then by (a), $f^{-1}(M)$ is a matching of G. Let us show that it covers all vertices of G. Let v be a vertex of G. Then the three edges incident to v are colored by a similar three edges of H. Since M is a perfect matching of H, one of these three edges must belong to M, hence $f^{-1}(M) \cap \partial_G(v) \neq \emptyset$. Thus, $f^{-1}(M)$ is a perfect matching of G.

(b) and (d): For the proof of (b) assume that $\chi'(H) = s$ and let $M_1, \dots, M_s$ be the color classes of H in an s-edge-coloring. Consider $f^{-1}(M_1), \dots, f^{-1}(M_s)$. Observe that due to (a), they are forming s matchings covering the edge-set of G. Thus, $\chi'(G) \leq s = \chi'(H)$. The proof of (d) is similar: let $k(H) = s$ and let $M_1, \dots, M_s$ be the s perfect matchings of H covering $E(H)$. Consider $f^{-1}(M_1), \dots, f^{-1}(M_s)$. Observe that due to (c), they are forming s perfect matchings covering the edge-set of G. Thus, $k(G) \leq s = k(H)$.

(e): Let $C = (F_1, \dots, F_6)$ be a Berge-Fulkerson cover of H. Consider the list $f^{-1}(C) = (f^{-1}(F_1), \dots, f^{-1}(F_6))$. Observe that due to (c) they are forming a list of six perfect matchings of G. Moreover, every edge of G belongs to at least two of these perfect matchings. Hence $f^{-1}(C)$ is a Berge-Fulkerson cover of G.

(f): Let C be an even subgraph of H. Let us show that any vertex v of G has even degree in $f^{-1}(C)$. Since H is cubic, C is a disjoint union of cycles. Assume that in f the three edges incident to v are colored with three edges incident to a vertex w of H. Then if w is isolated in C, then clearly v is isolated in $f^{-1}(C)$. On the other hand, if w has degree two in C, then v is of degree two in $f^{-1}(C)$. Thus, v always has even degree in $f^{-1}(C)$, or $f^{-1}(C)$ is an even subgraph of G.

(g): Let e be a bridge of G and let $(X, V(G) \setminus X)$ be a partition of $V(G)$, such that $\partial_G(X) = \{e\}$. Assume that the edge $f(e)$ is not a bridge in H. Then there is a cycle C in H that contains the edge $f(e)$. By (f) $f^{-1}(C)$ is an even subgraph of G that has non-empty intersection with $\partial_G(X)$. Since the intersection of an even subgraph with $\partial_G(X)$ always contains an even number of edges, we have that $\partial_G(X)$ contains at least two edges which contradicts our assumption.

(h): This follows from (g): if H has no bridge, then any edge of G cannot be a bridge, as otherwise its color in f will be a bridge in H.

(i): Assume that $\chi'_N(H) = s$, and let g be a normal s-edge-coloring of H. Consider a mapping h defined on the edge-set of G as follows: for any edge e of G, let $h(e) = g(f(e))$. Let us show that h is a normal s-edge-coloring of G. Let $e = vw$ be any edge of G. Assume that in f the three edges incident to v are colored by the three edges incident to a vertex u_1 of H, the three edges incident to w are colored by the three edges incident to a vertex u_2 of H.

If $u_1 = u_2$, then the edge e is poor in h. Thus, we can assume that $u_1 \neq u_2$. Since $e \in \partial_G(v) \cap \partial_G(w)$, we have that $u_1u_2 \in E(H)$ and $f(e) = u_1u_2$. Now, observe that since g is a normal edge-coloring, we have that if $f(e)$ is a poor edge in g, then e is a poor edge in h, and if $f(e)$ is a rich edge in g, then e is a rich edge in h. Thus, h is a normal s-edge-coloring of G. Hence $\chi'_N(G) \leq s = \chi'_N(H)$. The proof of the lemma is complete. □

We will need some results on claw-free bridgeless cubic graphs. Recall that a graph G is claw-free, if it does not contain 4 vertices, such that the subgraph of G induced on these vertices is isomorphic to $K_{1,3}$. In [2], arbitrary claw-free graphs are characterized. In [12], Oum has characterized simple, claw-free bridgeless cubic graphs. In order to formulate Oum's result, we need some definitions. In a claw-free simple cubic graph G any vertex belongs to 1, 2, or 3 triangles. If a vertex v belongs to 3 triangles of G, then the component of G containing v is isomorphic to K_4 (Figure 3). An induced subgraph of G that is isomorphic to $K_4 - e$ is called a diamond [12]. It can be easily checked that in a claw-free cubic graph no 2 diamonds intersect.

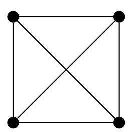

Figure 3: The graph K_4.

A string of diamonds of G is a maximal sequence $F_1, \ldots, F_k$ of diamonds, in which F_i has a vertex adjacent to a vertex of F_{i+1}, $1 \leq i \leq k-1$. A string of diamonds has exactly

2 vertices of degree 2, which are called the head and the tail of the string. Replacing an edge $e = uv$ with a string of diamonds with the head x and the tail y is to remove e and add edges (u, x) and (v, y).

If G is a connected claw-free simple cubic graph such that each vertex lies in a diamond, then G is called a ring of diamonds. It can be easily checked that each vertex of a ring of diamonds lies in exactly one diamond. As in [12], we require that a ring of diamonds contains at least 2 diamonds.

Proposition 2.3 (Oum [12])**.** *G is a connected claw-free simple bridgeless cubic graph, if and only if*

(1) G is isomorphic to K_4, or

(2) G is a ring of diamonds, or

(3) there is a connected bridgeless cubic graph H, such that G can be obtained from H by replacing some edges of H with strings of diamonds, and by replacing any vertex of H with a triangle.

The next auxiliary result allows us to relate coverings with even subgraphs to coverings with specific parity subgraphs. Like we stated in the introduction, some of the even subgraphs here might be empty.

Theorem 2.4 ([7, Theorem 3.3])**.** *For a graph G, the following two conditions are equivalent:*

(1) G contains five even subgraphs such that any edge of G belongs to exactly two of them;

(2) G contains four parity subgraphs such that each edge belongs to either one or two of the parity subgraphs.

Our final auxiliary result is a theorem proved by Giuseppe Mazzuoccolo which offers a new reformulation of Conjecture 1.3.

Theorem 2.5. *Conjecture 1.3 is equivalent to the statement that for all bridgeless claw-free cubic graphs we have $k(G) \leq 4$.*

Proof. Assume that for any claw-free bridgeless cubic graph G, we have $k(G) \leq 4$. Let us show that Conjecture 1.3 is also true. It is known that it suffices to prove Conjecture 1.3 for cubic graphs [18]. Let G be an arbitrary bridgeless cubic graph. Consider the cubic graph H obtained from G by replacing every vertex of G with a triangle. Observe that H is a claw-free bridgeless cubic graph. By our assumption, the edge-set of H can be covered with four perfect matchings. Observe that perfect matchings are parity subgraphs in cubic graphs, hence by Theorem 2.4, H admits a list of 5 even subgraphs covering each edge exactly twice.

In order to complete the proof, let us observe that if a cubic graph K admits a list of 5 even subgraphs covering each edge exactly twice and it contains a triangle T, then the graph K/T also admits a list of 5 even subgraphs covering each edge exactly twice. In order to see this, let $\mathcal{C} = (Ev_1, \dots, Ev_5)$ be the list of 5 even subgraphs covering the edges of K twice. Then it is easy to see that the edges of the 3-cut $\partial_K(T)$ are covered as follows: first edge belongs to Ev_1 and Ev_2, the second edge belongs to Ev_1 and Ev_3, and finally the third edge belongs to Ev_2 and Ev_3. Moreover, Ev_4 and Ev_5 do not intersect the 3-cut. One can always achieve this by renaming the even subgraphs. Now, if we consider the restrictions of $(Ev_1, \dots, Ev_5)$ to K/T, we will have that they are forming a list of 5 even subgraphs covering each edge of K/T exactly twice.

Applying this observation $|V(G)|$ times to H, we will get the statement for the original cubic graph G.

For the proof of the converse statement, let us assume that Conjecture 1.3 is true, and show that any claw-free bridgeless cubic graph G can be covered with four perfect matchings. We prove the latter statement by induction on $|V(G)|$. If $|V(G)| = 2$, the the statement is trivially true. Assume that it is true for all claw-free bridgeless cubic graphs with less n vertices and let us consider a claw-free bridgeless cubic graph G with $n \geq 4$ vertices.

Clearly, we can assume that G is connected. Let us show that we can assume that G is simple. On the opposite assumption, consider the vertices u and v that are joined with two edges. Let u' and v' be the the other neighbors of u and v, respectively. Consider a cubic graph G' obtained from $G - \{u, v\}$ by adding a possibly parallel edge $u'v'$. Observe that G' is a bridgeless cubic graph with $|V(G')| < n$. Moreover, it is claw-free. Thus, by induction hypothesis, G' can be covered with four perfect matchings. Now, it is easy to see that using these list of four perfect matchings of G' we can construct a list of four perfect matchings of G covering $E(G)$.

Thus, without loss of generality, we can assume that G is simple. Hence, we can apply Proposition 2.3. If G meets the first or the second condition of the proposition, then it is easy to see that G is 3-edge-colorable, hence it can be covered with three perfect matchings. Thus, we can assume that there is a connected bridgeless cubic graph H such that G can be obtained from H by replacing some edges of H with strings of diamonds and every vertex of H with a triangle.

Let us show that we can also assume that G has no string of diamonds. Assume it has one. Let it be S whose head and tails are u and v, respectively. Let u' and v' be the neighbors of u and v, respectively, that lie outside S. Consider a graph G' obtained from $G - V(S)$ by adding a possibly parallel edge $u'v'$. Observe that G' is a bridgeless cubic graph with $|V(G')| < n$. Moreover, it is claw-free. Thus, by induction hypothesis, G' can be covered with four perfect matchings. Now, it is easy to see that using these list of four perfect matchings of G' we can construct a list of four perfect matchings of G covering $E(G)$.

Thus, without loss of generality, we can assume that G can be obtained from the connected bridgeless cubic graph H by replacing its every vertex with a triangle. By Conjecture 1.3, H has a list of five even subgraphs covering its edges exactly twice. By Theorem 2.4, we have that H admits a cover with four joins such that each edge of H is covered once or twice. Let v any vertex of H and let $C = (T_1, T_2, T_3, T_4)$ be the cover of H with four joins. Since each edge of H is covered once or twice in C, we have that there is at most one join in C that contains all three edges incident to v. Thus, for any vertex v we have that either one of joins contains all three edges incident to v and the other three joins contain exactly one edge incident to v, or all joins contain exactly one edge incident to v. Now, it is not hard to see that these four joins covering H can be extended to four perfect matchings of G so that they cover G. The proof of the theorem is complete. $\square$

3 Coloring-hereditary classes of cubic graphs

In this section, we briefly discuss coloring-hereditary classes of cubic graphs. It is these classes that allowed us to come up with more conjectures related to Conjectures 1.1 and 1.4.

If G and H are two cubic graphs with $H \prec G$ or $G \prec H$, then we will say that G and H are comparable. A (not necessarily finite) set of cubic graphs is said to be an anti-chain, if any two cubic graphs from the set are not comparable. Let $\mathcal{C}$ be the class of all connected cubic graphs. If $\mathcal{M} \subseteq \mathcal{C}$ is a class of connected cubic graphs, then we will say that $\mathcal{M}$ is coloring-hereditary, if for any cubic graphs G and H, if $H \in \mathcal{M}$ and $H \prec G$, then $G \in \mathcal{M}$. Assume that $\mathcal{B} \subseteq \mathcal{M}$ is a subset of some coloring-hereditary class $\mathcal{M}$ of cubic graphs. We will say that $\mathcal{B}$ is a basis for $\mathcal{M}$, if $\mathcal{B}$ is an anti-chain and for any connected cubic graph G we have that $G \in \mathcal{M}$ if and only if there is a cubic graph $H \in \mathcal{B}$, such that $H \prec G$.

Our starting question is the following: does every coloring-hereditary class of cubic graphs admit a finite basis, that is, a basis with finitely many elements? It turns out that the answer to this question is negative. Let $\mathcal{I}$ be the infinite anti-chain of cubic graphs constructed in [14]. Consider the class $\mathcal{M}$ of connected cubic graphs G, such that for any G we have: $G \in \mathcal{M}$, if and only if there is a cubic graph $H \in \mathcal{I}$, such that $H \prec G$. It is easy to see that $\mathcal{M}$ is a coloring-hereditary class of cubic graphs without a finite basis.

Despite this, one may still look for interesting coloring-hereditary classes arising in graph theory, that admit a finite basis. Below, we discuss some such classes. The first class is $\mathcal{C}$-the class of all connected cubic graphs. Clearly, it is coloring-hereditary. Observe that any connected cubic graph admitting an S_{10}-coloring belongs to $\mathcal{C}$. On the other hand, Conjecture 1.4 predicts that any cubic graph from $\mathcal{C}$ admits an S_{10}-coloring. Thus, we can view Conjecture 1.4 as a statement that S_{10} forms a basis for $\mathcal{C}$.

Let $\mathcal{C}_b$ be the class of all connected bridgeless cubic graphs. Statement (h) of Lemma 2.2 implies that $\mathcal{C}_b$ is a coloring-hereditary class of cubic graphs. Observe that any connected cubic graph admitting a P_{10}-coloring belongs to $\mathcal{C}_b$. On the other hand, Conjecture 1.1 predicts that any bridgeless cubic graph from $\mathcal{C}_b$ admits a P_{10}-coloring. Thus, we can view Conjecture 1.1 as a statement that P_{10} forms a basis for $\mathcal{C}_b$.

Let $\mathcal{C}_3$ be the class of all connected 3-edge-colorable cubic graphs. Statement (b) of Lemma 2.2 implies that $\mathcal{C}_3$ is a coloring-hereditary class of cubic graphs. Let H be any connected 3-edge-colorable cubic graph. Observe that any cubic graph G is 3-edge-colorable if and only if $H \prec G$. Thus, H forms a basis for $\mathcal{C}_3$.

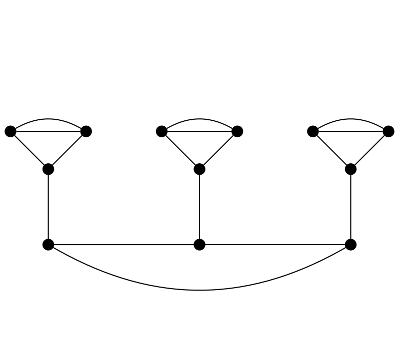

Figure 4: The graph S_{12}.

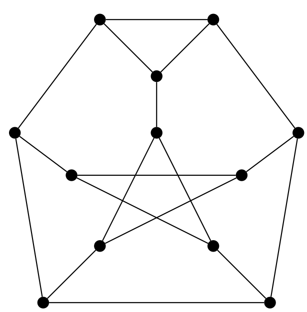

Figure 5: The graph P_{12}.

Let $\mathcal{C}_p$ be the class of all connected cubic graphs containing a perfect matching. Statement (c) of Lemma 2.2 implies that $\mathcal{C}_p$ is a coloring-hereditary class of cubic graphs. Observe that any connected cubic graph admitting an S_{12}-coloring (the graph from Figure 4) belongs to $\mathcal{C}_p$. On the other hand, we suspect that

Conjecture 3.1. *Any cubic graph with a perfect matching admits an S_{12}-coloring.*

Conjecture 3.1 predicts that all cubic graphs from $\mathcal{C}_p$ admit an S_{12}-coloring. Thus, we can view Conjecture 3.1 as a statement that S_{12} forms a basis for $\mathcal{C}_p$. Let us note that Conjecture 3.1 has been verified for claw-free cubic graphs in [5].

Let $\mathcal{C}^{(4)}$ be the class of all connected cubic graphs G with $k(G) \leq 4$. Statement (d) of Lemma 2.2 implies that $\mathcal{C}^{(4)}$ is a coloring-hereditary class of cubic graphs. Observe that any connected cubic graph admitting a P_{12}-coloring (the graph from Figure 4) belongs to $\mathcal{C}^{(4)}$. On the other hand, we suspect that

Conjecture 3.2. *Any cubic graph G with $k(G) \leq 4$ admits a P_{12}-coloring.*

Conjecture 3.2 predicts that all cubic graphs from $\mathcal{C}^{(4)}$ admit a P_{12}-coloring. Thus, we can view Conjecture 3.2 as a statement that P_{12} forms a basis for $\mathcal{C}^{(4)}$. Also, note that (e) of Lemma 2.2 implies that Conjecture 4.9 from [7] is a consequence of Conjecture 3.2.

4 The main results

In this section, we obtain our main results. First, we discuss the choice of graphs P_{12} and S_{12} in Conjectures 3.2 and 3.1, respectively. For this purpose, we recall the following two theorems that are proved in [11].

Theorem 4.1. *If G is a connected bridgeless cubic graph with $G \prec P_{10}$, then $G = P_{10}$.*

Theorem 4.2. *If G is a connected cubic graph with $G \prec S_{10}$, then $G = S_{10}$.*

The first theorem suggests that in Conjecture 1.1 the graph P_{10} cannot be replaced with any other connected bridgeless cubic graph. Similarly, the second theorem suggests that in Conjecture 1.4 the graph S_{10} cannot be replaced with other connected cubic graph. Now, we are going to obtain similar results for Conjectures 3.2 and 3.1.

Theorem 4.3. *Let G be a connected bridgeless cubic graph with $G \prec P_{12}$. Then either $G = P_{10}$ or $G = P_{12}$.*

Proof. Assume that f is a G-coloring of P_{12}. Consider the triangle T in P_{12}. Assume that the edges of T are e_1, e_2, e_3. Since these three edges are pairwise adjacent in P_{12}, we have that the edges $f(e_1), f(e_2), f(e_3)$ are pairwise adjacent in G. We have two cases to consider:

Case 1: There is a vertex v of G, such that $\partial_G(v) = \{f(e_1), f(e_2), f(e_3)\}$. Observe that in this case the edges of the 3-edge-cut $\partial_{P_{12}}(V(T))$ are colored by $f(e_1), f(e_2), f(e_3)$, respectively. Thus, if we contract T in P_{12}, we will get a G-coloring of P_{10}. Hence, by Theorem 4.1, $G = P_{10}$.

Case 2: The edges $f(e_1), f(e_2), f(e_3)$ form a triangle T_0 in G. Observe that in this case the edges of the 3-edge-cut $\partial_{P_{12}}(V(T))$ are colored by the edges of the 3-edge-cut $\partial_G(V(T_0))$. Thus, f induces a G/T_0-coloring of $P_{12}/T = P_{10}$. Hence, by Theorem 4.1, $G/T_0 = P_{10}$, which implies that $G = P_{12}$. The proof of the theorem is complete. □

Corollary 4.4. *Let G be a connected bridgeless cubic graph with $k(G) \leq 4$ and $G \prec P_{12}$. Then $G = P_{12}$.*

Theorem 4.5. *Let G be a connected cubic graph with $G \prec S_{12}$. Then either $G = S_{10}$ or $G = S_{12}$.*

Proof. Assume that f is a G-coloring of S_{12}. Consider the central triangle T in S_{12}, that is, the unique triangle T such that all edges of $\partial_{S_{12}}(V(T))$ are bridges. Assume that the edges of T are e_1, e_2, e_3. Since these three edges are pairwise adjacent in S_{12}, we have that the edges $f(e_1), f(e_2), f(e_3)$ are pairwise adjacent in G. We have two cases to consider:

Case 1: There is a vertex v of G, such that $\partial_G(v) = \{f(e_1), f(e_2), f(e_3)\}$. Observe that in this case the edges of the 3-edge-cut $\partial_{S_{12}}(V(T))$ are colored by $f(e_1), f(e_2), f(e_3)$, respectively. Thus, if we contract T in S_{12}, we will get a G-coloring of S_{10}. Hence, by Theorem 4.2, $G = S_{10}$.

Case 2: The edges $f(e_1), f(e_2), f(e_3)$ form a triangle T_0 in G. Observe that in this case the edges of the 3-edge-cut $\partial_{S_{12}}(V(T))$ are colored by the edges of the 3-edge-cut $\partial_G(V(T_0))$. Moreover, since all edges of $\partial_{S_{12}}(V(T))$ are bridges, by (g) of Lemma 2.2, we have that the three edges of $\partial_G(V(T_0))$ are bridges. This, in particular, means that each edge of T_0 is of multiplicity one in G. Observe that f induces a G/T_0-coloring of $S_{12}/T = S_{10}$. Hence, by Theorem 4.2, $G/T_0 = S_{10}$. Moreover, the new vertex v_{T_0} of G/T_0 corresponding to T_0, is incident to three bridges. Hence v_{T_0} is the unique cut-vertex of $G/T_0 = S_{10}$ that is incident to three bridges. This means that $G = S_{12}$. The proof of the theorem is complete. □

Corollary 4.6. *Let G be a connected cubic graph with a perfect matching such that $G \prec S_{12}$. Then $G = S_{12}$.*

In the next statement, we discuss the following problem: assume that a bridgeless cubic G graph admits a P_{10}-coloring such that one of the edges of P_{10} is not used. What can we say about G? We discuss the related problem for Conjecture 3.2 afterwards. Let us note that the following statement is proved by Eckhard Steffen.

Proposition 4.7. *Let G be a bridgeless cubic that admits a P_{10}-coloring f, such that for an edge e of P_{10}, we have: $f^{-1}(e) = \emptyset$. Then $\chi'(G) = 3$.*

Proof. ([16]) Assume that the edge e of P_{10} is not used in a P_{10}-coloring f of G. We have that there exist two perfect matchings M_1 and M_2 of P_{10}, such that $M_1 \cap M_2 = \{e\}$. By (c) of Lemma 2.2, we have that $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are perfect matchings in G. Since the edge e is not used in f, we have that the perfect matchings are edge-disjoint in G. Thus $\chi'(G) = 3$. The proof is complete. □

Next, we characterize the edges of P_{12}, which can be fictive in a P_{12}-coloring of a graph with $k(G) \leq 4$.

Proposition 4.8. *Let G be a bridgeless cubic graph and let T be the unique triangle of P_{12}.*

(a) If G admits a P_{12}-coloring f, such that for an edge $e \notin T$ of P_{12}, we have that $f^{-1}(e) = \emptyset$, then $\chi'(G) = 3$.

(b) There exist infinitely many bridgeless cubic graphs G with $k(G) = 4$, such that G admits a P_{12}-coloring f, such that for any edge $e \in T$, we have: $f^{-1}(e) = \emptyset$.

Proof. (a): We follow the approach of the proof of Proposition 4.7, that is, we find two perfect matchings of P_{12} whose intersection is e. Assume that $e \notin T$.

If $e \notin \partial_{P_{12}}(V(T))$, then we have that there exist two perfect matchings M_1 and M_2 of P_{10}, such that $M_1 \cap M_2 = \{e\}$. Now, these two perfect matchings can be uniquely extended to perfect matchings N_1 and N_2 of P_{12}. Observe that $N_1 \cap N_2 = \{e\}$.

On the other hand, if $e \in \partial_{P_{12}}(V(T))$, then one can find a perfect matching N_1 intersecting $\partial_{P_{12}}(V(T))$ in a single edge and a perfect matching N_2 intersecting $\partial_{P_{12}}(V(T))$ in three edges, such that $N_1 \cap N_2 = \{e\}$.

(b): Start with arbitrary 3-edge-colorable cubic graph H and consider the cubic graph G obtained from H by replacing every vertex of G with $P_{10} - v$. Since $P_{10} - v$ is not 3-edge-colorable, we have that G is not 3-edge-colorable, hence $k(G) \geq 4$. Let us show that we have equality here. Consider the three edges incident to v, and let it be our colors in a 3-edge-coloring of H. Now, color the remaining copies of $P_{10} - v$ in G by edges of $P_{10} - v$, so that each edge is colored with its copy. As a result, we get a P_{12}-coloring of G. Thus, by (d) of Lemma 2.2, we have $k(G) \leq 4$. Hence $k(G) = 4$. Moreover, in the P_{12}-coloring of G the edges of T are not used. The proof is complete. □

In the final part of the paper we establish some connections among the discussed conjectures.

Theorem 4.9. *Conjecture 3.1 implies Conjecture 1.4.*

Proof. Assume that Conjecture 3.1 is true. We claim that any cubic graph G admits an S_{10}-coloring. In this proof, we will assume the following notation for the edges of S_{12}: the three bridges of S_{12} are denoted by a, b, c, the edges of the unique contractible triangle of S_{12} are denoted by x, y, z, such that x and a, y and b, z and c are opposite edges. Finally, the edges of the end-block containing a vertex incident to a have the following labels: the edges incident to a are a_1 and a_2, and the parallel edges are a_3 and a_4. Similarly, we label other edges by b_1, b_2, b_3, b_4 and c_1, c_2, c_3, c_4.

Let G be a cubic graph. If G contains a perfect matching, then it has an S_{12}-coloring. Since S_{12} has an S_{10}-coloring, we have the statement in this case. Thus, without loss of generality, we can assume that G does not contain a perfect matching.

Consider the graph G_Δ obtained from G by replacing all vertices of G with triangles. Observe that G_Δ contains a perfect matching. An example of such a matching would be $E(G)$.

Thus, there exists a smallest subset $U \subseteq V(G)$, such that if we replace all vertices of U with triangles, we will get a cubic graph H containing a perfect matching.

By Conjecture 3.1, H admits an S_{12}-coloring f. Now, we claim that all triangles of H corresponding to vertices of U are colored with triangles of S_{12}.

Assume the opposite, that is, there is a triangle T corresponding to a vertex of H, such that $f(E(T)) = \partial_{S_{12}}(v)$ for some vertex v of S_{12}. Consider the graph H' obtained from H by contracting T. Observe that the resulting graph H' still has an S_{12}-coloring, hence by (c) of Lemma 2.2 it contains a perfect matching. However, this violates the definition of the set U, since we found a smaller subset of vertices, whose replacement with triangles was leading to a cubic graph containing a perfect matching.

Now, all triangles of H corresponding to vertices of U are colored with triangles of S_{12}. Let us show that all these triangles corresponding to vertices of U are colored with the central triangle of S_{12}, that is the only contractible triangle of S_{12}.

On the opposite assumption, assume that T, one of these triangles, is colored with other triangles of S_{12}. Without loss of generality, we can assume that the edges of this triangle of S_{12} are a_1, a_2, a_3. Thus the set of edges leaving T, are colored with a and a_4. Two of them are colored with a_4, and one is colored with a.

Let M be a perfect matching of S_{12} containing the edges a and a_3. By (c) of Lemma 2.2, we have that $F = f^{-1}(M)$ is a perfect matching in H. Now, observe that $|F \cap \partial_H(V(T))| = 1$. Consider the cubic graph H'' obtained from H by contracting T. Observe that $F \setminus (F \cap E(T))$ is a perfect matching of H''. This violates the definition of the set U, since we found a smaller subset of vertices, whose replacement with triangles was leading to a cubic graph containing a perfect matching.

Thus, all triangles of H corresponding to U are colored with the edges x, y, z of the central triangle of S_{12}.

Observe that G can be obtained from H by contracting all the triangles corresponding to U. Now, using the S_{12}-coloring of H, we obtain an S_{10}-coloring of G. Contract all triangles of H corresponding to U and the central triangle of S_{12} to obtain S_{10}, and re-color the edges of H having color x with the color a, the edges of H with color y with color b and finally, the edges of H with color z with color c, respectively. Since x, y, z form an even subgraph in S_{12}, by (f) of Lemma 2.2, the edges of $f^{-1}(\{x, y, z\})$ will form an even subgraph, that is vertex-disjoint union of cycles. Hence, after the re-coloring we obtain an S_{10}-coloring of G. The proof of the theorem is complete. □

Theorem 4.10. *Conjectures 1.3 and 3.2 imply Conjecture 1.1.*

Proof. Assume that Conjectures 1.3 and 3.2 are true, and let G be a bridgeless cubic graph. Let us show that G admits a P_{10}-coloring. If $k(G) \leq 4$, then by Conjecture 3.2 it has a P_{12}-coloring. Since P_{12} admits a P_{10}-coloring, we have that G admits a P_{10}-coloring. Thus, without loss of generality, we can assume that $k(G) \geq 5$.

Consider the graph H obtained from G by replacing all vertices of G with triangles. Observe that H is a claw-free bridgeless cubic graph. Hence by Conjecture 1.3 and Theorem 2.5, $k(H) \leq 4$. Thus, by Conjecture 3.2, H admits a P_{12}-coloring. Since P_{12} admits a P_{10}-coloring, we have that H admits a P_{10}-coloring f. Observe that since P_{10} is triangle-free, we have that for any triangle T of H there is a vertex v of P_{10}, such that $f(E(T)) = \partial_{P_{10}}(v)$. Thus, if we contract all the triangles of H that correspond to vertices of G, we will obtain a P_{10}-coloring of G. The proof of the theorem is complete. □

The diagram from Figure 6 explains the relationship among the main four conjectures discussed in the paper. The first arrow shows that P_{12}-conjecture implies P_{10}-conjecture if 5-CDC is true (Conjecture 1.3). The second arrow shows that the statement "P_{10}-conjecture implies S_{12}-conjecture" is the formulation of Conjecture 5.1. Finally, the third arrow shows that in Theorem 4.9 we showed that S_{12}-conjecture implies the S_{10}-conjecture.

5 Future work

In this section, we discuss some open problems and conjectures that are interesting in our point of view. In the previous section, we established a connection between Conjectures 3.2 and 1.1, and Conjectures 3.1 and 1.4. We suspect that this relationship can be extended to a linear order among these four conjectures. Related with this, we would like to offer:

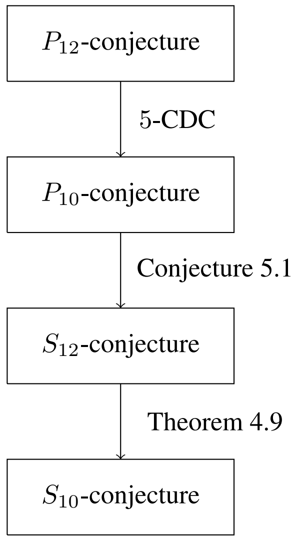

Figure 6: The relationship among the main conjectures.

Conjecture 5.1. *Conjecture 1.1 implies Conjecture 3.1.*

All coloring-hereditary classes that we discussed up to now either had or are conjectured to have a basis with one element. One may wonder whether there is a coloring-hereditary class of cubic graphs arising from an interesting graph theoretic property, such that the basis of the class contains at least two graphs. For a positive integer k let $\mathcal{C}_k$ be the class of connected cubic graphs G with $\chi'_N(G) \leq k$. Statement (i) of Lemma 2.2 implies that $\mathcal{C}_k$ is a coloring-hereditary class of cubic graphs. Recently, it was shown that for any simple cubic graph $\chi'_N(G) \leq 7$ [10]. By using this result, a simple inductive proof can be obtained for the following extension of this result:

Theorem 5.2. *Let G be a cubic graph admitting a normal k-edge-coloring for some integer k. Then $\chi'_N(G) \leq 7$.*

Theorem 5.2 suggests that $\mathcal{C}_k$ is meaningful when $k = 3, 4, 5, 6, 7$. Below, we discuss these classes for each of these values. When $k = 3$, $\mathcal{C}_k$ represents the class of connected 3-edge-colorable cubic graphs. Thus, our notation is consistent with that of Section 3. When $k = 4$, it can be easily seen that a cubic graph admits a normal 4-edge-coloring, if and only if it admits a 3-edge-coloring. Thus, $\mathcal{C}_4 = \mathcal{C}_3$. When $k = 5$, Jaeger has shown that a cubic graph admits a P_{10}-coloring if and only if it admits a normal 5-edge-coloring. On the other hand, we have that any cubic graph admitting a P_{10}-coloring, has to be bridgeless. Thus, if Conjecture 1.1 is true, then $\mathcal{C}_5 = \mathcal{C}_b$. Finally, when $k = 6$ or $k = 7$, we suspect that the bases of the classes $\mathcal{C}_6$ and $\mathcal{C}_7$ contain infinitely many cubic graphs. We are able to show that the basis of $\mathcal{C}_7$ must contain at least two graphs. Let $\mathcal{B}$ be any basis of $\mathcal{C}_7$. It can be easily seen that we can assume that it does not contain a 3-edge-colorable graph. Moreover, by a simple inductive proof, one can show that all elements of $\mathcal{B}$ can be assumed to be simple graphs. Now, let S_{16} be the graph from Figure 7. The following two results are proved in [5]:

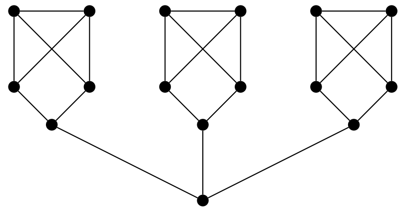

Figure 7: The graph S_{16}.

Theorem 5.3. *Let G be a simple graph with $G \prec S_{16}$. Then $G = S_{16}$.*

Theorem 5.4. *Let G be a simple graph with $G \prec P_{10}$. Then $G = P_{10}$.*

Theorems 5.3 and 5.4 suggest that the only way to color the graphs S_{16} and P_{10} with simple graphs is to take them in the basis $\mathcal{B}$. Thus, $\mathcal{B}$ must contain at least two graphs.

Finally, we would like to discuss some algorithmic problems. For a fixed connected cubic graph H consider a decision problem which we call the H-problem:

Problem 5.5 (H-problem)**.** Given a connected cubic graph G, the goal is to decide whether G admits an H-coloring.

Observe that when H is 3-edge-colorable, we have that H-problem is equivalent to testing 3-edge-colorability of the input graph G, which is NP-complete [6]. When $H = S_{10}$, we have that all instances of H-problem have a trivial "yes" answer provided that Conjecture 1.4 is true. Thus, this problem is polynomial time solvable if Conjecture 1.4 is true. When $H = S_{12}$, Conjecture 3.1 implies that H-problem is equivalent to testing the existence of a perfect matching in the input graph G. This is known to be polynomial-time solvable. When $H = P_{10}$, Conjecture 1.1 implies that H-problem is equivalent to testing bridgelessness of the input graph G. This problem is also polynomial time solvable. Finally, when $H = P_{12}$, Conjecture 3.2 implies that H-problem is equivalent to testing whether the input graph G can be covered with four perfect matchings. The latter problem is proved to be NP-complete in [3]. Thus, depending on the choice of H, the H-problem may or may not be NP-complete. Let $\mathcal{C}_{\mathrm{NP}}$ be the class of all connected cubic graphs H, for which the H-problem is NP-complete. We suspect that:

Conjecture 5.6. *$\mathcal{C}_{\mathrm{NP}}$ is a coloring-hereditary class of cubic graphs.*

We also would like to offer the following conjecture, which presents a dichotomy for H-problems:

Conjecture 5.7. *Let H be a connected cubic graph. Then:*

- *if H admits a P_{12}-coloring, then the H-problem is NP-complete;*
- *if H does not admit a P_{12}-coloring, then the H-problem is polynomial-time solvable.*

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The complement of a subspace in a classical polar space

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Abstract

In a polar space, embeddable into a projective space, we fix a subspace, that is contained in some hyperplane. The complement of that subspace resembles a slit space or a semiaffine space. We prove that under some assumptions the ambient polar space can be recovered in this complement.

Keywords: Polar space, projective space, semiaffine space, slit space, complement.

Math. Subj. Class.: 51A15, 51A45

1 Introduction

Cohen and Shult coined the term *affine polar space* in [4] as a polar space with some hyperplane removed. They prove that from such an affine reduct the ambient polar space can be recovered. In [9] we prove something similar for the complement of a subset in a projective space. Looking at the results of these two papers it is seen that an interesting case has been set aside: the complement of a subspace in a polar space. We are trying to fill this this gap here, although under several specific assumptions: we consider classical polar spaces, i.e. embeddable into projective spaces (cf. [2]), and our subspace is contained in a hyperplane.

A projective space with some subspace removed is called a slit space (cf. [5, 6, 8]) so, our complement can be seen as a generalized slit space. Singular subspaces in a polar space are projective spaces, in an affine polar space they are affine spaces (cf. [4]), while in our complement they are semiaffine or projective spaces. Adopting the terminology of [7], where the class of semiaffine spaces includes affine spaces, projective spaces and everything in between, we could say that singular subspaces of our complement are simply

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semiaffine spaces. This let us call our complement a *semiaffine polar space*. Anyway, it is clear that the complement we examine is affine in spirit. A natural parallelism is there and the subspace we remove can be viewed as the horizon.

As this paper is closely related to [4] and [9], it borrows some concepts, notations and reasonings from these two works. There are however new difficulties in this case. In contrast to [4], the horizon is not a hyperplane and thus, it induces a partial parallelism (cf. [8]). There are lines disjoint with the horizon in the ambient space and those lines, called non-affine, cannot be parallel to any line in the complement. If we had applied the definition of parallelism from [4] as it is, we would end up with non-affine lines in its equivalence classes. Therefore we use the Veblen condition to express parallelism in terms of incidence in the complement. This method unfortunately is viable only if we have at least 4 points per line in the polar space.

Roughly speaking, the points of the horizon are identified with equivalence classes of parallelism, or, in other words, with directions of lines. On the horizon of an affine polar space a deep point emerges as the point which could be reached by no line of the complement. If the removed subspace is not a hyperplane then there is no deep point but a new problem arises. Some lines on the horizon are recoverable in a standard way, as directions of planes. For the others there are no planes in the complement that would reach them. An analogy to a deep point is clear, so we call them deep lines. To overcome the problem we introduce the following relation: a line K is anti-euclidean to a line L iff there is no line intersecting K that is parallel to L. Based on this relation is a ternary collinearity of points on deep lines.

We do not know whether every subspace of a polar space is contained in a hyperplane. Any subspace can be extended to a maximal one, but does it have to be a hyperplane? If that is the case our assumptions could be weakened significantly.

2 Generalities

A point-line structure $\mathfrak{M} = \langle S, \mathcal{L} \rangle$, where the elements of S are called *points*, the elements of $\mathcal{L}$ are called *lines*, and where $\mathcal{L} \subset 2^S$, is said to be a *partial linear space*, or a *point-line space*, if two distinct lines share at most one point and every line is of size (cardinality) at least 2 (cf. [3]). A line of size 3 or more will be called *thick*. If all lines in $\mathfrak{M}$ are thick then $\mathfrak{M}$ is thick. $\mathfrak{M}$ is said to be *nondegenerate* if no point is collinear with all others, and it is called *singular* if any two of its points are collinear. It is called *Veblenian* iff for any two distinct lines L_1, L_2 through a point p and any two distinct lines K_1, K_2 not through the point p whenever each of L_1, L_2 intersects both of K_1, K_2, then K_1 intersects K_2. A *subspace* of $\mathfrak{M}$ is a subset $X \subseteq S$ that contains every line, which meets X in at least two points. A proper subspace of $\mathfrak{M}$ that shares a point with every line is said to be a *hyperplane*. If $\mathfrak{M}$ satisfies exchange axiom, then a *plane* of $\mathfrak{M}$ is a singular subspace of dimension 2. A partial linear space satisfying *one-or-all* axiom, that is

for every line L and a point $a \notin L$, a is collinear with one or all points on L,

will be called a *polar space*. The rank of a polar space is the maximal number n for which there is a chain of singular subspaces $\emptyset \neq X_1 \subset X_2 \subset \cdots \subset X_n$ ($n = -1$ if this chain is reduced to the empty set). For $a \in S$ by $a^\perp$ we denote the set of all points collinear with a, and for $X \subseteq S$ we put

$$X^\perp = \bigcap \{a^\perp : a \in X\}, \qquad \operatorname{rad} X = X \cap X^\perp.$$

As an immediate consequence of one-or-all axiom we get (cf. [4]):

Fact 2.1. *For any point $a \in S$ the set $a^\perp$ is a hyperplane of $\mathfrak{P}$.*

Following [10], a subset X of S is called

- *spiky* when every point $a \in X$ is collinear with some point $b \notin X$,
- *flappy* when for every line $L \subseteq X$ there is a point $a \notin X$ such that $L \subseteq a^\perp$.

2.1 Complement

Let $\mathfrak{M} = \langle S, \mathcal{L} \rangle$ be a thick partial linear space and let $\mathcal{W}$ be a proper subspace of $\mathfrak{M}$. By the *complement of $\mathcal{W}$ in $\mathfrak{M}$* we mean the structure

$$\mathfrak{D}_{\mathfrak{M}}(\mathcal{W}) := \langle S_{\mathcal{W}}, \mathcal{L}_{\mathcal{W}} \rangle,$$

where

$$S_{\mathcal{W}} := S \setminus \mathcal{W} \qquad \text{and} \qquad \mathcal{L}_{\mathcal{W}} := \{k \cap S_{\mathcal{W}} : k \in \mathcal{L} \wedge k \nsubseteq \mathcal{W}\}.$$

The subspace $\mathcal{W}$ will be called the *horizon of* $\mathfrak{D}_{\mathfrak{M}}(\mathcal{W})$. Note that the complement $\mathfrak{D}_{\mathfrak{M}}(\mathcal{W})$ is a partial linear space. Following a standard convention we call the points and lines of the complement $\mathfrak{D}_{\mathfrak{M}}(\mathcal{W})$ *proper*, and points and lines of $\mathcal{W}$ are said to be *improper*. Given a subspace X of $\mathfrak{D}_{\mathfrak{M}}(\mathcal{W})$ its *closure* $\overline{X}$ is a subspace of $\mathfrak{M}$ with $X \subseteq \overline{X}$. We say that two lines $K, L \in \mathcal{L}_{\mathcal{W}}$ are *parallel*, and we write

$$K \parallel_{\mathcal{W}} L \quad \text{iff} \quad \overline{K} \cap \overline{L} \cap \mathcal{W} \neq \emptyset. \tag{2.1}$$

This is always an equivalence relation. Its domain is $\mathcal{L}_{\mathcal{W}}$ only in case $\mathcal{W}$ is a hyperplane, or in other words, a line $L \in \mathcal{L}_{\mathcal{W}}$ with $L = \overline{L}$ cannot be parallel to any line. A line $L \in \mathcal{L}_{\mathcal{W}}$ with the property that $L \parallel_{\mathcal{W}} L$ will be called an *affine line*. The set of all affine lines, the domain of $\parallel_{\mathcal{W}}$, will be denoted by $\mathcal{L}^*$. For affine line L we write L^∞ for the point of $\overline{L}$ in $\mathcal{W}$, i.e. the point at infinity. A point $a \in \mathcal{W}$ is said to be a *deep point* if there is no line $L \in \mathcal{L}_{\mathcal{W}}$ such that $a = L^\infty$. A plane of $\mathfrak{D}_{\mathfrak{M}}(\mathcal{W})$ containing an affine line is said to be a *semiaffine plane*. By Π^∞ we denote the set of points at infinity of a semiaffine plane Π, i.e. $\Pi^\infty = \{M^\infty : M \in \mathcal{L}^* \text{ and } M \subseteq \Pi\}$. Note that Π^∞ is a line iff Π is an affine plane. A line $L \subseteq \mathcal{W}$ is said to be a *deep line* if there is no plane in $\mathfrak{D}_{\mathfrak{P}}(\mathcal{W})$ with $L = \Pi^\infty$.

3 Complement in a polar space

Let $\mathfrak{P} = \langle S, \mathcal{L} \rangle$ be a thick, nondegenerate polar space of rank at least 3. In the remainder of the paper we deal with $\mathfrak{D}_{\mathfrak{P}}(\mathcal{W})$, where $\mathcal{W}$ is a proper subspace contained in some hyperplane of $\mathfrak{P}$. Let us emphasize, that we do not mean one particular hyperplane and it is not fixed in our reasonings in any way. If there was a unique hyperplane H containing $\mathcal{W}$ we would be able to recover the ambient space applying Proposition 2.7 from [4], which says that every automorphism of the complement $\mathfrak{D}_{\mathfrak{P}}(H)$ can be uniquely extended to an automorphism of $\mathfrak{P}$. It is not however doable as there could be many hyperplanes containing $\mathcal{W}$ and none of them can be distinguished in terms of the complement.

In polar spaces deep points appear only on hyperplanes and there could be at most one deep point on a hyperplane.

Lemma 3.1.

(i) If $\mathcal{W}$ is a hyperplane, then there is at most one deep point on $\mathcal{W}$ and it is in $\operatorname{rad}\mathcal{W}$.

(ii) If $\mathcal{W}$ is not a hyperplane, then there are no deep points on $\mathcal{W}$, that is $\mathcal{W}$ is spiky.

Proof. (i): By Corollary 1.3 (ii) in [4].

(ii): Assume that a is a deep point in $\mathcal{W}$. Then $a^\perp \subseteq \mathcal{W}$, and by Fact 2.1 we get that $\mathcal{W}$ contains a hyperplane. A contradiction, as a hyperplane in $\mathfrak{P}$ is a maximal proper subspace (cf. [4, Lemma 1.1]). □

Lemma 3.2. *Let $\mathfrak{P}$ be an embeddable polar space and $K, L \in \mathcal{L}_\mathcal{W}$ be two distinct lines such that $K \parallel_\mathcal{W} L$. The subspace $\mathcal{W}$ can be extended to a hyperplane of $\mathfrak{P}$ not containing $\overline{K}$ and $\overline{L}$.*

Proof. If $\mathcal{W}$ is a hyperplane of $\mathfrak{P}$ then $\mathcal{W}$ itself is the required hyperplane.

Assume that $\mathcal{W}$ is not a hyperplane. Let H be a hyperplane containing $\mathcal{W}$, $\mathfrak{N}$ be a projective space embracing $\mathfrak{P}$, and f be an embedding of $\mathfrak{P}$ into $\mathfrak{N}$. Consider the projective subspace G spanned by $f(H)$. By Proposition 5.2 from [4] G is a hyperplane of $\mathfrak{N}$. If $f(\overline{K}), f(\overline{L}) \nsubseteq G$ then the hyperplane $H = f^{-1}(G \cap f(S))$ is the required one. Let $\mathcal{H}$ be the family of all hyperplanes in G containing $f(\mathcal{W})$.

Now, assume that $f(\overline{K}) \subseteq G$ and $f(\overline{L}) \nsubseteq G$. Take $a_K \in f(\overline{K}) \setminus f(\mathcal{W})$ and $a_L \in f(\overline{L}) \setminus f(\mathcal{W})$ and choose a hyperplane $G_0 \in \mathcal{H}$ with $a_K \notin G_0$. Note that $\overline{a_K, a_L}$ meets G in a_K. Take $b \in \overline{a_K, a_L}$ distinct from a_K and a_L. Assume that there is a line through b that intersects $f(\overline{L}) \setminus f(\mathcal{W})$ in some point c and meets G_0 in a point d. Note that $d \notin f(\overline{K})$ as otherwise we would have $a_K \in G_0$. Lines $\overline{a_L, d}$ and $f(\overline{K})$ are on a plane spanned by the triangle a_L, b, c. Therefore the line $\overline{a_L, d}$ intersects $f(\overline{K})$ in some point e distinct from d. Then $\overline{d, e} \subseteq G$, and thus $a_L \in G$, a contradiction. Hence, $G' = \langle G_0, b\rangle$ is a hyperplane of $\mathfrak{N}$ such that $f(\overline{L}) \nsubseteq G'$. We have also $\overline{a_K, b} \cap G_0 = \emptyset$ since $\overline{a_K, b} \nsubseteq G$. Thus $f(\overline{K}) \nsubseteq G'$. Finally, $H' := f^{-1}(G' \cap f(S))$ is the hyperplane we are looking for. The case with $f(\overline{K}) \nsubseteq G$ and $f(\overline{L}) \subseteq G$ goes the same way.

Now, let $f(\overline{K}) \subseteq G$ and $f(\overline{L}) \subseteq G$. As in the previous case we take $a_K \in f(\overline{K}) \setminus f(\mathcal{W})$, $a_L \in f(\overline{L}) \setminus f(\mathcal{W})$, but this time choose a hyperplane $G_0 \in \mathcal{H}$ with $a_K, a_L \notin G_0$. Let $b \notin G$. Note that $\overline{a_i, b} \cap G_0 = \emptyset$ for $i = K, L$. So, if we set $G' = \langle G_0, b\rangle$ then $f(\overline{K}) \nsubseteq G'$ and $f(\overline{L}) \nsubseteq G'$. Moreover, G' is a hyperplane of $\mathfrak{N}$. Again, $H' := f^{-1}(G' \cap f(S))$ is the required hyperplane. □

Lemma 3.3. *Let $K, L \in \mathcal{L}_\mathcal{W}$ be two distinct lines such that $K \parallel_\mathcal{W} L$. There is a sequence $\Pi_1, \dots, \Pi_n$ of planes in $\mathfrak{D}_\mathfrak{P}(\mathcal{W})$ such that $K^\infty = L^\infty \in \overline{\Pi_i}$ for $i = 1, \dots, n$ and $K \subseteq \Pi_1$, $L \subseteq \Pi_n$, and Π_j, Π_{j+1} share a line for $j = 1, \dots, n-1$.*

Proof. By Lemma 3.2 we can extend $\mathcal{W}$ to a hyperplane H of $\mathfrak{P}$ such that $K, L \nsubseteq H$. Take the point $a = K^\infty$. By (2.1) we have $a = L^\infty$. Now, take in $\mathfrak{P}$ the bundle of all the lines together with all the planes through a. This structure is actually the quotient space $a^\perp/a$, and it is, up to an isomorphism, a polar space (cf. [1, Lemma 2.1]), that we denote by $\mathfrak{P}'$. The set H', consisting of all the lines through a contained in H, is a hyperplane in $\mathfrak{P}'$ induced by H. Then $\mathfrak{D}_{\mathfrak{P}'}(H')$ is an affine polar space, that in itself is connected (cf. [4]). So there is in $\mathfrak{D}_{\mathfrak{P}'}(H')$ a sequence of intersecting lines joining K and L as points of $\mathfrak{D}_{\mathfrak{P}'}(H')$. However, lines of $\mathfrak{D}_{\mathfrak{P}'}(H')$ are planes of $\mathfrak{D}_\mathfrak{P}(H)$. As $\mathcal{W} \subseteq H$ these planes are also planes of $\mathfrak{D}_\mathfrak{P}(\mathcal{W})$. □

3.1 Parallelism

Let $K_1, K_2 \in \mathcal{L}_{\mathcal{W}}$. Then

$$K_1 \parallel^* K_2 \quad \text{iff} \quad \begin{array}{l} K_1 \cap K_2 = \emptyset \text{ and there are two distinct lines} \\ L_1, L_2 \in \mathcal{L}_{\mathcal{W}} \text{ crossing both of } K_1, K_2, \text{ such that} \\ L_1 \cap L_2 \neq \emptyset \text{ and } L_1 \cap L_2 \cap K_i = \emptyset \text{ for } i = 1, 2. \end{array} \tag{3.1}$$

In case there are are exactly 3 points per line in our polar space $\mathfrak{P}$, no two lines K_1, K_2 on an affine plane in $\mathfrak{D}_{\mathfrak{P}}(\mathcal{W})$ such that $K_1 \parallel_{\mathcal{W}} K_2$ satisfy the right hand side of (3.1), as the required lines L_1, L_2 had to be of size 4. This is why from now on we assume that

there are at least 4 points on every line of $\mathfrak{P}$.

Let $\parallel$ be the transitive closure of $\parallel^*$. It is clearly seen that $\parallel \subseteq \mathcal{L}^* \times \mathcal{L}^*$.

Lemma 3.4. *The relation $\parallel$ is reflexive on $\mathcal{L}^*$.*

Proof. Given a line $K_1 \in \mathcal{L}_{\mathcal{W}}$, considering that the rank of $\mathfrak{P}$ is at least 3, take a plane π containing K_1 in a maximal singular subspace through K_1. There are lines K_2, L_1, L_2 on π such that $K_1 \cap K_2 = \emptyset$ (that is $K_1^\infty = K_2^\infty$), $L_1 \neq L_2$, $L_1 \cap L_2 \neq \emptyset$, and $K_i \cap L_j \neq \emptyset$ for $i, j = 1, 2$. Thus $K_1 \parallel^* K_2$ by (3.1). This means that $K_1 \parallel K_2$ and $K_2 \parallel K_1$, which by transitivity implies that $K_1 \parallel K_1$. □

Proposition 3.5. *Let $\mathcal{W}$ be a subspace of $\mathfrak{P}$. The relation $\parallel_{\mathcal{W}}$ defined in* (2.1) *and the relation $\parallel$ coincide on the set of lines of $\mathfrak{D}_{\mathfrak{P}}(\mathcal{W})$.*

Proof. Let $K_1, K_2 \in \mathcal{L}_{\mathcal{W}}$. If $K_1 = K_2$, then $K_1 \parallel_{\mathcal{W}} K_2$ and $K_1 \parallel K_2$. So, assume that $K_1 \neq K_2$.

Consider the case where $K_1 \parallel_{\mathcal{W}} K_2$. By (2.1) it means that $\overline{K} \cap \overline{L} \cap W \neq \emptyset$, and consequently $K_1^\infty = K_2^\infty = a$ for some $a \in \mathcal{W}$. This implies that $K_1 \cap K_2 = \emptyset$. Assume that K_1 and K_2 are coplanar, and Π is the plane of $\mathfrak{D}_{\mathfrak{P}}(\mathcal{W})$ containing both of K_1, K_2. The plane $\overline{\Pi}$ is, up to an isomorphism, a projective plane, so it is Veblenian. Thus, by (3.1), $K_1 \parallel^* K_2$. If K_1 and K_2 are not coplanar, then by Lemma 3.3 there is a sequence of planes $\Pi_1, \dots, \Pi_n$ such that $K_1 \subseteq \Pi_1$, $K_2 \subseteq \Pi_n$, $a \in \overline{\Pi_i}$ for $i = 1, \dots, n$, and Π_j, Π_{j+1} share a line for $j = 1, \dots, n-1$. Let $\Pi_j \cap \Pi_{j+1} = M_j$. Note that $a \in \overline{M_1}, \dots, \overline{M_{n-1}}$ and M_j, M_{j+1} are coplanar. Therefore $M_j \parallel^* M_{j+1}$. Moreover, $K_1 \parallel^* M_1$ and $M_{n-1} \parallel^* K_2$ by the same reasons. So finally we get $K_1 \parallel K_2$.

Now, assume that $K_1 \parallel^* K_2$. Then K_1, K_2 are disjoint and coplanar. Thus $\overline{K_1}, \overline{K_2}$ meet in the closure of some plane, this means that they meet in $\mathcal{W}$. By (2.1) it gives $K_1 \parallel_{\mathcal{W}} K_2$. If $K_1 \parallel K_2$ then there is a sequence of proper lines $L_1, \dots, L_n$ such that $K_1 \parallel^* L_1 \parallel^* \cdots \parallel^* L_n \parallel^* K_2$. So, from the previous reasoning we get $K_1 \parallel_{\mathcal{W}} L_1 \parallel_{\mathcal{W}} \cdots \parallel_{\mathcal{W}} L_n \parallel_{\mathcal{W}} K_2$. As the relation $\parallel_{\mathcal{W}}$ is transitive we have $K_1 \parallel_{\mathcal{W}} K_2$. □

As an immediate consequence of Proposition 3.5 we get

Corollary 3.6. *Affine lines can be distinguished in the set $\mathcal{L}_{\mathcal{W}}$ as those parallel to themselves.*

3.2 Recovering

If $\mathcal{W}$ is a hyperplane it follows by [4, Proposition 2.7] that:

Proposition 3.7. *Let $\mathfrak{P}$ be a thick nondegenerate polar space of rank at least 2 and let H be its hyperplane. The polar space $\mathfrak{P}$ can be recovered in the complement $\mathfrak{D}_{\mathfrak{P}}(H)$.*

So, from now on we additionally assume that $\mathcal{W}$ is not a hyperplane. By Proposition 3.5 the relation $\|_{\mathcal{W}}$ can be expressed purely in terms of $\mathfrak{D}_{\mathfrak{P}}(\mathcal{W})$. Recall that our parallelism is partial: it is defined only on affine lines. However it is not a problem in view of Corollary 3.6. From Lemma 3.1(ii) there is a bijection between the sets $\mathcal{W} = \{L^\infty : L \in \mathcal{L}^*\}$ and $\{[L]_\| : L \in \mathcal{L}^*\}$. Thus we can recover $\mathcal{W}$ pointwise in a standard way:

points of the horizon $\mathcal{W}$ are identified with equivalence classes of parallelism i.e. directions of affine lines of the complement $\mathfrak{D}_{\mathfrak{P}}(\mathcal{W})$.

Let us introduce a relation $\sim\ \subseteq \mathcal{L}^* \times \mathcal{L}^*$ defined by the following condition:

$$K_1 \sim K_2 \quad \text{iff} \quad \text{for all } M \in \mathcal{L}^* \text{ if } M \cap K_1 \neq \emptyset \text{ then } M \nparallel K_2. \tag{3.2}$$

In the sense of Euclid's Fifth Postulate it could be read as *anti-euclidean parallelism.* A lot more useful for us is its derivative $\equiv\ \subseteq \mathcal{L}^*/_\| \times \mathcal{L}^*/_\|$ defined as follows:

$$[K_1]_\| \equiv [K_2]_\| \quad \text{iff} \quad \text{for all } M \in [K_1]_\|, N \in [K_2]_\| :\ M \sim N \text{ and } N \sim M. \tag{3.3}$$

Lemma 3.8. *Let M, N be two nonparallel affine lines. The following conditions are equivalent:*

(i) $[M]_\| \equiv [N]_\|$,

(ii) there is a deep line $L \subseteq \mathcal{W}$, such that $M^\infty, N^\infty \in L$.

Proof. (i) $\Rightarrow$ (ii): From one-or-all axiom, M^∞ must be collinear with at least one point of the line $\overline{N}$. Moreover, M^∞ cannot be collinear with a proper point of $\overline{N}$, as $[M]_\| \equiv [N]_\|$. Thus M^∞ is collinear with the unique improper point of $\overline{N}$, which is N^∞.

Let L be the line through M^∞, N^∞. Assume, that Π is a semiaffine plane with $L = \Pi^\infty$. Then, there are some affine lines $M_1, N_1 \subseteq \Pi$ with $M^\infty = M_1^\infty$ and $N^\infty = N_1^\infty$. So, either $M_1 \parallel N_1$ or M_1 and N_1 share a proper point. In view of (3.3), in both cases we get $[M]_\| \not\equiv [N]_\|$.

(ii) $\Rightarrow$ (i): Assume that $[M]_\| \not\equiv [N]_\|$. Due to (3.2) and (3.3) there is a proper point $a \in M$ and an affine line K such that $a \in K \parallel N$ (or the symmetrical case holds). This means that a and N^∞ are collinear in $\mathfrak{P}$. The one-or-all axiom implies, that either there are no other points on M that are collinear with N^∞, or N^∞ is collinear with all points on M. In the first case N^∞ is not collinear with M^∞, in the latter $\langle N^\infty, M\rangle \nsubseteq \mathcal{W}$ is the plane containing the line $\overline{M^\infty, N^\infty}$. □

One can note, that the relation $\equiv$ defined by (3.3) and the relation $\equiv$ introduced in [4] coincide, though their definitions are expressed differently. Besides, our relation is not transitive, but the reflexive closure of its analogue in [4] is an equivalence relation. This benefit is the result of some hyperplane properties (see Lemma 3.1(i)). Nevertheless, we can overcome this inconvenience and define ternary relation of collinearity on the horizon $\mathcal{W}$.

Lemma 3.9. *If K_1, K_2, K_3 are pairwise nonparallel affine lines such that $[K_i]_{\parallel} \equiv [K_{(i+1) \bmod 3}]_{\parallel}$ for $i = 1, 2, 3$, then points K_1^{∞}, K_2^{∞}, K_3^{∞} are on a line.*

Proof. Let $a = K_1^{\infty}, b = K_2^{\infty}, c = K_3^{\infty}$. By Lemma 3.8 there are improper lines $L = \overline{a, b}$, $M = \overline{b, c}$, $N = \overline{c, a}$. Let H be a hyperplane containing $\mathcal{W}$. If in $\mathfrak{D}_{\mathfrak{P}}(H)$ there is a plane, which closure contains one of the lines L, M or N, then we also have such a plane in $\mathfrak{D}_{\mathfrak{P}}(\mathcal{W})$, that contradicts Lemma 3.8. Thus, $L, M, N \subseteq H$ are deep lines in relation to $\mathfrak{D}_{\mathfrak{P}}(H)$. By Lemma 2.3 of [4] this means that each of L, M and N contains a point of $\operatorname{rad} H$. Let $d \in \operatorname{rad} H$. Then, by Corollary 1.3 of [4], $H = d^{\perp}$, $\{d\} = \operatorname{rad} H$, and d is the unique deep point of H. As we have $d \in L, M, N$, it must be $L = M = N$. □

Lemma 3.10. *Let K_1, K_2, K_3 be pairwise nonparallel affine lines. Points K_1^{∞}, K_2^{∞}, K_3^{∞} are on a line iff one of the following holds:*

(i) *there are affine lines $M_1 \parallel K_1$, $M_2 \parallel K_2$, $M_3 \parallel K_3$ such that M_1, M_2, M_3 form a triangle in $\mathfrak{D}_{\mathfrak{P}}(\mathcal{W})$,*

(ii) $[K_1]_{\parallel} \equiv [K_2]_{\parallel}$, $[K_2]_{\parallel} \equiv [K_3]_{\parallel}$, *and* $[K_3]_{\parallel} \equiv [K_1]_{\parallel}$.

Proof. Assume that K_1^{∞}, K_2^{∞}, K_3^{∞} are on a line L. If (i) does not hold, then there is no plane Π in $\mathfrak{D}_{\mathfrak{P}}(\mathcal{W})$ with $L = \Pi^{\infty}$. This means that L is a deep line and by Lemma 3.8 we get (ii).

Now, assume that (i) is the case. Take a plane Π spanned by the triangle M_1, M_2, M_3. Then $K_1, K_2, K_3 \subseteq \Pi$ and K_1^{∞}, K_2^{∞}, K_3^{∞} are on a line Π^{∞}. If (ii) is fulfilled then K_1^{∞}, K_2^{∞}, K_3^{∞} are on a line directly by Lemma 3.9. □

The meaning of Lemma 3.10 is that we are able to recover improper lines regardless of whether $\mathcal{W}$ is flappy or not. Let $\big[[K]_{\parallel}, [L]_{\parallel}\big]_{\equiv} := \big\{[M]_{\parallel} : [M]_{\parallel} \equiv [K]_{\parallel}, [L]_{\parallel}\big\}$. Then new lines can be grouped into two sets:

$$\mathcal{L}' := \Big\{\big[[K]_{\parallel}, [L]_{\parallel}\big]_{\equiv} : [K]_{\parallel} \equiv [L]_{\parallel} \text{ and } K \nparallel L\Big\},$$
$$\mathcal{L}'' := \big\{\Pi^{\infty} : \Pi \text{ is a semiaffine plane of } \mathfrak{D}_{\mathfrak{P}}(\mathcal{W})\big\}.$$

All our efforts in this paper essentially amount to the following isomorphism

$$\mathfrak{P} \cong \big\langle S_{\mathcal{W}} \cup \mathcal{L}^{*}/_{\parallel},\ \mathcal{L}_{\mathcal{W}} \cup \mathcal{L}' \cup \mathcal{L}'',\ \big|\big\rangle.$$

A new point $[K]_{\parallel}$ is incident to a line $L \in \mathcal{L}_{\mathcal{W}}$ iff $K \parallel L$. It is incident to a line $L \in \mathcal{L}'$ iff there is $M \in \mathcal{L}_{\mathcal{W}}$ such that $\big[[K]_{\parallel}, [M]_{\parallel}\big]_{\equiv} = L$. Eventually, it is incident to a line $L \in \mathcal{L}''$ iff $K \subseteq \Pi$ and $L = \Pi^{\infty}$.

Theorem 3.11. *Let $\mathfrak{P}$ be a nondegenerate, embeddable polar space of rank at least 3, with at least 4 points per line, and $\mathcal{W}$ be its subspace, that is contained in a hyperplane. The polar space $\mathfrak{P}$ can be recovered in the complement $\mathfrak{D}_{\mathfrak{P}}(\mathcal{W})$.*

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Unicyclic graphs with the maximal value of Graovac-Pisanski index*

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Abstract

Let G be a graph and let Γ be its group of automorphisms. Graovac-Pisanski index of G is $\mathrm{GP}(G) = \frac{|V(G)|}{2|\Gamma|} \sum_{u \in V(G)} \sum_{\alpha \in \Gamma} d(u, \alpha(u))$, where $d(u, v)$ is the distance from u to v in G. One can observe that $\mathrm{GP}(G) = 0$ if G has no nontrivial automorphisms, but it is not known which graphs attain the maximum value of Graovac-Pisanski index. In this paper we show that among unicyclic graphs on n vertices the n-cycle attains the maximum value of Graovac-Pisanski index.

Keywords: Graovac-Pisanski index, modified Wiener index, unicyclic graphs.

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1 Introduction

Wiener index, the sum of distances in a graph, is an important molecular descriptor. It was introduced by Wiener in 1949, see [18], and since then many other molecular descriptor have appeared. One of them is the Graovac-Pisanski index [8], originally known as the modified Wiener index. With this index an algebraic approach for generalizing the Wiener index was presented. Namely, as the Wiener index also the Graovac-Pisanski index is based on distances but its advantage is in considering also the symmetries of a graph, and it is known that symmetries of a molecule have an influence on its properties [14].

In his pioneering paper, Wiener showed a correlation of the Wiener index of alkanes with their boiling points [18]. It turns out that the Graovac-Pisanski index combines the symmetry and topology of molecules to obtain a good correlation with some physico-chemical properties of molecules. Recently, Črepnjak et al. showed that the Graovac-Pisanski index of some hydrocarbon molecules is correlated with their melting points [6].

This index also drew attention from theoretical point of view. Researchers are interested in the difference between the Wiener and Graovac-Pisanski index. This difference was computed in [9] for some families of polyhedral graphs. The Graovac-Pisanski index of nanostructures was studied in [1, 2, 15, 16, 17] and for some classes of fullerenes and fullerene-like molecules in [3, 11, 12]. In [13] the symmetry groups and Graovac-Pisanski index of some linear polymers were computed. Upper and lower bounds for Graovac-Pisanski index were considered in [11]. In [7] and [16] Graovac-Pisanski index was further considered from computational point of view. Exact formulae for the Graovac-Pisanski index for some graph operations are presented in [4]. Recently it was proved that for any connected bipartite graph, as well as for any connected graph on even number of vertices, the Graovac-Pisanski index is an integer number [5].

Let G be a connected graph. The *Graovac-Pisanski index* of G is defined as

$$\mathrm{GP}(G) = \frac{|V(G)|}{2|\operatorname{Aut}(G)|} \sum_{u \in V(G)} \sum_{\alpha \in \operatorname{Aut}(G)} \mathrm{dist}_G(u, \alpha(u)),$$

where $\operatorname{Aut}(G)$ is the group of automorphisms of G and $\mathrm{dist}_G(u, v)$ denotes the distance from u to v in G. However, in the paper we will use a result from [5] to compute this index. To explain the method we need some additional definitions. Let G be a graph, $u \in V(G)$ and $S \subseteq V(G)$. The *distance* of u in S, $w_S(u)$, is defined as

$$w_S(u) = \sum_{v \in S} \mathrm{dist}_G(u, v).$$

The group of automorphisms of G partitions $V(G)$ into orbits. We say that $u, v \in V(G)$ belong to the same *orbit* if there is an automorphism $\alpha \in \operatorname{Aut}(G)$ such that $\alpha(u) = v$. Let $V_1, V_2, \ldots, V_t$ be all the orbits of $\operatorname{Aut}(G)$ in G. Moreover, for every i, $1 \leq i \leq t$, let $v_i \in V_i$. That is, v_i's are the representatives of V_i's. It was shown in [5] that

$$\mathrm{GP}(G) = \frac{|V(G)|}{2} \sum_{i=1}^{t} w_{V_i}(v_i). \tag{1.1}$$

By (1.1), if a graph has no nontrivial automorphisms, that is if all its orbits consist of single vertices, then its Graovac-Pisanski index is 0. Hence, all graphs with no nontrivial automorphisms achieve the minimum value of Graovac-Pisanski index. More interesting is the opposite problem.

Problem 1.1. Find all graphs on n vertices with the maximum value of Graovac-Pisanski index.

This problem was solved for trees in [10]. By a long H on n vertices we denote a tree obtained from a path on $n-4$ vertices by attaching two pendent vertices to each endvertex of the path.

Theorem 1.2. *Let T be a tree on $n \geq 8$ vertices with the maximum value of Graovac-Pisanski index. Then T is either a path or a long H. Moreover,*

$$\mathrm{GP}(T) = \begin{cases} \frac{n^3-n}{8} & \text{if } n \text{ is odd,} \\ \frac{n^3}{8} & \text{if } n \text{ is even.} \end{cases}$$

For $n \leq 7$ there are also three other trees with the maximum value of Graovac-Pisanski index. However, they have the value of Graovac-Pisanski index as stated in Theorem 1.2.

In this paper we prove the following statement.

Theorem 1.3. *Let G be a unicyclic graph on n vertices with the maximum value of Graovac-Pisanski index. Then G is the n-cycle and*

$$\mathrm{GP}(C_n) = \begin{cases} \frac{n^3-n}{8} & \text{if } n \text{ is odd,} \\ \frac{n^3}{8} & \text{if } n \text{ is even.} \end{cases}$$

Observe that Graovac-Pisanski index for extremal trees and for extremal unicyclic graphs has the same value. We believe the following holds.

Conjecture 1.4. *Let G be a graph on n vertices, $n \geq 8$, with the maximum value of Graovac-Pisanski index. Then G is either a path, or a long H, or a cycle.*

To support this conjecture we performed some computer experiments. They showed the validity of the conjecture for $n = 8$ and $n = 9$. We believe that the maximal degree of extremal graphs is small (at most 3), thus for the cases $n = 10$ and $n = 11$ we limited our computer search to maximal degrees 5 and 4, respectively, and in these cases the conjecture was confirmed as well. The graphs from the conjecture are extremal also for $n \in \{5, 6, 7\}$, however when n equals 7 there exists an additional extremal graph.

2 Proof

In this section we prove several claims which imply Theorem 1.3. Obviously, if we consider graphs of order n, we do not need to consider the multiplicative term $\frac{n}{2}$ in (1.1). Therefore we define

$$\mathrm{GPa}(G) = \sum_{i=1}^{t} w_{V_i}(v_i), \tag{2.1}$$

where $V_1, V_2, \ldots, V_t$ are all the orbits of $\mathrm{Aut}(G)$ in G and $v_1, v_2, \ldots, v_t$ are their representatives, respectively. Then for given n, graphs on n vertices with the maximum value of GPa are the solutions of Problem 1.1.

For a cycle on n vertices, $\mathrm{GPa}(C_n) = w_V(v)$ where v is an arbitrary vertex of C_n and $V = V(C_n)$. This implies the following statement.

Proposition 2.1. *We have*

$$\mathrm{GPa}(C_n) = \begin{cases} \frac{n^2-1}{4} & \textit{if } n \textit{ is odd,} \\ \frac{n^2}{4} & \textit{if } n \textit{ is even.} \end{cases}$$

In what follows we generalize the GPa-parameter. Let $\mathcal{Z} = \{Z_1, \dots, Z_{t_\mathcal{Z}}\}$ be a partition of $V(G)$ and let $z_i \in Z_i$, $1 \leq i \leq t_\mathcal{Z}$. Then

$$\mathrm{GPa}^\mathcal{Z}(G) = \sum_{i=1}^{t_\mathcal{Z}} w_{Z_i}(z_i).$$

In our proofs, sets Z_i will usually be unions of orbits of $\mathrm{Aut}(G)$. Nevertheless, $\mathrm{GPa}^\mathcal{Z}(G)$ will depend on the choice of the representatives z_i.

To prove Theorem 1.3 we start with $\mathrm{GPa}(G)$, where G is an extremal unicyclic graph on n vertices different from the n-cycle. Then in a sequence of steps we modify either the graph or the partition and in each step we obtain a larger value of $\mathrm{GPa}^\mathcal{Z}$. Since we terminate this process with C_n and GPa, we get the result.

Hence, let G be a unicyclic graph on n vertices with the maximum value of Graovac-Pisanski index and such that G is not the n-cycle. Then G consists of a single cycle C and trees rooted at the vertices of the cycle. In what follows, orbits of vertices of C will be important.

We start with modifying the partition by merging together some orbits of vertices which have the same distance from C. We denote by $\mathcal{X}$ the new partition of $V(G)$, while the original partition into orbits is denoted by $\mathcal{V}$. Let v be a vertex of C. If $\{v\}$ is a trivial orbit of $\mathrm{Aut}(G)$, then orbits in the v-rooted tree form sets of the partition $\mathcal{X}$. But if $\{v\}$ is not a trivial orbit of $\mathrm{Aut}(G)$, we do the following. Let O_v be the orbit of $\mathrm{Aut}(G)$ containig v and let $O_v(G)$ be the set of vertices of u-rooted trees where $u \in O_v$. We partition the vertices of $O_v(G)$ according to their distance from C. Hence, O_v alone is one set of $\mathcal{X}$, another set of $\mathcal{X}$ contains those vertices of $O_v(G)$ which are adjacent to a vertex of C, etc. We have the following statement.

Lemma 2.2. *For arbitrary choice of the representatives of sets in $\mathcal{X}$ we have*

$$\mathrm{GPa}(G) \leq \mathrm{GPa}^\mathcal{X}(G).$$

Proof. Let X_i be a set from $\mathcal{X}$ and let x_i be an abitrary vertex of X_i. Observe that X_i is a union of several orbits of $\mathrm{Aut}(G)$. Let V_0 be an orbit of $\mathrm{Aut}(G)$ such that $V_0 \subseteq X_i$. Then $w_{V_0}(u)$ is the same for every $u \in V_0$. So let v_0 be a vertex of V_0 at the shortest distance from x_i. Then both x_i and v_0 are in the same tree rooted at a vertex of C. Assume that they are in a v-rooted tree T.

Let u be a vertex of V_0. If u is not in T then $\mathrm{dist}_G(x_i, u) = \mathrm{dist}_G(v_0, u)$ since $\mathrm{dist}_G(x_i, v) = \mathrm{dist}_G(v_0, v)$. So let u be a vertex in T. Let z be a vertex on the (unique) (v_0, u)-path at the shortest distance from v. Since v_0 is a vertex of V_0 at the shortest distance from x_i, the shortest (x_i, u)-path must contain z. Thus $\mathrm{dist}_G(v_0, u) \leq \mathrm{dist}_G(x_i, u)$ and so $w_{V_0}(v_0) \leq w_{V_0}(x_i)$. Consequently, $\mathrm{GPa}(G) \leq \mathrm{GPa}^\mathcal{X}(G)$ as required. □

Now we modify the graph G, and we consider a partition $\mathcal{Y}$ of the vertex set of the modified graph inherited from the partition $\mathcal{X}$ of G. So let v be a vertex of C. If $\{v\}$ is a trivial orbit of $\mathrm{Aut}(G)$ then we do not change the v-rooted tree, and its orbits form sets of the partition $\mathcal{Y}$. Hence, in this case the sets of $\mathcal{Y}$ coincide with the sets of $\mathcal{X}$ (and also with the orbits of $\mathcal{V}$). But if $\{v\}$ is not a trivial orbit of $\mathrm{Aut}(G)$ then we change the v-rooted tree. If the v-rooted tree has p vertices in G then we replace it by a path on p vertices rooted at the endvertex, which we again denote by v. Denote by F the graph which results when all these replacements are made. Since we did not change the cycle, we denote the cycle of F again by C. Let O_v be the orbit of $\mathrm{Aut}(G)$ containing v. By our assumption $|O_v| \geq 2$. Analogously as above, let $O_v(F)$ be the set of vertices of u-rooted trees where $u \in O_v$. Partition $O_v(F)$ into p disjoint sets of $\mathcal{Y}$ according to their distance from C.

Observe that for every $Y_i \in \mathcal{Y}$ and for every two vertices $y_i^1, y_i^2 \in Y_i$ we have $w_{Y_i}(y_i^1) = w_{Y_i}(y_i^2)$. Hence when computing $\mathrm{GPa}^{\mathcal{Y}}(F)$, we can choose the representatives y_i in Y_i arbitrarily. However, orbits of F may be strictly larger than the sets Y_i. This is caused by the fact that two non-isomorphic rooted trees may have the same numbers of vertices. Our next statement follows.

Lemma 2.3. *For arbitrary choice of representatives of sets in $\mathcal{Y}$ we have*

$$\mathrm{GPa}^{\mathcal{X}}(G) \leq \mathrm{GPa}^{\mathcal{Y}}(F).$$

Proof. Let H be a graph. A ray in H is a subgraph of H which is isomorphic to a path, its first vertex has degree at least 3 in H, its last vertex has degree 1 in H and all the other vertices have degree 2 in H.

We do not prove the inequality directly. Instead, we construct a sequence of graphs $G = G^0, G^1, \ldots, G^q = F$, each G^i with a partition $\mathcal{X}^i$, such that

$$\mathrm{GPa}^{\mathcal{X}^i}(G^i) \leq \mathrm{GPa}^{\mathcal{X}^{i+i}}(G^{i+1})$$

for a special choice of representatives in $\mathcal{X}^{i+1}$, where $0 \leq i \leq q-1$, $\mathcal{X}^0 = \mathcal{X}$ and $\mathcal{X}^q = \mathcal{Y}$. We remark that for every i, G^i will be a unicyclic graph with the cycle C such that if $O = \{v_1, \ldots, v_t\}$ is an orbit of vertices of C in G, then all v_j-rooted trees in G^i are mutually isomorphic, $1 \leq j \leq t$. If $t = 1$ then the v_1-rooted trees in $G, G^1, \ldots, F$ are mutually isomorphic and all $\mathcal{X}, \mathcal{X}^1, \ldots, \mathcal{Y}$ coincide on the vertex sets of these trees. However if $t \geq 2$, then the vertex set $O_{v_1}(G^i)$ of the v_j-rooted trees, $1 \leq j \leq t$, is partitioned in $\mathcal{X}^i$ according to the distance from C, and we assume that all the representatives of these sets are in the v_1-rooted tree. This assupmtion is possible since O is an orbit in G, and although O does not need to be an orbit of G^i, the vertices of O are nicely distributed along the cycle C in G^i.

So consider i, $0 \leq i < q$. We assume that G^i is already known and we construct G^{i+1}. For this, let $O = \{v_1, \ldots, v_t\}$ be an orbit of vertices of C in G, where $t \geq 2$. If the v_1-rooted tree (and so also v_j-rooted trees for $2 \leq j \leq t$) is a path rooted at the endvertex, then we are done with this orbit of G. So suppose that the v_1-rooted tree has at least two endvertices different from v_1, and consequently, at least two distinct rays starting at a common vertex. Let R_1 and R_2 be two rays starting at a vertex c such that $\mathrm{dist}_{G^i}(v_1, c)$ is maximum possible. We assume that R_1 is not shorter than R_2. If there is a representative x_j^i of X_j^i which is in R_2, then replace it by a vertex of X_j^i in R_1. Observe that this replacement does not change $\mathrm{GPa}^{\mathcal{X}^i}(G^i)$. Now delete R_2 from the v_1-rooted tree and attach it to the second vertex of R_1. Moreover, repeat the same procedure in all the other v_j-rooted trees, $2 \leq j \leq t$, and denote by G^{i+1} the resulting graph. Denote by T^i and T^{i+1} the v_1-rooted tree in G^i and G^{i+1}, respectively. If R_1 and R_2 have the same length, then this operation may create a new set in $\mathcal{X}^{i+1}$, because T^i may have smaller depth than T^{i+1}. (As is the custom, by depth we denote the largest distance from the root.) In such a case choose a representative of this new set in R_2. This is the unique case when a representative will be in R_2 in T^{i+1}.

Let $d = \operatorname{dist}_{G^i}(v_1, c)$ and let ℓ be the length of R_2. Assume that the indices of sets in $\mathcal{X}^i$ and $\mathcal{X}^{i+1}$ are chosen so that X_j^{i+1} in G^{i+1} was obtained from X_j^i in G^i and the representatives of X_j^i and X_j^{i+1} coincide whenever possible. Then $w_{X_j^i}(x_j^i)$ in G^i may differ from $w_{X_j^{i+1}}(x_j^{i+1})$ in G^{i+1} only if X_j^{i+1} contains vertices of v_k-rooted trees, $1 \leq k \leq t$, which are at distance $d+1, d+2, \dots, d+\ell+1$ from C. We distinguish three cases.

Case 1: X_j^{i+1} *contains vertices at distance* $d+1$ *from* C*.* Then in the v_1-rooted tree, X_j^{i+1} is smaller than X_j^i by exactly one vertex. Consequently $|X_j^i| - |X_j^{i+1}| = t$. Comparing to $\mathrm{GPa}^{\mathcal{X}^i}(G^i)$, $\mathrm{GPa}^{\mathcal{X}^{i+1}}(G^{i+1})$ is decreased by 2 due to a missing vertex in T^{i+1} and it is decreased by $(t-1)2(d+1)+c$ due to missing vertices in v_k-rooted trees $2 \leq k \leq t$. Here c represents the distances using the edges of C, that is $c = \sum_{k=2}^{t} \operatorname{dist}_G(v_1, v_k)$. Hence,

$$w_{X_j^{i+1}}(x_j^{i+1}) - w_{X_j^i}(x_j^i) = -2 - (t-1)2(d+1) - c. \tag{2.2}$$

Case 2: X_j^{i+1} *contains vertices at distance* $d+a$ *from* C*, where* $2 \leq a \leq \ell$*.* Then $|X_j^{i+1}| = |X_j^i|$ and comparing to $\mathrm{GPa}^{\mathcal{X}^i}(G^i)$, $\mathrm{GPa}^{\mathcal{X}^{i+1}}(G^{i+1})$, is decreased by 2 due to a shorter distance to a vertex of X_j^{i+1} in R_2. Hence,

$$w_{X_j^{i+1}}(x_j^{i+1}) - w_{X_j^i}(x_j^i) = -2. \tag{2.3}$$

Case 3: X_j^{i+1} *contains vertices at distance* $d+\ell+1$ *from* C*.* Then in the v_1-rooted tree, X_j^{i+1} is larger than X_j^i by exactly one vertex. Consequently $|X_j^{i+1}| - |X_j^i| = t$. Comparing to $\mathrm{GPa}^{\mathcal{X}^i}(G^i)$, $\mathrm{GPa}^{\mathcal{X}^{i+1}}(G^{i+1})$ is increased by $(t-1)2(d+\ell+1)+c$ due to new vertices in v_k-rooted trees, $2 \leq k \leq t$. Here c is the very same constant as in Case 1, that is $c = \sum_{k=2}^{t} \operatorname{dist}_G(v_1, v_k)$. In some cases, namely if X_j^i is not empty, $\mathrm{GPa}^{\mathcal{X}^i}$ is increased by at least 2 due to a new vertex in T^{i+1}, but we do not need to consider this contribution in our calculations. Hence,

$$w_{X_j^{i+1}}(x_j^{i+1}) - w_{X_j^i}(x_j^i) \geq (t-1)2(d+\ell+1) + c. \tag{2.4}$$

Since $w_{X_j^{i+1}}(x_j^{i+1}) = w_{X_j^i}(x_j^i)$ when $X_j^i \nsubseteq O_{v_i}(G^i)$, summing the expressions (2.2), (2.3) and (2.4) we get

$$\begin{aligned} \mathrm{GPa}^{\mathcal{X}^{i+1}}(G^{i+1}) - \mathrm{GPa}^{\mathcal{X}^i}(G^i) &\geq \big(-2 - (t-1)2(d+1) - c\big) - (\ell-1)2 \\ &\quad + \big((t-1)2(d+\ell+1) + c\big) \\ &= (t-2)2\ell \geq 0 \end{aligned}$$

since $t \geq 2$. □

Let $Y_i \in \mathcal{Y}$. Observe that if $Y_i \cap V(C) \neq \emptyset$, then $Y_i \subseteq V(C)$. Let $\mathcal{Y}'$ be those sets of $\mathcal{Y}$ which contain vertices of $V(C)$. We define a new partition $\mathcal{Z}$ of F as follows:

$$\mathcal{Z} = \mathcal{Y} \setminus \mathcal{Y}' \cup V(C).$$

That is, we merge together all sets Y_i containing vertices of $V(C)$. All the other sets of $\mathcal{Z}$ coincide with the sets of $\mathcal{Y}$. We have the following statement.

Lemma 2.4. *For arbitrary choice of representatives of sets in $\mathcal{Z}$ we have*

$$\mathrm{GPa}^{\mathcal{Y}}(F) \leq \mathrm{GPa}^{\mathcal{Z}}(F).$$

Proof. Observe that there are three types of sets in $\mathcal{Z}$. First, if $v_1 \in V(C)$ is a trivial orbit in G, then orbits of vertices of the v_1-rooted tree are sets of $\mathcal{Z}$. Second, if $\{v_1, \dots, v_t\} \subseteq V(C)$ is a non-trivial orbit in G, that is if $t \geq 2$, then the v_i-rooted trees are paths with endvertices v_i, and sets of vertices of $O_{v_1}(F)$ in $\mathcal{Z}$ contain vertices of these t rooted trees which are at the same distance from C. Finally, $\mathcal{Z}$ contains $V(C)$. If Z_i is a set of $\mathcal{Z}$ of the first type or of the second type and $u, v \in Z_i$, then $w_{Z_i}(u) = w_{Z_i}(v)$. Hence, to prove the statement it suffices to show that

$$\sum_{Y_i \in \mathcal{Y}'} w_{Y_i}(y_i) \leq w_{V(C)}(z) = \sum_{Y_i \in \mathcal{Y}'} w_{Y_i}(z)$$

where z is an arbitrary vertex of $V(C)$ and $\mathcal{Y}'$ is defined before Lemma 2.4. (Recall that y_i is a representative of Y_i in $\mathcal{Y}$.)

Thus, let $z \in V(C)$ and let $Y_i \in \mathcal{Y}'$. In what follows we show that $w_{Y_i}(y_i) \leq w_{Y_i}(z)$. We distinguish four cases.

Case 1: $|Y_i| = 1$. Since $w_{Y_i}(y_i) = 0 \leq \mathrm{dist}_F(z, y_i) = w_{Y_i}(z)$, we have $w_{Y_i}(y_i) \leq w_{Y_i}(z)$.

Case 2: $|Y_i| = 2$. let $Y_i = \{y_i, y\}$. Then by triangle inequality

$$w_{Y_i}(y_i) = \mathrm{dist}_F(y_i, y) \leq \mathrm{dist}_F(z, y_i) + \mathrm{dist}_F(z, y) = w_{Y_i}(z).$$

Hence, in the sequel we assume that $|Y_i| \geq 3$. Since Y_i is an orbit of vertices of C in G, there is a nontrivial rotational automorphism α in $\mathrm{Aut}(G)$ such that $\{\alpha^k(y_i) \mid k \in \mathbb{N}\} \subseteq Y_i$. Let r be the biggest order of a rotational automorphism of this type and let α be the corresponding automorphism. Observe that $r \geq 2$. Since $w_{Y_i}(u) = w_{Y_i}(v)$ for $u, v \in Y_i$, we assume that y_i is chosen so that $\mathrm{dist}_F(z, y_i)$ is smallest possible.

Case 3: *r is even.* Let $Y_i' = \{\alpha^k(y_i) \mid 0 \leq k < r\}$. We rename vertices of Y_i' as $\{y^0, y^1, \dots, y^{r-1}\}$ so that $\mathrm{dist}_F(y_i, y^k) \leq \mathrm{dist}_F(y_i, y^{k+1})$ whenever $0 \leq k < r-1$. Observe that $y^0 = y_i$, the vertices $y^{2\ell-1}$ and $y^{2\ell}$ have the same distance from y_i if $1 \leq \ell < r/2$ and y^{r-1} is the unique vertex of Y_i' with the largest distance from y_i. Since y_i is the vertex of Y_i with the smallest distance from z, we have

$$\mathrm{dist}_F(y^{2\ell-1}, y_i) + \mathrm{dist}_F(y_i, y^{2\ell}) = \mathrm{dist}_F(y^{2\ell-1}, z) + \mathrm{dist}_F(z, y^{2\ell})$$

for $1 \leq \ell < r/2$ and also

$$\mathrm{dist}_F(y_i, y^{r-1}) = \mathrm{dist}_F(y_i, z) + \mathrm{dist}_F(z, y^{r-1}) = \tfrac{1}{2}|V(C)|.$$

Hence, $w_{Y_i'}(y_i) = w_{Y_i'}(z)$. If $Y_i = Y_i'$, we are done. Therefore, in the sequel assume that there is also a reflexion β such that $\beta(Y_i') \subseteq Y_i$ and $\beta(Y_i') \cap Y_i' = \emptyset$. Then $Y_i = Y_i' \cup \beta(Y_i')$ and $|Y_i| = 2r$. Observe that all the vertices of $\beta(Y_i')$ are obtained from arbitrary one of them using α. Thus, let y_i^β be a vertex of $\beta(Y_i')$ with the smallest distance from y_i. Then using the same arguments as above we get

$$w_{\beta(Y_i')}(y_i^\beta) = w_{\beta(Y_i')}(y_i).$$

Since $w_{\beta(Y_i')}(y_i^\beta) = w_{Y_i'}(y_i)$, we get

$$w_{Y_i}(y_i) = 2w_{Y_i'}(y_i).$$

Analogously, if $y_i^{\beta z}$ is a vertex of $\beta(Y_i')$ at the smallest distance from z, we get

$$w_{\beta(Y_i')}(y_i^{\beta z}) = w_{\beta(Y_i')}(z).$$

Since $w_{\beta(Y_i')}(y_i^{\beta z}) = w_{Y_i'}(y_i)$, we obtain $w_{Y_i}(y_i) = w_{Y_i}(z)$.

Case 4: *r is odd.* Let $Y_i' = \{\alpha^k(y_i) \mid 0 \le k < r\}$. Then proceeding analogously as in Case 3 one gets

$$w_{Y_i'}(z) = w_{Y_i'}(y_i) + \operatorname{dist}_F(y_i, z),$$

and so $w_{Y_i}(y_i) \le w_{Y_i}(z)$ if $Y_i = Y_i'$. Hence, assume that there is a reflexion β such that $\beta(Y_i') \subseteq Y_i$ and $\beta(Y_i') \cap Y_i' = \emptyset$. Again, $Y_i = Y_i' \cup \beta(Y_i')$ and $|Y_i| = 2r$, and all the vertices of $\beta(Y_i')$ are obtained from arbitrary one of them using α. Thus, let y_i^β be a vertex of $\beta(Y_i')$ with the shortest distance from y_i. Then analogously as in Case 3 one gets

$$w_{Y_i'}(y_i) = w_{\beta(Y_i')}(y_i^\beta) = w_{\beta(Y_i')}(y_i) - \operatorname{dist}_F(y_i^\beta, y_i)$$

and so

$$w_{Y_i}(y_i) = 2w_{Y_i'}(y_i) + \operatorname{dist}_F(y_i^\beta, y_i).$$

Also, let $y_i^{\beta z}$ be a vertex of $\beta(Y_i')$ with the shortest distance from z. Then analogously as above we get

$$w_{Y_i'}(y_i) = w_{\beta(Y_i')}(y_i^{\beta z}) = w_{\beta(Y_i')}(z) - \operatorname{dist}_F(y_i^{\beta z}, z)$$

and so

$$w_{Y_i}(z) = 2w_{Y_i'}(y_i) + \operatorname{dist}_F(y_i^{\beta z}, z) + \operatorname{dist}_F(y_i, z).$$

Since

$$\operatorname{dist}_F(y_i^\beta, y_i) \le \operatorname{dist}_F(y_i^{\beta z}, y_i) \le \operatorname{dist}_F(y_i^{\beta z}, z) + \operatorname{dist}_F(y_i, z),$$

we get $w_{Y_i}(y_i) \le w_{Y_i}(z)$. □

Finally, we are in a position to prove the last lemma which implies Theorem 1.3. Observe that there is a strict inequality in Lemma 2.5.

Lemma 2.5. *We have*

$$\mathrm{GPa}^{\mathcal{Z}}(F) < \mathrm{GPa}(C_n).$$

Proof. Analogously as in Lemma 2.3, we prove the statement by a sequence of steps. Let $O^1, \dots, O^q$ be all orbits of vertices of C in G, such that for every $v \in O^i$ the v-rooted tree is nontrivial (i.e., it has more than one vertex). Observe that if the v-rooted tree is nontrivial in G, then the v-rooted tree in F is also nontrivial. Assume that $|O^1| \ge |O^2| \ge \cdots \ge |O^q|$.

We consecutively create unicyclic graphs $F = F^0, F^1, \ldots, F^q = C_n$ with partitions $\mathcal{Z} = \mathcal{Z}^0, \mathcal{Z}^1, \ldots, \mathcal{Z}^q$, respectively, and for every i, $0 \le i < q$, we show that $\mathrm{GPa}^{\mathcal{Z}^i}(F^i) < \mathrm{GPa}^{\mathcal{Z}^{i+1}}(F^{i+1})$. The graph F^{i+1} is obtained from F^i by moving the vertices of v-rooted trees, where $v \in O^{i+1}$, into the unique cycle C^i of F^i. Now we describe the process in detail.

Choose i, $0 \le i < q$. For $v \in O^{i+1}$, let $p+1$ be the number of vertices of v-rooted tree in F^i (or in G, since the numbers of vertices of v-rooted trees in G and F are the same). By our assumption $p \ge 1$. Orient the cycle C^i of F^i and for every vertex $u \in V(C^i)$ let u^f be the vertex following u on C^i. Let $v \in O^{i+1}$. Delete the p non-root vertices of the v-rooted tree from F^i and subdivide the edge vv^f exactly p times. Repeat this procedure for all vertices of O^{i+1} and denote by F^{i+1} the resulting unicyclic graph. The partition $\mathcal{Z}^{i+1}$ is exactly the same as $\mathcal{Z}^i$, the only exception is that instead of the set $V(C^i)$ and various sets partitioning $O_v(F^i)$ for $v \in O^{i+1}$ we have just the set $V(C^{i+1})$ in $\mathcal{Z}^{i+1}$. We assume that if a set of $\mathcal{Z}^i$ is identical with a set of $\mathcal{Z}^{i+1}$, then they have the same representatives.

Let $Z' \in \mathcal{Z}^i$ and $Z^* \in \mathcal{Z}^{i+1}$ such that $Z' = Z^*$. Then Z' is a collection of vertices of $O_u(F^i)$, i.e., of u-rooted trees for $u \in O^j$, where $j > i+1$. Since the distances between these vertices cannot be shorter in F^{i+1} than in F^i (they can be only larger due to the extension of C^i to C^{i+1}), we have $w_{Z'}(z') \le w_{Z^*}(z^*)$ where z' is a representative of Z' in F^i and z^* is a representative of Z^* in F^{i+1}. Hence, it suffices to check the contribution of $V(C^{i+1})$ in $\mathrm{GPa}^{\mathcal{Z}^{i+1}}(F^{i+1})$ and in $\mathrm{GPa}^{\mathcal{Z}^i}(F^i)$ the contribution of $V(C^i)$ and of the sets of non-root vertices of v-rooted trees for $v \in O^{i+1}$. Let $t = |O^{i+1}|$. Analogously as in the proof of Lemma 2.4 we distinguish four cases. In these cases, we set $c = |V(C^i)|$. Moreover, by δ_a we denote the parity of a. That is $\delta_a = 1$ if a is odd and $\delta_a = 0$ if a is even.

Case 1: $t = 1$. By Proposition 2.1, $V(C^i)$ contributes $\frac{1}{4}(c^2 - \delta_c)$ to $\mathrm{GPa}^{\mathcal{Z}^i}(F^i)$ and $V(C^{i+1})$ contributes $\frac{1}{4}((c+p)^2 - \delta_{c+p})$ to $\mathrm{GPa}^{\mathcal{Z}^{i+1}}(F^{i+1})$. Let $O^{i+1} = \{v_1\}$. Denote by T the v_1-rooted tree in F^i. Since T is a tree on $p+1$ vertices, the orbits of T contribute to $\mathrm{GPa}^{\mathcal{Z}^i}(F^i)$ at most $\frac{1}{4}((p+1)^2 - \delta_{p+1})$ by Theorem 1.2. So

$$\begin{aligned} 4\big(\mathrm{GPa}^{\mathcal{Z}^{i+1}}(F^{i+1}) - \mathrm{GPa}^{\mathcal{Z}^i}(F^i)\big) &\ge (c+p)^2 - \delta_{c+p} - c^2 + \delta_c - (p+1)^2 + \delta_{p+1} \\ &\ge (c+p)^2 - c^2 - (p+1)^2 - 1 \\ &= 2p(c-1) - 2 > 0 \end{aligned}$$

since $c \ge 3$ and $p \ge 1$.

Case 2: $t = 2$. In this case the v-rooted trees are paths whenever $v \in O^{i+1}$. Since the contribution to $\mathrm{GPa}^{\mathcal{Z}^i}(F^i)$ of j-th vertices of these paths (i.e., of vertices at distance j from the roots) is at most $j + c/2 + j$, the total contribution of non-root vertices of v-rooted trees, $v \in O^{i+1}$, is at most $2\binom{p+1}{2} + \frac{1}{2}cp$. Since the contribution of $V(C^i)$ is $\frac{1}{4}(c^2 - \delta_c)$ and the contribution of $V(C^{i+1})$ is $\frac{1}{4}((c+2p)^2 - \delta_{c+2p})$, where $\delta_{c+2p} = \delta_c$, we get

$$\begin{aligned} 4\big(\mathrm{GPa}^{\mathcal{Z}^{i+1}}(F^{i+1}) - \mathrm{GPa}^{\mathcal{Z}^i}(F^i)\big) &\ge (c+2p)^2 - \delta_c - c^2 + \delta_c - 8\tbinom{p+1}{2} - 2cp \\ &\ge (c+2p)^2 - c^2 - 4p^2 - 4p - 2cp \\ &= 2p(c-2) > 0 \end{aligned}$$

since $c \ge 3$ and $p \ge 1$.

In the remaining cases we may assume that there is a nontrivial rotational automorphism α of F^i such that when $v \in O^{i+1}$ then also $\alpha(v) \in O^{i+1}$. Let r be the biggest order of such a rotational automorphism α, and moreover, let α be such that the distance s between v and $\alpha(v)$ is the smallest possible. Then $c = r \cdot s$.

Case 3: *r is even.* Then $r \geq 2$. Let $v_0 \in O^{i+1}$. Denote $O' = \{\alpha^k(v_0) \mid 0 \leq k < r\}$. First assume that $|O^{i+1}| = 2r$. Hence there is also a reflexion β such that $\beta(O') \subseteq O^{i+1}$ and $\beta(O') \cap O' = \emptyset$. Let $v_1 \in O^{i+1}$ such that the distance t between v_0 and v_1 is the smallest possible. Then $t \leq s/2$ and $v_1 \in \beta(O')$. Observe that now $t \geq 1$ and $s \geq 2$. Since $c = rs$ is even, the contribution of $V(C^i)$ to $\mathrm{GPa}^{\mathcal{Z}^i}(F^i)$ is $\frac{1}{4}r^2s^2$. The contribution of $V(C^{i+1})$ to $\mathrm{GPa}^{\mathcal{Z}^{i+1}}(F^{i+1})$ is $\frac{1}{4}r^2(s+2p)^2$. Now we calculate the contribution of non-root vertices of v-rooted trees when $v \in O^{i+1}$. These trees are paths and the contribution of j-th vertices is

$$\begin{aligned}
&2(s+2j) + 2(2s+2j) + \cdots + 2\big((\tfrac{r}{2}-1)s+2j\big) + (\tfrac{r}{2}s+2j) \\
&\quad + (t+2j) + (s+t+2j) + \cdots + \big((\tfrac{r}{2}-1)s+t+2j\big) + (s-t+2j) \\
&\quad + (2s-t+2j) + \cdots + \big((\tfrac{r}{2}-1)s-t+2j\big) + (\tfrac{r}{2}s-t+2j) \\
&\qquad = 4(s+2j) + 4(2s+2j) + \cdots + 4\big((\tfrac{r}{2}-1)s+2j\big) + 2(\tfrac{r}{2}s+2j) + 2j \\
&\qquad = \tfrac{1}{2}r^2s + 2j(2r-1).
\end{aligned}$$

So the contribution of non-root vertices of v-rooted trees, $v \in O^{i+1}$, is

$$\sum_{j=1}^{p}\Big(\tfrac{1}{2}r^2s + 2j(2r-1)\Big) = \tfrac{1}{2}r^2sp + (p^2+p)(2r-1).$$

Hence

$$\begin{aligned}
\mathrm{GPa}^{\mathcal{Z}^{i+1}}(F^{i+1}) - \mathrm{GPa}^{\mathcal{Z}^i}(F^i) &\geq \tfrac{1}{4}r^2s^2 + r^2sp + r^2p^2 - \tfrac{1}{4}r^2s^2 \\
&\qquad - \tfrac{1}{2}r^2sp - 2rp^2 - 2rp + p^2 + p \\
&= p^2(r-1)^2 + p\big(r(\tfrac{1}{2}rs-2)+1\big) > 0
\end{aligned}$$

since $r \geq 2$, $s \geq 2$ and $p \geq 1$.

In the case when $|O^{i+1}| = r$, the contribution of $V(C^{i+1})$ is $\frac{1}{4}r^2(s+p)^2$ and the contribution of j-th vertices in v-rooted trees, $v \in O^{i+1}$, is

$$2(s+2j) + 2(2s+2j) + \cdots + 2\big((\tfrac{r}{2}-1)s+2j\big) + (\tfrac{r}{2}s+2j) = \tfrac{1}{4}r^2s + 2j(r-1).$$

So the contribution of non-root vertices of v-rooted trees, $v \in O^{i+1}$, is

$$\sum_{j=1}^{p}\Big(\tfrac{1}{4}r^2s + 2j(r-1)\Big) = \tfrac{1}{4}r^2sp + (p^2+p)(r-1).$$

Hence

$$\begin{aligned}
\mathrm{GPa}^{\mathcal{Z}^{i+1}}(F^{i+1}) - \mathrm{GPa}^{\mathcal{Z}^i}(F^i) &\geq \tfrac{1}{4}r^2s^2 + \tfrac{1}{2}r^2sp + \tfrac{1}{4}r^2p^2 - \tfrac{1}{4}r^2s^2 \\
&\qquad - \tfrac{1}{4}r^2sp - rp^2 - rp + p^2 + p \\
&= p^2(\tfrac{r}{2}-1)^2 + p\big(r(\tfrac{1}{4}rs-1)+1\big) > 0
\end{aligned}$$

since in this case $r \geq 4$, $s \geq 1$ and $p \geq 1$.

Case 4: *r is odd.* Then $r \geq 3$. Let $v_0 \in O^{i+1}$ and $O' = \{\alpha^k(v_0) \mid 0 \leq k < r\}$. First assume that $|O^{i+1}| = 2r$. Hence there is also a reflexion β such that $\beta(O') \subseteq O^{i+1}$ and $\beta(O') \cap O' = \emptyset$. Let $v_1 \in O^{i+1}$ such that the distance t between v_0 and v_1 is the smallest possible. Then $t \leq s/2$ and $v_1 \in \beta(O')$. The contribution of $V(C^i)$ to $\mathrm{GPa}^{\mathcal{Z}^i}(F^i)$ is $\frac{1}{4}(r^2s^2 - \delta_{rs})$. The contribution of $V(C^{i+1})$ to $\mathrm{GPa}^{\mathcal{Z}^{i+1}}(F^{i+1})$ is $\frac{1}{4}\big(r^2(s+2p)^2 - \delta_{r(s+2p)}\big)$. Now we calculate the contribution of non-root vertices of v-rooted trees when $v \in O^{i+1}$. These trees are paths and the contribution of j-th vertices is

$$\begin{aligned}
&2(s+2j) + 2(2s+2j) + \cdots + 2(\tfrac{r-1}{2}s+2j)\\
&\quad + (t+2j) + (s+t+2j) + \cdots + (\tfrac{r-1}{2}s+t+2j)\\
&\quad + (s-t+2j) + (2s-t+2j) + \cdots + (\tfrac{r-1}{2}s-t+2j)\\
&\qquad = 4(s+2j) + 4(2s+2j) + \cdots + 4(\tfrac{r-1}{2}s+2j) + (t+2j)\\
&\qquad = \tfrac{1}{2}(r^2-1)s + 2j(2r-1) + t.
\end{aligned}$$

So the contribution of non-root vertices of v-rooted trees, $v \in O^{i+1}$, is

$$\begin{aligned}
\sum_{j=1}^{p}\Big(\tfrac{1}{2}(r^2-1)s + 2j(2r-1) + t\Big) &= \tfrac{1}{2}(r^2-1)sp + (p^2+p)(2r-1) + pt\\
&\leq \tfrac{1}{2}r^2sp + (p^2+p)(2r-1)
\end{aligned}$$

since $-\frac{1}{2}sp + pt \leq 0$. Hence

$$\begin{aligned}
\mathrm{GPa}^{\mathcal{Z}^{i+1}}(F^{i+1}) - \mathrm{GPa}^{\mathcal{Z}^i}(F^i) &\geq \tfrac{1}{4}r^2s^2 + r^2sp + r^2p^2 - \tfrac{1}{4}\delta_{rs} - \tfrac{1}{4}r^2s^2 + \tfrac{1}{4}\delta_{rs}\\
&\quad - \tfrac{1}{2}r^2sp - 2rp^2 - 2rp + p^2 + p\\
&= p^2(r-1)^2 + p\big(r(\tfrac{1}{2}rs-2)+1\big) > 0
\end{aligned}$$

since $r \geq 3$, $s \geq 2$ and $p \geq 1$.

In the case when $|O^{i+1}| = r$, the contribution of $V(C^{i+1})$ to $\mathrm{GPa}^{\mathcal{Z}^{i+1}}(F^{i+1})$ is $\frac{1}{4}\big(r^2(s+p)^2 - \delta_{r(s+p)}\big)$ and the contribution of j-th vertices in v-rooted trees, $v \in O^{i+1}$, is

$$2(s+2j) + 2(2s+2j) + \cdots + 2(\tfrac{r-1}{2}s+2j) = \tfrac{1}{4}(r^2-1)s + 2j(r-1).$$

So the contribution of non-root vertices of v-rooted trees, $v \in O^{i+1}$, is

$$\sum_{j=1}^{p}\Big(\tfrac{1}{4}(r^2-1)s + 2j(r-1)\Big) = \tfrac{1}{4}(r^2-1)sp + (p^2+p)(r-1).$$

Hence

$$\begin{aligned}
\mathrm{GPa}^{\mathcal{Z}^{i+1}}(F^{i+1}) - \mathrm{GPa}^{\mathcal{Z}^i}(F^i) &\geq \tfrac{1}{4}r^2s^2 + \tfrac{1}{2}r^2sp + \tfrac{1}{4}r^2p^2 - \tfrac{1}{4} - \tfrac{1}{4}r^2s^2\\
&\quad - \tfrac{1}{4}(r^2-1)sp - rp^2 - rp + p^2 + p\\
&= p^2(\tfrac{r}{2}-1)^2 + p\big(r(\tfrac{1}{4}rs-1) + \tfrac{1}{4}s + 1\big) - \tfrac{1}{4} > 0
\end{aligned}$$

since in this case $r \geq 3$, $s \geq 1$ and $p \geq 1$. (Observe that the second bracket is at least $\frac{2}{4}$ if $r = 3$ and $s = 1$.) □

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On separable abelian p-groups

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Abstract

An S-ring (a Schur ring) is said to be *separable* with respect to a class of groups $\mathcal{K}$ if every algebraic isomorphism from the S-ring in question to an S-ring over a group from $\mathcal{K}$ is induced by a combinatorial isomorphism. A finite group is said to be *separable* with respect to $\mathcal{K}$ if every S-ring over this group is separable with respect to $\mathcal{K}$. We provide a complete classification of abelian p-groups separable with respect to the class of abelian groups.

Keywords: Isomorphisms, Schur rings, p-groups.

Math. Subj. Class.: 05E30, 05C60, 20B35

1 Introduction

Let G be a finite group. A subring of the group ring $\mathbb{Z}G$ is called an S*-ring* (a *Schur ring*) over G if it is determined in a natural way by a special partition of G (the exact definition is given in Section 2). The classes of the partition are called the *basic sets* of the S-ring. The concept of the S-ring goes back to Schur and Wielandt. They used S-rings to study a permutation group containing a regular subgroup [19, 20]. For more details on S-rings and their applications we refer the reader to [13].

Let $\mathcal{A}$ and $\mathcal{A}'$ be S-rings over groups G and G' respectively. An *algebraic isomorphism* from $\mathcal{A}$ to $\mathcal{A}'$ is a ring isomorphism inducing a bijection between the basic sets of $\mathcal{A}$ and the basic sets of $\mathcal{A}'$. Another type of an isomorphism of S-rings comes from graph theory. A *combinatorial isomorphism* from $\mathcal{A}$ to $\mathcal{A}'$ is defined to be an isomorphism of the corresponding Cayley schemes (see Subsection 2.2). Every combinatorial isomorphism induces the algebraic one. However, the converse statement is not true (the corresponding examples can be found in [6]).

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Let $\mathcal{K}$ be a class of groups. Following [3], we say that an S-ring $\mathcal{A}$ is *separable* with respect to $\mathcal{K}$ if every algebraic isomorphism from $\mathcal{A}$ to an S-ring over a group from $\mathcal{K}$ is induced by a combinatorial one. We call a finite group *separable* with respect to $\mathcal{K}$ if every S-ring over G is separable with respect to $\mathcal{K}$ (see [18]).

The importance of separable S-rings comes from the following observation. Suppose that an S-ring $\mathcal{A}$ is separable with respect to $\mathcal{K}$. Then $\mathcal{A}$ is determined up to isomorphism in the class of S-rings over groups from $\mathcal{K}$ only by the tensor of its structure constants (with respect to the basis of $\mathcal{A}$ corresponding to the partition of the underlying group).

Given a group G denote the class of groups isomorphic to G by $\mathcal{K}_G$. If G is separable with respect to $\mathcal{K}_G$ then the isomorphism of two Cayley graphs over G can be verified efficiently by using the Weisfeiler-Leman algorithm [12]. In the sense of [10] this means that the Weisfeiler-Leman dimension of the class of Cayley graphs over G is at most 3. More information concerned with separability and the graph isomorphism problem is presented in [3, 17].

Denote the classes of cyclic and abelian groups by $\mathcal{K}_C$ and $\mathcal{K}_A$ respectively. The cyclic group of order n is denoted by C_n. In the present paper we are interested in abelian groups and especially in abelian p-groups which are separable with respect to $\mathcal{K}_A$. The problem of determining of all groups separable with respect to a given class $\mathcal{K}$ seems quite complicated even for $\mathcal{K} = \mathcal{K}_C$. Examples of cyclic groups which are non-separable with respect to $\mathcal{K}_C$ were found in [6]. In [5] it was proved that cyclic p-groups are separable with respect to $\mathcal{K}_C$. We prove that a similar statement is also true for $\mathcal{K}_A$.

Theorem 1.1. *For every prime p a cyclic p-group is separable with respect to $\mathcal{K}_A$.*

The result obtained in [18] implies that an abelian group of order $4p$ is separable with respect to $\mathcal{K}_A$ for every prime p. From [9] it follows that for every group G of order at least 4 the group $G \times G$ is non-separable with respect to $\mathcal{K}_{G\times G}$. One can check that a normal subgroup of a group separable with respect to $\mathcal{K}_A$ is separable with respect to $\mathcal{K}_A$ (see also Lemma 2.5). The above discussion shows that a non-cyclic abelian p-group separable with respect to $\mathcal{K}_A$ is isomorphic to $C_p \times C_{p^k}$ or $C_p \times C_p \times C_{p^k}$, where $p \in \{2, 3\}$ and $k \geq 1$. The separability of the groups from the first family was proved in [17]. In the present paper we study the question on the separability of the groups from the second family.

Theorem 1.2. *The group $C_p \times C_p \times C_{p^k}$, where $p \in \{2, 3\}$ and $k \geq 1$, is separable with respect to $\mathcal{K}_A$ if and only if $k = 1$.*

As an immediate consequence of Theorem 1.1, Theorem 1.2, and the above mentioned results, we obtain a complete classification of abelian p-groups separable with respect to $\mathcal{K}_A$.

Theorem 1.3. *An abelian p-group is separable with respect to $\mathcal{K}_A$ if and only if it is cyclic or isomorphic to one of the following groups:*

$$C_2 \times C_{2^k}, \quad C_3 \times C_{3^k}, \quad C_2^3, \quad C_3^3,$$

where $k \geq 1$.

Throughout the paper we write for short “separable” instead of “separable with respect to $\mathcal{K}_A$”. The text is organized in the following way. Section 2 contains a background of

S-rings. Section 3 is devoted to S-rings over cyclic p-groups. We finish Section 3 with the proof of Theorem 1.1. In Section 4 we prove Theorem 1.2.

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Notation.

- The ring of rational integers is denoted by $\mathbb{Z}$.
- Let $X \subseteq G$. The element $\sum_{x \in X} x$ of the group ring $\mathbb{Z}G$ is denoted by $\underline{X}$.
- The order of $g \in G$ is denoted by $|g|$.
- The set $\{x^{-1} : x \in X\}$ is denoted by X^{-1}.
- The subgroup of G generated by X is denoted by $\langle X \rangle$; we also set $\mathrm{rad}(X) = \{g \in G : gX = Xg = X\}$.
- If $m \in \mathbb{Z}$ then the set $\{x^m : x \in X\}$ is denoted by $X^{(m)}$.
- Given a set $X \subseteq G$ the set $\{(g, xg) : x \in X, g \in G\}$ of edges of the Cayley graph $\mathrm{Cay}(G, X)$ is denoted by $R(X)$.
- The group of all permutations of a set Ω is denoted by $\mathrm{Sym}(\Omega)$.
- The subgroup of $\mathrm{Sym}(G)$ induced by right multiplications of G is denoted by G_{right}.
- For a set $\Delta \subseteq \mathrm{Sym}(G)$ and a section $S = U/L$ of G we set

$$\Delta^S = \{f^S : f \in \Delta, S^f = S\},$$

  where $S^f = S$ means that f permutes the L-cosets in U and f^S denotes the bijection of S induced by f.
- If a group K acts on a set Ω then the set of all orbtis of K on Ω is denoted by $\mathrm{Orb}(K, \Omega)$.
- If $H \leq G$ then the normalizer of H in G is denoted by $N_G(H)$.
- If $K \leq \mathrm{Sym}(\Omega)$ and $\alpha \in \Omega$ then the stabilizer of α in K is denoted by K_α.
- The cyclic group of order n is denoted by C_n.

2 S-rings

In this section we give a background of S-rings. The most of definitions and statements presented here are taken from [13, 17].

2.1 Definitions and basic facts

Let G be a finite group and $\mathbb{Z}G$ the group ring over the integers. The identity element of G is denoted by e. A subring $\mathcal{A} \subseteq \mathbb{Z}G$ is called an *S-ring* over G if there exists a partition $\mathcal{S} = \mathcal{S}(\mathcal{A})$ of G such that:

(1) $\{e\} \in \mathcal{S}$,

(2) if $X \in \mathcal{S}$ then $X^{-1} \in \mathcal{S}$,

(3) $\mathcal{A} = \mathrm{Span}_{\mathbb{Z}}\{\underline{X} : X \in \mathcal{S}\}$.

The elements of $\mathcal{S}$ are called the *basic sets* of $\mathcal{A}$ and the number $|\mathcal{S}|$ is called the *rank* of $\mathcal{A}$. Given $X, Y, Z \in \mathcal{S}$ the number of distinct representations of $z \in Z$ in the form $z = xy$ with $x \in X$ and $y \in Y$ is denoted by $c_{X,Y}^Z$. If X and Y are basic sets of $\mathcal{A}$ then $\underline{X}\,\underline{Y} = \sum_{Z \in \mathcal{S}(\mathcal{A})} c_{X,Y}^Z \underline{Z}$. So the integers $c_{X,Y}^Z$ are structure constants of $\mathcal{A}$ with respect to the basis $\{\underline{X} : X \in \mathcal{S}\}$. It is easy to verify that given basic sets X and Y the set XY is also basic whenever $|X| = 1$ or $|Y| = 1$.

A set $X \subseteq G$ is said to be an $\mathcal{A}$-*set* if $\underline{X} \in \mathcal{A}$. A subgroup $H \leq G$ is said to be an $\mathcal{A}$-*subgroup* if H is an $\mathcal{A}$-set. One can check that for every $\mathcal{A}$-set X the groups $\langle X \rangle$ and $\mathrm{rad}(X)$ are $\mathcal{A}$-subgroups.

A section U/L is said to be an $\mathcal{A}$-*section* if U and L are $\mathcal{A}$-subgroups. If $S = U/L$ is an $\mathcal{A}$-section then the module

$$\mathcal{A}_S = \mathrm{Span}_{\mathbb{Z}} \left\{\underline{X}^{\pi} : X \in \mathcal{S}(\mathcal{A}), X \subseteq U\right\},$$

where $\pi \colon U \to U/L$ is the canonical epimorphism, is an S-ring over S.

If $K \leq \mathrm{Aut}(G)$ then the set $\mathrm{Orb}(K, G)$ forms a partition of G that defines an S-ring $\mathcal{A}$ over G. In this case $\mathcal{A}$ is called *cyclotomic* and denoted by $\mathrm{Cyc}(K, G)$.

Let G be abelian. Then from Schur's result [19] it follows that $X^{(m)} \in \mathcal{S}(\mathcal{A})$ for every $X \in \mathcal{S}(\mathcal{A})$ and every m coprime to $|G|$. We say that $X, Y \in \mathcal{S}(\mathcal{A})$ are *rationally conjugate* if $Y = X^{(m)}$ for some m coprime to $|G|$.

2.2 Isomorphisms and schurity

Throughout this and the next two subsections $\mathcal{A}$ and $\mathcal{A}'$ are S-rings over groups G and G' respectively. A bijection $f \colon G \to G'$ is called *a (combinatorial) isomorphism* from $\mathcal{A}$ over to $\mathcal{A}'$ if

$$\{R(X)^f : X \in \mathcal{S}(\mathcal{A})\} = \{R(X') : X' \in \mathcal{S}(\mathcal{A}')\},$$

where $R(X)^f = \{(g^f, h^f) : (g, h) \in R(X)\}$. If there exists an isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ we write $\mathcal{A} \cong \mathcal{A}'$. The group $\mathrm{Iso}(\mathcal{A})$ of all isomorphisms from $\mathcal{A}$ onto itself has a normal subgroup

$$\mathrm{Aut}(\mathcal{A}) = \{f \in \mathrm{Iso}(\mathcal{A}) : R(X)^f = R(X) \text{ for every } X \in \mathcal{S}(\mathcal{A})\}.$$

This subgroup is called the *automorphism group* of $\mathcal{A}$. Note that $\mathrm{Aut}(\mathcal{A}) \geq G_{right}$. If S is an $\mathcal{A}$-section then $\mathrm{Aut}(\mathcal{A})^S \leq \mathrm{Aut}(\mathcal{A}_S)$. An S-ring $\mathcal{A}$ over G is said to be *normal* if $G_{right} \trianglelefteq \mathrm{Aut}(\mathcal{A})$. One can check that

$$N_{\mathrm{Aut}(\mathcal{A})}(G_{right})_e = \mathrm{Aut}(\mathcal{A}) \cap \mathrm{Aut}(G). \tag{2.1}$$

Now let K be a subgroup of $\mathrm{Sym}(G)$ containing G_{right}. As Schur proved in [19], the $\mathbb{Z}$-submodule

$$V(K, G) = \mathrm{Span}_{\mathbb{Z}}\{\underline{X} : X \in \mathrm{Orb}(K_e, G)\},$$

is an S-ring over G. An S-ring $\mathcal{A}$ over G is called *schurian* if $\mathcal{A} = V(K, G)$ for some K such that $G_{right} \leq K \leq \mathrm{Sym}(G)$. Not every S-ring is schurian. The first example of a non-schurian S-ring was found by Wielandt in [20, Theorem 25.7]. It is easy to see that $\mathcal{A}$ is schurian if and only if

$$\mathcal{S}(\mathcal{A}) = \mathrm{Orb}(\mathrm{Aut}(\mathcal{A})_e, G). \tag{2.2}$$

Every cyclotomic S-ring is schurian. More precisely, if $\mathcal{A} = \mathrm{Cyc}(K, G)$ for some $K \leq \mathrm{Aut}(G)$ then $\mathcal{A} = V(G_{right} \rtimes K, G)$.

2.3 Algebraic isomorphisms and separability

A bijection $\varphi\colon \mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{A}')$ is called an *algebraic isomorphism* from $\mathcal{A}$ to $\mathcal{A}'$ if

$$c_{X,Y}^{Z} = c_{X^{\varphi},Y^{\varphi}}^{Z^{\varphi}}$$

for all $X, Y, Z \in \mathcal{S}(\mathcal{A})$. The mapping $\underline{X} \to \underline{X^{\varphi}}$ is extended by linearity to the ring isomorphism of $\mathcal{A}$ and $\mathcal{A}'$. This ring isomorphism we denote also by φ. If there exists an algebraic isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ then we write $\mathcal{A} \cong_{\mathrm{Alg}} \mathcal{A}'$. An algebraic isomorphism from $\mathcal{A}$ to itself is called an *algebraic automorphism* of $\mathcal{A}$. The group of all algebraic automorphisms of $\mathcal{A}$ is denoted by $\mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A})$.

Every isomorphism f of S-rings preserves the structure constants and hence f induces the algebraic isomorphism φ_f. However, not every algebraic isomorphism is induced by a combinatorial one (see [6]). Let $\mathcal{K}$ be a class of groups. An S-ring $\mathcal{A}$ is defined to be *separable* with respect to $\mathcal{K}$ if every algebraic isomorphism from $\mathcal{A}$ to an S-ring over a group from $\mathcal{K}$ is induced by a combinatorial isomorphism.

Put

$$\mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A})_0 = \{\varphi \in \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A}) : \varphi = \varphi_f \text{ for some } f \in \mathrm{Iso}(\mathcal{A})\}.$$

It is easy to see that $\varphi_f = \varphi_g$ for $f, g \in \mathrm{Iso}(\mathcal{A})$ if and only if $gf^{-1} \in \mathrm{Aut}(\mathcal{A})$. Therefore

$$|\,\mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A})_0| = |\,\mathrm{Iso}(\mathcal{A})|/|\,\mathrm{Aut}(\mathcal{A})|. \tag{2.3}$$

One can verify that for every group G the S-ring of rank 2 over G and $\mathbb{Z}G$ are separable with respect to the class of all finite groups. In the former case there exists the unique algebraic isomorphism from the S-ring of rank 2 over G to the S-ring of rank 2 over a given group of order $|G|$ and this algebraic isomorphism is induced by every bijection. In the latter case every basic set is singleton and hence every algebraic isomorphism is induced by an isomorphism in a natural way.

Let $\varphi\colon \mathcal{A} \to \mathcal{A}'$ be an algebraic isomorphism. One can check that φ is extended to a bijection between $\mathcal{A}$- and $\mathcal{A}'$-sets and hence between $\mathcal{A}$- and $\mathcal{A}'$-sections. The images of an $\mathcal{A}$-set X and an $\mathcal{A}$-section S under these extensions are denoted by X^{φ} and S^{φ} respectively. If S is an $\mathcal{A}$-section then φ induces the algebraic isomorphism $\varphi_S\colon \mathcal{A}_S \to \mathcal{A}'_{S'}$, where $S' = S^{\varphi}$. The above bijection between the $\mathcal{A}$- and $\mathcal{A}'$-sets is, in fact, an isomorphism of the corresponding lattices. One can check that

$$\langle X^{\varphi}\rangle = \langle X\rangle^{\varphi} \quad \text{and} \quad \mathrm{rad}(X^{\varphi}) = \mathrm{rad}(X)^{\varphi}$$

for every $\mathcal{A}$-set X (see [4, Equation (10)]). Since $c_{X,Y}^{\{e\}} = \delta_{Y,X^{-1}}|X|$, where $X, Y \in \mathcal{S}(\mathcal{A})$ and $\delta_{Y,X^{-1}}$ is the Kronecker delta, we conclude that $|X| = c_{X,X^{-1}}^{\{e\}}$, $(X^{-1})^{\varphi} = (X^{\varphi})^{-1}$, and $|X| = |X^{\varphi}|$ for every $\mathcal{A}$-set X. In particular, $|G| = |G'|$.

2.4 Cayley isomorphisms

A group isomorphism $f\colon G \to G'$ is called a *Cayley isomorphism* from $\mathcal{A}$ to $\mathcal{A}'$ if $\mathcal{S}(\mathcal{A})^f = \mathcal{S}(\mathcal{A}')$. If there exists a Cayley isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ we write $\mathcal{A} \cong_{\mathrm{Cay}} \mathcal{A}'$. Every Cayley isomorphism is a (combinatorial) isomorphism, however the converse statement is not true.

2.5 Algebraic fusions

Let $\mathcal{A}$ be an S-ring over G and $\Phi \leq \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A})$. Given $X \in \mathcal{S}(\mathcal{A})$ put $X^{\Phi} = \bigcup_{\varphi \in \Phi} X^{\varphi}$. The partition

$$\{X^{\Phi} : X \in \mathcal{S}(\mathcal{A})\}$$

defines an S-ring over G called the *algebraic fusion* of $\mathcal{A}$ with respect to Φ and denoted by $\mathcal{A}^{\Phi}$. Suppose that $\Phi = \{\varphi_f : f \in K\}$ for some $K \leq \mathrm{Iso}(\mathcal{A})$ and $\mathcal{A}$ is schurian. Then one can verify that

$$\mathcal{A}^{\Phi} = V(\mathrm{Aut}(\mathcal{A})K, G).$$

In particular, the following statement holds.

Lemma 2.1. *Let $\mathcal{A}$ be a schurian S-ring over G and $K \leq \mathrm{Iso}(\mathcal{A})$. Then $\mathcal{A}^{\Phi}$, where $\Phi = \{\varphi_f : f \in K\}$, is also schurian.*

2.6 Wreath and tensor products

Let $\mathcal{A}$ be an S-ring over a group G and $S = U/L$ an $\mathcal{A}$-section. The S-ring $\mathcal{A}$ is called the *S-wreath product* if $L \trianglelefteq G$ and $L \leq \mathrm{rad}(X)$ for all basic sets X outside U. In this case we write

$$\mathcal{A} = \mathcal{A}_U \wr_S \mathcal{A}_{G/L}.$$

The S-wreath product is called *non-trivial* or *proper* if $e \neq L$ and $U \neq G$. If $U = L$ we say that $\mathcal{A}$ is the *wreath product* of $\mathcal{A}_L$ and $\mathcal{A}_{G/L}$ and write $\mathcal{A} = \mathcal{A}_L \wr \mathcal{A}_{G/L}$.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be S-rings over groups G_1 and G_2 respectively. Then the set

$$\mathcal{S} = \mathcal{S}(\mathcal{A}_1) \times \mathcal{S}(\mathcal{A}_2) = \{X_1 \times X_2 : X_1 \in \mathcal{S}(\mathcal{A}_1), X_2 \in \mathcal{S}(\mathcal{A}_2)\}$$

forms a partition of $G = G_1 \times G_2$ that defines an S-ring over G. This S-ring is called the *tensor product* of $\mathcal{A}_1$ and $\mathcal{A}_2$ and denoted by $\mathcal{A}_1 \otimes \mathcal{A}_2$.

Lemma 2.2. *The tensor product of two separable S-rings is separable.*

Proof. As noted in [18, Lemma 2.6], the statement of the lemma follows from [1, Theorem 1.20]. □

Lemma 2.3 ([17, Lemma 4.4])**.** *Let $\mathcal{A}$ be the S-wreath product over an abelian group G for some $\mathcal{A}$-section $S = U/L$. Suppose that $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are separable and $\mathrm{Aut}(\mathcal{A}_U)^S = \mathrm{Aut}(\mathcal{A}_S)$. Then $\mathcal{A}$ is separable. In particular, the wreath product of two separable S-rings is separable.*

Let Ω be a finite set. Permutation groups $K,\ K' \leq \mathrm{Sym}(\Omega)$ are called 2-*equivalent* if $\mathrm{Orb}(K, \Omega^2) = \mathrm{Orb}(K', \Omega^2)$. A permutation group $K \leq \mathrm{Sym}(\Omega)$ is called 2-*isolated* if it is the only group which is 2-equivalent to K.

Lemma 2.4. *Let $\mathcal{A}$ be the S-wreath product over an abelian group G for some $\mathcal{A}$-section $S = U/L$. Suppose that $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are separable, $\mathcal{A}_U$ is schurian, and the group $\mathrm{Aut}(\mathcal{A}_S)$ is 2-isolated. Then $\mathcal{A}$ is separable.*

Proof. Since $\mathcal{A}_U$ is schurian, the groups $\mathrm{Aut}(\mathcal{A}_U)^S$ and $\mathrm{Aut}(\mathcal{A}_S)$ are 2-equivalent. Indeed,

$$\mathrm{Orb}(\mathrm{Aut}(\mathcal{A}_U)^S, S^2) = \mathrm{Orb}(\mathrm{Aut}(\mathcal{A}_S), S^2) = \{R(X) : X \in \mathcal{S}(\mathcal{A}_S)\}.$$

This implies that $\mathrm{Aut}(\mathcal{A}_U)^S = \mathrm{Aut}(\mathcal{A}_S)$ because $\mathrm{Aut}(\mathcal{A}_S)$ is 2-isolated. Therefore the conditions of Lemma 2.3 hold and $\mathcal{A}$ is separable. □

Lemma 2.5. *Let H be a normal subgroup of a group G, $\mathcal{B}$ an S-ring over H, $\varphi \in \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{B}) \setminus \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{B})_0$. Then there exists $\psi \in \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A}) \setminus \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A})_0$, where $\mathcal{A} = \mathcal{B} \wr \mathbb{Z}(G/H)$, such that $\psi^H = \varphi$.*

Proof. Define ψ as follows: $X^\psi = X^\varphi$ for $X \in \mathcal{S}(\mathcal{A}_H)$ and $X^\psi = X$ for $X \in \mathcal{S}(\mathcal{A}) \setminus \mathcal{S}(\mathcal{A}_H)$. Let us prove that $\psi \in \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A})$. To do this it suffices to check that $c_{X^\psi,Y^\psi}^{Z^\psi} = c_{X,Y}^Z$ for all $X, Y, Z \in \mathcal{S}(\mathcal{A})$. Suppose that $X, Y \in \mathcal{S}(\mathcal{A}_H)$. If $Z \in \mathcal{S}(\mathcal{A}_H)$ then $c_{X^\psi,Y^\psi}^{Z^\psi} = c_{X^\varphi,Y^\varphi}^{Z^\varphi} = c_{X,Y}^Z$. If $Z \notin \mathcal{S}(\mathcal{A}_H)$ then $Z^\psi \notin \mathcal{S}(\mathcal{A}_H)$ and hence $c_{X^\psi,Y^\psi}^{Z^\psi} = c_{X,Y}^Z = 0$.

Now suppose that exactly one of the sets X, Y, say X, lies inside H. Then $Y^\psi = Y$ and $X \cup X^\psi \subseteq H \leq \mathrm{rad}(Y)$. So $\underline{X}\underline{Y} = \underline{X^\psi}\underline{Y} = |X|\underline{Y}$. This implies that $c_{X^\psi,Y^\psi}^{Z^\psi} = c_{X,Y}^Z = |X|$ whenever $Z = Y$ and $c_{X^\psi,Y^\psi}^{Z^\psi} = c_{X,Y}^Z = 0$ otherwise.

Finally, suppose that $X, Y \notin \mathcal{S}(\mathcal{A}_H)$. In this case $X^\psi = X$ and $Y^\psi = Y$. If $Z \notin \mathcal{S}(\mathcal{A}_H)$ then $Z^\psi = Z$ and hence $c_{X^\psi,Y^\psi}^{Z^\psi} = c_{X,Y}^Z$. If $Z \in \mathcal{S}(\mathcal{A}_H)$ then Z and Z^ψ enter the element $\underline{XY}$ with the same coefficients because $H = \mathrm{rad}(X) \cap \mathrm{rad}(Y)$. Therefore $c_{X^\psi,Y^\psi}^{Z^\psi} = c_{X,Y}^Z$. Thus, $\psi \in \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A})$.

If ψ is induced by an isomorphism then [4, Lemma 3.4] implies that $\psi^H = \varphi$ is also induced by an isomorphism. We obtain a contradiction with the assumption of the lemma and the lemma is proved. □

3 S-rings over cyclic p-groups

In this section we prove Theorem 1.1. Before the proof we recall some results on S-rings over cyclic p-groups. The most of them can be found in [7, 8]. Throughout the section p is an odd prime, G is a cyclic p-group and $\mathcal{A}$ is an S-ring over G. We say that $X \in \mathcal{S}(\mathcal{A})$ is *highest* if X contains a generator of G. Put $\mathrm{rad}(\mathcal{A}) = \mathrm{rad}(X)$, where X is highest. Note that $\mathrm{rad}(\mathcal{A})$ does not depend on the choice of X because every two basic sets are rationally conjugate and hence have the same radicals.

Lemma 3.1. *The S-ring $\mathcal{A}$ is schurian and one of the following statements holds for $\mathcal{A}$:*

(1) $|\mathrm{rad}(\mathcal{A})| = 1$ *and* $\mathrm{rk}(\mathcal{A}) = 2$*;*

(2) $|\mathrm{rad}(\mathcal{A})| = 1$, *$\mathcal{A}$ is normal, and $\mathcal{A} = \mathrm{Cyc}(K, G)$ for some $K \leq K_0$, where K_0 is the subgroup of $\mathrm{Aut}(G)$ of order $p - 1$;*

(3) $|\mathrm{rad}(\mathcal{A})| > 1$ *and $\mathcal{A}$ is the proper generalized wreath product.*

Proof. The S-ring $\mathcal{A}$ is schurian by the main result of [16]. The other statements of the lemma follow from [8, Theorem 4.1, Theorem 4.2 (1)] and [7, Lemma 5.1, Equation (1)]. □

Lemma 3.2. *Let S be an $\mathcal{A}$-section with $|S| \geq p^2$. The following statements hold:*

(1) If Statement (2) of Lemma 3.1 holds for $\mathcal{A}$ then Statement (2) of Lemma 3.1 holds for $\mathcal{A}_S$;

(2) *If* $\mathrm{rk}(\mathcal{A}_S) = 2$ *then* $\mathrm{Aut}(\mathcal{A})^S = \mathrm{Sym}(S)$.

Proof. Statement (1) of the lemma follows from [8, Corollary 4.4] and Statement (2) of the lemma follows from [8, Theorem 4.6 (1)]. □

Lemma 3.3. *Suppose that Statement (2) of Lemma 3.1 holds for* $\mathcal{A}$. *Then* $\mathrm{Aut}(\mathcal{A})$ *is* 2*-isolated.*

Proof. By [15, Lemma 8.2], it suffices to prove that $\mathrm{Aut}(\mathcal{A})_e$ has a faithful regular orbit. The S-ring $\mathcal{A}$ is normal. So (2.1) implies that $\mathrm{Aut}(\mathcal{A})_e \leq \mathrm{Aut}(G)$. Let $X \in \mathcal{S}(\mathcal{A})$ be highest. Since $\mathcal{A}$ is cyclotomic, each element of X is a generator of G. If $f \in \mathrm{Aut}(\mathcal{A})_e$ fixes some $x \in X$ then f is trivial because $f \in \mathrm{Aut}(G)$ and x is a generator of G. Besides, $\mathcal{A}$ is schurian and hence $X \in \mathrm{Orb}(\mathrm{Aut}(\mathcal{A})_e, G)$ by (2.2). Therefore X is a regular orbit of $\mathrm{Aut}(\mathcal{A})_e$. The group $\mathrm{Aut}(G)$ is cyclic because p is odd. So both of the groups $\mathrm{Aut}(\mathcal{A})_e$ and $\mathrm{Aut}(\mathcal{A})_e^X$ are cyclic groups of order $|X|$. Thus, X is a faithful regular orbit of $\mathrm{Aut}(\mathcal{A})_e$ and the lemma is proved. □

Lemma 3.4. *Suppose that Statement (2) of Lemma 3.1 holds for* $\mathcal{A}$ *and* φ *is an algebraic isomorphism from* $\mathcal{A}$ *to an* S*-ring* $\mathcal{A}'$ *over an abelian group* G'. *Then* G' *is cyclic.*

Proof. By the hypothesis,

$$\mathcal{A} = \mathrm{Cyc}(K, G) \text{ for some } K \leq \mathrm{Aut}(G) \text{ with } |K| \leq p - 1.$$

The group $E = \{g \in G : |g| = p\}$ is an $\mathcal{A}$-subgroup of order p because $\mathcal{A}$ is cyclotomic. The group $E' = E^{\varphi}$ is an $\mathcal{A}'$-subgroup of order p by the properties of an algebraic isomorphism. Assume that G' is non-cyclic. Then there exists $X' \in \mathcal{S}(\mathcal{A}')$ containing an element of order p outside E'. Let $X \in \mathcal{S}(\mathcal{A})$ such that $X^{\varphi} = X'$. The set X consists of elements of order greater than p because G is cyclic and all elements of order p from G lie inside E. The identity element e of G enters the element $\underline{X}^p$ with a coefficient dividing by p because $x^p \neq e$ for each $x \in X$. The identity element e' of G' enters the element $(\underline{X}')^p$ with a coefficient which is not divided by p because $(x')^p = e'$ for some $x' \in X'$ and $|X'| \leq p - 1$. Since φ is an algebraic isomorphism, we have

$$(\underline{X}^p)^{\varphi} = (\underline{X}')^p \quad \text{and} \quad \{e\}^{\varphi} = \{e'\}.$$

This implies that e and e' must enter $\underline{X}^p$ and $(\underline{X}')^p$ respectively with the same coefficients, a contradiction. Therefore G' is cyclic and the lemma is proved. □

Lemma 3.5. *Suppose that* $|\mathrm{rad}(\mathcal{A})| > 1$. *Then there exists an* $\mathcal{A}$*-section* $S = U/L$ *such that* $\mathcal{A}$ *is the proper* S*-wreath product,* $|\mathrm{rad}(\mathcal{A}_U)| = 1$, *and* $|L| = p$.

Proof. From [17, Lemma 5.2] it follows that there exists an $\mathcal{A}$-section U/L_1 such that $\mathcal{A}$ is the proper U/L_1-wreath product and $|\mathrm{rad}(\mathcal{A}_U)| = 1$. Let L be a subgroup of L_1 of order p. Then the lemma holds for $S = U/L$. □

Lemma 3.6 ([5, Theorem 1.3])**.** *Every* S*-ring over a cyclic* p*-group is separable with respect to* $\mathcal{K}_C$.

Proof of the Theorem 1.1. The statement of the theorem for $p \in \{2, 3\}$ was proved in [17, Lemma 5.5]. Further we assume that $p \geq 5$. Let $\mathcal{A}$ be an S-ring over a cyclic p-group G of order p^k, where $k \geq 1$. Prove that $\mathcal{A}$ is separable. We proceed by induction on k. If $k = 1$ then G is the unique up to isomorphism group of order p and the statement of the theorem follows from Lemma 3.6.

Let $k \geq 2$. One of the statements of Lemma 3.1 holds for $\mathcal{A}$. If Statement (1) of Lemma 3.1 holds for $\mathcal{A}$ then $\mathrm{rk}(\mathcal{A}) = 2$ and hence $\mathcal{A}$ is separable. Suppose that Statement (2) of Lemma 3.1 holds for $\mathcal{A}$. Let φ be an algebraic isomorphism from $\mathcal{A}$ to an S-ring $\mathcal{A}'$ over an abelian group G'. Due to Lemma 3.4, the group G' is cyclic. So φ is induced by an isomorphism by Lemma 3.6. Therefore $\mathcal{A}$ is separable.

Now suppose that Statement (3) of Lemma 3.1 holds for $\mathcal{A}$. Then $\mathcal{A} = \mathcal{A}_U \wr_S \mathcal{A}_{G/L}$ for some $\mathcal{A}$-section $S = U/L$ with $L > e$ and $U < G$. The S-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are separable by the induction hypothesis. Due to Lemma 3.5 we may assume that $\mathrm{rad}(\mathcal{A}_U) = e$ and $|L| = p$. In this case $\mathrm{rk}(\mathcal{A}_U) = 2$ or Statement (2) of Lemma 3.1 holds for $\mathcal{A}_U$. If $\mathrm{rk}(\mathcal{A}_U) = 2$ or $|S| = 1$ then $U = L$ and $\mathcal{A}$ is separable by Lemma 2.3.

Assume that Statement (2) of Lemma 3.1 holds for $\mathcal{A}_U$. If $|S| \geq p^2$ then Statement (2) of Lemma 3.1 holds for $\mathcal{A}_S$ by Statement (1) of Lemma 3.2. Lemma 3.3 implies that $\mathrm{Aut}(\mathcal{A}_S)$ is 2-isolated. The S-ring $\mathcal{A}_U$ is cyclotomic and hence it is schurian. Therefore $\mathcal{A}$ is separable by Lemma 2.4.

It remains to consider the case when $|S| = p$. In this case $|U| = p^2$. If $\mathrm{rad}(X) > L$ for every $X \in \mathcal{S}(\mathcal{A})$ outside U then $\mathrm{rad}(X) \geq U$ for every $X \in \mathcal{S}(\mathcal{A})$ outside U because G is cyclic. This yields that $\mathcal{A} = \mathcal{A}_U \wr \mathcal{A}_{G/U}$ and hence $\mathcal{A}$ is separable by Lemma 2.3.

Suppose that there exists $X \in \mathcal{S}(\mathcal{A})$ outside U with $\mathrm{rad}(X) = L$. The remaining part of the proof is divided into two cases.

Case 1: $\langle X \rangle < G$. In this case put $S_1 = \langle X \rangle / L$. The S-ring $\mathcal{A}$ is the S_1-wreath product and $|S_1| \geq p^2$. Note that $|\mathrm{rad}(\mathcal{A}_{S_1})| = 1$ because $\mathrm{rad}(X) = L$. So Statement (1) or Statement (2) of Lemma 3.1 holds for $\mathcal{A}_{S_1}$. In the former case $\mathrm{Aut}(\mathcal{A}_{\langle X \rangle})^{S_1} = \mathrm{Aut}(\mathcal{A}_{S_1}) = \mathrm{Sym}(S_1)$ by Statement (2) of Lemma 3.2 and $\mathcal{A}$ is separable by Lemma 2.3. In the latter case $\mathrm{Aut}(\mathcal{A}_{S_1})$ is 2-isolated by Lemma 3.3. Since $\mathcal{A}_{\langle X \rangle}$ is schurian, the conditions of Lemma 2.4 hold for S_1 and $\mathcal{A}$ is separable by Lemma 2.4.

Case 2: $\langle X \rangle = G$. In this case $|\mathrm{rad}(\mathcal{A}_{G/L})| = 1$ because $\mathrm{rad}(X) = L$. Let $\pi \colon G \to G/L$ be the canonical epimorphism. Clearly, $\pi(U)$ is an $\mathcal{A}_{G/L}$-subgroup and $\pi(X)$ lies outside $\pi(U)$. So $\mathrm{rk}(\mathcal{A}_{G/L}) > 2$ and hence Statement (2) of Lemma 3.1 holds for $\mathcal{A}_{G/L}$.

Let φ be an algebraic isomorphism from $\mathcal{A}$ to an S-ring $\mathcal{A}'$ over an abelian group G'. Put $U' = U^{\varphi}$ and $L' = L^{\varphi}$. Clearly,

$$L' \leq U'. \tag{3.1}$$

The algebraic isomorphism φ induces the algebraic isomorphism φ_U from $\mathcal{A}_U$ to $\mathcal{A}_{U'}$, where $U' = U^{\varphi}$. From Lemma 3.4 it follows that

$$U' \cong C_{p^2}. \tag{3.2}$$

Also φ induces the algebraic isomorphism $\varphi_{G/L}$ from $\mathcal{A}_{G/L}$ to $\mathcal{A}_{G'/L'}$. Lemma 3.4 implies that G'/L' is cyclic. Since $|L'| = |L| = p$, we conclude that

$$G' \cong C_{p^k} \quad \text{or} \quad G' \cong C_p \times C_{p^{k-1}}.$$

However, in the latter case L' is not contained in a cyclic group of order p^2 because G'/L' is cyclic. This contradicts to (3.1) and (3.2). So G' is cyclic and φ is induced by an isomorphism by Lemma 3.6. Therefore $\mathcal{A}$ is separable and the theorem is proved. □

4 Proof of Theorem 1.2

Proposition 4.1. *The group C_p^3 is separable for $p \in \{2, 3\}$.*

Before we prove Propostion 4.1 we give the lemma providing a description of S-rings over these groups.

Lemma 4.2. *Let $\mathcal{A}$ be an S-ring over C_p^3, where $p \in \{2, 3\}$. Then $\mathcal{A}$ is schurian and one of the following statements holds:*

(1) $\mathrm{rk}(\mathcal{A}) = 2$;

(2) $\mathcal{A}$ is the tensor product of smaller S-rings;

(3) $\mathcal{A}$ is the proper S-wreath product of two S-rings with $|S| \leq p$;

(4) $p = 3$ and $\mathcal{A} \cong_{\mathrm{Cay}} \mathcal{A}_i$, where $\mathcal{A}_i$ is one of the 14 exceptional S-rings whose parameters are listed in Table 1.

Remark 4.3. In Table 1 the notation k^m means that an S-ring have exactly m basic sets of size k.

Table 1: Parameters of the 14 exceptional S-rings $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{14}$.

S-ring	rank	sizes of basic sets
$\mathcal{A}_1$	3	$1, 13^2$
$\mathcal{A}_2$	4	1, 6, 8, 12
$\mathcal{A}_3$	4	$1, 2, 12^2$
$\mathcal{A}_4$	5	$1, 4^2, 6, 12$
$\mathcal{A}_5$	5	$1, 2, 8^3$
$\mathcal{A}_6$	6	$1, 2, 6^4$
$\mathcal{A}_7$	7	$1, 2, 4^4, 8$
$\mathcal{A}_8$	7	$1, 2, 3^2, 6^3$
$\mathcal{A}_9$	8	$1, 2, 4^6$
$\mathcal{A}_{10}$	9	$1, 2^3, 4^5$
$\mathcal{A}_{11}$	10	$1, 2^5, 4^4$
$\mathcal{A}_{12}$	10	$1^3, 3^6, 6$
$\mathcal{A}_{13}$	11	$1^3, 3^8$
$\mathcal{A}_{14}$	14	$1, 2^{13}$

Proof. The statement of the lemma can be checked with the help of the GAP package COCO2P [11]. □

Proof of the Proposition 4.1. From [17, Theorem 1, Lemma 5.5] it follows that the group C_p^k is separable for $p \in \{2,3\}$ and $k \leq 2$. Let $\mathcal{A}$ be an S-ring over $G \cong C_p^3$, where $p \in \{2,3\}$. Then one of the statements of Lemma 4.2 holds for $\mathcal{A}$. If Statement (1) of Lemma 4.2 holds for $\mathcal{A}$ then, obviously, $\mathcal{A}$ is separable. If Statement (2) of Lemma 4.2 holds for $\mathcal{A}$ then $\mathcal{A}$ is separable by Lemma 2.2. Suppose that Statement (3) of Lemma 4.2 holds for $\mathcal{A}$. Then $\mathcal{A}$ is the proper schurian S-wreath product for some $\mathcal{A}$-section $S = U/L$ with $|S| \leq 3$. Since $\mathcal{A}$ is schurian, $\mathcal{A}_U$ is also schurian. Note that $\mathrm{Aut}(\mathcal{A}_S)$ is 2-isolated becasue $|S| \leq 3$. Therefore $\mathcal{A}$ is separable by Lemma 2.4.

Suppose that Statement (4) of Lemma 4.2 holds for $\mathcal{A}$ and φ is an algebraic isomorphism from $\mathcal{A}$ to an S-ring $\mathcal{A}'$ over an abelian group G'. Clearly, if $\mathcal{A}'$ is separable then φ^{-1} is induced by an isomorphism and hence φ is also induced by an isomorphism. If $G' \cong C_{p^3}$ then $\mathcal{A}'$ is separable by Theorem 1.1; if $G' \cong C_p \times C_{p^2}$ then $\mathcal{A}'$ is separable by [17, Theorem 1]; if $G' \cong C_p^3$ and one of the Statements (1)–(3) of Lemma 4.2 holds for $\mathcal{A}'$ then $\mathcal{A}'$ is separable by the previous paragraph. So in the above cases φ is induced by an isomorphism. Thus, we may assume that $G' \cong C_p^3$ and Statement (4) of Lemma 4.2 holds for $\mathcal{A}'$.

Two algebraically isomorphic S-rings have the same rank and sizes of basic sets. So information from Table 1 implies that $\mathcal{A}_i \ncong_{\mathrm{Alg}} \mathcal{A}_j$ whenever $i \neq j$. Therefore we may assume that

$$\mathcal{A} = \mathcal{A}' = \mathcal{A}_i$$

for some $i \in \{1, \ldots, 14\}$. Using the package COCO2P again, one can find that

$$|\,\mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A}_j)| = |\,\mathrm{Iso}(\mathcal{A}_j)|/|\,\mathrm{Aut}(\mathcal{A}_j)|$$

for every $j \in \{1, \ldots, 14\}$. In view of (2.3) this yields that $\mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A}_j) = \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A}_j)_0$ for every $j \in \{1, \ldots, 14\}$. So $\varphi \in \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{A}_i)_0$ and hence φ is induced by an isomorphism. Thus, $\mathcal{A}$ is separable and the proposition is proved. □

Proposition 4.4. *The group $C_p \times C_p \times C_{p^k}$ is non-separable for $p \in \{2,3\}$ and $k \geq 2$.*

Proof. In view of Lemma 2.5 to prove that the group $C_p \times C_p \times C_{p^k}$ is non-separable for $p \in \{2,3\}$ and $k \geq 2$ it is sufficient to construct an S-ring $\mathcal{A}$ over $C_p \times C_p \times C_{p^2}$, $p \in \{2,3\}$, and an algebraic isomorphism φ from $\mathcal{A}$ to itself which is not induced by an isomorphism.

Let $G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, where $|a| = |b| = p$ and $|c| = p^2$. Put $A = \langle a \rangle$, $B = \langle b \rangle$, $C = \langle c \rangle$, $c_1 = c^p$, and $C_1 = \langle c_1 \rangle$. Firstly consider the case $p = 2$. Let $f \in \mathrm{Aut}(G)$ such that

$$f\colon (a, b, c) \to (a, bac_1, ca)$$

and $\mathcal{A} = \mathrm{Cyc}(\langle f \rangle, G)$. It easy to see that $|f| = 2$ and the basic sets of $\mathcal{A}$ are the following

$$T_0 = \{e\}, \quad T_1 = \{a\}, \quad T_2 = \{c_1\}, \quad T_3 = \{ac_1\},$$
$$X_1 = cA, \quad X_2 = c^3A,$$
$$Y_1 = b\langle ac_1 \rangle, \quad Y_2 = ba\langle ac_1 \rangle,$$
$$Z_1 = bcC_1, \quad Z_2 = bcaC_1.$$

Define a permutation φ on the set $\mathcal{S}(\mathcal{A})$ as follows:

$$T_0^\varphi = T_0, \quad T_1^\varphi = T_1, \quad T_2^\varphi = T_3, \quad T_3^\varphi = T_2,$$
$$X_1^\varphi = X_1, \quad X_2^\varphi = X_2,$$

$$Y_1^{\varphi} = Z_1, \quad Y_2^{\varphi} = Z_2, \quad Z_1^{\varphi} = Y_1, \quad Z_2^{\varphi} = Y_2.$$

It easy to see that $|\varphi| = 2$. The straightforward check implies that φ is an algebraic isomorphism from $\mathcal{A}$ to itself. Let us check, for example, that $c_{Y_1,Y_2}^{T_2} = c_{Y_1^{\varphi},Y_2^{\varphi}}^{T_2^{\varphi}}$. We have $\underline{Y_1}\,\underline{Y_2} = 2a + 2c_1$ and $\underline{Y_1}^{\varphi}\underline{Y_2}^{\varphi} = \underline{Z_1}\,\underline{Z_2} = 2a + 2ac_1$. So $c_{Y_1,Y_2}^{T_2} = c_{Y_1^{\varphi},Y_2^{\varphi}}^{T_2^{\varphi}} = 2$.

Note that $\mathcal{A}$ corresponds to a Kleinian quasi-thin scheme of index 4 in the sense of [14]. The S-ring $\mathcal{A}$ is cyclotomic and hence it is schurian. Assume that φ is induced by an isomorphism. Then the algebraic fusion $\mathcal{A}^{\langle\varphi\rangle}$ is schurian by Lemma 2.1. However, computer calculations made by using the package COCO2P [11] (see also [21]) imply that $\mathcal{A}^{\langle\varphi\rangle}$ is non-schurian, a contradiction. Therefore, φ is not induced by an isomorphism and hence $\mathcal{A}$ is non-separable.

Now let $p = 3$. Let $f_1, f_2, f_3 \in \mathrm{Aut}(G)$ such that

$$f_1\colon (a,b,c) \to (a^{-1}, b^{-1}, c^{-1}),\ f_2\colon (a,b,c) \to (a,b,cc_1),\ f_3\colon (a,b,c) \to (a,ba,c).$$

The direct check implies that $|f_1| = 2$, $|f_2| = |f_3| = 3$, and f_1, f_2, f_3 pairwise commute. Put $K = \langle f_1\rangle \times \langle f_2\rangle \times \langle f_3\rangle$ and $\mathcal{A} = \mathrm{Cyc}(K, G)$. The basic sets of $\mathcal{A}$ are the following:

$$T_0 = \{e\},\ T_1 = \{a, a^{-1}\},\ T_2 = \{c_1, c_1^{-1}\},\ T_3 = \{ac_1, a^{-1}c_1^{-1}\},\ T_4 = \{a^{-1}c_1, ac_1^{-1}\},$$
$$X_1 = cC_1 \cup c^{-1}C_1, \quad X_2 = caC_1 \cup c^{-1}a^{-1}C_1, \quad X_3 = ca^{-1}C_1 \cup c^{-1}aC_1,$$
$$Y_1 = bA \cup b^{-1}A, \quad Y_2 = bc_1A \cup b^{-1}c_1^{-1}A, \quad Y_3 = b^{-1}c_1A \cup bc_1^{-1}A,$$
$$Z_1 = \{bc, b^{-1}c^{-1}\}(A \times C_1), \quad Z_2 = \{b^{-1}c, bc^{-1}\}(A \times C_1).$$

Let φ be a permutation on the set $\mathcal{S}(\mathcal{A})$ such that $T_3^{\varphi} = T_4$, $T_4^{\varphi} = T_3$, and $X^{\varphi} = X$ for every $X \in \mathcal{S}(\mathcal{A}) \setminus \{T_3, T_4\}$. Clearly, $|\varphi| = 2$. Note that for every $X, Y \in \mathcal{S}(\mathcal{A}) \setminus \{T_3, T_4\}$ the elements $\underline{T_3}$ and $\underline{T_4}$ enter with non-zero coefficients the element $\underline{XY}$ only in the following cases: $X = Y = Z_i$; $X = X_i, Y = X_j,\ i \neq j$; $X = Y_i, Y = Y_j,\ i \neq j$. The straightforward check using this observation implies that φ is an algebraic isomorphism from $\mathcal{A}$ to itself.

If φ is induced by an isomorphism then $\mathcal{A}^{\langle\varphi\rangle}$ is schurian by Lemma 2.1. However, $\mathcal{A}^{\langle\varphi\rangle}$ coincides with the non-schurian S-ring constructed in [2, pp. 8–10] in case of $G \cong C_3 \times C_3 \times C_9$, a contradiction. Thus, φ is not induced by an isomorphism and hence $\mathcal{A}$ is non-separable. The proposition is proved. □

Theorem 1.2 is an immediate consequence of Proposition 4.1 and Proposition 4.4.

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Strong geodetic problem on complete multipartite graphs*

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Abstract

The strong geodetic problem is to find the smallest number of vertices such that by fixing one shortest path between each pair, all vertices of the graph are covered. In this paper we study the strong geodetic problem on complete bipartite graphs. Some results for complete multipartite graphs are also derived. Finally, we prove that the strong geodetic problem restricted to (general) bipartite graphs is NP-complete.

Keywords: Geodetic problem, strong geodetic problem, (complete) bipartite graphs, (complete) multipartite graphs.

Math. Subj. Class.: 05C12, 05C70, 68Q17

1 Introduction

The strong geodetic problem was introduced in [1] as follows. Let $G = (V, E)$ be a graph. For a set $S \subseteq V$, and for each pair of vertices $\{x, y\} \subseteq S$, $x \neq y$, define $\widetilde{g}(x, y)$ as a *selected fixed* shortest path between x and y. We set

$$\widetilde{I}(S) = \{\widetilde{g}(x, y) : x, y \in S\},$$

and $V(\widetilde{I}(S)) = \bigcup_{\widetilde{P} \in \widetilde{I}(S)} V(\widetilde{P})$. If $V(\widetilde{I}(S)) = V$ for some $\widetilde{I}(S)$, then the set S is called a *strong geodetic set*. This means that the selected fixed geodesics between vertices from S cover all vertices of the graph G. If G has just one vertex, then its vertex is considered the unique strong geodetic set. The *strong geodetic problem* is to find a minimum strong

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geodetic set of G. The size of a minimum strong geodetic set is the *strong geodetic number* of G and is denoted by $\operatorname{sg}(G)$. A strong geodetic set of size $\operatorname{sg}(G)$ is also called an *optimal strong geodetic set*.

In the first paper [1], it was proved that the problem is NP-complete. The invariant has also been determined for complete Apollonian networks [1], thin grids and cylinders [14], and balanced complete bipartite graphs [12]. Some properties of the strong geodetic number of Cartesian product of graphs have been studied in [13]. Recently, a concept of strong geodetic cores has been introduced and applied to the Cartesian product graphs [9]. An edge version of the problem was defined and studied in [16].

The strong geodetic problem is just one of the problems which aim to cover all vertices of a graph with shortest paths. Another such problem is the *geodetic problem*, in which we determine the smallest number of vertices such that the geodesics between them cover all vertices of the graph [3, 5, 10, 11]. Note that we may use more than one geodesic between the same pair of vertices. Thus this problem seems less complex than the strong geodetic problem. The geodetic problem is known to be NP-complete on general graphs [2], on chordal and bipartite weakly chordal graphs [6], on co-bipartite graphs [7], and on graphs with maximal degree 3 [4]. However, it is polynomial on co-graphs and split graphs [6], on proper interval graphs [8], on block-cactus graphs and monopolar chordal graphs [7]. Moreover, the geodetic number of complete bipartite (and multipartite) graphs is straightforward to determine, i.e. $\operatorname{sg}(K_{n,m}) = \min\{n, m, 4\}$ [11].

Recall from [12] that the strong geodetic problem on a complete bipartite graph can be presented as a (non-linear) optimization problem as follows. Let (X, Y) be the bipartition of $K_{n,m}$ and $S \cup T$, $S \subseteq X$, $T \subseteq Y$, its strong geodetic set. Let $|S| = s$ and $|T| = t$. Thus, $\operatorname{sg}(K_{n,n}) \leq s + t$. With geodesics between vertices from S we wish to cover vertices in $Y - T$. Vice versa, with geodesics between vertices from T we are covering vertices in $X - S$. The optimization problem for $\operatorname{sg}(K_{n,m})$ reads as follows:

$$\begin{aligned} \min \quad & s + t \\ \text{subject to:} \quad & 0 \leq s \leq n \\ & 0 \leq t \leq m \\ & \binom{t}{2} \geq n - s \\ & \binom{s}{2} \geq m - t \\ & s, t \in \mathbb{Z}. \end{aligned} \tag{1.1}$$

This holds due to the fact that every geodesic in a complete bipartite graph is either of length 0, 1 (an edge), or 2 (a path with both endvertices in the same part of the bipartition). If a strong geodetic set S has k vertices in one part of the bipartition, then geodesics between those vertices can cover at most $\binom{k}{2}$ vertices in the other part.

The exact value is known for balanced complete bipartite graphs: if $n \geq 6$, then

$$\operatorname{sg}(K_{n,n}) = \begin{cases} 2\left\lceil \frac{-1+\sqrt{8n+1}}{2} \right\rceil, & 8n-7 \text{ is not a perfect square,} \\ 2\left\lceil \frac{-1+\sqrt{8n+1}}{2} \right\rceil - 1, & 8n-7 \text{ is a perfect square.} \end{cases}$$

See [12, Theorem 2.1].

In the following section, we generalize the above result to all complete bipartite graphs.

To conclude the introduction, we state the following interesting and surprisingly important fact.

Lemma 1.1 (Shifting Lemma). *Let $K_{n_1,\dots,n_r}$ be a complete multipartite graph with the multipartition $X_1,\dots,X_r$, $|X_i| = n_i$ for $i \in [r]$. Let $S = S_1 \cup \dots \cup S_r$ be an optimal strong geodetic set, with $S_i \subseteq X_i$ and $|S_i| = s_i$ for $i \in [r]$.*

If $s_1 \le s_2, s_3$ and $s_2 < n_2$, $s_3 < n_3$, then there exist $x \in S_1$ and $y \in S_2 \cup S_3$, such that $S \cup \{y\} - \{x\}$ is also an optimal strong geodetic set.

Proof. Let $G = K_{n_1,\dots,n_r}$, $|X_i| = n_i$, $|S_i| = s_i$ for $i \in [r]$. Suppose $S_i \neq \emptyset, X_i$ for $i \in \{1, 2, 3\}$. Without loss of generality, $s_1 = \min\{s_1, s_2, s_3\}$, and let geodesics between vertices of S_1 cover fewer vertices in $X_2 - S_2$ than in $X_3 - S_3$.

Select vertices $x \in S_1$ and $y \in X_2 - S_2$. Geodesics between vertices from S_1 can be fixed in such a way that no vertex in X_2 is covered with a geodesic containing x. This is trivial for $s_1 \in \{1, 2, 3\}$, and follows from $\left\lfloor \frac{\binom{s_1}{2}}{2} \right\rfloor \le \binom{s_1-1}{2}$ for $s_1 \ge 4$.

Now consider $T = S \cup \{y\} - \{x\}$, $|T| = |S|$. We will prove that T is a strong geodetic set of G. Fix geodesics between vertices in T in the same way as in S, except those containing x or y. As $x \notin T$, at most $s_1 - 1$ vertices U in $V(G) - X_1 - X_2$ are uncovered. But geodesics containing y can cover the vertex x, as well as $s_2 - 1$ other vertices in $V(G) - X_2$. As we have $s_2 - 1 \ge s_1 - 1$, those geodesics can be fixed in such a way that U is covered. □

Proposition 1.2. *For every complete multipartite graph there exist an optimal strong geodetic set such that its intersection with all but two parts of the multipartition is either empty or the whole part.*

Proof. Let $G = K_{n_1,\dots,n_r}$, $|X_i| = n_i$, be a multipartite graph. The Shifting Lemma states that every strong geodetic set with three or more parts of size not equal to 0 or n_i can be transformed into a strong geodetic set of the same size, where one of these parts becomes smaller and one larger. After repeating this procedure on other such triples, at most two parts can have size different from 0 or n_i. □

The rest of the paper is organized as follows. In the next section, some further results about the strong geodetic number of complete bipartite graphs are obtained. In Section 3 we discuss the strong geodetic problem on complete multipartite graphs. Finally, in Section 4 the complexity of the strong geodetic problem on multipartite and complete multipartite graphs is discussed.

2 On complete bipartite graphs

In this section, we give a complete description of the strong geodetic number of a complete bipartite graph. Instead of giving an explicit formula for $\mathrm{sg}(K_{n,m})$, we classify the triples (n, m, k) for which $\mathrm{sg}(K_{n,m}) = k$.

Define

$$f(\alpha, \beta) = \alpha - 1 + \binom{\max\{\beta - 1, 2\}}{2}.$$

Theorem 2.1. *For positive integers n, m and k, $(n, m) \notin \{(1,1),(2,2)\}$, $\operatorname{sg}(K_{n,m}) = k$ if and only if*

$$n < k \ \&\ m = f(k,n) \quad \textit{or} \quad m < k \ \&\ n = f(k,m) \quad \textit{or}$$
$$f(k,i-1) \le m \le f(k,i) \ \&\ f(k,k-i-1) \le n \le f(k,k-i) \textit{ for some } i, 0 \le i \le k.$$

Note that for the two exceptional cases, we have $\operatorname{sg}(K_{1,1}) = 2$ and $\operatorname{sg}(K_{2,2}) = 3$.

Example 2.2. The strong geodetic numbers of small complete bipartite graphs can be found in Table 1.

Table 1: The strong geodetic numbers $\operatorname{sg}(K_{n,m})$ for some small complete bipartite graphs.

m \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	2	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	2	3	3	4	5	6	7	8	9	10	11	12	13	14	15
3	3	3	3	4	5	6	7	8	9	10	11	12	13	14	15
4	4	4	4	4	4	4	5	6	7	8	9	10	11	12	13
5	5	5	5	4	5	5	5	5	5	5	6	7	8	9	10
6	6	6	6	4	5	6	6	6	6	6	6	6	6	6	6
7	7	7	7	5	5	6	7	7	7	7	7	7	7	7	7
8	8	8	8	6	5	6	7	8	8	8	8	8	8	8	8
9	9	9	9	7	5	6	7	8	8	8	9	9	9	9	9
10	10	10	10	8	5	6	7	8	8	8	9	9	9	9	10
11	11	11	11	9	6	6	7	8	9	9	9	9	9	9	10
12	12	12	12	10	7	6	7	8	9	9	9	10	10	10	10
13	13	13	13	11	8	6	7	8	9	9	9	10	10	10	10
14	14	14	14	12	9	6	7	8	9	9	9	10	10	10	10
15	15	15	15	13	10	6	7	8	9	10	10	10	10	10	10

Figure 1 shows the positions of all 201 pairs (n, m) for which $\operatorname{sg}(K_{n,m}) = 12$. We can notice the "parabolas" corresponding to $m = f(k,n)$ and $n = f(k,m)$, as well as the "intersecting rectangles" corresponding to $f(k,i-1) \le m \le f(k,i)$, $f(k,k-i-1) \le n \le f(k,k-i)$.

Proof of Theorem 2.1. It is not difficult to see that $\operatorname{sg}(K_{n,m}) = 2$ if and only if $(n,m) \in \{(1,1),(1,2),(2,1)\}$, and that $\operatorname{sg}(K_{2,2}) = 3$. So assume that $k \ge 3$ and $\max\{n,m\} \ge 3$.

The statement follows from the following (note that the sum $s_1 + s_2$ equals k for every (s_1, s_2) that appears below). Note that all different optimal solutions are described here, hence some of the conditions overlap.

1. If $n \le 3$ and $m = f(k,n) = k$, or $m = f(k,i-1)$ and $f(k,k-i-1) < n \le f(k,k-i)$ for $i \le 4$, or $m = f(k,i-1)$ and $n = f(k,k-i-1)$ for $i \le 4$, then $(0,k)$ is an optimal solution. Symmetrically, if $m \le 3$ and $n = f(k,m) = k$, or $f(k,i-1) \le m \le f(k,i)$ and $n = f(k,k-i-1)$ for $i \ge k-4$, then $(k,0)$ is an optimal solution.

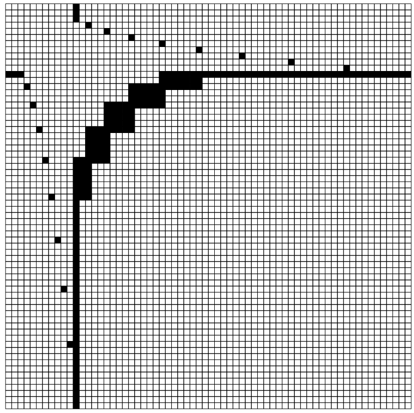

Figure 1: All pairs (n, m) for which $\mathrm{sg}(K_{n,m}) = 12$.

2. If $3 \le n < k$ and $m = f(k, n)$, then $(n, k-n)$ is an optimal solution. Symmetrically, if $3 \le m < k$ and $n = f(k, m)$, then $(k - m, m)$ is an optimal solution.
3. If $f(k, i-1) < m \le f(k, i)$ and $f(k, k-i-1) < n \le f(k, k-i)$ for $4 \le i \le k-4$, or $m = f(k, i-1)$ and $f(k, k-i-1) < n \le f(k, k-i)$ for $i \ge 3$, or $f(k, i-1) < m \le f(k, i)$ and $n = f(k, k-i-1)$ for $i \le k-3$, or $m = f(k, i-1)$ and $n = f(k, k-i-1)$ for $3 \le i \le k-3$, then $(i, k-i)$ is an optimal solution.
4. If $f(k, i-1) < m \le f(k, i)$ and $n = f(k, k-i-1)$ for $i \le k-4$, or $m = f(k, i-1)$ and $n = f(k, k-i-1)$ for $2 \le i \le k-4$, then $(i+1, k-i-1)$ is an optimal solution.
5. If $m = f(k, i-1)$ and $f(k, k-i-1) < n \le f(k, k-i)$ for $i \ge 4$, or $m = f(k, i-1)$ and $n = f(k, k-i-1)$ for $4 \le i \le k-2$, then $(i-1, k-i+1)$ is an optimal solution.

It is easy to see that the above solutions give rise to the strong geodetic sets of size k. For example, in the first case, the part of the bipartition of size m is a strong geodetic set with parameters $(0, k)$. What remains to be proved is $\mathrm{sg}(K_{n,m}) \ge k$ for each case. This can be shown by a simple case analysis. As the reasoning is similar in all cases, we demonstrate only two of them. Let X be the part of the bipartition of size n and Y the part of size m. Also, let $S = S_1 \cup S_2$, where $S_1 \subseteq X$, $S_2 \subseteq Y$, be some strong geodetic set.

- The case $k > n \ge 3$ and $m = k - 1 + \binom{n-1}{2} = k - n + \binom{n}{2}$: If $S_1 = X$, then geodesics between these vertices cover at most $\binom{n}{2}$ vertices in Y, so at least $k - n$ vertices in Y must also lie in a strong geodetic set. Hence, $|S| \ge n - (k - n) = k$.

  If $S_1 \ne X$, geodesics between these vertices cover at most $\binom{n-1}{2}$ vertices in Y, so at least $k - 1$ vertices from Y must lie in a strong geodetic set. Hence, $|S| \ge |S_1| + (k - 1)$. If $S_1 \ne \emptyset$ or $|S_2| \ge k$, we have $|S| \ge k$. Otherwise, $S = S_2$ and contains exactly $k-1$ vertices. But then the remaining vertices in Y are not covered.
- The case where $f(k, i-1) < m \le f(k, i)$ and $f(k, k-i-1) < n \le f(k, k-i)$

for $4 \leq i \leq k-4$: We can write

$$n = k-1+\binom{k-i-2}{2}+l, \quad l \in \{1,\dots,k-i-2\},$$
$$m = k-1+\binom{i-2}{2}+j, \quad j \in \{1,\dots,i-2\}.$$

Suppose $|S| \leq k-1$. If $|S_1| \leq i-2$, these vertices cover at most $\binom{i-2}{2}$ vertices in X, thus at least k vertices remain uncovered and $|S| \geq k$. Hence, $|S_1| \geq i-1$. Similarly, $|S_2| \geq k-i-1$.

If $|S_1| = i-1$, then $\binom{i-1}{2}$ vertices in Y are covered. As $k+j-i+1$ are left uncovered, it holds that $|S_2| \geq k-i+2$ and thus $|S| \geq k+1$.

If $|S_2| = k-i-1$, then $\binom{k-i-1}{2}$ vertices in X are covered. As $l+i+1$ are left uncovered, it holds that $|S_1| \geq i+2$ and thus $|S| \geq k+1$.

Hence $|S_1| \geq i$ and $|S_2| \geq k-i$ and thus $|S| \geq k$. □

The first condition from Theorem 2.1 can be simplified as follows.

Corollary 2.3. *If $n \geq 3$ and $m > \binom{n}{2}$, then $\mathrm{sg}(K_{n,m}) = m+1-\binom{n-1}{2}$. If $n \leq 3$ and $m > n$, then $\mathrm{sg}(K_{n,m}) = m$.*

When $m \leq \binom{n}{2}$, Theorem 2.1 is harder to apply. Note, however, that the theorem suggests that m is approximately equal to $k-1+\binom{i-1}{2}$, and n is approximately equal to $k-1+\binom{k-i-1}{2}$. Furthermore, note that we can rewrite the system of equations (with known m, n and variables k, i) $m = k-1+\binom{i-1}{2}$, $n = k-1+\binom{k-i-1}{2}$ as a polynomial equation of degree 4 for k (say by subtracting the two equations, solving for i, and plugging the result into one of the equations), and solve it explicitly. It seems that one of the four solutions is always very close to $\mathrm{sg}(K_{m,n})$. Denote the minimal distance between $\mathrm{sg}(K_{m,n})$ and a solution k of $m = k-1+\binom{i-1}{2}$, $n = k-1+\binom{k-i-1}{2}$ by $e(m,n)$. Then our data is indicated in Table 2.

Table 2: The difference between the exact and estimated values of $\mathrm{sg}(K_{m,n})$ for different values of n.

n	10	100	1000	10000	100000
$\max\{e(m,n) : n \leq m \leq \binom{n}{2}\}$	1.094	1.774	1.941	1.983	1.995

We conjecture the following.

Conjecture 2.4. *If $n \leq m \leq \binom{n}{2}$, then $e(m,n) < 2$.*

If the conjecture is true, $\mathrm{sg}(K_{m,n})$ is among the (at most 16) positive integers that are at distance < 2 from one of the four solutions of the system $m = k-1+\binom{i-1}{2}$, $n = k-1+\binom{k-i-1}{2}$. For each of these (at most) 16 candidates, there are at most three (consecutive) i's for which $f(k,i-1) \leq m \leq f(k,i)$, found easily by solving the quadratic equation $m = k-1+\binom{i-1}{2}$. For each such i, check if $f(k,k-i-1) \leq n \leq f(k,k-i)$. This allows for computation of $\mathrm{sg}(K_{m,n})$ with a constant number of operations.

3 On complete multipartite graphs

The optimization problem (1.1) can be generalized to complete multipartite graphs. However, solving such a program seems rather difficult. Hence, we present an approximate program which gives a nice lower bound for the strong geodetic number of a complete multipartite graph. If i vertices from one part are in a strong geodetic set, geodesics between them cover at most $\binom{i}{2}$ other vertices. In the following, we do not take into account the condition that they can only cover vertices in other parts, and that the number of selected vertices must be an integer. Recall the notation $\langle 1^{m_1}, \ldots, k^{m_k} \rangle$ which describes a partition with m_i parts of size i, $1 \leq i \leq k$. Let G be a complete multipartite graph corresponding to the partition $\pi = \langle 1^{m_1}, \ldots, k^{m_k} \rangle$ and let a_{ij} denote the number of parts of size j with exactly i vertices in the strong geodetic set. Thus we must have $\sum_{i=0}^{j} a_{ij} = m_j$ and $\sum_{j=1}^{k} \sum_{i=0}^{j} \binom{i}{2} a_{ij} \geq \sum_{j=1}^{k} \sum_{i=0}^{j} (j-i) a_{ij}$. The second condition simplifies to $\sum_{j=1}^{k} \sum_{i=1}^{j} \binom{i+1}{2} a_{ij} \geq \sum_{j=1}^{k} \sum_{i=0}^{j} j a_{ij} = \sum_{j=1}^{k} j m_j = n$. As a_{0j}'s do not appear in it anymore, we also simplify the first condition to $\sum_{i=1}^{j} a_{ij} \leq m_j$ and get

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{k} \sum_{i=1}^{j} i a_{ij} \\
\text{subject to:} \quad & \sum_{i=1}^{j} a_{ij} \leq m_j \\
& \sum_{j=1}^{k} \sum_{i=1}^{j} \binom{i+1}{2} a_{ij} \geq n \\
& 0 \leq a_{ij} \leq m_j .
\end{aligned}
\tag{3.1}
$$

As the sequence $\binom{k}{2} - k$ is increasing for $k \geq 3$, it is better to select more vertices in a bigger part. Hence, the optimal solution is

$$
\begin{aligned}
a_{k,k} &= m_k \\
a_{k-1,k-1} &= m_{k-1} \\
&\vdots \\
a_{l+1,l+1} &= m_{l+1} \\
a_{l,l} &= \frac{l m_l + \cdots + 1 m_1 - \binom{k}{2} m_k - \cdots - \binom{l+1}{2} m_{l+1}}{\binom{l+1}{2}},
\end{aligned}
$$

where l is the smallest positive integer such that $\binom{k+1}{2} m_k + \cdots + \binom{l+2}{2} m_{l+1} \leq k m_k + \cdots + 1 m_1 = |V(K_{\langle 1^{m_1}, \ldots, k^{m_k} \rangle})|$, which is equivalent to $l m_l + \cdots + 1 m_1 \geq \binom{k}{2} m_k + \cdots + \binom{l+1}{2} m_{l+1}$, and

$$
\operatorname{sg}(K_{\langle 1^{m_1}, \ldots, k^{m_k} \rangle}) \geq \left\lceil k m_k + \cdots + (l+1) m_{l+1} + \frac{l m_l + \cdots + m_1 - \binom{k}{2} m_k - \cdots - \binom{l+1}{2} m_{l+1}}{\frac{l+1}{2}} \right\rceil .
$$

The result is particularly interesting in the case when $\pi = \langle k^m \rangle$, i.e. when we observe a multipartite graph with m parts of size k, as we get $l = k$ and

$$\operatorname{sg}(K_{\langle k^m \rangle}) \geq \left\lceil \frac{2km}{k+1} \right\rceil .$$

On the other hand, considering a strong geodetic set consisting only of the whole parts of the bipartition yields an upper bound. At least $l \in \mathbb{Z}$, where $l(k + \binom{k}{2}) \geq mk$, parts must be in a strong geodetic set. Hence,

$$\operatorname{sg}(K_{\langle k^m \rangle}) \leq \left\lceil \frac{2m}{k+1} \right\rceil \cdot k.$$

This implies the following result.

Proposition 3.1. *If $k, n \in \mathbb{N}$ and $(k+1) \mid 2m$, then $\operatorname{sg}(K_{\langle k^m \rangle}) = \frac{2mk}{k+1}$.*

4 Complexity results for multipartite graphs

The strong geodetic problem can be naturally formed as a decision problem.

Problem 4.1 (STRONG GEODETIC SET)**.**
Input: a graph G, an integer k.
Question: does a graph G have a strong geodetic set of size at most k?

The strong geodetic problem on general graphs is known to be NP-complete [1]. In the following we prove that it is also NP-complete on multipartite graphs.

The reduction uses the dominating set problem. Recall that a set $D \subseteq V(G)$ is a dominating set in the graph G if every vertex in $V(G) - D$ has a neighbor in D.

Problem 4.2 (DOMINATING SET)**.**
Input: a graph G, an integer k.
Question: does a graph G have a dominating set of size at most k?

The dominating set problem is known to be NP-complete on bipartite graphs [15]. The idea of the following proof is similar to the proof that the ordinary geodetic problem restricted to chordal bipartite graphs is NP-complete [6].

Theorem 4.3. STRONG GEODETIC SET *restricted to bipartite graphs is NP-complete.*

Proof. To prove NP-completeness, we describe a polynomial reduction of DOMINATING SET on bipartite graphs to STRONG GEODETIC SET on bipartite graphs. Let (G, k) be an input for DOMINATING SET, and (X, Y) a bipartition of the graph G. Define a graph G',

$$V(G') = V(G) \cup \{u_1, u_2\} \cup \{x' : x \in X\} \cup \{y' : y \in Y\},$$

with the edges $E(G)$, $u_1 \sim u_2$, and $x \sim u_2 \sim x'$ for all $x \in X$, $y \sim u_1 \sim y'$ for all $y \in Y$. Define the sets

$$X' = X \cup \{u_1\} \cup \{x' : x \in X\},$$
$$Y' = Y \cup \{u_2\} \cup \{y' : y \in Y\},$$

and observe that (X', Y') is a bipartition of the graph G'. Define the parameter $k' = k + |V(G)|$.

Suppose D is a dominating set of the graph G of size at most k. Define

$$D' = D \cup \{x' : x \in X\} \cup \{y' : y \in Y\}.$$

Notice that $|D'| \leq k'$. For each $x \in X \cap D$, fix geodesics $x \sim y \sim u_1 \sim y', y \in N_G(x)$. Similarly, for each $y \in Y \cap D$, fix $y \sim x \sim u_2 \sim x', x \in N_G(y)$. As D is a dominating set, these geodesics cover all vertices in $V(G)$. Additionally, fix geodesics $x \sim u_2 \sim x'$ for some $x \in X$, and $y \sim u_1 \sim y'$ for some $y \in Y$, to cover the vertices u_1, u_2. Hence, D' is a strong geodetic set of the graph G'.

Conversely, suppose D' is a strong geodetic set of G' of size at most k'. Vertices $\{x' : x \in X\} \cup \{y' : y \in Y\}$ are all simplicial, hence they all belong to D'. Geodesics between them cannot cover any vertices in $V(G)$, thus $V(G) \cap D' \neq \emptyset$. Let $D = D' \cap V(G)$. Clearly, $|D| \leq k$. Consider $x \in V(G) - D$. Thus x is an inner point of some y, z-geodesic. At most one of y, z does not belong to D. The structure of the graph ensures that at least one of y, z is a neighbor of x. Hence, D is a dominating set of the graph G. □

Corollary 4.4. STRONG GEODETIC SET *restricted to multipartite graphs is NP-complete.*

In the following we consider the complexity of STRONG GEODETIC SET on *complete* multipartite graphs. Proposition 1.2 gives rise to the following algorithm.

Let G be a graph and $(X_1, \dots, X_r)$ its multipartition. Denote $n_i = |X_i|$, $i \in [r]$. For all $\{i, j\} \subseteq \binom{[r]}{2}$, for all subsets R of $[r] - \{i, j\}$, for all $s_i \in \{0, \dots, n_i\}$, for all $s_j \in \{0, \dots, n_j\}$, set $S_i \subseteq X_i$ of size s_i, and $S_j \subseteq X_j$ of size s_j. Check if $S_i \cup S_j \cup \bigcup_{k \in R} X_k$ is a strong geodetic set for G. The answer is the size of the smallest strong geodetic set.

The time complexity of this algorithm is $O(n^2 r^2 2^r)$. This confirms the already known result that STRONG GEODETIC SET restricted to complete bipartite graphs is in P, which is an easy consequence of Theorem 2.1. Moreover, it is now clear that the problem is solvable in quadratic time. The same holds for complete r-partite graphs (when r is fixed). But for a general complete multipartite graph (when the size of the multipartition is part of the input), the algorithm tells us nothing about complexity.

But we also observe an analogy between the STRONG GEODETIC SET problem on complete multipartite graphs and the KNAPSACK PROBLEM, which is known to be NP-complete [17]. Recall that in this problem, we are given a set of items with their weights and values, and we need to determine which items to put in a backpack, so that a total weight is smaller that a given bound and a total value is as large as possible. The approximate reduction from the STRONG GEODETIC SET on complete multipartite graphs to the KNAPSACK PROBLEM is the following. Let $(X_1, \dots, X_r)$ be the parts of the complete multipartite graph. The items $x_1, \dots, x_r$ represent those parts, a value if x_i is $\binom{|X_i|}{2}$ and the weight is $|X_i|$. Thus selecting the items such that their total value is as large as possible and the total weight as small as possible, is almost the same as finding the smallest strong geodetic set of the complete multipartite graph (as Proposition 1.2 states that at most two parts in the strong geodetic set are selected only partially). We were not able to find a reduction from the KNAPSACK PROBLEM to the STRONG GEODETIC SET on complete multipartite graphs. But due to the connection with the KNAPSACK PROBLEM, it seems that the problem is not polynomial. Hence we pose

Conjecture 4.5. STRONG GEODETIC SET *restricted to complete multipartite graphs is NP-complete.*

However, as already mentioned, determining the strong geodetic number of complete r-partite graphs for fixed r is polynomial. Using a computer program (implemented in Mathematica [18]) we derive the results shown in Table 3.

Table 3: The strong geodetic numbers for some small complete multipartite graphs.

π	$\mathrm{sg}(K_\pi)$
$\langle 1\rangle$	1
$\langle 2\rangle$	2
$\langle 1^2\rangle$	2
$\langle 3\rangle$	3
$\langle 1,2\rangle$	2
$\langle 1^3\rangle$	3
$\langle 4\rangle$	4
$\langle 1,3\rangle$	3
$\langle 2^2\rangle$	3
$\langle 1^2,2\rangle$	3
$\langle 1^4\rangle$	4
$\langle 5\rangle$	5
$\langle 1,4\rangle$	4
$\langle 2,3\rangle$	3
$\langle 1^2,3\rangle$	3
$\langle 1,2^2\rangle$	4
$\langle 1^3,2\rangle$	4
$\langle 1^5\rangle$	5

π	$\mathrm{sg}(K_\pi)$
$\langle 6\rangle$	6
$\langle 1,5\rangle$	5
$\langle 2,4\rangle$	4
$\langle 1^2,4\rangle$	4
$\langle 3^2\rangle$	3
$\langle 1,2,3\rangle$	3
$\langle 1^3,3\rangle$	3
$\langle 2^3\rangle$	4
$\langle 1^2,2^2\rangle$	4
$\langle 1^4,2\rangle$	5
$\langle 1^6\rangle$	6
$\langle 7\rangle$	7
$\langle 1,6\rangle$	6
$\langle 2,5\rangle$	5
$\langle 1^2,5\rangle$	5
$\langle 3,4\rangle$	4
$\langle 1,2,4\rangle$	4
$\langle 1^3,4\rangle$	4

π	$\mathrm{sg}(K_\pi)$
$\langle 1,3^2\rangle$	4
$\langle 2^2,3\rangle$	4
$\langle 1^2,2,3\rangle$	4
$\langle 1^4,3\rangle$	4
$\langle 1,2^3\rangle$	5
$\langle 1^3,2^2\rangle$	5
$\langle 1^5,2\rangle$	6
$\langle 1^7\rangle$	7

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Archimedean toroidal maps and their minimal almost regular covers*

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Abstract

The automorphism group of a map acts naturally on its flags (triples of incident vertices, edges, and faces). An Archimedean map on the torus is called almost regular if it has as few flag orbits as possible for its type; for example, a map of type (4.8^2) is called almost regular if it has exactly three flag orbits. Given a map of a certain type, we will consider other more symmetric maps that cover it. In this paper, we prove that each Archimedean toroidal map has a unique minimal almost regular cover. By using the Gaussian and Eisenstein integers, along with previous results regarding equivelar maps on the torus, we construct these minimal almost regular covers explicitly.

Keywords: Maps, polytopes, groups, covers, Gaussian and Eisenstein integers.

Math. Subj. Class.: 52B15, 51M20, 52C22

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1 Introduction

Throughout the last few decades there have been many results about polytopes and maps that are highly symmetric, but that are not necessarily regular. In particular, there has been recent interest in the study of discrete objects using combinatorial, geometric, and algebraic approaches, with the topic of symmetries of maps receiving a lot of interest.

There is a great history of work surrounding maps on the Euclidean plane or on the 2-dimensional torus. When working with discrete symmetric structures on a torus, many of the ideas follow the concepts introduced by Coxeter and Moser in [5, 6], where they present a classification of regular (reflexible and irreflexible) maps on the torus. Such "toroidal" maps can be seen as quotients of regular tessellations of the Euclidean plane. More recently, Brehm and Kühnel [1] classified the equivelar maps on the two dimensional torus. Several more results have appeared about highly symmetric maps (see [4, 24] for example), and about highly symmetric tessellations of tori in larger dimensions (see [15, 16]).

There is also much interest in finding minimal regular covers of different families of maps and polytopes (see for example [10, 17, 22]). In a previous paper [7], two of the authors constructed the minimal rotary cover of any equivelar toroidal map. Here we extend this idea to toroidal maps that are no longer equivelar, and construct minimal toroidal covers of the Archimedean toroidal maps with maximal symmetry. We call these covers *almost regular*; they will no longer be regular (or chiral), but instead will have the same number of flag orbits as their associated tessellation of the Euclidean plane (see the definition in Section 2).

Our main results can be summarized by the following theorem.

Theorem 1.1. *Each Archimedean map on the torus has a minimal almost regular cover on the torus, this cover is unique and can be constructed explicitly.*

The paper is organized as follows. Section 2 contains the necessary background on maps and their symmetries, including the definition of an almost regular Archimedean map. In Section 3, almost regular Archimedean toroidal maps are characterized in terms of their lifts to the planar tessellations and the translation subgroups generating respective quotients. Section 4 contains our main results (Theorems 4.2 – 4.4) regarding the relationship between maps and their minimal almost regular covers; these theorems together constitute the Main Theorem.

2 Preliminares

In this section we provide definitions and results necessary for our main theorems; many of these ideas, as well as further details, can be found in [2, 16, 25].

A finite graph X embedded in a compact 2-dimensional manifold S such that every connected component of $S \setminus X$ (which is called a *face*) is homeomorphic to an open disc is called a *map* (on the surface S).

In this paper, we consider Archimedean maps on a flat 2-dimensional torus, which we call *Archimedean toroidal maps*. A map $\mathcal{M}$ on the torus is *Archimedean* if the faces of $\mathcal{M}$ are regular polygons (in the canonical flat metric on the torus) and every pair of vertices of $\mathcal{M}$ can be mapped into each other by an isometry of the torus (here an isometry of the torus is a distance preserving diffeomorphism of the torus, again with respect to the canonical flat metric). A map $\mathcal{M}$ is equivelar of (Schläfli) type $\{p, q\}$ if all of its vertices are q-valent, and all of its faces are regular p-gons. If the map is Archimedean, it can be described, as in [9], by the arrangement of polygons around a vertex, where a map of type $(p_1.p_2 \dots p_k)$ has k polygons (a p_1-gon, p_2-gon, …, and a p_k-gon) in the given order incident to each vertex. A particular vertex structure of a map is called a *type*. We note here that there is some debate over how the word Archimedean should be used. In some settings it means just that all of the faces are regular polygons and all of the vertex figures are the same. When you add the requirement that, for any pair of vertices, there exists a symmetry mapping one to the other, these maps are sometimes called *uniform* (for example, see [8, 20, 21, 23]).

As we will see below, Archimedean toroidal maps arise naturally as quotients of tessellations of the Euclidean plane with regular polygons; these tessellations are called *Archimedean*, and they are described in the same way as Archimedean maps. The following classical theorem gives a complete classification of planar Archimedean tessellations.

Theorem 2.1 (Classification of Archimedean tessellations on the plane [9])**.** *There are only* 11 *tessellations on the plane by regular polygons so that any vertex can be mapped to every other vertex by the symmetry of the tessellation. These are the following tessellations:*

$$\{3,6\},\ \{4,4\},\ \{6,3\},\ (4.8^2),\ (3.12^2),\ (3.6.3.6),\ (3.4.6.4),$$
$$(4.6.12),\ (3^2.4.3.4),\ (3^4.6),\ (3^3.4^2)$$

(see Figure 1).

The equivelar Archimedean tessellations of type $\{3,6\}$, $\{4,4\}$, and $\{6,3\}$ and the corresponding toroidal maps were considered in [7]. In this paper we will mainly work with the non-equivelar Archimedean tessellations of the Euclidean plane; denote $\mathcal{A}$ to be the set of all non-equivelar tessellations.

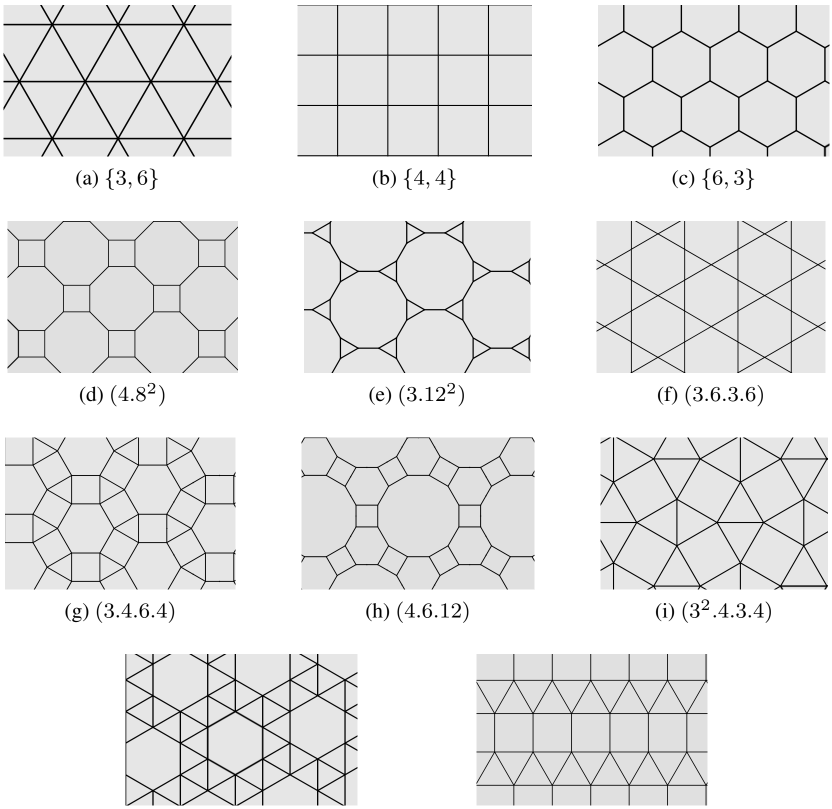
Furthermore, for reference, on each such tessellation we can place a Cartesian coordinate system with the origin at a vertex of the tessellation. For the tessellations $\{4,4\}$, $(3^2.4.3.4)$, and (4.8^2), the coordinate system is further specified by assuming that the vectors $\mathbf{e_1} := (1,0)$ and $\mathbf{e_2} := (0,1)$ represent the shortest possible translational symmetries of the tessellation. Similarly, for the remaining Archimedean tessellations, other than $(3^3.4^2)$, we assume that the vectors $\mathbf{e_1} := (1,0)$ and $\mathbf{e_2} := \big(1/2, \sqrt{3}/2\big)$ represent the shortest possible translational symmetries.

This choice of coordinate system will allow us to utilize the geometry of the Gaussian and Eisenstein integers in our results (see Subsection 4.1). Given an Archimedean tessellation τ, we call $(\mathbf{e_1}, \mathbf{e_2})$ *the basis for* τ. For the tessellation $(3^3.4^2)$ the basis will be specified separately in the later sections.

Let $(\mathbf{e_1}, \mathbf{e_2})$ be the basis for $\tau \in \mathcal{A} \setminus \{(3^3.4^2)\}$. The set $\{\lambda\mathbf{e_1} + \mu\mathbf{e_2} : \lambda, \mu \in \mathbb{Z}\}$ forms the vertex set of a regular tessellation which we call *the tessellation associated with* τ, and which we denote by τ^*. By construction, $(\mathbf{e_1}, \mathbf{e_2})$ is also the basis for τ^*. To clarify this notation, we note here that for example if τ is of type $\{6,3\}$, then τ^* is of type $\{3,6\}$.

Given an Archimedean tessellation τ, denote by T_τ the maximal group of translations that preserve τ. As we already mentioned, Archimedean toroidal maps can be seen as quotients of planar Archimedean tessellations. These quotients can be written explicitly in terms of possible subgroups of T_τ, due to the following theorem.

Theorem 2.2 (Archimedean toroidal maps are quotients [19])**.** *Let $\mathcal{M}$ be an Archimedean map on the torus. Then there exists an Archimedean tessellation τ of the Euclidean plane and a subgroup $G \le T_\tau$ so that $\mathcal{M} = \tau/G$.*



(a) $\{3, 6\}$ (b) $\{4, 4\}$ (c) $\{6, 3\}$

(d) (4.8^2) (e) (3.12^2) (f) $(3.6.3.6)$

(g) $(3.4.6.4)$ (h) $(4.6.12)$ (i) $(3^2.4.3.4)$

(j) $(3^4.6)$ (k) $(3^3.4^2)$

Figure 1: Archimedean tessellations.

This theorem shows that, for each type, there is a one-to-one correspondence between Archimedean toroidal maps and translation subgroups of T_τ; clearly, the pair of generators of every such subgroup should be comprised of non-collinear vectors.

We point out here that the converse of Theorem 2.2 is also true, as clearly any map on the torus that is obtained as a quotient of an Archimedean tessellation by a translation subgroup is an Archimedean toroidal map.

Let $\mathcal{M}$ be an Archimedean toroidal map; by Theorem 2.2, it can be written as $\tau / \langle \mathbf{a}, \mathbf{b} \rangle$, where τ is the planar Archimedean tessellation (of the same type as $\mathcal{M}$) and $\langle \mathbf{a}, \mathbf{b} \rangle \leq T_\tau$ is the translation subgroup with generators $\mathbf{a}, \mathbf{b} \in T_\tau$. We use the standard notation $\tau_{\mathbf{a},\mathbf{b}} := \tau / \langle \mathbf{a}, \mathbf{b} \rangle$ for the map $\mathcal{M}$. Note that the pair $\mathbf{a}, \mathbf{b}$ is not uniquely defined by $\mathcal{M}$, but the quotient is independent of possible choices.

A *flag* of a planar tessellation τ is a triple of an incident vertex, edge, and face of the tessellation. We can then define a flag of a toroidal map $\tau_{\mathbf{a},\mathbf{b}}$ as the orbit of a flag under the group $\langle \mathbf{a}, \mathbf{b} \rangle$. We note that when the map is combinatorially equivalent to an abstract polytope (see [16]), this is equivalent to a flag equaling a triple of an incident vertex, edge, and face of the map itself. Two flags of a map on the torus are said to be *adjacent* if they lift to flags in the plane that differ in exactly one element.

Let $\mathcal{N} = \tau/H$ and $\mathcal{M} = \tau/G$ be Archimedean maps on the torus, where H is a subgroup of G. Then there is a surjective function $\eta\colon \mathcal{N} \to \mathcal{M}$ that preserves adjacency and sends vertices of $\mathcal{N}$ to vertices of $\mathcal{M}$ (and edges to edges, and faces to faces). The function η is called a *covering* of the map $\mathcal{M}$ by the map $\mathcal{N}$. This is denoted by $\mathcal{N} \searrow \mathcal{M}$, and we say that $\mathcal{N}$ is a *cover* of $\mathcal{M}$. We can use the notion of covering to create a partial order $\leq$ on any non-empty set $\mathcal{S}$ of toroidal maps, where $\mathcal{M} \leq \mathcal{N}$ if and only if $\mathcal{N}$ is a cover of $\mathcal{M}$. A *minimal cover* of a map $\mathcal{M} \in \mathcal{S}$ is minimal with respect to this partial order in $\mathcal{S}$. We note here that this notion of covering can also be generalized to maps on different surfaces, and to abstract polytopes of higher ranks. If $\mathcal{S}$ is the set of all regular maps that cover a given map $\mathcal{M}$, then the minimal elements of the partial order are the minimal regular covers of $\mathcal{M}$, as studied in [10, 18] for example.

For a map $\mathcal{M} = \tau/G$, where $G = \langle \mathbf{a}, \mathbf{b} \rangle$, we call the parallelogram spanned by the vectors $\mathbf{a}, \mathbf{b}$ a *fundamental region* of $\mathcal{M}$. Then a fundamental region for the covering map $\mathcal{N} = \tau/H$, $H \leq G$, can be viewed as k fundamental regions of $\mathcal{M}$ 'glued together' (see Figure 2). It is easy to show that the number k is equal to the index $[G : H]$ of the subgroup H in G.

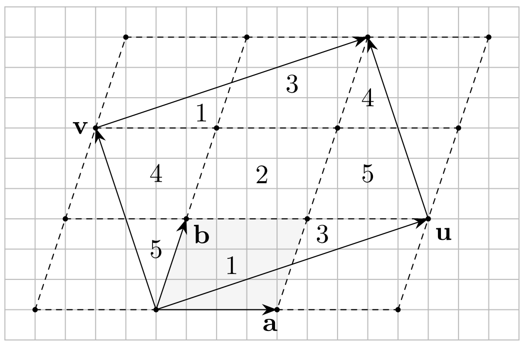


Figure 2: $\{4,4\}_{\mathbf{u},\mathbf{v}} \searrow \{4,4\}_{\mathbf{a},\mathbf{b}}$ is a 5-sheeted covering, and the covering map $\{4,4\}_{\mathbf{u},\mathbf{v}}$ is obtained by gluing together 5 fundamental regions of $\{4,4\}_{\mathbf{a},\mathbf{b}}$.

2.1 Symmetries and automorphisms of tessellations and maps

In this section we follow [3, 11, 12] and [13] in our notation and definitions.

Let τ be a tessellation of the Euclidean plane, and let $\mathrm{Aut}(\tau)$ be its symmetry group (the collection of isometries of the Euclidean plane that preserve the tessellation). Let G and H be two subgroups of $\mathrm{Aut}(\tau)$ generated by two linearly independent translations. The maps τ/G and τ/H are isomorphic if G and H are conjugate in $\mathrm{Aut}(\tau)$.

A symmetry $\gamma \in \mathrm{Aut}(\tau)$ induces an automorphism of a toroidal map τ/G if and only if it normalizes G, that is $\gamma G \gamma^{-1} = G$; denote $\mathrm{Norm}_{\mathrm{Aut}(\tau)}(G)$ for the group of elements in $\mathrm{Aut}(\tau)$ that normalize G. Geometrically, such γ maps fundamental regions of τ/G to fundamental regions of τ/G.

Finally, the we define the automorphism group $\mathrm{Aut}(\tau/G)$ as the group induced by the normalizer $\mathrm{Norm}_{\mathrm{Aut}(\tau)}(G)$; in other words $\mathrm{Aut}(\tau/G) \cong \mathrm{Norm}_{\mathrm{Aut}(\tau)}(G)/G$. We will also denote the collection of symmetries $\mathrm{Norm}_{\mathrm{Aut}(\tau)}(G)$ as simply $\mathrm{Sym}(\tau/G)$.

We note here that an automorphism of a map can equivalently be defined as an automorphism of the underlying graph that can be extended to a homeomorphism of the surface.

A map $\mathcal{M}$ is called *regular* if its automorphism group acts transitively on the set of flags. A map $\mathcal{M}$ is called *chiral* if its automorphism group has two orbits on flags with adjacent flags lying in different orbits. A map $\mathcal{M}$ is called *rotary* if it is either regular or chiral.

For the toroidal maps of type $\{4,4\}$, $\{3,6\}$, and $\{6,3\}$ the minimum possible number of flag orbits is one, given by regular maps of those types (which have been previously classified, see also Subsection 2.2). For other types of maps on the torus, the minimum number of flag orbits will not be one. However, we still would like to understand the maps of each type that achieve the fewest possible number of flag orbits.

An Archimedean map on the torus is called *almost regular* if it has the same number of flag orbits under the action of its automorphism group as the Archimedean tessellation on the plane of the same type has under the action of its symmetry group.

2.2 Regular and chiral toroidal maps

The classification in the next sections depends heavily on the classification of regular and chiral toroidal maps. Here we summarize the relevant details about toroidal maps of type $\{4,4\}$ and $\{3,6\}$ that are needed in our results. The results in this subsection, and much more can all be found in [13].

Let τ be a tessellation of the Euclidean plane of type $\{4,4\}$ or $\{3,6\}$. Then $\mathrm{Aut}(\tau)$ is of the form $T_\tau \rtimes \mathbf{S}$, where $\mathbf{S}$ is the stabilizer of a vertex of τ, which we can assume to be the origin without loss; $\mathbf{S}$ is called *a point stabilizer*.

Then let $\mathcal{M} = \tau/G$ be a toroidal map. Notice that every translation in T_τ induces an automorphism of $\mathcal{M}$ (where the elements of G induce the trivial automorphism). Define χ as the central inversion of the Euclidean plane, that is the symmetry that sends any vector $\mathbf{u}$ to $-\mathbf{u}$. Then, as seen in Lemma 6 of [13], $\mathrm{Aut}(\mathcal{M})$ is induced by a group $\mathbf{K}$ so that $T_\tau \rtimes \langle\chi\rangle \le \mathbf{K} \le \mathrm{Aut}(\tau)$.

Furthermore, there is a bijection between such groups $\mathbf{K}$ and subgroups $\mathbf{K}'$ of $\mathbf{S}$ containing χ, and thus one needs to determine which symmetries in the point stabilizer $\mathbf{S}$ normalize G. Finally, the number of flag orbits of the toroidal map $\mathcal{M}$ is the index of $\mathrm{Norm}_{\mathrm{Aut}(\tau)}(G)$ in $\mathrm{Aut}(\tau)$, which is the same as the index of $\mathbf{K}'$ in $\mathbf{S}$.

First let us consider toroidal maps of type $\{4,4\}$; let τ be the regular tessellation of the Euclidean plane of this type, and $(\mathbf{e_1},\mathbf{e_2})$ be the basis for τ. The point stabilizer $\mathbf{S}$ is generated by two reflections R_1 and R_2, where R_1 is reflection across the line spanned by $\mathbf{e_1}+\mathbf{e_2}$, sending vectors (x,y) to (y,x), and R_2 is reflection across the line spanned by $\mathbf{e_1}$, sending vectors (x,y) to $(x,-y)$. There are exactly three conjugacy classes of proper subgroups $\mathbf{K}'$ of $\mathbf{S}$ containing χ but not equal to $\langle\chi\rangle$. In other words there are exactly five possible point stabilizers: all of $\mathbf{S}$, only $\langle\chi\rangle$, and finally the three groups described next. The three subgroups are $\mathbf{K}'$ are $\langle\chi,R_1\rangle$, $\langle\chi,R_2\rangle$, and $\langle\chi,R_1R_2\rangle$, and each has index 2 in $\mathbf{S}$, where $\langle\chi\rangle$ has index 4 in $\mathbf{S}$.

We note that it is important for our classification to notice that a toroidal map of type $\{4,4\}$ is regular if and only if $\mathbf{K}'$ contains both R_1 and R_2, as well as R_1R_2 which is the rotation by $\pi/2$ around the origin. This occurs only in the two well known families of regular toroidal maps, $\{4,4\}_{(a,0),(0,a)}$ and $\{4,4\}_{(a,a),(a,-a)}$, both of which have squares as fundamental regions. The chiral toroidal maps of type $\{4,4\}$ also have squares as their fundamental regions, but have no reflections in their automorphism groups. Finally, the remaining classes of toroidal maps have fundamental regions that are not squares.

Next, let us consider toroidal maps of type $\{3,6\}$; let τ be the regular tessellation of the Euclidean plane of this type. We use the previously described basis of $\mathbf{e_1} = (1, 0)$ and $\mathbf{e_2} = \big(1/2, \sqrt{3}/2\big)$ to describe the symmetries of these maps.

The point stabilizer $\mathbf{S}$ is again generated by two reflections R_1 and R_2, where R_1 is reflection across the line spanned by $\mathbf{e_1} + \mathbf{e_2}$, sending vectors (x, y) to (y, x), and R_2 is reflection across the line spanned by $\mathbf{e_1}$, sending vectors (x, y) to $(x + y, -y)$.

Now there are exactly two conjugacy classes of proper subgroups $\mathbf{K}'$ of $\mathbf{S}$ containing χ but not equal to $\langle\chi\rangle$. These subgroups $\mathbf{K}'$ are $\langle\chi, R_1R_2\rangle$, with index 2 in $\mathbf{S}$, and $\langle\chi, R_2\rangle$ with index 3 in $\mathbf{S}$, where $\langle\chi\rangle$ has index 6 in $\mathbf{S}$.

We note that it is important for our classification to notice that a toroidal map of type $\{3, 6\}$ is regular if and only if $\mathbf{K}'$ contains both R_1 and R_2, as well as R_1R_2 which is the rotation by $\pi/3$ around the origin. This occurs only in the two well known families of regular toroidal maps, $\{3, 6\}_{(a,0),(0,a)}$ and $\{3, 6\}_{(a,a),(2a,-a)}$. For those two families the fundamental regions are parallelograms composed of two regular triangles. The chiral toroidal maps of type $\{3, 6\}$ also have parallelograms composed of two regular triangles as their fundamental regions, but have no reflections in their automorphism groups; this is similar to the type $\{4, 4\}$.

3 Almost regular maps

In this section we consider Archimedean tessellations of the torus with as much symmetry as possible. As we already mentioned, one natural way to understand the symmetry of a map is to consider the action of its automorphism group on its flags. Here we want to understand the maps on the torus with as few flag orbits as possible.

Theorem 3.1 (Regular to almost regular maps)**.** *For*

$$\mathcal{A}_{reg} := \big\{(4.8^2),\ (3.6.3.6),\ (3.12^2),\ (4.6.12),\ (3.4.6.4),\ (3^2.4.3.4)\big\}\,,$$

let $\tau \in \mathcal{A}_{reg}$ be an Archimedean tessellation of one of these types. Then $\tau_{\mathbf{u},\mathbf{v}}$ is an almost regular Archimedean map if and only if $(\tau^*)_{\mathbf{u},\mathbf{v}}$ is a regular map on the torus, with τ^* being the regular tessellation associated with τ.

The proof of this theorem will follow from the following six propositions, each separately dealing with a type of map. In each case we will use the translational symmetries to simplify the problem by considering a fundamental region of τ/T_τ.

Proposition 3.2 (Almost regular maps of type (4.8^2))**.** *A map $\mathcal{M} = \tau/G$ on the torus of type (4.8^2) is almost regular (with three flag orbits) if and only if τ^*/G is regular.*

Proof. Notice first that τ^* is of type $\{4, 4\}$, and let $(\mathbf{e_1}, \mathbf{e_2})$ be the basis for τ (and hence for τ^*).

Assume that a map $\mathcal{M}$ on the torus of type (4.8^2) has exactly three flag orbits. For this to be the case, there must be the following symmetries in $\mathrm{Sym}(\mathcal{M})$, as shown in Figure 3:

- reflection across a line in the direction $\mathbf{e_1} + \mathbf{e_2}$ through the center of a square of the map;
- reflection across a line in the direction $\mathbf{e_1}$ through the center of a square and the edge of an octagon.

It was summarized in Subsection 2.2 that the existence of these symmetries is enough to show that τ^*/G is regular.

Conversely, if τ^*/G is regular, then the fundamental region of $\mathcal{M}$ is a square, and each of the previous listed symmetries are elements of $\mathrm{Sym}(\mathcal{M})$. Furthermore, every translational symmetry in $\mathrm{Sym}(\tau/T_\tau)$ is also in $\mathrm{Sym}(\tau/G)$, and thus $\mathcal{M}$ has only three flag orbits. Note that the translations in $\mathrm{Sym}(\tau/T_\tau)$ act on the flags in 24 flag orbits, and then the listed symmetries force there to only be three orbits.

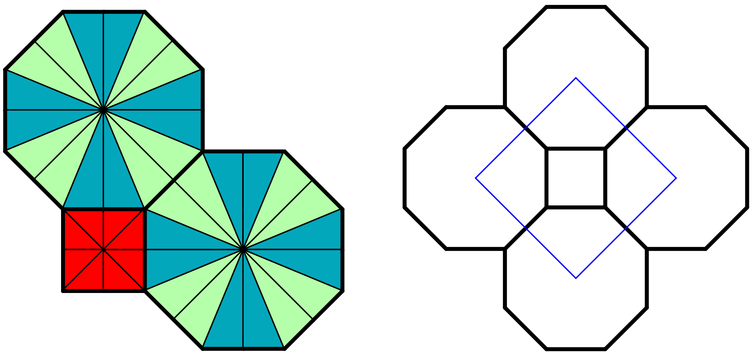


Figure 3: Minimum number of flag orbits for τ of type (4.8^2), and a fundamental region of τ/T_τ, where the vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ form the boundary of the fundamental region.

On the left of Figure 3, and other figures to follow, all the faces incident to single vertex are shown, and the flags in these faces, which can be seen as triangles in the barycentric subdivision of the tessellation, are colored based on their orbit. On the right of Figure 3, the fundamental region of τ/T_τ is shown in blue, with the underlying tessellation shown in black. One can see, for example, that this fundamental region contains 24 flags of τ. $\square$

The proofs of the following five propositions is similar to the proof of Proposition 3.2.

Proposition 3.3 (Almost regular maps of type $(3.6.3.6)$)**.** *A map $\mathcal{M} = \tau/G$ on the torus of type $(3.6.3.6)$ is almost regular (with two flag orbits) if and only if τ^*/G is regular.*

Proof. Notice first that τ^* is of type $\{3, 6\}$, and let $(\mathbf{e_1}, \mathbf{e_2})$ be the basis for τ.

Assume that a map $\mathcal{M}$ on the torus of type $(3.6.3.6)$ has exactly two flag orbits. For this to be the case, there must be the following symmetries in $\mathrm{Sym}(\mathcal{M})$, as shown in Figure 4:

- reflection across a line in the direction $\mathbf{e_1} + \mathbf{e_2}$ going through the centers of a hexagon and an adjacent triangle;
- reflection across a line in the direction $\mathbf{e_1}$ going through the centers of two hexagons incident to the same vertex.

As summarized in Subsection 2.2, the existence of these symmetries is enough to conclude that τ^*/G is regular.

Conversely, if τ^*/G is regular, then each of the previous listed symmetries are elements of $\mathrm{Sym}(\mathcal{M})$. Furthermore, every translational symmetry in $\mathrm{Sym}(\tau/T_\tau)$ is also in $\mathrm{Sym}(\tau/G)$, and thus $\mathcal{M}$ has only two flag orbits. Note that the translations in $\mathrm{Sym}(\tau/T_\tau)$ act on the flags in 24 flag orbits, and then the listed symmetries force there to only be two orbits. $\square$

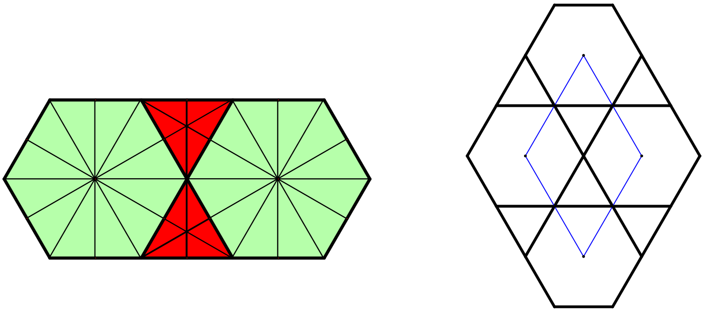


Figure 4: Minimum number of flag orbits for τ of type $(3.6.3.6)$, and a fundamental region of τ/T_τ (as above, the vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ form the boundary of the fundamental region).

Proposition 3.4 (Almost regular maps of type (3.12^2))**.** *A map $\mathcal{M} = \tau/G$ on the torus of type (3.12^2) is almost regular (with three flag orbits) if and only if τ^*/G is regular.*

Proof. Notice again that τ^* is of type $\{3, 6\}$, and let $(\mathbf{e_1}, \mathbf{e_2})$ be the basis for τ.

Assume that a map $\mathcal{M}$ on the torus of type (3.12^2) has exactly three flag orbits. For this to be the case, there must be the following symmetries in $\mathrm{Sym}(\mathcal{M})$, as shown in Figure 5:

- reflection across a line in the direction $\mathbf{e_1} + \mathbf{e_2}$ going through the centers of a 12-gon and an adjacent triangle;
- reflection across a line in the direction $\mathbf{e_1}$ going through the centers of two adjacent 12-gons.

As in the previous proposition, the existence of these symmetries again forces τ^*/G to be regular.

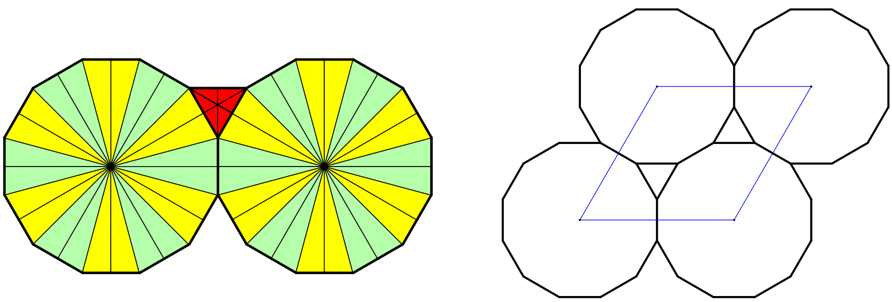


Figure 5: Minimum number of flag orbits for τ of type (3.12^2), and a fundamental region of τ/H (as above, the vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ form the boundary of the fundamental region).

Conversely, if τ^*/G is regular, then each of the previous three listed symmetries are elements of $\mathrm{Sym}(\mathcal{M})$. Furthermore, every translational symmetry in $\mathrm{Sym}(\tau/T_\tau)$ is also in $\mathrm{Sym}(\tau/G)$, and thus $\mathcal{M}$ has only three flag orbits. Note that the translations in $\mathrm{Sym}(\tau/T_\tau)$ act on the flags in 36 flag orbits, and then the listed symmetries force there to only be three orbits. □

Proposition 3.5 (Almost regular maps of type $(4.6.12)$)**.** *A map $\mathcal{M} = \tau/G$ on the torus of type $(4.6.12)$ is almost regular (with six flag orbits) if and only if τ^*/G is regular.*

Proof. Notice again that τ^* is of type $\{3, 6\}$, and let $(\mathbf{e_1}, \mathbf{e_2})$ be the basis for τ.

Assume that a map $\mathcal{M}$ on the torus of type $(4.6.12)$ has exactly six flag orbits. For this to be the case, there must be the following symmetries in $\mathrm{Sym}(\mathcal{M})$:

- reflection across a line in the direction $\mathbf{e_1} + \mathbf{e_2}$ going through the centers of a 12-gon and an adjacent hexagon;
- reflection across a line in the direction $\mathbf{e_1}$ going through the centers of a 12-gon and an adjacent square.

As in the previous proposition, the existence of these symmetries again forces τ^*/G to be regular.

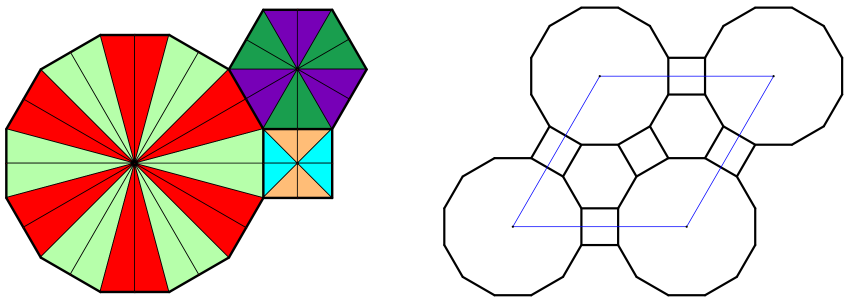


Figure 6: Minimum number of flag orbits for τ of type $(4.6.12)$, and a fundamental region of τ/T_τ (as above, the vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ form the boundary of the fundamental region).

Conversely, if τ^*/G is regular, then each of the previous three listed symmetries are elements of $\mathrm{Sym}(\mathcal{M})$. Furthermore, every translational symmetry in $\mathrm{Sym}(\tau/T_\tau)$ is also in $\mathrm{Sym}(\tau/G)$, and thus $\mathcal{M}$ has only six flag orbits. Note that the translations in $\mathrm{Sym}(\tau/T_\tau)$ act on the flags in 72 flag orbits, and then the listed symmetries force there to only be six orbits. □

Proposition 3.6 (Almost regular maps of type $(3.4.6.4)$)**.** *A map $M = \tau/G$ on the torus of type $(3.4.6.4)$ is almost regular (with four flag orbits) if and only if τ^*/G is regular.*

Proof. Notice again that τ^* is of type $\{3, 6\}$. Let $(\mathbf{e_1}, \mathbf{e_2})$ be the basis for τ. Assume that a map $\mathcal{M}$ on the torus of type $(3.4.6.4)$ has exactly four flag orbits. For this to be the case, there must be the following symmetries in $\mathrm{Sym}(\mathcal{M})$:

- reflection across a line in the direction $\mathbf{e_1} + \mathbf{e_2}$ going through the centers of a hexagon and a triangle sharing an incident vertex;
- reflection across a line in the direction $\mathbf{e_1}$ going through the centers of a 12-gon and an adjacent square.

Again the existence of these symmetries again forces τ^*/G to be regular. Conversely, if τ^*/G is regular, then each of the previous three listed symmetries are elements of $\mathrm{Sym}(\mathcal{M})$. Furthermore, every translational symmetry in $\mathrm{Sym}(\tau/T_\tau)$ is also in $\mathrm{Sym}(\tau/G)$, and thus $\mathcal{M}$ has only four flag orbits. Note that the translations in $\mathrm{Sym}(\tau/T_\tau)$ act on the flags in 48 flag orbits, and then the listed symmetries force there to only be four orbits. □

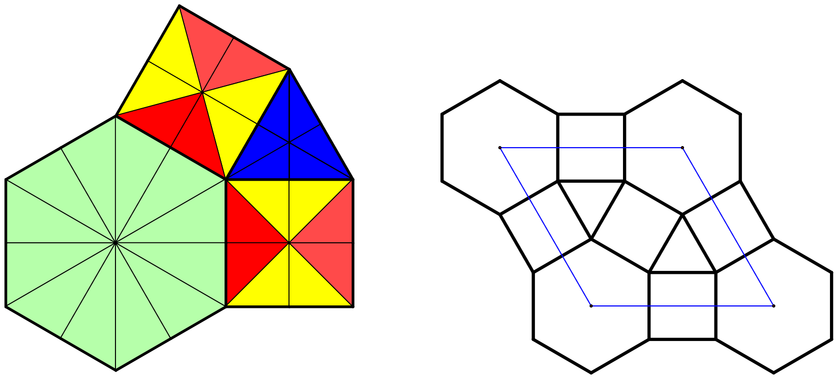


Figure 7: Minimum number of flag orbits for τ of type $(3.4.6.4)$, and a fundamental region of τ/T_τ (as above, the vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ form the boundary of the fundamental region).

Proposition 3.7 (Almost regular maps of type $(3^2.4.3.4)$)**.** *A map $\mathcal{M} = \tau/G$ on the torus of type $(3^2.4.3.4)$ is almost regular (with five flag orbits) if and only if τ^*/G is regular.*

Proof. Notice again that τ^* is of type $\{4,4\}$. Let $(\mathbf{e_1}, \mathbf{e_2})$ be the basis for τ.

Assume that a map $\mathcal{M}$ on the torus of type $(3^2.4.3.4)$ has exactly five flag orbits. For this to be the case, there must be a rotation by $\pi/2$ around the center of a square in $\mathrm{Sym}(\mathcal{M})$. This symmetry can be represented by R_1R_2 as described in Subsection 2.2, and thus τ^*/G is either regular or chiral. However, $\mathrm{Sym}(\mathcal{M})$ must also contain a reflection across the edge of adjacent triangles in the direction of $\mathbf{e_1} + \mathbf{e_2}$. This means that the 8 flags in the fundamental region of τ^*/T_τ are all in the same orbit, and thus τ^*/G is regular.

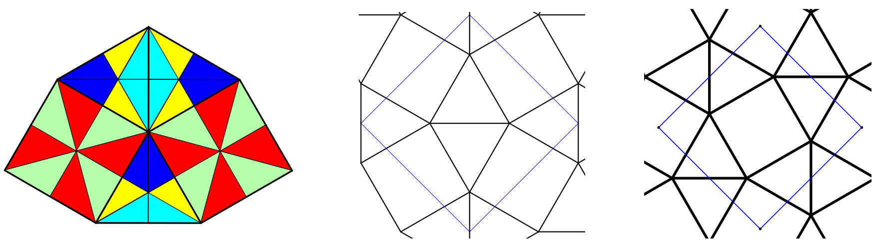


Figure 8: Minimum number of flag orbits for τ of type $(3^2.4.3.4)$, and a fundamental region of τ/T_τ drawn in two equivalent ways so that to show existence of mirror symmetries (second picture) and rotational symmetry (third picture); as above, the vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ form the boundary of the fundamental region.

Conversely, if τ^*/G is regular, then there is a rotation by $\pi/2$ around the center of a square, as well as a reflection across the edge of adjacent triangles in $\mathrm{Sym}(M)$. Furthermore, every translational symmetry in $\mathrm{Sym}(\tau/T_\tau)$ is also in $\mathrm{Sym}(\tau/G)$, and thus M has only five flag orbits. □

Theorem 3.8 (Rotary to almost regular toroidal map)**.** *Let τ be the Archimedean tessellation of type $(3^4.6)$. Then $\tau_{\mathbf{u},\mathbf{v}}$ is an almost regular Archimedean map (with ten flag orbits) if and only if $\tau^*_{\mathbf{u},\mathbf{v}}$ is a rotary map on the torus.*

Proof. Suppose that the map $\tau_{\mathbf{u},\mathbf{v}}$ has exactly ten flag orbits. For this to be the case, there must be a rotation by $\pi/3$ around the center of a hexagon in $\mathrm{Sym}(\tau_{\mathbf{u},\mathbf{v}})$. The existence of this symmetry forces $\tau^*/\langle \mathbf{u}, \mathbf{v}\rangle$ to be rotary. Note, these are the only reflexive symmetries in $\mathrm{Sym}(\tau_{\mathbf{u},\mathbf{v}})$.

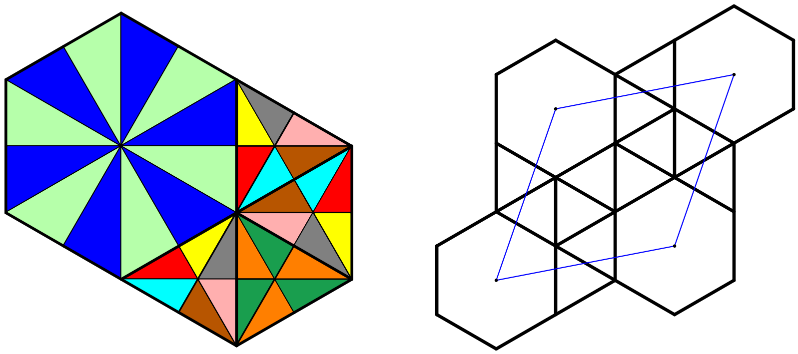


Figure 9: Minimum number of flag orbits for τ of type $(3^4.6)$, and a fundamental region of τ/H (as above, the vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ form the boundary of the fundamental region).

Conversely, if $\tau^*_{\mathbf{u},\mathbf{v}}$ is rotary, then there is a rotation by $\pi/3$ around the center of a hexagon in $\mathrm{Sym}(\tau_{\mathbf{u},\mathbf{v}})$. Furthermore, every translational symmetry in $\mathrm{Sym}(\tau/T_\tau)$ is also in $\mathrm{Sym}(\tau_{\mathbf{u},\mathbf{v}})$, and thus $\tau_{\mathbf{u},\mathbf{v}}$ has only ten flag orbits. Note that the translations in $\mathrm{Sym}(\tau/T_\tau)$ act on the flags in 60 flag orbits, and then the listed symmetry forces there to only be ten orbits. □

Theorems 3.1 and 3.8 provide us with a fair understanding of how almost regular maps of type $\mathcal{A}_{\text{reg}}$ and $(3^4.6)$ look: for each of them the associated map on the torus must be regular, respectfully rotary.

The only remaining Archimedean tessellation not covered by the previous two results is $(3^3.4^2)$. Since the translation subgroup of the symmetry group of $(3^3.4^2)$ does not coincide with the symmetry group of one of the regular planar tessellations (as it is for all tessellations in $\mathcal{A}_{\text{reg}}$), and does not contain the rotation subgroup of a regular planar tessellation (as it is for the tessellation $(3^4.6)$), we have to deal with $(3^3.4^2)$ separately and with different techniques.

In order to state a complete characterization of almost regular maps of type $(3^3.4^2)$, we introduce the following notation: write $(\mathbf{e_1}, \mathbf{e_2})$ for the positively-oriented basis of the plane $\mathbb{E}^2$ represented by the shortest non-parallel translations that are in the symmetry group of $(3^3.4^2)$. Recall also that T_τ stands for the maximal translation subgroup of the symmetry group $\mathrm{Sym}(\tau)$ of a given Archimedean tessellation τ.

Theorem 3.9 (Almost regular maps of type $(3^3.4^2)$)**.** *Let* $\tau = (3^3.4^2)$. *Then* $\mathcal{M} = \tau/G$, *with* $G < T_\tau$, *is an almost regular Archimedean map (with five flag orbits) if and only if the translation subgroup* G *is of the form*

$$\left\langle c\,\mathbf{e_1}, -d\,\mathbf{e_1} + 2d\,\mathbf{e_2}\right\rangle,\ \textit{or}\ \left\langle c\,\mathbf{e_1}, -d\,\mathbf{e_1} + (c+2d)\,\mathbf{e_2}\right\rangle \tag{3.1}$$

for non-zero integers c *and* d.

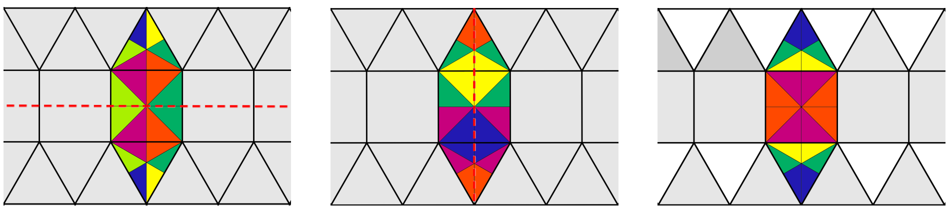
Remark 3.10. Observe that the statement above, in fact, does not depend on the choice of the basis for $\mathbb{E}^2$. The explicit coordinate form (3.1) was used only to simplify later search for minimal almost regular covers in the proof of Theorem 4.4.

Remark 3.11. The groups listed in (3.1) are never isomorphic.

Proof. The beginning of the proof is similar to the proofs given for the tessellations in $\mathcal{A} \setminus \{(3^3.4^2)\}$. Assume that a map $\mathcal{M}$ has exactly five flag orbits. For this to be the case, there must be the following symmetries in $\mathrm{Sym}(\mathcal{M})$:

- reflections across a horizontal line through the center of a square (see Figure 10(a));
- reflections across a vertical line through the center of a square (see Figure 10(b)).

We will write $\mathbf{h}$, respectively $\mathbf{v}$, for a reflection across a horizontal, resp. vertical, line through the center of a square, where a horizontal line is in the direction of $\mathbf{e_1}$.



(a) Reflection across the horizontal line; 10 flag orbits in the fundamental region.

(b) Reflection across the vertical line; 10 flag orbits in the fundamental region.

(c) The minimal number of flag orbits in the fundamental region.

Figure 10: Flag orbits in the fundamental region of $(3^3.4^2)/T_{(3^3.4^2)}$.

Let us find all possible subgroups $G < T_\tau$ such that the listed symmetries preserve G by conjugation. Suppose G is generated by a pair of non-parallel vectors $\mathbf{a}, \mathbf{b} \in T_\tau$, and assume that $\mathbf{h} \circ \mathbf{u} \circ \mathbf{h}^{-1} \in G$ and $\mathbf{v} \circ \mathbf{u} \circ \mathbf{v}^{-1} \in G$ for every $\mathbf{u} \in G$.

Because the basis $(\mathbf{e_1}, \mathbf{e_2})$ of $\mathbb{E}^2$ was chosen in such a way that both $\mathbf{e_1}$ and $\mathbf{e_2}$ are the symmetries of τ which generate the group T_τ (i.e. $\langle \mathbf{e_1}, \mathbf{e_2} \rangle = T_\tau$), there exist two pairs of integers a_1, a_2 and b_1, b_2 such that

$$\mathbf{a} = a_1\mathbf{e_1} + a_2\mathbf{e_2} = (a_1, a_2), \quad \mathbf{b} = b_1\mathbf{e_1} + b_2\mathbf{e_2} = (b_1, b_2).$$

Note that if $2a_1 + a_2 = 2b_1 + b_2 = 0$, then $\mathbf{a}$ and $\mathbf{b}$ are parallel, which is impossible. Hence, without loss of generality we can assume $2a_1 + a_2 \neq 0$; this technical assumption will be used later in the proof.

In order to understand the structure of the group G, observe that

$$\begin{aligned} R_v(\mathbf{e_1}) &:= \mathbf{v} \circ \mathbf{e_1} \circ \mathbf{v}^{-1} = -\mathbf{e_1}, \quad & R_v(\mathbf{e_2}) &= \mathbf{e_2} - \mathbf{e_1}, \\ R_h(\mathbf{e_1}) &:= \mathbf{h} \circ \mathbf{e_1} \circ \mathbf{h}^{-1} = \mathbf{e_1}, \quad & R_h(\mathbf{e_2}) &= \mathbf{e_1} - \mathbf{e_2}. \end{aligned} \tag{3.2}$$

For an element $\mathbf{u} \in T_\tau$, the action of R_v and R_h on $\mathbf{u}$ is defined using (3.2) by linearity.

Since the reflection across a vertical line through the center of a square preserves G by conjugation, the group G must contain the vector $R_v(\mathbf{a}) = (-a_1 - a_2, a_2)$, as it is easy to compute from (3.2). Hence G contains the vector $\mathbf{a} - R_v(\mathbf{a}) = (2a_1 + a_2, 0)$, which is not zero since $2a_1 + a_2 \neq 0$ by our assumption. Therefore, G contains a proper non-trivial subgroup G' of vectors with vanishing second coordinate. Pick $\mathbf{c} = (c_1, 0)$ with $c_1 > 0$ to be a generator of G'. Observe that, in fact, $\mathbf{c}$ is the shortest vector among all vectors in G with positive first coordinate and vanishing second coordinate.

Let $\mathbf{d} \in G$ be a vector such that $G = \langle \mathbf{c}, \mathbf{d} \rangle$. Moreover, since $\langle \mathbf{c}, \mathbf{d} \rangle = \langle \mathbf{c}, \mathbf{d} + k\,\mathbf{c} \rangle$ for every $k \in \mathbb{Z}$, by picking an appropriate k we may assume that $\mathbf{d}$ is chosen in such a way that in coordinates $\mathbf{d} = (d_1, d_2)$ we have

$$d_1 \in \left[-\frac{d_2}{2}, c_1 - \frac{d_2}{2} \right). \tag{3.3}$$

Now we use the second symmetry from our list: since the conjugation by a reflection across a horizontal line through the center of a square preserves G, the vectors $(2d_1 + d_2, 0)$ and $(2d_1 + d_2 - c_1, 0)$ must be in G'. Indeed, (3.2) and linearity of R_h gives us $R_h(\mathbf{d}) = (d_1 + d_2, -d_2)$, and this vector must be in G. Hence, $R_h(\mathbf{d}) + \mathbf{d} = (2d_1 + d_2, 0) \in G' < G$. Similarly, $R_h(\mathbf{d}) + \mathbf{d} - \mathbf{c} = (2d_1 + d_2 - c_1, 0) \in G' < G$.

Recall that $\mathbf{c} = (c_1, 0)$ was the shortest vector in G' among all vectors with positive first coordinate. Therefore, as $(2d_1 + d_2, 0) \in G'$ we must have either $2d_1 + d_2 = 0$, or $2d_1 + d_2 \geqslant c_1$ (note that $2d_1 + d_2$ is necessarily non-negative by assumption (3.3)). In the latter case it then follows from (3.3) that $0 \leqslant 2d_1 + d_2 - c_1 < c_1$, which is possible, again by minimality of c_1, only if $2d_1 + d_2 - c_1 = 0$.

Therefore, either $d_2 = -2d_1$, or $d_2 = -2d_1 + c_1$, and hence setting $c := c_1$, $d := -d$ we obtain that the group G must be of one of the types in (3.1). One implication in Theorem 3.9 is proven.

Conversely, it is straightforward to see that a group of one of the types in (3.1) is preserved by conjugation with both $\mathbf{h}$ and $\mathbf{v}$, and hence the symmetry group $\mathrm{Sym}(\tau/G)$ contains both $\mathbf{h}$ and $\mathbf{v}$, which implies that τ/G has only five flag orbits, and thus is an almost regular Archimedean map. □

4 Minimal covers of Archimedean toroidal maps

In this section we will prove the Main Theorem. This will be done by combination of three statements — Theorem 4.2, 4.3 and 4.4. Prior the proofs, in the following subsection we recall some of the facts about the Gaussian and the Eisenstein integers — a number-theoretic tool that provides a natural 'language' for our main results.

4.1 Gaussian and Eisenstein integers

The Gaussian and Eisenstein integers provide an essential ingredient for understanding the Archimedean toroidal maps. The plots of these domains in the complex plane are the vertex sets of regular tessellations: a tessellation of type $\{4,4\}$ for the Gaussian integers and $\{3,6\}$ for the Eisenstein integers. We will follow [7] and [14] in our notation.

The *Gaussian integers* $\mathbb{Z}[i]$ are defined as the set $\{a + bi : a, b \in \mathbb{Z}\} \subset \mathbb{C}$, where i is the imaginary unit, with the standard addition and multiplication of complex numbers. Similarly, the *Eisenstein integers* $\mathbb{Z}[\omega]$ are defined as $\{a + b\omega : a, b \in \mathbb{Z}\} \subset \mathbb{C}$, where $\omega := (1 + i\sqrt{3})/2$. We adopt a unifying notation $\mathbb{Z}[\sigma]$ with $\sigma \in \{i, \omega\}$ to denote either of these two sets. Since we are dealing with different types of integers, to avoid confusion we will call *rational integers* the elements of $\mathbb{Z}$.

As in [7], in this paper we use the following notation: given $\alpha = a + b\sigma \in \mathbb{Z}[\sigma]$, we call its *conjugate* the number $\overline{\alpha} := a + b\overline{\sigma}$, where $\overline{\sigma}$ is the conjugate complex number to $\sigma \in \mathbb{C}$. Also we call $\operatorname{Re}\alpha := a$ and $\operatorname{Im}\alpha := b$ the *real* and *imaginary* parts, respectively. Note here that if $\sigma = i$, then this is the traditional notion of 'real part' and 'imaginary part' of a complex number. However, if $\sigma = \omega$, then the 'traditional' real and imaginary parts of $a + b\omega$ are $a + b/2$ and $b\sqrt{3}/2$, respectively.

For every $\alpha = a + b\sigma \in \mathbb{Z}[\sigma]$, we assign the *norm* $N(\alpha) := \alpha\overline{\alpha}$. An integer $\alpha \in \mathbb{Z}[\sigma] \setminus \{0\}$ *divides* $\beta \in \mathbb{Z}[\sigma]$ if and only if there is $\gamma \in \mathbb{Z}[\sigma]$ such that $\beta = \alpha\gamma$. Recall that, in the ring of Gaussian integers, the *units* are only $\pm 1, \pm i$, while in the ring of Eisenstein integers the units are only $\pm 1, \pm\omega, \pm\overline{\omega}$. Two integers $\alpha, \beta \in \mathbb{Z}[\sigma]$ are called *associated* if $\alpha = \beta\varepsilon$ for some unit ε.

Let us recall the concept of a greatest common divisor for rings of integers. An integer $\gamma \in \mathbb{Z}[\sigma]$ is a *greatest common divisor (GCD)* of $\alpha, \beta \in \mathbb{Z}[\sigma]$ with $N(\alpha) + N(\beta) \neq 0$, if γ divides both α and β and for every γ' with the same property it follows that γ' divides γ. It is well-known that both $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$ are Unique Factorization Domains (see [14]), that is the rings with unique (up to associates) factorization into primes. Hence, a greatest common divisor is well-defined, again up to associates. Because of that, we write $\gamma = \operatorname{GCD}(\alpha, \beta)$ implying that γ is defined up to multiplication by an associate. We also agree that if there is a rational integer n among associates to $\operatorname{GCD}(\alpha, \beta)$, then we specifically take $\operatorname{GCD}(\alpha, \beta) := |n|$. For example, $\operatorname{GCD}(3, 6i) \in \{3, -3, 3i, -3i\}$, and thus by our convention $\operatorname{GCD}(3, 6i) = 3$.

The power of Gaussian and Eisenstein integers is coming from the natural identification of these sets with the vertex set of a regular tessellation $\{4, 4\}$ or $\{3, 6\}$. In particular, we can identify the basis $(\mathbf{e_1}, \mathbf{e_2})$ (see Section 2) with the ordered pair $(1, \sigma)$ from $\mathbb{Z}[\sigma]$. This identification leads to the group homomorphism of $T_{\{4,4\}}$ (resp. $T_{\{3,6\}}$) with $\mathbb{Z}[i]$ (resp. $\mathbb{Z}[\omega]$) — where the latter groups are regarded as Abelian groups with respect to addition. From this point of view, we will identify every two vectors $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$ with the complex numbers $\alpha = a_1 + a_2\sigma$ and $\beta = b_1 + b_2\sigma$. Finally, write $\tau_{\alpha,\beta} := \tau_{\mathbf{a},\mathbf{b}}$, and, for further brevity, $\tau_\eta := \tau_{\eta,\sigma\eta}$.

Note here that the pair of vectors η, $\sigma\eta$ span a square if $\sigma = i$, or a rhombus with angle $\pi/3$ if $\sigma = \omega$. Therefore, if η and η' are equal up to associates, then $\tau_\eta = \tau_{\eta'}$.

4.2 Proof of the Main Theorem

In Theorems 4.2 – 4.4 we will prove that each Archimedean map on the torus has a unique minimal almost regular cover on the torus, which we will construct explicitly. To accomplish these proofs, we will use some known results for equivelar toroidal maps (see [7]).

Proposition 4.1 (Covering correspondence of maps and their associates)**.** *Let τ be an Archimedean tessellation of the plane, not of type $(3^3.4^2)$, and G and H two subgroups of T_τ. Then τ/H covers τ/G if and only if τ^*/H covers τ^*/G.*

Proof. Suppose $\tau/H \searrow \tau/G$. Then H is a subgroup of G, and both of them are subgroups of T_τ. By construction, the groups T_τ and T_{τ^*} are equal, and so the same subgroup structure holds in T_{τ^*}, which means that τ^*/H covers τ^*/G. Conversely, if τ^*/H covers τ^*/G, then H is a subgroup of G, and G is a subgroup of T_{τ^*}. The latter group is again equal to T_τ by the very definition of τ^*. Hence, $\tau/H \searrow \tau/G$. □

Observe that for every $\tau \in \mathcal{A} \setminus \{(3^3.4^2)\}$ we have a well-defined basis $(\mathbf{e_1}, \mathbf{e_2})$ such that $\langle \mathbf{e_1}, \mathbf{e_2} \rangle = T_\tau = T_{\tau^*}$, where $\tau^* = \{p, q\}$ is the regular tessellation associated to τ. Therefore, every $\mathbf{u} \in T_\tau$ with the coordinates (u_1, u_2) in the basis $(\mathbf{e_1}, \mathbf{e_2})$ can be identified by the homomorphism discussed in Subsection 4.1 with the integer $u_1 + u_2\sigma \in \mathbb{Z}[\sigma]$ (here σ depends on τ^*). We will use this equivalent language instead of the vector language in order to state the following two theorems which are the first two main results of the paper.

Theorem 4.2 (Minimal almost regular covers for toroidal maps of type $\mathcal{A}_{\text{reg}}$)**.** *Let $\tau_{\alpha,\beta}$ be an Archimedean map given as a quotient of an Archimedean tessellation $\tau \in \mathcal{A}_{\text{reg}}$ by a translation subgroup $\langle \alpha, \beta \rangle < T_\tau$ generated by two integers $\alpha, \beta \in \mathbb{Z}[\sigma] \setminus \{0\}$ with $\alpha/\beta \notin \mathbb{Z}$. Then the map τ_η with*

$$\eta = \begin{cases} \frac{\operatorname{Im}(\overline{\alpha}\beta)}{N(1+\sigma)\, c}\,(1+\sigma), & \textit{if } N(1+\sigma) \textit{ divides } \frac{\operatorname{Re}\alpha}{c} - \frac{\operatorname{Im}\alpha}{c} \textit{ and } \frac{\operatorname{Re}\beta}{c} - \frac{\operatorname{Im}\beta}{c}, \\ \frac{\operatorname{Im}(\overline{\alpha}\beta)}{c}, & \textit{otherwise}, \end{cases}$$

where $c = \operatorname{GCD}(\operatorname{Re}\alpha, \operatorname{Im}\alpha, \operatorname{Re}\beta, \operatorname{Im}\beta)$ is a unique minimal almost regular cover of $\tau_{\alpha,\beta}$. Moreover, the number k_{min} of fundamentals regions of $\tau_{\alpha,\beta}$ glued together in order to the fundamental region of τ_η is equal to

$$k_{\text{min}} = \begin{cases} \frac{|\operatorname{Im}(\overline{\alpha}\beta)|}{N(1+\sigma)\, c^2}, & \textit{if } N(1+\sigma) \textit{ divides } \frac{\operatorname{Re}\alpha}{c} - \frac{\operatorname{Im}\alpha}{c} \textit{ and } \frac{\operatorname{Re}\beta}{c} - \frac{\operatorname{Im}\beta}{c}, \\ \frac{|\operatorname{Im}(\overline{\alpha}\beta)|}{c^2}, & \textit{otherwise.} \end{cases}$$

Proof. This theorem is a direct consequence of [7, Theorem 3.6]. Indeed, let τ^* be the regular tessellation associated to the Archimedean tessellation τ. By Theorem 3.1, there is one-to-one correspondence between the translation subgroups of T_{τ^*} that generate regular maps on the torus and translation subgroups of T_τ that generate almost regular maps on the torus. By Proposition 4.1, any such correspondence preserves the covering order and, in particular, sends minimal elements to minimal elements. Therefore, τ^*/H is the minimal regular cover of τ^*/G, where $G := \langle \alpha, \beta \rangle$, if and only if τ/H is the minimal almost regular cover of τ/G. By [7, Theorem 3.6], every map τ^*/G has a unique minimal regular cover τ^*/H, where $H < T_{\tau^*}$ can be given explicitly in terms of number-theoretical properties of $\alpha, \beta \in \mathbb{Z}[\sigma]$. Hence, the same holds for τ/G, which yields existence and uniqueness of a minimal almost regular cover. The explicit form of H from [7, Theorem 3.6] translates verbatim into the explicit form given in Theorem 4.2; this finishes the proof. □

Theorem 4.3 (Minimal almost regular covers for toroidal maps of type $(3^4.6)$)**.** *Let $\tau_{\alpha,\beta}$ be an Archimedean map given as a quotient of an Archimedean tessellation $\tau = (3^4.6)$ by a translation subgroup $\langle \alpha, \beta \rangle < T_\tau$ generated by two integers $\alpha, \beta \in \mathbb{Z}[\omega] \setminus \{0\}$ with $\alpha/\beta \notin \mathbb{Z}$. Then the map τ_η with*

$$\eta = \frac{\operatorname{Im}(\overline{\alpha}\beta)}{N(\gamma)}\gamma$$

where $\gamma = \operatorname{GCD}(\alpha, \beta)$ is a unique minimal almost regular cover of $\tau_{\alpha,\beta}$. Moreover, the number k_{min} of fundamentals regions of $\tau_{\alpha,\beta}$ glued together in order to obtain the fundamental region of τ_η is equal to

$$k_{\text{min}} = \frac{|\operatorname{Im}(\overline{\alpha}\beta)|}{N(\gamma)}.$$

Proof. The proof is similar to the proof of Theorem 4.2, where instead of [7, Theorem 3.6] we use [7, Theorem 3.5]. □

Recall that we associate to the tessellation of type $(3^3.4^2)$ the basis $(\mathbf{e_1}, \mathbf{e_2})$ comprised of a pair of translations that are in the symmetry group of $(3^3.4^2)$ such that $\mathbf{e_1}$ connects the centers of two adjacent squares and $\mathbf{e_2}$ is the shortest translation vector forming an acute angle to $\mathbf{e_1}$. Everywhere below we assume that the coordinate representation of a translation from $T_{(3^3.4^2)}$ is given with respect to the basis $(\mathbf{e_1}, \mathbf{e_2})$.

Theorem 4.4 (Minimal almost regular covers for toroidal maps of type $(3^3.4^2)$)**.** *Let $\tau_{\mathbf{a},\mathbf{b}}$ be an Archimedean toroidal map given as a quotient of the tessellation $\tau = (3^3.4^2)$ by a translation subgroup $\langle \mathbf{a}, \mathbf{b}\rangle < T_\tau$ generated by two vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ with $\Delta := a_1b_2 - a_2b_1 \neq 0$. Then for $\tau_{\mathbf{a},\mathbf{b}}$ there exists and is unique a minimal almost regular cover $\tau_{\mathbf{u},\mathbf{v}}$ generated by the subgroup*

$$\langle \mathbf{u}, \mathbf{v}\rangle = \begin{cases} \big\langle (c, 0), (-d_1, c + 2d_1)\big\rangle, & \text{provided } \frac{a_2}{g_1} - \frac{2a_1+a_2}{g_2} \text{ and } \frac{b_2}{g_1} - \frac{2b_1+b_2}{g_2} \text{ are even},\\ \big\langle (c, 0), (-d_2, 2d_2)\big\rangle, & \text{otherwise,} \end{cases}$$

where

$$g_1 = \mathrm{GCD}(a_2, b_2), \quad g_2 = \mathrm{GCD}(2a_1 + a_2, 2b_1 + b_2),$$
$$c = \frac{\Delta}{g_1}, \quad d_1 = -\frac{\Delta}{2}\left(\frac{1}{g_1} + \frac{1}{g_2}\right), \quad d_2 = -\frac{\Delta}{g_2}.$$

Moreover, the number $k_{\min}$ of fundamentals domains of $\tau_{\mathbf{a},\mathbf{b}}$ glued together in order to obtain the fundamental region of $\tau_{\mathbf{u},\mathbf{v}}$ is equal to

$$k_{\min} = \begin{cases} \left|\frac{\Delta}{g_1g_2}\right|, & \text{if } \frac{a_2}{g_1} - \frac{2a_1+a_2}{g_2} \text{ and } \frac{b_2}{g_1} - \frac{2b_1+b_2}{g_2} \text{ are even},\\ 2\left|\frac{\Delta}{g_1g_2}\right|, & \text{otherwise.} \end{cases}$$

Proof. Our strategy in proving Theorem 4.4 will be the following: we explicitly describe *all* almost regular covers for $\tau_{\mathbf{a},\mathbf{b}}$ and then determine the one which is the smallest (under the covering relation).

Suppose that an Archimedean map $\tau_{\mathbf{u},\mathbf{v}}$ is a cover of $\tau_{\mathbf{a},\mathbf{b}}$. This is equivalent of saying that the group $\langle \mathbf{u}, \mathbf{v}\rangle$ is a proper subgroup of $\langle \mathbf{a}, \mathbf{b}\rangle$, which in algebraic terms is equivalent to existence of a linear integer relation between the generators of both groups:

$$\begin{cases} n_1\mathbf{a} + m_1\mathbf{b} = \mathbf{u},\\ n_2\mathbf{a} + m_2\mathbf{b} = \mathbf{v}, \end{cases} \tag{4.1}$$

for some integers n_1, n_2, m_1, m_2 with $n_1m_2 \neq n_2m_1$. (The last condition guarantees that $\mathbf{u}$ and $\mathbf{v}$ are, in fact, non-parallel.) If, on top, $\tau_{\mathbf{u},\mathbf{v}}$ is almost regular, then the generators $\mathbf{u}, \mathbf{v}$ might be chosen to be of one of the types in (3.1) (see Theorem 3.9). We now consider these two cases one by one.

Case 1: suppose $\mathbf{u} = c\,\mathbf{e_1}$ and $\mathbf{v} = -d\,\mathbf{e_1} + 2d\,\mathbf{e_2}$ for some non-zero integers c and d. Then, in order to find the generators of such type, we must solve the following system of vector Diophantine equations

$$\begin{cases} n_1\mathbf{a} + m_1\mathbf{b} = c\,\mathbf{e_1}, \\ n_2\mathbf{a} + m_2\mathbf{b} = -d\,\mathbf{e_1} + 2d\,\mathbf{e_2}, \end{cases} \tag{4.2}$$

for the variables n_1, n_2, m_1, m_2 treating c and d as parameters.

The first equation in (4.2) in coordinates is equivalent to the system of linear Diophantine equations

$$\begin{cases} n_1a_1 + m_1b_1 = c, \\ n_1a_2 + m_1b_2 = 0. \end{cases} \tag{4.3}$$

By the standard methods the full family of solutions for (4.3) is

$$n_1 = \frac{b_2}{g_1}k, \quad m_1 = -\frac{a_2}{g_1}k, \quad c = \frac{\Delta}{g_1}k, \quad k \in \mathbb{Z}^*, \tag{4.4}$$

where we recall that $\Delta = a_1b_2 - a_2b_1$ and $g_1 = \mathrm{GCD}(a_2, b_2)$ (here $\mathbb{Z}^*$ stands for the set of all non-zero integers).

Similarly, the second equation in (4.2) in coordinates reads

$$\begin{cases} n_2a_1 + m_2b_1 = -d, \\ n_2a_2 + m_2b_2 = 2d; \end{cases} \tag{4.5}$$

Multiplying the first equation by 2 and adding the second we get

$$n_2(2a_1 + a_2) + m_2(2b_1 + b_2) = 0,$$

from which we conclude, after straightforward cancellations, that the full family of solutions for (4.5) is

$$n_2 = \frac{2b_1 + b_2}{g_2}s, \quad m_2 = -\frac{2a_1 + a_2}{g_2}s, \quad d = -\frac{\Delta}{g_2}s, \quad s \in \mathbb{Z}^*, \tag{4.6}$$

where $g_2 = \mathrm{GCD}(2a_1 + a_2, 2b_1 + b_2)$.

Concluding Case 1 from obtained solution (4.4) and (4.6), we see that $\tau_{\mathbf{u},\mathbf{v}}$ is an almost regular Archimedean cover of $\tau_{\mathbf{a},\mathbf{b}}$ and is of the first type in (3.1) if and only if

$$\langle \mathbf{u}, \mathbf{v} \rangle = \left\langle \left(\frac{\Delta}{g_1}k, 0\right), \left(\frac{\Delta}{g_2}s, -2\frac{\Delta}{g_2}s\right) \right\rangle =: G_{k,s}.$$

Finally, observe that for every given pair of non-zero integers k and s the almost regular map $\tau/G_{k,s}$ covers $\tau/G_{1,1}$. Hence, by definition of a minimal cover, $\tau/G_{1,1}$ is the minimal almost regular cover (for $\tau_{\mathbf{a},\mathbf{b}}$) of the first type in (3.1). Note that the full family of toroidal maps $\tau/G_{k,s}$ does not form a totally ordered set with respect to covering; however, this poset has a unique minimal element. We will see a similar type of covering behavior later.

Finishing Case 1, we compute the number of fundamental regions of $\langle \mathbf{a}, \mathbf{b} \rangle$ one should glue together in order to obtain the fundamental region of $G_{1,1}$. This is done by comparing areas of those regions. In the standard basis in $\mathbb{E}^2$ the area of the fundamental region of $\langle \mathbf{a}, \mathbf{b} \rangle$ is equal to

$$A_0 := |\mathbf{a} \times \mathbf{b}| = |\Delta| \cdot |\mathbf{e_1} \times \mathbf{e_2}|.$$

Similarly, by using (4.4) and (4.6) we compute the area of the fundamental region of $G_{1,1}$:

$$A_1 := |c\,\mathbf{e_1} \times (-d\,\mathbf{e_1} + 2d\,\mathbf{e_2})| = \frac{2\Delta^2}{|g_1 g_2|} \cdot |\mathbf{e_1} \times \mathbf{e_2}| \,.$$

Therefore, the number we are looking for is equal to

$$\frac{A_1}{A_0} = 2\left|\frac{\Delta}{g_1 g_2}\right|. \tag{4.7}$$

Observe that

$$2(a_1 b_2 - a_2 b_1) = (2a_1 + a_2)b_1 - (2b_1 + b_2)a_2, \tag{4.8}$$

and hence the right hand side in (4.7) is an integer. Of course, the same conclusion likewise follows from the geometric meaning of A_1/A_0.

Case 2: suppose $\mathbf{u} = c\,\mathbf{e_1}$ and $\mathbf{v} = -d\,\mathbf{e_1} + (c+2d)\mathbf{e_2}$ for some non-zero integer c and an integer d; this is the second type in (3.1). We proceed similarly to Case 1, with a bit more involved computation.

Again, in order to find all almost regular covers of the second type we have to find all solutions of the system

$$\begin{cases} n_1\mathbf{a} + m_1\mathbf{b} = c\,\mathbf{e_1}, \\ n_2\mathbf{a} + m_2\mathbf{b} = -d\,\mathbf{e_1} + (c+2d)\mathbf{e_2}, \end{cases}$$

for integers n_1, n_2, m_1, m_2 treating c and d as parameters. Similarly to Case 1, the solutions to the first equation in this system have the following form:

$$n_1 = \frac{b_2}{g_1}k, \quad m_1 = -\frac{a_2}{g_1}k, \quad c = \frac{\Delta}{g_1}k, \quad k \in \mathbb{Z}^*. \tag{4.9}$$

The second equation from the system above in coordinates reads:

$$\begin{cases} n_2 a_1 + m_2 b_1 = -d, \\ n_2 a_2 + m_2 b_2 = c + 2d. \end{cases} \tag{4.10}$$

Again, multiplying the first equation by 2 and adding the second one we obtain, by using (4.9),

$$n_2(2a_1 + a_2) + m_2(2b_1 + b_2) = c = \frac{\Delta}{g_1}k. \tag{4.11}$$

This is a linear nonhomogeneous Diophantine equation in n_2 and m_2. The standard theory of linear Diophantine equations tells us that the any solution of (4.11) is the sum of a partial solution to the given equation and of the general solution of the corresponding homogeneous equation $n_2(2a_1+a_2)+m_2(2b_1+b_2) = 0$. Moreover, a necessary condition for (4.11) to have a solution is that

$$g_2 \text{ divides } \frac{\Delta}{g_1}k. \tag{4.12}$$

Using (4.8) it is straightforward to check that $\Delta/(g_1g_2) \in \mathbb{Z}^*$ if and only if

$$\frac{a_2}{g_1} - \frac{2a_1 + a_2}{g_2} \quad \text{and} \quad \frac{b_2}{g_1} - \frac{2b_1 + b_2}{g_2} \quad \text{are even.} \tag{4.13}$$

Therefore, if condition (4.13) is met, then the necessary condition (4.12) is satisfied for every integer k. On the other hand, if (4.13) is violated, then (4.12) can be only satisfied if k is even (as it follows from (4.8)). Let us consider these two sub-cases.

Sub-case 2.1: Suppose that condition (4.13) is violated; hence k is necessarily even. Then it is straightforward to check that the pair of integers

$$n_2' = \frac{b_2}{2g_1}k, \quad m_2' = -\frac{a_2}{2g_1}k$$

provides a partial solution of equation (4.11). Therefore, in Sub-case 2.1 the full family of solutions of (4.11), and thus of (4.10), has the form

$$\begin{aligned} n_2 &= \frac{b_2}{2g_1}k + \frac{2b_1 + b_2}{g_2}s, \\ m_2 &= -\frac{a_2}{2g_1}k - \frac{2a_1 + a_2}{g_2}s, \qquad k \in 2\mathbb{Z},\ s \in \mathbb{Z}. \\ d &= -\frac{\Delta}{2g_1}k - \frac{\Delta}{g_2}s, \end{aligned} \tag{4.14}$$

Concluding Sub-case 2.1 of Case 2 by combining (4.9) and (4.14), we see that, provided condition (4.13) is not met, $\tau_{\mathbf{u},\mathbf{v}}$ is an almost regular Archimedean cover of $\tau_{\mathbf{a},\mathbf{b}}$ and is of the second type in (3.1) if and only if

$$\langle \mathbf{u}, \mathbf{v} \rangle = \left\langle \left(2\frac{\Delta}{g_1}l, 0\right), \left(\frac{\Delta}{g_1}l + \frac{\Delta}{g_2}s, -2\frac{\Delta}{g_2}s\right) \right\rangle =: H_{l,s},$$

where $l, s \in \mathbb{Z}^*$.

Now let us check the covering relations. Note that for a given pair of non-zero integers l and s, the almost regular map $\tau/H_{l,s}$ non-trivially covers $\tau/G_{1,1}$. Therefore, in the poset of coverings of almost regular maps of the form $\tau/G_{k,s}$ (see Case 1) and $\tau/H_{l,s}$ the quotient of τ by $G_{1,1}$ is the unique minimal element.

Sub-case 2.2: assume condition (4.13) is satisfied. This is equivalent of saying that both pairs a_1/g_1, $(2a_1 + a_2)/g_2$ and b_2/g_1, $(2b_1 + b_2)/g_2$ consist of integers with the same parity. As we showed above, this is also equivalent to $\Delta/(g_1g_2) \in \mathbb{Z}^*$.

If k is even, then we can run verbatim the same arguments as in Sub-case 2.1, with the same conclusion. Hence we can assume that k is odd.

But if k is odd and condition (4.13) is met, then the pair of numbers

$$n_2' = \frac{b_2}{2g_1}k + \frac{2b_1 + b_2}{2g_2}, \quad m_2' = -\frac{a_2}{2g_1}k - \frac{2a_1 + a_2}{2g_2}$$

are necessarily full integers, and moreover provide a partial solution to (4.11). Therefore, we can write down a complete solution to (4.11):

$$\begin{aligned} n_2 &= \frac{b_2}{2g_1}k + \frac{2b_1 + b_2}{2g_2} + \frac{2b_1 + b_2}{g_2}s = \frac{b_2}{2g_1}k + \frac{2b_1 + b_2}{2g_2}(2s + 1), \\ m_2 &= -\frac{a_2}{2g_1}k - \frac{2a_1 + a_2}{2g_2} - \frac{2a_1 + a_2}{2g_2}s = -\frac{a_2}{2g_1}k - \frac{2a_1 + a_2}{2g_2}(2s + 1), \end{aligned}$$

where $s \in \mathbb{Z}$.

Substituting this solution in either of the equations in (4.10), we obtain

$$d = -\frac{\Delta}{2g_1}k - \frac{\Delta}{2g_2}(2s+1),$$

where both k and $2s+1$ are some odd integers.

Similarly to as we did before, concluding Sub-case 2.2 of Case 2, we see that, provided condition (4.13) is satisfied and k is odd (the case k was already discussed), $\tau_{\mathbf{u},\mathbf{v}}$ is an almost regular Archimedean cover of $\tau_{\mathbf{a},\mathbf{b}}$ and is of the second type in (3.1) if and only if

$$\langle \mathbf{u}, \mathbf{v} \rangle = \left\langle \left(\frac{\Delta}{g_1}(2l+1), 0 \right), \left(\frac{\Delta}{2g_1}(2l+1) + \frac{\Delta}{2g_2}(2s+1), -\frac{\Delta}{g_2}(2s+1) \right) \right\rangle =: F_{l,s},$$

where $l, s \in \mathbb{Z}$. Observe that $\tau/F_{l,s}$ covers $\tau/F_{0,0}$ for any pair $l, s \in \mathbb{Z}^*$.

Finally, comparing the groups

$$G_{1,1} = \left\langle \left(\frac{\Delta}{g_1}, 0 \right), \left(\frac{\Delta}{g_2}, -2\frac{\Delta}{g_2} \right) \right\rangle \quad \text{and} \quad F_{0,0} = \left\langle \left(\frac{\Delta}{g_1}, 0 \right), \left(\frac{\Delta}{2g_1} + \frac{\Delta}{2g_2}, -\frac{\Delta}{g_2} \right) \right\rangle,$$

we conclude that the almost regular map $\tau/G_{1,1}$ non-trivially covers the almost regular map $\tau/F_{0,0}$.

Therefore, summing up the results of Case 1 and Case 2, we obtain that if condition (4.13) is satisfied, then $\tau/F_{0,0}$ is a minimal almost regular Archimedean map that covers $\tau_{\mathbf{a},\mathbf{b}}$. Otherwise, $\tau/G_{1,1}$ is a minimal almost regular cover. In both cases these minimal covers are unique elements in the corresponding posets of possible almost regular covers. This almost finishes the proof of Theorem 4.4. The only thing that is left to check is that in Sub-case 2.2 of Case 2, provided k is odd, the number of fundamental regions of $\langle \mathbf{a}, \mathbf{b} \rangle$ one should glue together to obtain the fundamental region of $F_{0,0}$ is equal to

$$\left| \frac{\Delta}{g_1 g_2} \right|$$

(note that this is an integer). This computation is similar to the one found at the end of Case 1, and is thus omitted; this completes the proof of Theorem 4.4. □

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Symplectic semifield spreads of $\mathrm{PG}(5, q^t)$, q even

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Abstract

Let $q > 2{\cdot}3^{4t}$ be even. We prove that the only symplectic semifield spread of $\mathrm{PG}(5, q^t)$, whose associate semifield has center containing $\mathbb{F}_q$, is the Desarguesian spread. Equivalently, a commutative semifield of order q^{3t}, with middle nucleus containing $\mathbb{F}_{q^t}$ and center containing $\mathbb{F}_q$, is a field. We do that by proving that the only possible $\mathbb{F}_q$-linear set of rank $3t$ in $\mathrm{PG}(5, q^t)$ disjoint from the secant variety of the Veronese surface is a plane of $\mathrm{PG}(5, q^t)$.

Keywords: Semifields, spreads, symplectic polarity, linear sets, Veronese variety.

Math. Subj. Class.: 05B25, 51E15, 51E23, 14M12

1 Introduction

Let $\mathrm{PG}(r-1, q)$ be the projective space of dimension $r-1$ over the finite field $\mathbb{F}_q$ of order q. An $(n-1)$-*spread* $\mathcal{S}$ of $\mathrm{PG}(2n-1, q)$, which we will call simply *spread* from now on, is a partition of the point-set in $(n-1)$-dimensional subspaces. With any spread $\mathcal{S}$ it is associated a translation plane $A(\mathcal{S})$ of order q^n via the André-Bruck-Bose construction (see e.g. [7, Section 5.1]). Translation planes associated with different spreads of $\mathrm{PG}(2n-1, q)$ are isomorphic if and only if there is a collineation of $\mathrm{PG}(2n-1, q)$ mapping one spread to the other (see [1] or [16, Chapter 1]). A spread $\mathcal{S}$ is said to be *Desarguesian* if $A(\mathcal{S})$ is isomorphic to $\mathrm{AG}(2, q^n)$ and hence a plane coordinatized by the field of order q^n. The spread $\mathcal{S}$ is said to be a *semifield spread* if $A(\mathcal{S})$ is a plane of Lenz-Barlotti class V and this is equivalent to saying that $A(\mathcal{S})$ is coordinatized by a semifield.

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A *finite semifield* $\mathbb{S} = (\mathbb{S}, +, \star)$ is a finite algebra satisfying all the axioms for a skew-field except (possibly) associativity of multiplication. The subsets

$$\begin{aligned} \mathbb{N}_l &= \{a \in \mathbb{S} : (a \star b) \star c = a \star (b \star c),\ \forall b, c \in \mathbb{S}\}, \\ \mathbb{N}_m &= \{b \in \mathbb{S} : (a \star b) \star c = a \star (b \star c),\ \forall a, c \in \mathbb{S}\}, \\ \mathbb{N}_r &= \{c \in \mathbb{S} : (a \star b) \star c = a \star (b \star c),\ \forall a, b \in \mathbb{S}\} \text{ and} \\ \mathcal{K} &= \{a \in \mathbb{N}_l \cap \mathbb{N}_m \cap \mathbb{N}_r : a \star b = b \star a,\ \forall b \in \mathbb{S}\} \end{aligned}$$

are fields and are known, respectively, as the *left nucleus*, the *middle nucleus*, the *right nucleus* and the *center* of the semifield. A finite semifield is a vector space over its nuclei and its center.

If $A(\mathcal{S})$ is coordinatized by the semifield $\mathbb{S}$, then $\mathbb{S}$ has order q^n and its *left nucleus* contains $\mathbb{F}_q$.

Semifields are studied up to an equivalence relation called *isotopy*, which corresponds to the study of semifield planes up to isomorphisms (for more details on semifields see, e.g., [7]).

The spread $\mathcal{S}$ is said to be *symplectic* if the elements of $\mathcal{S}$ are totally isotropic with respect to a *symplectic polarity* of $\mathrm{PG}(2n-1, q)$. If $A(\mathcal{S})$ is coordinatized by the semifield $\mathbb{S}$, then $\mathbb{S}$ is called symplectic semifield and if its *center* contains $\mathbb{F}_s \leq \mathbb{F}_q$, then from $\mathbb{S}$ we get by the cubical array (see [13]) a semifield isotopic to a commutative semifield with *middle nucleus* containing $\mathbb{F}_q$ and *center* containing $\mathbb{F}_s$ ([11]).

Let q be even. For $n = 2$, there is the following remarkable theorem due to Cohen and Ganley.

Theorem 1.1 ([6]). *A commutative semifield of order q^2 with middle nucleus containing $\mathbb{F}_q$ is a field.*

For $n > 2$, the only known commutative semifields, that are not a field, are the Kantor-Williams symplectic pre-semifields of order q^n and $n > 1$ odd ([12]) and their commutative Knuth derivatives ([11]). Symplectic semifield spreads in characteristic 2 with odd dimension over $\mathbb{F}_2$ give arise to $\mathbb{Z}_4$-linear codes and extremal line sets in Euclidean spaces ([4]).

Most of the above mentioned results are obtained with an algebraic approach, whereas ours is mainly geometric. For small n, the study of semifield spreads has shown to be a good way to classify semifields.

Let $M(n, \mathbb{F}_q)$ be the set of all $n \times n$ matrices over $\mathbb{F}_q$. Without loss of generality, we may always assume that $S(\infty) := \{(\mathbf{0}, \mathbf{y}) : \mathbf{y} \in \mathbb{F}_q^n\}$ and $S(0) := \{(\mathbf{x}, \mathbf{0}) : \mathbf{x} \in \mathbb{F}_q^n\}$ belong to $\mathcal{S}$, hence we may write $\mathcal{S} = \{S(A) : A \in \mathbb{C}\} \cup S(\infty)$, with $S(A) := \{(\mathbf{x}, \mathbf{x}A) : \mathbf{x} \in \mathbb{F}_q^n\}$, with $\mathbb{C} \subset M(n, \mathbb{F}_q)$ such that $|\mathbb{C}| = q^n$ and $\mathbb{C}$ contains the zero matrix. The set $\mathbb{C}$ is called the *spread set* associated with $\mathcal{S}$. In order to have a semifield spread, the non-zero elements of $\mathbb{C}$ must be invertible and $\mathbb{C}$ must be a subgroup of the additive group of $M(n, \mathbb{F}_q)$ ([7, Section 5.1]), hence $\mathbb{C}$ is a vector space over some subfield of $\mathbb{F}_q$. If we choose the symplectic polarity induced by the alternating bilinear form $\beta((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2\mathbf{y}_2)) = \mathbf{x}_1\mathbf{y}_2^T - \mathbf{y}_1\mathbf{x}_2^T$, $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{F}_q^n$, then the subspace $S(A) \in \mathcal{S}$ is totally isotropic if and only if A is symmetric. The symmetric matrices form an $\frac{n(n+1)}{2}$-dimensional subspace of $M(n, \mathbb{F}_q)$ that then induces a $\mathrm{PG}\left(\frac{n(n+1)}{2} - 1, q\right)$. The rank-1 symmetric matrices form the Veronese variety $\mathcal{V}$ of degree 2 of $\mathrm{PG}\left(\frac{n(n+1)}{2} - 1, q\right)$ (this is the so called determinantal representation of the Veronese variety of degree 2, see [8, Example 2.6]). Hence the singular symmetric matrices form the $(n-2)$-th secant variety, say $\mathcal{V}_{n-2}$, of the Veronese variety. If $\mathbb{C}$ is an $\mathbb{F}_s$-vector space, $q = s^t$, then $\dim_{\mathbb{F}_s} \mathbb{C} = nt$ and it defines a subset L of $\mathrm{PG}\left(\frac{n(n+1)}{2} - 1, q\right)$ called $\mathbb{F}_s$*-linear set* of rank nt (for a complete overview on linear sets see [18]). So to a symplectic semifield spread of $\mathrm{PG}(2n-1, q)$ there corresponds an $\mathbb{F}_s$-linear set L, $q = s^t$, of $\mathrm{PG}\left(\frac{n(n+1)}{2} - 1, q\right)$ of rank tn such that $L \cap \mathcal{V}_{n-2} = \emptyset$ (see also [15]). We recall the associated semifield has left nucleus containing $\mathbb{F}_q$ and if $\mathbb{F}_s$ is the maximum subfield with respect to L is linear, then the center of the semifield is isomorphic to $\mathbb{F}_s$. So the isotopic commutative semifield we get has middle nucleus containing $\mathbb{F}_q$ and center isomorphic to $\mathbb{F}_s$.

In this article, we are focused on the case $n = 3$, i.e., on symplectic semifield spreads of $\mathrm{PG}(5, q)$, when q is even. In such a case, only two non-sporadic examples are known: the Desarguesian spread and one of its cousin (see [10]), so they are both obtained by slicing the so called Desarguesian spread of $\mathcal{Q}^+(7, q)$. In the former case, the associated translation plane is the Desarguesian plane, hence it is coordinatized by the finite field of order q^3 and the relevant linear set is actually linear on $\mathbb{F}_q$. In the latter case, the semifield spread is associated to a spread set $\mathbb{C}$ that is an $\mathbb{F}_2$-linear set L of $\mathrm{PG}(5, q)$, where $\mathbb{F}_2$ is the maximum subfield of $\mathbb{F}_q$ for which L is linear, and the associate semifield has order q^3 and center $\mathbb{F}_2$.

In [5], it is proven that the only symplectic semifield spread of $\mathrm{PG}(5, q^2)$, $q > 2^{14}$, whose associate semifield has center containing $\mathbb{F}_q$, is the Desarguesian spread, meaning that a commutative semifield of order q^6, with middle nucleus containing $\mathbb{F}_{q^2}$ and center containing $\mathbb{F}_q$ is a field, provided q is not too small. That was done by studying the intersection of the five non-equivalent $\mathbb{F}_q$-linear sets of $\mathrm{PG}(5, q^2)$ with the secant variety $\mathcal{V}_1$ of the Veronese variety and the only one that can have empty intersection with $\mathcal{V}_1$ is a plane. A classification of the $\mathbb{F}_q$-linear sets of $\mathrm{PG}(5, q^t)$ of rank $3t$ is not feasible, as the number of non-equivalent ones quickly grows with t. In fact, the present paper, we had a slightly different approach which allowed us to generalize the result of [5] in $\mathrm{PG}(5, q^t)$ for any t: by field reduction, a $\mathrm{PG}(5, q^t)$ can be seen as $\mathrm{PG}(6t-1, q)$, a linear set of rank $3t$ as a subspace $\cong \mathrm{PG}(3t-1, q)$ and $\mathcal{V}_1$ an algebraic variety, say $\mathcal{V}_1^t$, of codimension t in $\mathrm{PG}(6t-1, q)$. Hence, we have studied when a subspace of dimension $3t-1$ can have empty intersection with $\mathcal{V}_1^t$ (over $\mathbb{F}_q$), regardless the geometric feature of the linear set in $\mathrm{PG}(5, q^t)$.

2 Preliminary results

2.1 $\mathbb{F}_q$-linear sets and the $\mathbb{F}_q$-linear representation of $\mathrm{PG}(r-1, q^t)$

The set $L \subset \mathrm{PG}(V, \mathbb{F}_{q^t}) = \mathrm{PG}(r-1, q^t)$, with V an r-dimensional vector space over $\mathbb{F}_{q^t}$, is said to be an $\mathbb{F}_q$*-linear* set of rank m if it is defined by the non-zero vectors of an $\mathbb{F}_q$-vector subspace U of V of dimension m, i.e.

$$L = L_U = \{\langle \mathbf{u} \rangle_{\mathbb{F}_{q^t}} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\}.$$

If $r = m$ and $\langle L_U \rangle = \mathrm{PG}(r-1, q^t)$, then $L_U \cong \mathrm{PG}(r-1, q)$. In this case, L_U is said to be a *subgeometry* (of order q) of $\mathrm{PG}(r-1, q^t)$. Throughout this paper, we shall extensively use the following result: a subset Σ of $\mathrm{PG}(r-1, q^t)$ is a subgeometry of order q if and only if there exists an $\mathbb{F}_q$-linear collineation σ of $\mathrm{PG}(r-1, q^t)$ of order t such that $\Sigma = \operatorname{Fix}\sigma$, where $\operatorname{Fix}\sigma$ is the set of points fixed by σ. This is a straightforward consequence of the fact that there is just one conjugacy class of $\mathbb{F}_q$-linear collineations of order t in $\mathrm{P\Gamma L}(r, q^t)$, namely that of

$$\varsigma\colon (x_0, x_1, \dots, x_{r-1}) \mapsto (x_0^q, x_1^q, \dots, x_{r-1}^q).$$

In particular, all subgeometries $\cong \mathrm{PG}(r-1, q)$ of $\mathrm{PG}(r-1, q^t)$ are projectively equivalent to the subgeometry induced by $\{(x_0, x_1, \dots, x_{r-1}) : x_i \in \mathbb{F}_q\}$. A subspace Π of $\mathrm{PG}(r-1, q^t)$ defines a subspace of $\operatorname{Fix}\sigma \cong \mathrm{PG}(r-1, q)$ of the same dimension if and only if $\Pi = \Pi^\sigma$ (see [14, Lemma 1]). It will be more convenient for us to explicitly state the following equivalent result.

Notation. Let $\mathbb{F}$ be any field containing $\mathbb{F}_q$. Throughout the paper we will denote by $\Pi(\mathbb{F})$ the unique subspace of $\mathrm{PG}(r-1, \mathbb{F})$ containing Π.

Lemma 2.1. *If we consider* $\mathrm{PG}(r-1, q)$ *embedded as a subgeometry of* $\mathrm{PG}(r-1, q^t)$ *and* Π *is a subspace of* $\mathrm{PG}(r-1, q)$ *of dimension* $s-1$*, then the subspace* $\Pi(\mathbb{F}_{q^t})$ *of* $\mathrm{PG}(r-1, q^t)$ *containing* Π *has dimension* $s-1$ *as well.*

Analogously, if $\mathcal{W}$ is an algebraic variety of $\mathrm{PG}(r-1, q^t)$, then $\mathcal{W} \cap \operatorname{Fix}\sigma \subset \mathcal{W} \cap \mathcal{W}^\sigma \cap \dots \cap \mathcal{W}^{\sigma^{t-1}}$ and hence $\mathcal{W} \cap \operatorname{Fix}\sigma$ has the same dimension and degree of $\mathcal{W}$ if and only if $\mathcal{W} = \mathcal{W}^\sigma$.

Remark 2.2. An algebraic variety $\mathcal{W}$ is said to be a variety of $\mathrm{PG}(r-1, q)$ if it consists of the set of zeros of polynomials $f_1, f_2, \dots, f_k \in \mathbb{F}_q[x_0, x_1, \dots, x_{r-1}]$, and we will write $\mathcal{W} = V(f_1, f_2, \dots, f_k)$. By *dimension* and *degree* of $\mathcal{W}$ we will mean the dimension and degree of the variety when considered as variety of $\mathrm{PG}(r-1, \overline{\mathbb{F}_q})$, with $\overline{\mathbb{F}_q}$ the algebraic closure of $\mathbb{F}_q$.

In the remaining part of this section, we will describe the setting we adopt to study the $\mathbb{F}_q$-linear sets of $\mathrm{PG}(V, \mathbb{F}_{q^t}) = \mathrm{PG}(r-1, q^t)$.

When we regard V as an $\mathbb{F}_q$-vector space, $\dim_{\mathbb{F}_q} V = rt$ and hence $\mathrm{PG}(V, q) = \mathrm{PG}(rt-1, q)$. Furthermore, a point $\langle v\rangle_{\mathbb{F}_{q^t}} \in \mathrm{PG}(r-1, q^t)$ corresponds to the $(t-1)$-dimensional subspace of $\mathrm{PG}(rt-1, q)$ given by $\{\lambda v : \lambda \in \mathbb{F}_{q^t}\}$. This is the so-called $\mathbb{F}_q$-*linear representation* of $\langle v\rangle_{\mathbb{F}_{q^t}}$ and the set $\mathcal{S}$, consisting of the $(t-1)$-subspaces of $\mathrm{PG}(rt-1, q)$ that are the linear representation of the points of $\mathrm{PG}(r-1, q^t)$, is a partition of the point set of $\mathrm{PG}(rt-1, q)$. Such a partition $\mathcal{S}$ is called *Desarguesian spread* of $\mathrm{PG}(rt-1, q)$. In this setting, a linear set L_U is the subset of the Desarguesian spread $\mathcal{S}$ with non-empty intersection with the projective subspace Π_U of $\mathrm{PG}(rt-1, q)$ induced by U.

We shall adopt the following cyclic representation of $\mathrm{PG}(rt-1, q)$ in $\mathrm{PG}(rt-1, q^t)$. Let $\mathrm{PG}(rt-1, q^t) = \mathrm{PG}(V', q^t)$, with V' the standard rt-dimensional vector space over $\mathbb{F}_{q^t}$ and let e_i the i-th element of the canonical base of V'. Consider the semi-linear collineation σ with accompanying automorphism $x \mapsto x^q$ and such that $e_i \mapsto e_{i+r}$, where the subscript are taken mod rt. Then σ is an $\mathbb{F}_q$-linear collineation of order t and $\operatorname{Fix}\sigma = \{(\mathbf{x}, \mathbf{x}^q, \dots, \mathbf{x}^{q^{t-1}}) : \mathbf{x} = (x_0, x_1, \dots, x_{r-1}), x_i \in \mathbb{F}_{q^t}, \mathbf{x} \neq \mathbf{0}\}$ is isomorphic to $\mathrm{PG}(rt-1, q)$. The elements of $\mathcal{S}$ are the subspaces $\Pi_P := \langle P, P^\sigma, \dots, P^{\sigma^{t-1}}\rangle \cap \operatorname{Fix}\sigma$, with $P \in \Pi_0 \cong \mathrm{PG}(r-1, q^t)$ and Π_0 defined by $x_i = 0\ \forall i > r-1$ (see [14]). Let Π_i be $\Pi_0^{\sigma^i}$. In the following, we shall identify a point P of $\Pi_0 = \mathrm{PG}(r-1, q^t)$ with the spread element Π_P. We observe that P is just the projection of Π_P from $\langle \Pi_1, \Pi_2, \dots, \Pi_{t-1}\rangle$ on Π_0. If L_U is a linear set of rank m, then it is induced by an $(m-1)$-dimensional subspace $\Pi_U \subset \mathrm{PG}(rt-1, q) = \mathrm{Fix}\,\sigma$ and it can be viewed both as the subset of Π_0 that is the projection of Π_U from $\langle \Pi_1, \Pi_2, \dots, \Pi_{t-1}\rangle$ on Π_0 as well as the subset of $\mathcal{S}$ consisting of the elements Π_P such that $\Pi_P \cap \Pi_U \neq \emptyset$. We stress out that we have defined the subspaces Π_U and Π_P as subspaces of $\mathrm{Fix}\,\sigma = \mathrm{PG}(rt-1, q)$. Let $\mathbb{F}$ be any field containing $\mathbb{F}_{q^t}$, then the projection of $\Pi_U(\mathbb{F})$ on Π_0 from $\langle \Pi_1, \Pi_2, \dots, \Pi_{t-1}\rangle$ is $\langle L_U\rangle_{\mathbb{F}}$.

Let $\mathcal{H}$ be a hypersurface of $\mathrm{PG}(r-1, q^t)$ and let $f \in \mathbb{F}_{q^t}[x_0, x_1, \dots, x_{r-1}]$ a polynomial defining $\mathcal{H}$, i.e., $\mathcal{H} = V(f)$. In the linear representation of $\mathrm{PG}(r-1, q^t) = \Pi_0$, the points of $\mathcal{H}$ correspond to the spread elements Π_P such that $P \in \mathcal{H}$, hence it is the intersection of the variety $V(f, f^\sigma, \dots, f^{\sigma^{t-1}})$ of $\mathrm{PG}(rt-1, q^t)$ with $\mathrm{Fix}\,\sigma$, where, by abuse of notation, we extend the action of σ also to polynomials. We observe that the variety $V(f, f^\sigma, \dots, f^{\sigma^{t-1}})$ is the *join* of the varieties $\mathcal{H}, \mathcal{H}^\sigma, \dots, \mathcal{H}^{\sigma^{t-1}}$ (see [8, Chapter 8]) and hence it has dimension $t(\dim \mathcal{H} + 1) - 1 = t(r-1) - 1 = tr - t - 1$ and degree $\deg(\mathcal{H})^t$. We observe that $V(f, f^\sigma, \dots, f^{\sigma^{t-1}})$ it is defined by t polynomials and $\dim V(f, f^\sigma, \dots, f^{\sigma^{t-1}}) = tr - t - 1 = \dim \mathrm{PG}(rt-1, q^t) - t$, hence $V(f, f^\sigma, \dots, f^{\sigma^{t-1}})$ is a *complete intersection* (see [8, Example 11.8]). We will denote the join of the varieties $\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_k$ by $\mathrm{Join}(\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_k)$.

Let $T_P(\mathcal{W})$ be the tangent space to the algebraic variety $\mathcal{W}$ at the point $P \in \mathcal{W}$.

Proposition 2.3 (Terracini's Lemma [20])**.** *Let* $\mathcal{W} = \mathrm{Join}(\mathcal{Y}_1, \mathcal{Y}_2)$ *and let* $P = \langle P_1, P_2\rangle \in \mathcal{W}$ *with* $P_i \in \mathcal{Y}_i$*. Then* $\langle T_{P_1}(\mathcal{Y}_1), T_{P_2}(\mathcal{Y}_2)\rangle \subseteq T_P(\mathcal{W})$*.*

The variety $V(f, f^\sigma, \dots, f^{\sigma^{t-1}})$ is the join of the varieties $\mathcal{H}^{\sigma^i}$, $i = 0, 1, \dots, t-1$. We recall that $\mathcal{H}^{\sigma^i}$ is a hypersurface of Π_i, hence $T_{P_i}(\mathcal{H}^{\sigma^i})$ is a hypersurface of Π_i for a *non-singular point* $P_i \in \mathcal{H}^{\sigma^i}$. By $\Pi_i \cap \langle \Pi_j, j \neq i\rangle = \emptyset$, we get

$$\dim\langle T_{P_0}(\mathcal{H}), T_{P_1}(\mathcal{H}^\sigma), \dots, T_{P_{t-1}}(\mathcal{H}^{\sigma^{t-1}})\rangle = rt - 1 - t$$

for non-singular points $P_0, P_1, \dots, P_{t-1}$. Since for a non-singular point $P \in V(f, f^\sigma, \dots, f^{\sigma^{t-1}})$, $\dim T_P(V(f, f^\sigma, \dots, f^{\sigma^{t-1}})) = rt - 1 - t$, we have

$$\langle T_{P_0}(\mathcal{H}), T_{P_1}(\mathcal{H}^\sigma), \dots, T_{P_{t-1}}(\mathcal{H}^{\sigma^{t-1}})\rangle = T_P(V(f, f^\sigma, \dots, f^{\sigma^{t-1}}))$$

for a non-singular $P \in V(f, f^\sigma, \dots, f^{\sigma^{t-1}})$.

Let $\mathrm{Sing}(\mathcal{W})$ be the set of the singular points of a variety $\mathcal{W}$; we recall that $\mathrm{Sing}(\mathcal{W})$ is a subvariety of $\mathcal{W}$. From the discussion above, it is clear that

$$\mathrm{Sing}(V(f, f^\sigma, \dots, f^{\sigma^{t-1}})) = \bigcup_{i=0}^{t-1} S_i,$$

with $S_i = \mathrm{Join}(\mathrm{Sing}(\mathcal{H}^{\sigma^i}), \mathcal{H}^{\sigma^j}, j \neq i)$.

2.2 The Veronese surface and its secant variety

In this section we denote by $\mathbb{P}^{n-1}$ the $(n-1)$-dimensional projective space over a generic field $\mathbb{F}$.

The Veronese map of degree 2

$$v_2\colon (x_0, x_1, x_2) \in \mathbb{P}^2 \longmapsto (\dots, \mathbf{x}^l, \dots) \in \mathbb{P}^5$$

is such that $\mathbf{x}^l$ ranges over all monomials of degree 2 in x_0, x_1, x_2. The image $\mathcal{V} := v_2(\mathbb{P}^2)$ is the *quadric Veronese surface*, a variety of dimension 2 and degree 4. A section $H \cap \mathcal{V}$, where H is a hyperplane of $\mathbb{P}^5$, consists of the points of $v_2(\mathcal{C})$, where $\mathcal{C}$ is a conic of $\mathbb{P}^2$.

If we use the so-called determinantal representation of $\mathcal{V}$ (see [8, Example 2.6]), then we can take $\mathbb{P}^5$ as induced by the subspace of $M(3, \mathbb{F})$ consisting of symmetric matrices and $v_2(x_0, x_1, x_2) = A$ such that $a_{ij} = x_i x_j$, i.e., $\mathcal{V}$ consists of the rank 1 matrices of $M(3, \mathbb{F})$.

Hence, the secant variety of $\mathcal{V}$, say $\mathcal{V}_1$, consists of the symmetric matrices of rank at most 2, i.e., $\mathcal{V}_1$ consists of the singular symmetric 3×3 matrices. So $\mathcal{V}_1$ is a hypersurface of $\mathbb{P}^5$ of degree 3. It is well known that the singular points of $\mathcal{V}_1$ are the points of $\mathcal{V}$.

The automorphism group $\hat{G}$ of $\mathcal{V}$ is the lifting of $G = \mathrm{PGL}(3, \mathbb{F})$ acting in the obvious way: $v_2(p)^{\hat{g}} = v_2(p^g)$ $\forall g \in \mathrm{PGL}(3, \mathbb{F})$. The group $\hat{G}$ obviously fixes $\mathcal{V}_1$.

The maximal subspaces contained in $\mathcal{V}_1$ are planes and they are of three types: the span of $v_2(\ell)$, with ℓ a line of $\mathbb{P}^2$, the tangent planes $T_P(\mathcal{V})$ for $P \in \mathcal{V}$, and, when the characteristic of $\mathbb{F}$ is even, the *nucleus plane* π_N.

Let the characteristic of $\mathbb{F}$ be even. The plane π_N of $\mathbb{P}^5$ consists of the symmetric matrices with zero diagonal, hence π_N is contained in $\mathcal{V}_1$. By the Jacobi's formula, $\frac{\partial}{\partial a_{ij}} \det A = \mathrm{tr}(\mathrm{adj}(A) \frac{\partial A}{\partial a_{ij}})$, where $\mathrm{tr}(M)$ is the trace of a matrix M and $\mathrm{adj}(M)$ is the adjoint matrix of M. Let E_{ij} be the 3×3 matrix with 1 in the ij-position and 0 elsewhere, so we have $\frac{\partial}{\partial a_{ij}} \det A = \mathrm{tr}(\mathrm{adj}(A) \frac{\partial A}{\partial a_{ij}}) = \mathrm{tr}(\mathrm{adj}(A)(E_{ij} + E_{ji})) = 0$ $\forall i \neq j$. It follows that a hyperplane is tangent to $\mathcal{V}_1$ if and only if it contains π_N. Also, each point of π_N is the nucleus of a point of a unique conic $v_2(\ell)$.

If $P \in \mathcal{V}_1$, then the tangent hyperplane H to $\mathcal{V}_1$ at P is such that $H \cap \mathcal{V} = v_2(\ell^2)$, where $\ell = \langle p_1, p_2 \rangle$ if $P \notin \pi_N$ and hence $P \in \langle v_2(p_1), v_2(p_2) \rangle$, or ℓ is such that P is the nucleus of $v_2(\ell)$ if $P \in \pi_N$. The tangent plane at $v_2(p)$ to $\mathcal{V}$ is the intersection of three hyperplanes K_1, K_2, K_3 such that $K_i \cap \mathcal{V} = v_2(\ell_i \cup \ell_i')$, where ℓ_i, ℓ_i' are lines through p.

If $\mathbb{F}$ is an algebraically closed field, then any subspace of $\mathbb{P}^5$ of dimension at least 1 has non-empty intersection with $\mathcal{V}_1$. If $\mathbb{F} = \mathbb{F}_q$, then there are subspaces of larger dimension disjoint from $\mathcal{V}_1$ and, by the Chevalley-Warning Theorem, we know that they can have dimension at most 2. For q even we have the following result.

Theorem 2.4 ([5]). *Let $q \geq 4$ be even, then there exists one orbit of planes under the action of $\hat{G}$ disjoint from $\mathcal{V}_1$.*

3 Proof of the main result

Through this section, we assume q to be even. Let $\overline{\mathbb{F}_q}$ be the algebraic closure of $\mathbb{F}_q$.

We adopt the $\mathbb{F}_q$-linear representation of $\mathrm{PG}(5, q^t)$, i.e., we regard the points of $\mathrm{PG}(5, q^t)$ as elements of a Desarguesian spread of $\mathrm{PG}(6t-1, q)$ and L_U as the subset of the spread with non-empty intersection with a $(3t-1)$-dimensional subspace Π_U of $\mathrm{PG}(6t-1, q)$; also, we consider $\mathrm{PG}(6t-1, q)$ as subgeometry of $\mathrm{PG}(6t-1, q^t)$ (cf. Section 2). Let f be the polynomial with coefficients in $\mathbb{F}_2$ such that $\mathcal{V}_1 = V(f)$, hence the $\mathbb{F}_q$-linear representation of $\mathcal{V}_1$ is $V(f, f^\sigma, \dots, f^{\sigma^{t-1}}) \cap \mathrm{Fix}\,\sigma$. Let $\mathcal{V}_1^t$ be $V(f, f^\sigma, \dots, f^{\sigma^{t-1}})$.

We have that $\mathcal{V}_1 \cap L_U = \emptyset \Leftrightarrow \mathcal{V}_1^t \cap \Pi_U = \emptyset \Leftrightarrow \mathcal{V}_1^t \cap \mathrm{Fix}\,\sigma \cap \Pi_U(\mathbb{F}_{q^t}) = \emptyset$. Let $\mathcal{W}$ be $\Pi_U(\overline{\mathbb{F}_q}) \cap \mathcal{V}_1^t$. We observe that $\mathcal{W} = \mathcal{W}^\sigma$, hence $\dim \mathcal{W} = \dim \mathcal{W} \cap \mathrm{Fix}\,\sigma$. We stress out that $\mathcal{W}$ is defined by polynomials in $\mathbb{F}_{q^t}[x_0, x_1, \ldots, x_{6t-1}]$ but it might not contain any $\mathbb{F}_{q^t}$-rational point. The linear representation of π_N is the $(3t-1)$-dimensional subspace Π_N of $\mathrm{Fix}\,\sigma$ that is partitioned by the spread elements $\{\Pi_P : P \in \pi_N\}$. As $L_U \cap \pi_N = \emptyset$, we must have $\Pi_U \cap \Pi_N = \emptyset$ and hence, by Lemma 2.1, $\Pi_U(\mathbb{F}_{q^t}) \cap \Pi_N(\mathbb{F}_{q^t}) = \emptyset$ and $\Pi_U(\overline{\mathbb{F}_q}) \cap \Pi_N(\overline{\mathbb{F}_q}) = \emptyset$.

Theorem 3.1. *Let $P \in \mathcal{W}$, then* $\dim T_P(\mathcal{V}_1^t) \cap \Pi_U(\overline{\mathbb{F}_q}) = \dim T_P(\mathcal{V}_1^t) - 3t$.

Proof. The subspace $\Pi_U(\overline{\mathbb{F}_q})$ has codimension $3t$, hence

$$\dim T_P(\mathcal{V}_1^t) \cap \Pi_U(\overline{\mathbb{F}_q}) \geq \dim T_P(\mathcal{V}_1^t) - 3t.$$

Let $P \in \langle P_0, P_1, \ldots, P_{t-1}\rangle$ with $P_i \in \Pi_i(\overline{\mathbb{F}_q})$. We have that

$$T_P(\mathcal{V}_1^t) = \langle T_{P_0}(\mathcal{V}_1), T_{P_1}(\mathcal{V}_1^{\sigma}), \ldots, T_{P_{t-1}}(\mathcal{V}_1^{\sigma^{t-1}})\rangle$$

and $\pi_N^{\sigma^i} \subset T_{P_i}(\mathcal{V}_1^{\sigma^i})$ $\forall i$, hence $\Pi_N(\overline{\mathbb{F}_q}) \subset T_P(\mathcal{V}_1^t)$. Since $\Pi_U(\overline{\mathbb{F}_q}) \cap \Pi_N(\overline{\mathbb{F}_q}) = \emptyset$ and $\dim \Pi_N(\overline{\mathbb{F}_q}) = 3t-1$, we have $\dim T_P(\mathcal{V}_1^t) \cap \Pi_U(\overline{\mathbb{F}_q}) \leq \dim T_P(\mathcal{V}_1^t) - 3t$, hence the statement follows. □

Corollary 3.2. *We have* $\dim \mathcal{W} = 2t-1$, *hence $\mathcal{W}$ is a complete intersection.*

Proof. If P is non-singular for $\mathcal{V}_1^t$, then $\dim T_P(\mathcal{V}_1^t) = \dim(\mathcal{V}_1^t) = 5t-1$, whereas $\dim T_P(\mathcal{V}_1^t) > 5t-1$ for $P \in \mathrm{Sing}(\mathcal{V}_1^t)$. As $\mathcal{W} = \mathcal{V}_1^t \cap \Pi_U(\overline{\mathbb{F}_q})$, $T_P(\mathcal{W}) = T_P(\mathcal{V}_1^t) \cap \Pi_U(\mathbb{F}_{q^t})$. By Theorem 3.1,

$$\dim T_P(\mathcal{V}_1^t) \cap \Pi_U(\overline{\mathbb{F}_q}) = \dim T_P(\mathcal{V}_1^t) - 3t \geq 2t-1,$$

and

$$\dim T_P(\mathcal{V}_1^t) \cap \Pi_U(\overline{\mathbb{F}_q}) > 2t-1$$

only if $P \in \mathrm{Sing}(\mathcal{V}_1^t)$. Hence $\dim \mathcal{W} = 2t-1$. We observe that $2t-1 = \dim \Pi_U(\overline{\mathbb{F}_q}) - t$, hence $\mathcal{W}$ is a complete intersection. □

Corollary 3.3. $\mathrm{Sing}(\mathcal{W}) = \mathrm{Sing}(\mathcal{V}_1^t) \cap \Pi_U(\overline{\mathbb{F}_q})$.

Proof. By Theorem 3.1, $\dim T_P(\mathcal{W}) = \dim T_P(\mathcal{V}_1^t) - 3t$, hence $\dim T_P(\mathcal{W}) > \dim \mathcal{W} = 2t-1$ if and only if $\dim T_P(\mathcal{V}_1^t) > 5t-1 = \dim(\mathcal{V}_1^t)$, i.e., $P \in \mathrm{Sing}(\mathcal{V}_1^t)$. □

If a variety $\mathcal{Y}$ is a complete intersection and $\dim \mathcal{Y} - \dim \mathrm{Sing}(\mathcal{Y}) \geq 2$, then $\mathcal{Y}$ is *normal* (see [19, Chapter 2, Section 5.1] for the general definition of normal varieties). An important tool for our proof is the following reformulation of the Hartshorne connectedness theorem ([9]).

Theorem 3.4 ([3, Theorem 2.1])**.** *If $\mathcal{Y}$ is a normal complete intersection, then $\mathcal{Y}$ is absolutely irreducible.*

Theorem 3.5. *If $\mathcal{W}$ is reducible and $L_U \cap \mathcal{V}_1 = \emptyset$, then L_U is a plane which is isomorphic to* $\mathrm{PG}(2, q^t)$ *disjoint from $\mathcal{V}_1$.*

Proof. If $\mathcal{W}$ is reducible, then $\mathcal{W}$ is not normal and hence $\dim \mathrm{Sing}(\mathcal{W}) = \dim \mathcal{W} - 1 = 2t-2$. A point $P \in \mathcal{W}$ is singular if and only if $P \in \mathrm{Sing}(\mathcal{V}_1^t) \cap \Pi_U(\overline{\mathbb{F}_q})$. We have $\mathrm{Sing}(\mathcal{V}_1^t) = \bigcup_{i=0}^{t-1} S_i$, with

$$S_i = \mathrm{Join}(\mathrm{Sing}(\mathcal{V}_1^{\sigma^i}), \mathcal{V}_1^{\sigma^j}, j \neq i) = \mathrm{Join}(\mathcal{V}^{\sigma^i}, \mathcal{V}_1^{\sigma^j}, j \neq i)$$

(see Section 2), so $S_0^{\sigma^i} = S_i$ and hence

$$\dim \mathrm{Sing}(\mathcal{V}_1^t) \cap \Pi_U(\overline{\mathbb{F}_q}) = \dim S_0 \cap \Pi_U(\overline{\mathbb{F}_q}) = 2t-2.$$

Let $P \in S_0 \cap \Pi_U(\overline{\mathbb{F}_q})$ with $P = \langle P_0, P_1, \ldots, P_{t-1}\rangle$, $P_0 \in \mathcal{V}$, $P_i \in \mathcal{V}_1^{\sigma^i}$, $i = 1, 2, \ldots, t-1$, then the tangent space $T_P(S_0 \cap \Pi_U(\overline{\mathbb{F}_q}))$ is

$$\begin{aligned}\langle T_{P_0}(\mathcal{V}), T_{P_1}(\mathcal{V}_1^{\sigma}), \ldots, T_{P_{t-1}}(\mathcal{V}_1^{\sigma^{t-1}})\rangle \cap \Pi_U(\overline{\mathbb{F}_q}) \\ = K_1^* \cap K_2^* \cap K_3^* \cap H_1^* \cap \cdots \cap H_{t-1}^* \cap \Pi_U(\overline{\mathbb{F}_q}),\end{aligned}$$

with K_i^*, H_j^* hyperplanes of $\mathrm{PG}(6t-1, q^t)$ such that K_i^* projects on the hyperplane K_i of Π_0 for $i = 1, 2, 3$, H_j^* projects on the hyperplane H_j of Π_j $\forall j = 1, 2, \ldots, t-1$, $K_1 \cap K_2 \cap K_3 = T_{P_0}(\mathcal{V})$ and $H_j = T_{P_j}(\mathcal{V}_1^{\sigma^j})$. We can take K_1, K_2, K_3 such that $K_1 \cap \mathcal{V} = v_2(\ell_1^2)$, $K_2 \cap \mathcal{V} = v_2(\ell_2^2)$ and $K_3 \cap \mathcal{V} = v_2(\ell_1 \cup \ell_2)$. Hence, $K_1^* \cap K_2^* \cap H_1^* \cap \cdots \cap H_{t-1}^*$ contains Π_N and so $\dim K_2^* \cap K_3^* \cap H_1^* \cap \cdots \cap H_{t-1}^* \cap \Pi_U(\overline{\mathbb{F}_q})$ is the smallest possible, i.e., $2t-2$. Hence,

$$K_1^* \supseteq K_2^* \cap K_3^* \cap H_1^* \cap \cdots \cap H_{t-1}^* \cap \Pi_U(\overline{\mathbb{F}_q})$$

and the projection of $\Pi_U(\overline{\mathbb{F}_q})$ on Π_0 is a subspace Π_0' such that the tangent space of P_0 at $\mathcal{V} \cap \Pi_0'$ has codimension 2 in Π_0'. So either the codimension of $\Pi_0' \cap \mathcal{V}$ in Π_0' is 2 or $\Pi_0' \cap \mathcal{V}$ has codimension 3 in Π_0' but it has singular points. Suppose we are in the latter case. The Veronese variety $\mathcal{V}$ is smooth, hence Π_0' can by a 3 or 4-dimensional subspace of Π_0. If Π_0' is a hyperplane of Π_0 and $\Pi_0' \cap \mathcal{V}(\overline{\mathbb{F}_q})$ has singular points, then $\Pi_0' \cap \mathcal{V}$ is is either $v_2(\ell^2)$ or $v_2(\ell_1 \cup \ell_2)$. In the first case, Π_0' contains π_N. A plane $\cong \mathrm{PG}(2, q^t)$ is a $\mathbb{F}_q$-linear set of rank $3t$, so $\Pi_0'(\mathbb{F}_{q^t}) \cong \mathrm{PG}(4, q^t)$ contains two linear sets of rank $3t$ that must intersect by Grassmann, i.e., $L_U \cap \mathcal{V}_1 \neq \emptyset$. If $\Pi_0' \cap \mathcal{V} = v_2(\ell_1 \cup \ell_2)$, then Π_0' contains the tangent space at $\mathcal{V}$ of the point $P = v_2(\ell_1 \cap \ell_2)$ and it is the unique tangent space at $\mathcal{V}$ contained in Π_0'. Let τ be the collineation induced by the field automorphism $x \mapsto x^{q^t}$, then both Π_0' and $\mathcal{V}(\overline{\mathbb{F}_q})$ are fixed by τ, hence $T_P(\mathcal{V})^\tau = T_P(\mathcal{V})$ and, by Lemma 2.1, $T_P(\mathcal{V})$ contains a $\mathrm{PG}(2, q^t)$. Again, by Grassmann, $L_U \cap \mathcal{V}_1 \neq \emptyset$. Suppose that Π_0' is a 3-dimensional space, so it contains 4 points counted with their multiplicity and at least one of them is multiple. If P is a multiple point and it is $\mathbb{F}_{q^t}$-rational, i.e., $P = P^\tau$, then Π_0' contains a line tangent to $\mathcal{V}$ at P that it is fixed by τ and hence contains a $\mathrm{PG}(1, q^t)$, so, by Grassmann, $L_U \cap \mathcal{V}_1 \neq \emptyset$. So a multiple point P must be $\mathbb{F}_{q^{st}}$-rational, but also $P^\tau \in \Pi_0' \cap \mathcal{V}$ would be, hence $s = 2$ and we have $\Pi_0' \cap \mathcal{V} = \{P, P^\tau\}$, with $P \in \Pi_0'(q^{2t})$. The line joining P and P^τ is set-wise fixed by τ and so it contains a $\mathrm{PG}(1, q^t)$, yielding again $L_U \cap \mathcal{V}_1 \neq \emptyset$. So suppose that the codimension of $\Pi_0' \cap \mathcal{V}(\overline{\mathbb{F}_q})$ in Π_0' is 2. Hence Π_0' is either a 3-dimensional space or a plane. Suppose that Π_0' is a 3-dimensional space and so $\dim \Pi_0' \cap \mathcal{V}(\overline{\mathbb{F}_q}) = 3 - 2 = 1$. Since $\Pi_0' \cap \mathcal{V}(\overline{\mathbb{F}_q})$ is the Veronese embedding of the intersection of two distinct conics, Π_0' contains the Veronese embedding of a line ℓ and it cannot contain the embedding of any other line. Hence $v_2(\ell)^\tau \subset \Pi_0'$ implies $v_2(\ell)^\tau = v_2(\ell)$ and so $\langle v_2(\ell)\rangle$ contains a plane $\cong \mathrm{PG}(2, q^t)$. By Grassmann, $L_U \cap \mathcal{V}_1 \neq \emptyset$. Hence $\Pi'_0(q^t)$ is a plane and so $L_U = \Pi'_0(q^t)$. □

Theorem 3.6. *If $\mathcal{W}$ is absolutely irreducible and $q > 2 \cdot 3^{4t}$, then $\mathcal{W} \cap \mathrm{Fix}\,\sigma$ has at least one point.*

Proof. By [2, Corollary 7.4], an absolutely irreducible algebraic variety of $\mathrm{PG}(n-1, q)$ with dimension r and degree δ for $q > \max\{2(r+1)\delta^2, 2\delta^4\}$ has at least one $\mathbb{F}_q$-rational point. By $r = 2t - 1$ and $\delta \leq 3^t = \deg \mathcal{V}_1^t$, we have the statement. □

We conclude the section with our main result.

Theorem 3.7. *Let $q > 2 \cdot 3^{4t}$ be even. The only symplectic semifield spread of $\mathrm{PG}(5, q^t)$ whose associate semifield has center containing $\mathbb{F}_q$, is the Desarguesian spread.*

Proof. By Theorems 3.6 and 3.5, we have that the only $\mathbb{F}_q$-linear set of rank $3t$ disjoint from $\mathcal{V}_1$ is a plane. The planes disjoint from $\mathcal{V}_1$ form a unique orbit under the action of $\hat{G}$ (see Theorem 2.4). In this case, the linear set is $\mathbb{F}_{q^t}$-linear as well, hence the semifield associated to the spread is 3-dimensional over its center. By [17], in even characteristic this implies that the semifield is a field, hence the spread is Desarguesian. □

Corollary 3.8. *Let $q > 2 \cdot 3^{4t}$ be even. Then a commutative semifield of order q^{3t}, with middle nucleus containing $\mathbb{F}_{q^t}$ and center containing $\mathbb{F}_q$, is a field.*

Remark 3.9. We emphasize that the hypothesis of even characteristic is crucial for all our arguments: only for even q the variety $\mathcal{V}_1$ contains the plane π_N, and using $L \cap \pi_N = \emptyset$ we can prove that $\mathcal{W}$ is a complete intersection, i.e. $\mathcal{W}$ has codimension t, and the singular points of $\mathcal{W}$ are just the ones coming from $\mathcal{V}$.

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On the Hamilton-Waterloo problem: the case of two cycles sizes of different parity*

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Abstract

The Hamilton-Waterloo problem asks for a decomposition of the complete graph of order v into r copies of a 2-factor F_1 and s copies of a 2-factor F_2 such that $r+s = \left\lfloor \frac{v-1}{2} \right\rfloor$. If F_1 consists of m-cycles and F_2 consists of n cycles, we say that a solution to (m,n)-$\mathrm{HWP}(v;r,s)$ exists. The goal is to find a decomposition for every possible pair (r,s). In this paper, we show that for odd x and y, there is a solution to $(2^k x, y)$-$\mathrm{HWP}(vm;r,s)$ if $\gcd(x,y) \geq 3$, $m \geq 3$, and both x and y divide v, except possibly when $1 \in \{r,s\}$.

Keywords: 2-factorizations, Hamilton-Waterloo problem, Oberwolfach problem, cycle decomposition, resolvable decompositions.

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1 Introduction

The Oberwolfach problem asks for a decomposition of the complete graph K_v into $\frac{v-1}{2}$ copies of a 2-factor F. To achieve this decomposition, v needs to be odd, because the vertices must have even degree. The problem with v even asks for a decomposition of K_v into $\frac{v-2}{2}$ copies of a 2-factor F, and one copy of a 1-factor. The uniform Oberwolfach problem (all cycles of the 2-factor have the same size) has been completely solved by

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Alspach and Haggkvist [1] and Alspach, Schellenberg, Stinson and Wagner [2]. The non-uniform Oberwolfach problem has been studied as well, and a survey of results up to 2006 can be found in [8]. Furthermore, one can refer to [6, 7, 9, 23, 24] for more recent results.

In [19] Liu first worked on the generalization of the Oberwolfach problem to equipartite graphs. He was seeking to decompose the complete equipartite graph $K_{(m:n)}$ with n partite sets of size m each into $\frac{(n-1)m}{2}$ copies of a 2-factor F. For such a decomposition to exist $(n-1)m$ has to be even. In [14] Hoffman and Holliday worked on the equipartite generalization of the Oberwolfach problem when $(n-1)m$ is odd, decomposing into $\frac{(n-1)m-1}{2}$ copies of a 2-factor F, and one copy of a 1-factor. The uniform Oberwolfach problem over equipartite graphs has since been completely solved by Liu [20] and Hoffman and Holliday [14]. For the non-uniform case, Bryant, Danziger and Pettersson [7] completely solved the case when the 2-factor is bipartite. In particular, Liu showed the following.

Theorem 1.1 ([20]). *For $m \geq 3$ and $u \geq 2$, $K_{(h:u)}$ has a resolvable C_m-factorization if and only if hu is divisible by m, $h(u-1)$ is even, m is even if $u = 2$, and $(h, u, m) \notin \{(2,3,3),(6,3,3),(2,6,3),(6,2,6)\}$.*

The Hamilton-Waterloo problem is a variation of the Oberwolfach problem, in which we consider two 2-factors, F_1 and F_2. It asks for a factorization of K_v when v is odd or $K_v - I$ (I is a 1-factor) when v is even into r copies of F_1 and s copies of F_2 such that $r+s = \left\lfloor \frac{v-1}{2} \right\rfloor$, where F_1 and F_2 are two 2-regular graphs on v vertices. Most of the results for the Hamilton-Waterloo problem are uniform, meaning F_1 consists of cycles of size m (C_m-factors), and F_2 consists of cycles of size n (C_n-factors). If there is a decomposition of K_v into r C_m-factors and s C_n-factors we say that a solution to (m,n)-$\mathrm{HWP}(v; r, s)$ exists. The case where both m and n are odd positive integers and v is odd is almost completely solved by [11, 12]; and if m and n are both even, then the problem again is almost completely solved (see [5, 6]). However, if m and n are of differing parities, then we only have partial results. Most of the work has been done in the case where one of the cycle sizes is constant. The case of $(m,n) = (3,4)$ is solved in [4, 13, 21, 25]. Other cases which have been studied include $(m,n) = (3,v)$ [18], $(m,n) = (3,3x)$ [3], and $(m,n) = (4,n)$ [16, 21] .

In this paper, we consider the case of m and n being of different parity. This case has gained attention recently, where it has been shown that the necessary conditions are sufficient for a solution to (m,n)-$\mathrm{HWP}(v; r, s)$ to exist whenever $m \mid n$, $v > 6n > 36m$, and $s \geq 3$ [10]. We provide a complementary result to this in our main theorem, which covers cases in which $m \nmid n$ and solves a major portion of the problem.

Theorem 1.2. *Let x, y, v, k and m be positive integers such that:*

(i) $v, m \geq 3$,

(ii) x, y *are odd,*

(iii) $\gcd(x, y) \geq 3$,

(iv) x *and* y *divide* v,

(v) 4^k *divides* v.

Then there exists a solution to $(2^k x, y)$-$\mathrm{HWP}(vm; r, s)$ for every pair r, s with $r + s = \lfloor (vm-1)/2 \rfloor$, $r, s \neq 1$.

2 Preliminaries

The *complete cyclic multipartite graph* $C_{(x:n)}$ is the graph with n partite sets of size x, where two vertices (g, i) and (h, j) are neighbors if and only if $i - j = \pm 1 \pmod{n}$, with subtraction being done modulo n. The *directed complete cyclic multipartite graph* $\overrightarrow{C}_{(x:n)}$ is the graph with n parts of size x, with arcs of the form $\big((g, i), (h, i+1)\big)$ for every $0 \leq g, h \leq x - 1, 0 \leq i \leq n - 1$.

One of the main tools in [17] is a Lemma that combines decompositions of $C_{(x:k)}$ to obtain decompositions of $K_{(v:m)}$. We present a version of the Lemma for uniform decompositions, as those are the focus of this manuscript.

Lemma 2.1 ([17]). *Let m, x, y, and v be positive integers. Let $s_1, \dots, s_{\frac{m-1}{2}}$ be non-negative integers. Suppose the following conditions are satisfied:*

- *There exists a decomposition of K_m into C_n-factors.*
- *For every $1 \leq t \leq \frac{m-1}{2}$ there exists a decomposition of $C_{(v:n)}$ into s_t C_{xn}-factors and r_t C_{yn}-factors.*

Let

$$s = \sum_{t=1}^{\frac{(m-1)}{2}} s_t \qquad \text{and} \qquad r = \sum_{t=1}^{\frac{(m-1)}{2}} r_t.$$

Then there exists a decomposition of $K_{(v:m)}$ into s C_{xn}-factors and r C_{yn}-factors.

In order to decompose $\overrightarrow{C}_{(x:n)}$, x and n odd, into C_n-factors and C_{xn}-factors, the authors of [17] labeled the vertices by $\mathbb{Z}_x \times \{0, \dots, n-1\}$. They build a 2-factor F by providing n permutations of G. The ith permutation is used to connect vertices in column $i-1$ to vertices in column i, in particular the n-th permutation is used to connect vertices in column $n-1$ to vertices in column 0. It must be said that these permutations were used implicitly in [17], as no permutation language was used for this part of the construction.

Notice that in general, if the columns are labeled by an abelian group G, f is the ith permutation and $g \in G$, in the 2-factor F, vertex $(g, i-1)$ is connected to vertex $(f(g), i)$. Let $\mathcal{F}$ be the composition of all n permutations of the 2-factor F, such that $(\mathcal{F}(g), 0)$ is the vertex at which we finish if we start at vertex $(g, 0)$ and move through F until we reach column 0 again. In the constructions in [17], G is abelian, and $g - \mathcal{F}(g)$ depends only on $\mathcal{F}$ and not on g. If this is the case, the length of the cycles of F is n times the order of the element $g - \mathcal{F}(g)$.

Lemma 2.2. *Assume F is a 2-factor built with the permutation $\mathcal{F}$, and $g - \mathcal{F}(g)$ depends only on $\mathcal{F}$. If q is the order of $g - \mathcal{F}(g)$, then F is a $\overrightarrow{C}_{qn}$-factor of $\overrightarrow{C}_{(xy4^k:n)}$.*

As we will need to use the permutations of $\mathbb{Z}_x$, we will introduce them. For $\alpha \in \mathbb{Z}_x$, let f_α be the permutation that adds α to every element of $\mathbb{Z}_x$, i.e. $f_\alpha(g) = g + \alpha$. Let $H(\alpha, \beta)$ be the 2-factor made with the following permutations:

- f_α from column $i-1$ to column i if $1 \leq i \leq n - 3/2$;
- $f_{-\alpha}$ from column $i-1$ to column i if $n - 3/2 + 1 \leq i \leq n - 3$;
- f_α from column $n-3$ to column $n-2$;

- $f_{-2\alpha}$ from column $n-2$ to column $n-1$ (this is a permutation because x is odd);
- f_β from column $n-1$ to column 0.

In [17] the first $n-3$ permutations were different, but the end result was the same. Notice that $\mathcal{F}(g) = g - \alpha + \beta$. For every $r \in \{0, 1, 2, \ldots, x-3, x-2, x\}$, the authors of [17] gave permutations ϕ of $\mathbb{Z}_x$ that satisfied:

(a) $\phi(\alpha) = \alpha$ for r elements of $\mathbb{Z}_x$;

(b) $\gcd(\alpha - \phi(\alpha), x) = 1$ for the remaining $x - r$ elements of $\mathbb{Z}_x$.

Then, the decomposition of $\overrightarrow{C}_{(x:n)}$ was given by the 2-factors $H(\alpha, \phi(\alpha))$, $\alpha \in \mathbb{Z}_x$.

In order for such a decomposition to work, for every $\alpha, \beta \in \mathbb{Z}_x$ the permutations f_α, f_β needed to satisfy $f_\alpha = f_\beta$ if and only if $\alpha = \beta$, as otherwise some arcs would be repeated in the factor $H(\alpha, \phi(\alpha))$ and the factor $H(\beta, \phi(\beta))$.

Then, in [17], decompositions of $\overrightarrow{C}_{(x:n)}$, $\overrightarrow{C}_{(y:n)}$, and $\overrightarrow{C}_{(4:n)}$ were combined using a graph product and permutations of $\mathbb{Z}_x \times \mathbb{Z}_y \times \mathbb{Z}_4$ to decompose $\overrightarrow{C}_{(4xy:n)}$. Instead of doing so, we will use group products to label the vertices of $\overrightarrow{C}_{(4^k xy:n)}$, although we will make use of permutations of the group product.

In Section 3, we give permutations of $\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}$, and show that they satisfy the necessary conditions to be used for decompositions. In Section 4, we use multivariate bijections to give decompositions of $\overrightarrow{C}_{(4^k xy:n)}$ into $\overrightarrow{C}_{2^k xk}$-factors and $\overrightarrow{C}_{yk}$-factors. Finally, in Section 5, we use these decompositions to prove our main results.

3 The permutation $f_\alpha(a, b)$ of $\mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}$

Consider the group $G = \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}$, an element $\alpha = (\alpha_1, \alpha_2)$ and the function $f_\alpha(a, b) = (-b + \alpha_1, a - b + \alpha_2)$.

Lemma 3.1. *f_α is a permutation of G.*

Proof. As $|G|$ is finite, it is enough to prove that f_α is an injective function.

Assume $f_\alpha(a, b) = f_\alpha(c, d)$. Then

$$(-b + \alpha_1, a - b + \alpha_2) = (-d + \alpha_1, c - d + \alpha_2).$$

The equality $-b + \alpha_1 = -d + \alpha_1$ implies $b = d$. Using $b = d$, the equality $a - b + \alpha_2 = c - d + \alpha_2$ implies $a = c$. Therefore, f_α is a permutation of G. □

Lemma 3.2. $f_\beta(f_\alpha^2(a, b)) = (a, b) - \alpha + \beta$.

Proof. We will prove this lemma by computing $f_\beta(f_\alpha^2(a, b))$.

$$\begin{aligned} f_\alpha(a, b) &= (-b + \alpha_1, a - b + \alpha_2) \\ f_\alpha^2(a, b) &= f(-b + \alpha_1, a - b + \alpha_2) \\ &= (-a + b - \alpha_2 + \alpha_1, -b + \alpha_1 - a + b - \alpha_2 + \alpha_2) \\ &= (-a + b - \alpha_2 + \alpha_1, \alpha_1 - a) \end{aligned}$$

$$\begin{aligned} f_\beta(f_\alpha^2(a,b)) &= f_\beta(-a+b-\alpha_2+\alpha_1, \alpha_1-a) \\ &= (a-\alpha_1+\beta_1, -a+b-\alpha_2+\alpha_1-\alpha_1+a+\beta_2) \\ &= (a-\alpha_1+\beta_1, b-\alpha_2+\beta_2) \\ &= (a,b)-\alpha+\beta. \end{aligned}$$ □

Letting $\beta = \alpha$ in Lemma 3.2 yields $f_\alpha^3(a,b) = (a,b)$.

Corollary 3.3. $f_\alpha^3(a,b) = (a,b)$.

As it was mentioned in Section 2, we need to show that if $\alpha \neq \beta$, then $f_\alpha(a,b) \neq f_\beta(a,b)$ for every $(a,b) \in \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}$; so that each arc is used exactly once. The statement of the following lemma is an equivalent claim.

Lemma 3.4. $f_\alpha(a,b) = f_\beta(a,b)$ *for some* $(a,b) \in \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}$ *if and only if* $\alpha = \beta$.

Proof. Assume $f_\alpha(a,b) = f_\beta(a,b)$. Then

$$(-b+\alpha_1, a-b+\alpha_2) = (-b+\beta_1, a-b+\beta_2).$$

Hence, $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. Therefore $\alpha = \beta$. □

4 Decomposing $\overrightarrow{C}_{(4^k xy:n)}$ into $\overrightarrow{C}_{yn}$-factors and $\overrightarrow{C}_{x2^k n}$-factors

Let $G = \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}$ and label each column of $\overrightarrow{C}_{(4^k xy:n)}$ with the elements of the group $G \times \mathbb{Z}_x \times \mathbb{Z}_y$.

Let $R = G \times \mathbb{Z}_x \times \mathbb{Z}_y$. For every $\lambda \in R$, let $\lambda = (\alpha, \beta, \gamma)$, with $\alpha \in G$, $\beta \in \mathbb{Z}_x$ and $\gamma \in \mathbb{Z}_y$. For $\alpha \in G$, let f_α be defined as in Section 3. For $\beta \in \mathbb{Z}_x$ let f_β be the permutation of $\mathbb{Z}_x$ defined by $f_\beta(a) = a + \beta$. Similarly, for $\gamma \in \mathbb{Z}_y$ let f_γ be the permutation of $\mathbb{Z}_y$ defined by $f_\gamma(a) = a + \gamma$. Finally, for $\lambda = (\alpha, \beta, \gamma) \in R$ let f_λ be the permutation of R defined by $f_\lambda(a,b,c) = (f_\alpha(a), f_\beta(b), f_\gamma(c))$.

Let φ be a permutation of R, and for each $\lambda \in G$ let $H_{4^k xy}(\lambda, \varphi(\lambda))$ be the 2-factor formed with the following permutations:

1. f_λ from column i to $i+1$ if $1 \leq i \leq n - 3/2$;
2. f_λ^{-1} from column i to $i+1$ if $n - 3/2 + 1 \leq i \leq n - 3$;
3. f_λ from column $n-2$ to column $n-1$;
4. $f(\alpha, -2\beta, -2\gamma)$ from column $n-1$ to column n;
5. $f_{\varphi(\alpha)}$ from column n to column 1.

Notice that if you start in column 1 at vertex (a,b,c) the first time you reach column 1 again you reach vertex

$$(a,b,c) - (\alpha,\beta,\gamma) + \varphi(\alpha,\beta,\gamma) = (a,b,c) - \lambda + \varphi(\lambda).$$

Hence, we can apply Lemma 2.2 to obtain the length of the cycles in the 2-factor.

Let $\lambda \in R$. If $\lambda - \varphi(\lambda) = (a,b,0)$ with $a \in \pm\{(1,0),(0,1),(1,1)\}$ and $\gcd(b,x) = 1$, then by Lemma 2.2 the 2-factor $H_{4^k}(\lambda, \varphi(\lambda))$ is a $\overrightarrow{C}_{2^k xn}$-factor. If $\lambda - \varphi(\lambda) = (0,0,c)$ with $\gcd(c,y) = 1$, Lemma 2.2 implies that $H_{4^k}(\lambda, \varphi(\lambda))$ is a $\overrightarrow{C}_{yn}$-factor.

Therefore, to obtain a decomposition of $\overrightarrow{C}_{(4^k xy:n)}$ into r $\overrightarrow{C}_{2^k xn}$-factors and s $\overrightarrow{C}_{yn}$-factors, we need a permutation φ satisfying

(A) $\lambda - \varphi(\lambda) = (a, b, 0)$ with $a \in \pm\{(1,0),(0,1),(1,1)\}$ and $\gcd(b, x) = 1$ for r elements $\lambda \in R$;

(B) $\lambda - \varphi(\lambda) = (0, 0, c)$ with $\gcd(c, y) = 1$ for $s = 4^k xy - r$ elements $\lambda \in R$.

In order to obtain the permutation φ, consider the subgroup $2G$ of G of index 4, and let

$$K = \{(0,0),(1,0),(0,1),(1,1)\}.$$

Notice that K is a set of representatives of the cosets of $2G$ in G. Let $\epsilon \in 2G$, and let ϕ be a permutation of G. If $g, \phi(g) \in \epsilon + K$, then either $g = \phi(g)$ or $|g - \phi(g)| = 2^k$ because $g - \phi(g) \in \pm K$. Hence, we can obtain φ by providing 4^{k-1} permutations ρ_ϵ of $K \times \mathbb{Z}_x \times \mathbb{Z}_y$ satisfying

(A′) $\lambda - \rho_\epsilon(\lambda) = (a, b, 0)$ with $a \in \pm\{(1,0),(0,1),(1,1)\}$ and $\gcd(b, x) = 1$ for r_ϵ elements $\lambda \in K \times \mathbb{Z}_x \times \mathbb{Z}_y$;

(B′) $\lambda - \rho_\epsilon(\lambda) = (0, 0, c)$ with $\gcd(c, y) = 1$ for $s_\epsilon = 4xy - r_\epsilon$ elements $\lambda \in K \times \mathbb{Z}_x \times \mathbb{Z}_y$;

having $r = \sum_{\epsilon \in 2G} r_\epsilon$, and having φ act in each $(\epsilon + K) \times \mathbb{Z}_x \times \mathbb{Z}_y$ as ρ_ϵ, i.e. if $g = (\epsilon, c, d) + (\mu, 0, 0)$, with $\mu \in K$, $\varphi(g) = (\epsilon, c, d) + \rho_\epsilon(\mu, c, d)$. Notice that if $a \in K$, $a \in \pm\{(1,0),(0,1),(1,1)\}$ if and only if $a \neq (0,0)$.

In [17], for every $r \in \{0, 2, 3, \ldots, 4xy - 3, 4xy - 2, 4xy\}$, permutations ϕ of $\mathbb{Z}_4 \times \mathbb{Z}_x \times \mathbb{Z}_y$ were given satisfying:

(A″) $\lambda - \phi(\lambda) = (a, b, 0)$, with $a \neq 0$ and $\gcd(b, x) = 1$ for r elements $\lambda \in \mathbb{Z}_4 \times \mathbb{Z}_x \times \mathbb{Z}_y$;

(B″) $\lambda - \phi(\lambda) = (0, 0, c)$, with $\gcd(c, y) = 1$ for the remaining $4xy - r$ elements $\lambda \in \mathbb{Z}_4 \times \mathbb{Z}_x \times \mathbb{Z}_y$.

Let $\pi\colon K \to \mathbb{Z}_4$ be a bijection such that $\pi(0,0) = 0$, and let $\psi\colon K \times \mathbb{Z}_x \times \mathbb{Z}_y \to \mathbb{Z}_4 \times \mathbb{Z}_x \times \mathbb{Z}_y$ be the bijection that fixes the coordinates of $\mathbb{Z}_x$ and $\mathbb{Z}_y$, and that behaves like π in the coordinate of K. Then if ϕ_ϵ is a permutation of $\mathbb{Z}_4 \times \mathbb{Z}_x \times \mathbb{Z}_y$, satisfying Conditions (A″) and (B″) with r_ϵ and s_ϵ, $\rho_\epsilon = \psi^{-1}\phi_\epsilon\psi$ is a permutation of $K \times \mathbb{Z}_x \times \mathbb{Z}_y$ satisfying Conditions (A′) and (B′) with r_ϵ and s_ϵ.

If we wanted either $x = 1$ or $y = 1$, we would need to change Conditions (A) and (B), but it is easy to see that the necessary permutations to decompose $\overrightarrow{C}_{(4^k xy:n)}$ exist.

Therefore we have the following.

Lemma 4.1. *Let $r \notin \{1, 4^k xy - 1\}$, then there is a decomposition of $\overrightarrow{C}_{(4^k xy:n)}$ into r $\overrightarrow{C}_{2^k xn}$-factors and $s = 4^k xy - r$ C_{yn}-factors.*

5 Main results

The complete solution to the uniform case of the Oberwolfach problem will be vital to the proof of our main result.

Theorem 5.1 ([1, 2, 15, 22]). *K_v can be decomposed into C_m-factors (and a 1-factor if v is even) if and only if $v \equiv 0 \pmod{m}$, $(v, m) \neq (6, 3)$ and $(v, m) \neq (12, 3)$.*

We now apply the results from Section 4 to produce the following important result for the uniform equipartite version of the Hamilton-Waterloo problem where the two factor types consist of cycle sizes of distinct parities.

Theorem 5.2. *Let x, y, z, v, m, k be positive integers $v, m, k \geq 3$ satisfying the following:*

(i) $v, m \geq 3$,

(ii) $k \geq 2$,

(iii) x, y, z *odd,*

(iv) $z \geq 3$,

(v) $\gcd(x, y) = 1$,

(vi) $vm \equiv 0 \pmod{4^k xyz}$, $v \equiv 0 \pmod{4^k xy}$,

(vii) $\frac{v(m-1)}{4^k xy}$ *is even,*

(viii) $\left(\frac{v}{4^k xy}, m, z\right) \notin \{(2,3,3), (6,3,3), (2,6,3), (6,2,6)\}$,

then there is a decomposition of $K_{(v:m)}$ into r $C_{2^k xz}$-factors and s C_{yz}-factors, for any $s, r \neq 1$.

Proof. Let $v_1 = v/4^k xy$. Consider $K_{(v_1:m)}$. Item (vi) ensures that z divides $v_1 m$; and items (vii), (i), and (viii) give us $v_1(m-1)$ is even, $m \neq 2$, and

$$\left(\frac{v}{4^k xy}, m, z\right) \notin \{(2,3,3), (6,3,3), (2,6,3), (6,2,6)\}.$$

Thus by Theorem 1.1 there is a decomposition of $K_{(v_1:m)}$ into C_z-factors.

Replace each vertex in $a \in K_{(v_1:m)}$ by $4^k xy$ vertices (a, b), with $0 \leq b \leq 4^k xy - 1$, having an edge between (a_1, b_1) and (a_2, b_2) if and only if there was an edge between a_1 and a_2. This yields $K_{(v:m)}$. Even more, each C_z-factor becomes a copy of $\frac{v_1 m}{z} C_{(4^k xy:z)}$. By Lemma 4.1, we have that each $\frac{v_1 m}{z} C_{(4^k xy:z)}$ can be decomposed into r_p $C_{2^k xz}$-factors and s_p C_{yz}-factors as long as $r_p, s_p \neq 1$. Choosing s_p such that $\sum_p s_p = s$ and $s_p, r_p \neq 1$, provides a decomposition of $K_{(v:m)}$ into r $C_{2^k xz}$-factors and s C_{yz}-factors by Lemma 2.1. □

The next lemma, given in [17] shows how to find solutions to the Hamilton-Waterloo problems by combining solutions for the problem on complete graphs and solutions for the problem on equipartite graphs.

Lemma 5.3 ([17])**.** *Let m and v be positive integers. Let F_1 and F_2 be two 2-factors on vm vertices. Suppose the following conditions are satisfied:*

- *There exists a decomposition of $K_{(v:m)}$ into s_α copies of F_1 and r_α copies of F_2.*
- *There exists a decomposition of mK_v into s_β copies of F_1 and r_β copies of F_2.*

Then there exists a decomposition of K_{vm} into $s = s_\alpha + s_\beta$ copies of F_1 and $r = r_\alpha + r_\beta$ copies of F_2.

We are now in a position to provide a proof of the main theorem.

Theorem 5.4. *Let x, y, v, k and m be positive integers such that:*

(i) $v, m \geq 3$,

(ii) x, y *are odd,*

(iii) $\gcd(x, y) \geq 3$,

(iv) x *and* y *divide* v,

(v) 4^k *divides* v.

Then there exists a solution to $(2^k x, y)$-$\mathrm{HWP}(vm; r, s)$ *for every pair* r, s *with* $r + s = \lfloor (vm-1)/2 \rfloor$, $r, s \neq 1$.

Proof. Let r and s be positive integers with $r + s = \lfloor (vm-1)/2 \rfloor$ and $r, s \neq 1$. Write $r = r_\alpha + r_\beta$ and $s = s_\alpha + s_\beta$, where $r_\alpha, r_\beta, s_\alpha, s_\beta$ are positive integers that satisfy $r_\alpha, s_\alpha \neq 1$, $r_\alpha + s_\alpha = v(m-1)/2$, $r_\beta + s_\beta = \lfloor (v-1)/2 \rfloor$, and $r_\beta, s_\beta \in \{0, \lfloor (v-1)/2 \rfloor\}$.

Start by decomposing K_{vm} into $K_{(v:m)} \oplus mK_v$. Let $z = \gcd(x, y)$, $x_1 = x/z$, $y_1 = y/z$. By Theorem 5.2 there is a decomposition of $K_{(v:m)}$ into r_α $C_{2^k x_1 z}$-factors and s_α $C_{y_1 z}$-factors. This is a decomposition of $K_{(v:m)}$ into r_α $C_{2^k x}$-factors and s_α C_y-factors. By Theorem 5.1 there is a decomposition of mK_v into r_β $C_{2^k x}$-factors and s_β C_y-factors. Lemma 5.3 shows that all of this together yields a decomposition of K_{vm} into r C_x-factors and s C_y-factors. □

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The finite embeddability property for IP loops and local embeddability of groups into finite IP loops

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Abstract

We prove that the class of all loops with the inverse property (IP loops) has the Finite Embeddability Property (FEP). As a consequence, every group is locally embeddable into finite IP loops. The first one of these results is obtained as a consequence of a more general embeddability theorem, contributing to a list of problems posed by T. Evans in 1978, namely, that every finite partial IP loop can be embedded into a finite IP loop.

Keywords: Group, IP loop, finite embeddability property, local embeddability.

Math. Subj. Class.: 20E25, 20N05, 05B07, 05B15, 05C25, 05C45

The *Finite Embeddability Property* (briefly FEP), was introduced by Henkin [17] for general algebraic systems already in 1956. For groupoids (i.e., algebraic structures $(G, \cdot)$ with a single binary operation), which is sufficient for our purpose, it reads as follows: A class $\mathbf{K}$ of groupoids has the FEP if for every algebra $(G, \cdot) \in \mathbf{K}$ and each nonempty finite subset $X \subseteq G$ there is a *finite* algebra $(H, *) \in \mathbf{K}$ extending $(X, \cdot)$, i.e., $X \subseteq H$ and $x \cdot y = x * y$ for all $x, y \in X$, such that $x \cdot y \in X$. Using this notion an earlier result of Henkin [16] can be stated as follows: The class of all abelian groups has the FEP (see also Grätzer [14]).

A more general notion of local embeddability can be traced back to even earlier papers by Mal'tsev [21, 22] (see also the posthumous monograph [23]). It was explicitly

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(re)introduced and studied in detail mainly for groups by Gordon and Vershik [28]: A groupoid $(G, \cdot)$ is *locally embeddable into a class of groupoids* $\mathbf{M}$ if for every nonempty finite set $X \subseteq G$ there is $(H, *) \in \mathbf{M}$ such that $X \subseteq H$ and $x \cdot y = x * y$ for all $x, y \in X$ satisfying $x \cdot y \in X$. Informally this means that every finite cut-out from the multiplication table of $(G, \cdot)$ can be embedded into an algebra from $\mathbf{M}$. A standard model-theoretic argument shows that this condition is equivalent to the embeddability of $(G, \cdot)$ into an *ultraproduct* of algebras from $\mathbf{M}$ (for the ultraproduct construction see, e.g., Chang, Keisler [3]).

Thus a class $\mathbf{K}$ has the FEP if and only if every $(G, \cdot) \in \mathbf{K}$ is locally embeddable into the class $\mathbf{K}_{\text{fin}}$ of all finite members in $\mathbf{K}$. As proved by Evans [9], for a variety (equational class) $\mathbf{K}$ this is equivalent to the condition that every finitely presented algebra in $\mathbf{K}$ is residually finite, i.e., embeddable into a direct product of finite algebras from $\mathbf{K}$. The groups locally embeddable into (the class of all) finite groups were called *LEF groups* in [28]. The authors also noticed that, unlike the abelian ones, not all groups are LEF, in other words, the class of all groups doesn't have the FEP. As examples of finitely presented groups which are not residaully finite, hence not locally embeddable into finite groups, can serve the Baumslag-Solitar groups $BS(m, n) = \langle a, b \mid a^{-1} b^m a = b^n \rangle$ for $|m|$, $|n| > 1$, $|m| \neq |n|$ (see Meskin [24]). A complete list of minimal partial Latin squares embeddable into a closely related infinite group but not embeddable into any finite group, even under a weaker concept of embedding, was recently described by Dietrich and Wanless [6].

This immediately raises the question of finding some classes of finite grupoids into which all the groups were locally embeddable and which, at the same time, would be "as close to groups as possible". The question is of interest for various reasons: The class of all LEF groups properly extends the class of all locally residually finite groups and plays an important role in dynamical systems, cellular automata, etc. (see, e.g., Ceccherini-Silberstein, Coornaert [2], Gordon, Vershik [28]).

Glebsky and Gordon [13] have shown that a group is locally embeddable into finite semigroups if and only if it is an LEF group. It follows that looking for a class of finite groupoids into which one could locally embed all the groups one has to sacrifice the associativity condition. They also noticed that the results about extendability of partial Latin squares to (complete) Latin squares imply that every group is locally embeddable into finite quasigroups. Refining slightly the original argument they have shown that every group can even be locally embedded into finite loops (see also their survey article [12]).

A further decisive step in this direction was done by Ziman [30]. Building upon the methods of extension of partial Latin squares preserving some symmetry conditions (see Cruse [4] and Lindner [20]), he has shown that the class of all loops with *antiautomorphic inverses*, i.e., loops with two-sided inverses satisfying the identity $(xy)^{-1} = y^{-1}x^{-1}$ (briefly *AAIP loops*), has the FEP (though he didn't use this notion explicitly). As a consequence, every group is locally embeddable into finite AAIP loops.

Quasigroups and loops experts consider the class of all AAIP loops still as a "rather far going extension" of the class of all groups. On the other hand, they find the class of all loops with the *inverse property*, i.e., loops with two-sided inverses satisfying the identities $x^{-1}(xy) = y = (yx)x^{-1}$ (briefly IP loops), which is a proper subclass of the class of all AAIP loops, a much more moderate extension of the class of all groups (Drápal [8]). In the present paper we are going to show that Ziman's result can indeed be strengthened in this sense. Using mainly graph-theoretical methods and Steiner triple systems, we will prove that the class of all IP loops still has the FEP. As a consequence, every group is locally embeddable into finite IP loops.

When the original version of this paper was already submitted, A. Drápal turned our attention towards the point that the problem we are solving was implicitly formulated in Evans' paper [10] in the last line of the table on page 798. At the same time he remarked that we have proved even more, namely that every finite partial IP loop can be embedded into a finite IP loop. Later on, the same point was made by the anonymous referee. We will discuss these issues in the next section, after introducing the respective notions and formulating our results more precisely.

For basic definitions and facts about quasigroups and loops the reader is referred to the monographs Belousov and Belyavskaya [1] and Pflugfelder [26].

1 Formulation of the main results, discussion and plan of the proof

In order to guarantee that the class of all IP loops forms a variety, we define an *IP loop* as an algebra $\left(L,\cdot,1,{}^{-1}\right)$ with a binary operation of multiplication $\cdot$, a distinguished element 1 denoting the unit, and a unary operation ${}^{-1}$ of taking inverses, satisfying the identities

$$1x = x = x1, \qquad \text{and} \qquad x^{-1}(xy) = y = (yx)x^{-1}.$$

Then the identities $x^{-1}x = 1 = xx^{-1}$ and $\left(x^{-1}\right)^{-1} = x$ easily follow. Also it is clear that, for any $a, b \in L$, the equations $ax = b$, $ya = b$ have unique solutions $x = a^{-1}b$, $y = ba^{-1}$ in L. Since the unit element 1 and the inverse map $x \mapsto x^{-1}$ in every IP loop are uniquely determined by the multiplication $\cdot$, referring to an IP loop $\left(L,\cdot,1,{}^{-1}\right)$ as just $(L,\cdot)$ is unambiguous. However, usually it will be denoted simply by L.

A *partial IP loop* $(P,\cdot)$ is a set P endowed with a partial binary operation $\cdot$ defined on a subset $D(P,\cdot) \subseteq P \times P$, called the *domain* of the operation $\cdot$, satisfying the following three conditions:

(1) there is an element $1 \in P$, called the *unit* of P, such that $(1,x), (x,1) \in D(P,\cdot)$ and $1x = x1 = x$ for all $x \in P$;

(2) for each $x \in P$ there is a unique $y \in P$, called the *inverse* of x and denoted by $y = x^{-1}$, such that $(x,y), (y,x) \in D(P,\cdot)$ and $xy = yx = 1$;

(3) for any $x, y \in P$ such that $(x,y) \in D(P,\cdot)$, we have $\left(x^{-1}, xy\right), \left(xy, y^{-1}\right) \in D(P,\cdot)$ and $x^{-1}(xy) = y$, $(xy)y^{-1} = x$.

In most cases we will denote a partial IP loop $(P,\cdot)$ as P and its domain as $D(P)$, only; the more unambiguous notation $(P,\cdot)$ and $D(P,\cdot)$ will be used mainly in case we need to distinguish the operations on two or more (partial) IP loops.

A partial IP loop $(Q,*)$ is called an *extension* of a partial IP loop $(P,\cdot)$ if $P \subseteq Q$, $D(P,\cdot) \subseteq D(Q,*)$ and $x \cdot y = x * y$ for each pair $(x,y) \in D(P)$. Suppressing the names of the operations, we write $P \le Q$ or $Q \ge P$. Alternatively we say that the partial IP loop P is *embedded* in the partial IP loop Q. Obviously, the relation $\le$ between partial IP loops is reflexive, antisymmetric and transitive.

Our main results are the following three theorems.

Theorem 1.1. *Every finite partial IP loop P can be embedded into some finite IP loop L.*

Given an IP loop L and a finite set $X \subseteq L$, we can form the finite partial IP loop

$$P = X \cup \{1\} \cup X^{-1},$$

where $X^{-1} = \{x^{-1} : x \in X\}$, by restricting the original loop operation on L to the set

$$D(P) = \{(x, y) \in P \times P : xy \in P\}.$$

Then, obviously, $P \leq L$. Thus Theorem 1.1 readily implies our next, already announced result.

Theorem 1.2. *The class of all IP loops has the Finite Embeddability Property. Equivalently, every finitely presented IP loop is residually finite.*

As a special case of Theorem 1.2 we obtain

Theorem 1.3. *Every group can be locally embedded into the class of all finite IP loops. Equivalently, every group can be embedded into some ultraproduct of finite IP loops.*

The second, equivalent formulation of Theorem 1.2 answers in affirmative the question posed by Evans [10] in the IP loop row and R. F. column of the table on page 798.

In order to discuss the relation of our results to Evans' table we recall the following three abbreviations used in [10]. Unlike the author, who used them for various classes of algebras, we apply them just to the class of all IP loops.

E_1: Every finite partial IP loop can be embedded into some finite IP loop.

E_2: Every finite partial IP loop can be embedded into some (finite or infinite) IP loop.

E_3: Every finite partial IP loop which can be embedded into some IP loop, can be embedded into some finite IP loop.

Obviously, condition E_3 is equivalent to the FEP for IP loops, and condition E_1 is equivalent to the conjunction $E_2 \wedge E_3$, depicted (among other relations) in the chart on the top of page 798 in [10] (the equivalence $E_1 \wedge E_2 \Leftrightarrow E_3$ on page 797 is clearly a typo).

Theorem 1.1 seems to contradict the sign "X" (meaning "No") in the IP loop row and E_1 column of the table in the middle of page 798 in [10]. Evans, however, considered a seemingly weaker notion of a partial IP loop (hence a stronger concept of embeddability) there. Paraphrasing and slightly adapting his definition to apply to our situation, we obtain a rather vague formulation: "A partial IP loop is a set P in which the operations of multiplication and taking inverses are defined on some subsets $D(P) \subseteq P \times P$ and $D'(P) \subseteq P$, respectively, which satisfies the defining IP loop identities, insofar as they can be applied to the partial operations on P" (cf. [10, §3, page 796]). In particular it is not clear (though not crucial) whether P has to contain the unit element 1 or not. Thus, at least at a glance, it seems possible that there could exist some finite partial IP loop in his sense, which is not embeddable into any finite IP loop. However, the responses "X" (i.e. "No") to E_1 and "$\surd$" (meaning "Yes") to E_2 in Evans' table, together with our Theorem 1.2 responding E_3 affirmatively, still contradict the equivalence $E_1 \Leftrightarrow E_2 \wedge E_3$, regardless of the details of the definition of a partial IP loop. Unfortunately, Evans provided neither any counterexample to E_1 nor any proof or reference in favor of E_2 in [10].

A possible clue to resolving this problem lies in the PhD thesis [27] by Evans' student C. Treash from 1969. Her definition of the concept of an *incomplete IP loop* on page 27 is namely equivalent to that of our partial IP loop (cf. our Lemma 2.1). According to her Theorem 1, stated and proved on page 28: *Every (finite or infinite) incomplete IP loop can be embedded into some IP loop*, which implies E_2 as a special case. Thus Evans' negative response to E_1 seems to be indeed a shortcoming or just another typo.

After this digression we are returning to the main theme of our paper.

From the course of our arguments it is clear that it suffices to prove just Theorem 1.1. We divide its proof into three steps consisting of the three propositions below. Their formulation requires some additional notions and notation.

In the absence of associativity there is no obvious way how to define the order of an element. Nonetheless, the sets of elements of order 2 and 3, respectively, can still be defined for any partial IP loop P:

$$O_2(P) = \{x \in P : x \neq 1,\ (x,x) \in D(P) \text{ and } xx = 1\},$$
$$O_3(P) = \{x \in P : x \neq 1,\ (x,x),(x,xx),(xx,x) \in D(P) \text{ and } x(xx) = (xx)x = 1\}.$$

In other words, for $x \neq 1$ in P we have $x \in O_2(P)$ if and only if $x^{-1} = x$, and $x \in O_3(P)$ if and only if $(x,x) \in D(P)$ and $x^{-1} = xx$. The number of elements of the sets $O_2(P)$, $O_3(P)$ in a finite partial IP loop P will be denoted by $o_2(P)$, $o_3(P)$, respectively. The number of elements of any finite set A is denoted by $\#A$.

Proposition 1.4. *Let $(P,\cdot)$ be a finite partial IP loop. Then there exists a finite partial IP loop $(Q,*)$ such that $P \leq Q$ and $3 \mid o_3(Q)$.*

A pair (x,y) in a partial IP loop P will be called a *gap* if $(x,y) \notin D(P)$. The set of all gaps in P will be denoted by

$$\Gamma(P,\cdot) = (P \times P) \setminus D(P,\cdot) = \{(x,y) \in P \times P : (x,y) \notin D(P,\cdot)\},$$

or just briefly by $\Gamma(P)$. Obviously, both $D(P)$, $\Gamma(P)$ are binary relation on the set P, and a partial IP loop P is an IP loop if and only if it contains no gaps, i.e., $\Gamma(P) = \emptyset$.

Proposition 1.5. *Let P be a finite partial IP loop such that $3 \mid o_3(P)$. Then there exists a finite partial IP loop $Q \geq P$ satisfying the following four conditions:*

(4) $3 \mid o_3(Q)$, $\#Q \geq 10$, $\#Q \equiv 4 \pmod 6$ *and* $\Gamma(Q) \subseteq O_2(Q) \times O_2(Q)$.

Proposition 1.6. *Let P be a finite partial IP loop satisfying the above conditions (4), such that $\Gamma(P) \neq \emptyset$. Then there exists a finite partial IP loop $Q \geq P$ satisfying the conditions (4), as well, such that $\#\Gamma(Q) < \#\Gamma(P)$.*

Theorem 1.1 follows from Propositions 1.4, 1.5 and 1.6. Indeed, if P is a finite partial IP loop (such that $\Gamma(P) \neq \emptyset$, because otherwise there is nothing to prove) then, using Proposition 1.4, we can find a finite partial IP loop $Q \geq P$ such that $3 \mid o_3(Q)$. If $\Gamma(Q) = \emptyset$ then $L = Q$ is already a finite IP loop extending P, and we are done. Otherwise, applying Proposition 1.5, we obtain a finite partial IP loop $Q_1 \geq Q$ satisfying conditions (4) from Proposition 1.5. If $\Gamma(Q_1) = \emptyset$ then we are done, again. Otherwise, we can apply Proposition 1.6 and get a finite partial IP loop $Q_2 \geq Q_1$ satisfying conditions (4), as well, such that $\#\Gamma(Q_2) < \#\Gamma(Q_1)$. Iterating this step finitely many times we finally arrive at some finite partial IP loop Q_n extending P such that $\Gamma(Q_n) = \emptyset$. Then $L = Q_n \geq P$ is a finite IP loop we have been looking for.

Thus it is enough to prove Propositions 1.4, 1.5 and 1.6. This will take place in the next four sections.

2 Some preliminary results

In this section we list the auxiliary results we will use in the proofs of Propositions 1.4, 1.5 and 1.6.

Lemma 2.1. *Let P be a partial IP loop and $x, y, z \in P$. Then the following six conditions are equivalent:*

(i) $(x, y) \in D(P)$ *and* $xy = z$*;*

(ii) $\left(z, y^{-1}\right) \in D(P)$ *and* $zy^{-1} = x$*;*

(iii) $\left(x^{-1}, z\right) \in D(P)$ *and* $x^{-1}z = y$*;*

(iv) $\left(y, z^{-1}\right) \in D(P)$ *and* $yz^{-1} = x^{-1}$*;*

(v) $\left(z^{-1}, x\right) \in D(P)$ *and* $z^{-1}x = y^{-1}$*;*

(vi) $\left(y^{-1}, x^{-1}\right) \in D(P)$ *and* $y^{-1}x^{-1} = z^{-1}$*.*

Proof. Using property (3) from the definition of partial IP loops and (if necessary) the fact that $\left(a^{-1}\right)^{-1} = a$ for any $a \in P$, we can get the following cycle of implications:

$$\text{(i)} \implies \text{(ii)} \implies \text{(v)} \implies \text{(vi)} \implies \text{(iv)} \implies \text{(iii)} \implies \text{(i)}.$$

We show just the first implication, leaving the remaining ones to the reader. If $(x, y) \in D(P)$ and $xy = z$ then, according to (3),

$$\left(z, y^{-1}\right) = \left(xy, y^{-1}\right) \in D(P)$$

and $zy^{-1} = x$. □

In particular, we will frequently use the fact that, as a consequence of Lemma 2.1, the conditions $(x, y) \in D(P)$ and $\left(y^{-1}, x^{-1}\right) \in D(P)$ are equivalent for any $x, y \in P$.

In the generic case all the six equivalent conditions in Lemma 2.1 are different. There are just two kinds of exceptions: first the trivial ones, when at least one of the elements x, y, z equals the unit 1 (which never produce gaps), and second, if $x = y \in O_3(P)$, when the six conditions reduce to just two:

(i_3) $(x, x) \in D(P)$ and $xx = x^{-1}$,

(ii_3) $\left(x^{-1}, x^{-1}\right) \in D(P)$ and $x^{-1}x^{-1} = x$.

Since the first condition is satisfied by the definition of order 3 elements, so is the second one, hence this situation produces any gap, neither.

From now on we will preferably use a more relaxed language: when writing $xy = z$ for elements x, y, z of some partial IP loop P we will automatically assume that $(x, y) \in D(P)$, without mentioning it explicitly.

The number of gaps in any finite partial IP loop P is related to the size of P and that of the set $O_3(P)$ of order three elements through a congruence modulo 6.

Lemma 2.2. *Let P be a finite partial IP loop. Then*

$$\#\Gamma(P) \equiv (\#P - 1)(\#P - 2) - o_3(P) \pmod 6.$$

Proof. We know that $(x,1),(1,x),\big(x,x^{-1}\big) \in D(P)$ for any $x \in P$. At the same time, $(a,a) \in D(P)$ for all $a \in O_3(P)$, as well. Except for these pairs, there are other $(\#P-1)(\#P-2)-o_3(P)$ pairs which can be either in $D(P)$ or in $\varGamma(P)$. Those which are in $D(P)$ can be split into sixtuples according to Lemma 2.1, hence their number is divisible by 6, proving the above congruence. □

We will also use the Dirac's theorem from [7], giving a sufficient condition for the existence of a Hamiltonian cycle in a graph. For our purpose, the term *graph* always means an undirected graph without loops and multiple edges. For the basic graph-theoretical concepts the reader is referred to Diestel [5].

Lemma 2.3. *Let G be a graph with $n \geq 3$ vertices in which every vertex has degree at least $n/2$. Then G has a Hamiltonian cycle.*

3 Extensions of partial IP loops and the proof of Proposition 1.4

All the three Propositions 1.4, 1.5 and 1.6 deal with extensions $(Q,*)$ of a partial IP loop $(P,\cdot)$, which can be combined using two more specific types of this construction: first, extensions preserving (the domain of) the binary operation $\cdot$ on the original partial IP loop P and extending the base set of P, and, second, extensions preserving the base set P and extending (the domain of) the binary operation on P. In symbols, the extending partial IP loop $Q \geq P$ satisfies $D(P) = D(Q) \cap (P \times P)$ in the first case, while $P = Q$ and $D(P,\cdot) \subset D(Q,*)$ in the second. We start with the first type of extensions.

Let P, Q be two partial IP loops such that $P \cap Q = \{1\}$, i.e., their base sets have just the unit element 1 in common. Then, obviously, the set $P \cup Q$ can be turned into a partial IP loop, which we denote by $P \sqcup Q$, extending both P and Q, with domain

$$D(P \sqcup Q) = D(P) \cup D(Q),$$

i.e., preserving the original operations on both P and Q, and leaving undefined all the products xy, yx, for $x \in P \setminus \{1\}$, $y \in Q \setminus \{1\}$. The partial IP loop $P \sqcup Q$ is called the *direct sum* of the partial IP loops P and Q.

Let us fix the notation for some particular cases of this construction, considered as extensions of the IP loop P fixed in advance. In all the particular cases below A denotes a nonempty set disjoint from P. Then the set $A \cup \{1\}$ will be turned into a partial IP loop $\big(A \cup \{1\},\cdot\big)$, depending on some map $\sigma\colon A \to A$.

Let $\sigma\colon A \to A$ be an involution, i.e., $\sigma(\sigma(a)) = a$ for any $a \in A$. Then the *minimal partial IP loop* $[A,\sigma]$ has the base set $A \cup \{1\}$ and the partial binary operation given by $1 \cdot 1 = 1$, and

$$1a = a1 = a, \qquad a\sigma(a) = \sigma(a)a = 1,$$

for any $a \in A$, leaving the operation result ab undefined for any other pair of elements $a, b \in A$. The reader is asked to realize that $[A,\sigma]$ is indeed a partial IP loop, and that it is minimal (concerning its domain) among all partial IP loops with the base set $A \cup \{1\}$, which satisfy

$$a^{-1} = \sigma(a)$$

for any $a \in A$. Then, obviously,

$$O_2[A,\sigma] = \{a \in A : \sigma(a) = a\},$$

i.e., order two elements in $[A, \sigma]$ coincide with the fixpoints of the map σ. The direct sum of the partial IP loops P and $[A, \sigma]$ is denoted by

$$P[A, \sigma] = P \sqcup [A, \sigma].$$

Order two elements in $P[A, \sigma]$ split into two disjoint easily recognizable parts

$$O_2(P[A, \sigma]) = O_2(P) \cup O_2[A, \sigma].$$

If $\sigma = \mathrm{id}_A \colon A \to A$ is the identity on A, we write

$$P[A, \mathrm{id}_A] = P[A],$$

in which case

$$O_2(P[A]) = O_2(P) \cup A.$$

If $A = \{a_1, \ldots, a_n\}$ is finite, we write

$$P[A] = P[a_1, \ldots, a_n].$$

In particular, if $A = \{a\}$ is a singleton (and $\sigma = \mathrm{id}_A$ is the unique map $A \to A$), then

$$P[\{a\}] = P[a].$$

If $A = \{a, a'\}$ where $a \neq a'$, and σ is the transposition exchanging a and a', we denote

$$P[A, \sigma] = P[a \leftrightarrow a'].$$

From among the second type of extensions of a partial IP loop P, preserving its base set P and extending just (the domain of) its operation the simplest ones attempt at filling in just a single gap in P. This type of extension will be called a *simple extension through the relation* $xy = z$. More precisely, having $x, y, z \in P$ such that $(x, y) \in \Gamma(P)$, we want to put $xy = z$. From Lemma 2.1 it follows that then we have to satisfy the remaining five relations, too. This is possible only if all the pairs (x, y), $\left(z, y^{-1}\right)$, $\left(x^{-1}, z\right)$, etc., occurring there are gaps in P. This is a sufficient condition, as well, since in that case we can define all the products as required by Lemma 2.1. Thus filling in the gap (x, y) enforces to fill in some other related gaps, too. In that case we automatically assume that the remaining five relations are defined in accord with Lemma 2.1.

Iterating simple extensions through particular relations we have to check in each step whether any new relation $uv = w$ (and its equivalent forms) does not interfere not only with the pairs in $D(P)$ but also with the gaps already filled in by previous simple extensions. In other words, we are interested in situations when we can fill in a whole set of gaps at once.

If $(P, \cdot)$ is a partial IP loop and $*$ is a partial operation on the set P with domain $T \subseteq P \times P$, such that $T \subseteq \Gamma(P)$ then, since $D(P) \cap T = \emptyset$, we can extend the original operation $\cdot$ to the set $D(P) \cup T$ by putting $xy = x * y$ for $(x, y) \in T$. The resulting structure will be called the *extension of the IP loop P through the operation* $*$ and denoted by P^*. The next lemma tells us when such an extension gives us a partial IP loop, again. In its formulation x^{-1} denotes the inverse of the element $x \in P$ with respect to the original operation $\cdot$ in P.

Lemma 3.1. *Let $(P,\cdot)$ be a partial IP loop and $*$ be a partial binary operation on the set P with domain $T \subseteq \Gamma(P)$. Then the extension of the operation $\cdot$ through the operation $*$ to the set $D(P)\cup T$ yields a partial IP loop extending P if and only if T and $*$ satisfy the following condition:*

(5) for any $x,y,z\in P$, if $(x,y)\in T$ and $x*y=z$ then also all the pairs

$$\left(z,y^{-1}\right),\ \left(x^{-1},z\right),\ \left(y,z^{-1}\right),\ \left(z^{-1},x\right),\ \left(y^{-1},x^{-1}\right)$$

belong to T and satisfy all the relations

$$z*y^{-1}=x,\qquad x^{-1}*z=y,\qquad y*z^{-1}=x^{-1},$$
$$z^{-1}*x=y^{-1},\qquad y^{-1}*x^{-1}=z^{-1}.$$

Proof. In view of Lemma 2.1, condition (5) obviously is necessary. By the same reason, condition (5) implies that each of the particular relations $xy=x*y$, for $(x,y)\in T$, can be separately added to P. Since $T\subseteq\Gamma(P)$, no particular relation $xy=x*y$ can interfere with the remaining added relations $uv=u*v$. □

The following simple combination of both the types of extensions will be used in the proof of Proposition 1.4.

Let A be a set (disjoint from P) and $\sigma\colon A\to A$ be a fixpointfree involution (i.e., $\sigma(a)\neq a$ for every $a\in A$). Then $[A,\sigma]_3$ denotes the extension of the minimal partial IP loop $[A,\sigma]$ through (just) the additional relations

$$aa=\sigma(a)$$

for any $a\in A$. Formally, $[A,\sigma]_3$ is the extension of $[A,\sigma]$ through the operation $*$ defined on the set $T=\{(a,a):a\in A\}$ by $a*a=\sigma(a)$ for any $a\in A$. It is clear that each pair (a,a) is indeed a gap in $[A,\sigma]$ and that the condition (5) from Lemma 3.1 is satisfied. Hence $[A,\sigma]_3$ is a partial IP loop extending $[A,\sigma]$ in which

$$a^{-1}=\sigma(a)=aa$$

for each $a\in A$, i.e., every element $a\in A$ has order three. For the direct sum

$$P[A,\sigma]_3=P\sqcup[A,\sigma]_3$$

we have

$$O_3(P[A,\sigma]_3)=O_3(P)\cup A.$$

If $A=\{a,a'\}$, where $a\neq a'$, then the denotations $[A,a\leftrightarrow a']_3$ and

$$P[a\leftrightarrow a']_3=P[A,a\leftrightarrow a']_3$$

are already self-explanatory, and similarly for $[A,a\leftrightarrow a',b\leftrightarrow b']_3$ and

$$P[a\leftrightarrow a',b\leftrightarrow b']_3=P[A,a\leftrightarrow a',b\leftrightarrow b']_3$$

where the set A consists of four distinct elements a, a', b, b'.

Proof of Proposition 1.4. Let a, a', b, b' be four distinct elements not belonging to P. Let us form the extensions $Q=P[a\leftrightarrow a']_3$ and $R=P[a\leftrightarrow a',b\leftrightarrow b']_3$. Obviously,

$$o_3(Q)=o_3(P)+2\qquad\text{and}\qquad o_3(R)=o_3(P)+4.$$

Since one of the numbers $o_3(P)$, $o_3(P)+2$, $o_3(P)+4$ is divisible by 3, one of the partial IP loops P, Q, R has the desired property. □

4 The proof of Proposition 1.5

A more subtle combination of the two types of extensions introduced in Section 3 will be required in the proof of Proposition 1.5.

Proof of Proposition 1.5. Let P be a finite partial IP loop such that $3 \mid o_3(P)$, and A be a finite set disjoint from P with the number of its elements satisfying

$$\#A \geq \max\{5(\#P) - 1, \#\Gamma(P)/2\} \qquad \text{and} \qquad 10 \leq \#P + \#A \equiv 4 \pmod 6.$$

First we construct the minimal extension $P[A]$, in which every element a of A has order two, while

$$O_3(P[A]) = O_3(P).$$

Hence the partial IP loop $P[A]$ has the base set $P \cup A$ with the required number of elements and the same number of elements of order three as P.

Next, we construct an extension of $P[A]$ in which all the original gaps in $\Gamma(P)$ will be filled. We take $T = \Gamma(P) \subseteq \Gamma(P[A])$ and introduce a binary operation $*$ on T, assigning to each pair of gaps $(x, y), (x^{-1}, y^{-1}) \in T$ a (self-inverse) element

$$x * y = y^{-1} * x^{-1} = (x * y)^{-1}$$

from A. At the same time we arrange that (with the above exception) $x*y \neq u*v$ whenever (x, y) and (u, v) are different gaps in P. This is possible, as $\#A \geq \#\Gamma(P)/2$. Since all the pairs $(x, y) \in P[A]$ such that $x \in A$ or $y \in A$, except for $(1, a)$, $(a, 1)$ and (a, a) where $a \in A$, are gaps in $P[A]$, condition (5) of Lemma 3.1 is obviously satisfied. Thus we can construct the partial IP loop $P[A]^*$, extending $P[A]$ through the operation $*$. It still has the base set $P \cup A$, while

$$\Gamma(P[A]^*) \cap (P \times P) = \emptyset.$$

At the same time, $D(P[A]^*) \cap (A \times A) = \{(a, a) : a \in A\}$, so that $ab = 1 \in P$ for any $(a, b) \in D(P[A]^*) \cap (A \times A)$.

Finally, we construct an extension Q of $P[A]^*$ with the same base set $P \cup A$, such that

$$\Gamma(Q) \subseteq O_2(Q) \times O_2(Q).$$

As all the elements of A are of order two, and $P[A]^*$ has no gap $(x, y) \in P \times P$, it suffices to manage that $(x, a), (a, x) \in D(Q)$ for all $a \in A$, $x \in P \setminus O_2(P)$, $x \neq 1$.

We will proceed by an induction argument. To this end we represent the set

$$P \setminus (O_2(P) \cup \{1\}) = \{x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}\},$$

in such a way that each pair of mutually inverse elements $x, x^{-1} \in P \setminus (O_2(P) \cup \{1\})$ occurs in this list exactly once. To start with we put $Q_0 = P[A]^*$. Now we assume that, for some $0 \leq k < n$, we already have an IP loop $Q_k \geq P[A]^*$ with the same base set $P \cup A$, satisfying the following three conditions:

(6) $au, va \in A$ for any $a \in A$, $u, v \in P \setminus \{1\}$ such that $(a, u), (v, a) \in D(Q_k)$,

(7) $ab \in P$ for any $(a, b) \in D(Q_k) \cap (A \times A)$, and

(8) $(x_l, a), (a, x_l) \in D(Q_k)$ for all $0 \leq l \leq k$, $a \in A$.

Observe that Q_0 trivially satisfies all these conditions (with $k = 0$), and condition (8) jointly with Lemma 2.1 imply that $\big(x_l^{-1}, a\big), \big(a, x_l^{-1}\big) \in D(Q_k)$ for all $0 \le l \le k$, $a \in A$, too. For $x = x_{k+1}$, we have to fill in all the gaps in Q_k in which x occurs, preserving all the conditions (6), (7), (8) with k replaced by $k+1$. That way all the gaps in Q_k containing x^{-1} will be filled in, as well.

Let us introduce the sets

$$L_x = \{a \in A : (a, x) \in \Gamma(Q_k)\} \qquad \text{and} \qquad R_x = \{a \in A : (x, a) \in \Gamma(Q_k)\}.$$

Then Lemma 2.1 implies that $(a, x) \in \Gamma(Q_k)$ if and only if $\big(x^{-1}, a\big) \in \Gamma(Q_k)$ for each $a \in A$, hence $L_x = R_{x^{-1}}$ and $R_x = L_{x^{-1}}$.

Claim 1. *We have* $\#L_x = \#R_x$.

Proof. Since $xu = v$ implies $v^{-1}x = u^{-1}$ for any $u, v \in P \cup A$, we have a bijection between the sets

$$(P \cup A) \setminus L_x = \{u \in P \cup A : (x, u) \in D(Q_k)\},$$
$$(P \cup A) \setminus R_x = \big\{v \in P \cup A : \big(v^{-1}, x\big) \in D(Q_k)\big\},$$

which implies that the sets L_x and R_x have the same number of elements. □

Thus there exists a bijective map $\eta\colon L_x \to R_x$ (with inverse map $\eta^{-1}\colon R_x \to L_x$); latter on we will specify some additional requirements concerning it. We intend to use η in defining the extending operation $*$ on the set

$$T_x = \big(L_x \times \{x\}\big) \cup \big(\{x\} \times R_x\big) \cup \big(R_x \times \{x^{-1}\}\big) \cup \big(\{x^{-1}\} \times L_x\big) \cup \big\{(a, \eta(a)) : a \in L_x\big\} \cup \big\{(\eta(a), a) : a \in L_x\big\}$$

by putting

$$a * x = \eta(a)$$

for any $a \in L_x$. Then we have to satisfy the remaining five conditions of Lemma 3.1, i.e. (remembering that the elements of A are self-inverse),

$$a * \eta(a) = x, \quad x^{-1} * a = \eta(a), \quad \eta(a) * a = x^{-1}, \quad \eta(a) * x^{-1} = x * \eta(a) = a.$$

The substitution $b = \eta(a)$ into the last two relations yields

$$b * x^{-1} = x * b = \eta^{-1}(b)$$

for any $b \in R_x$. Thus we have to guarantee that each pair $(a, \eta(a))$, where $a \in L_x$, will be a gap in Q_k. Since $(a, a) \in D(Q_k)$ for all $a \in A$, this will imply $\eta(a) \ne a$ for $a \in L_x \cap R_x$ (if any). Additionally, η should avoid any "crossing", i.e., the situation that

$$\eta(a) = b \qquad \text{and} \qquad \eta(b) = a$$

for some distinct $a, b \in L_x \cap R_x$. This namely, according to Lemma 2.1, would imply that $a * b = x = b * a$, and, since $(a * b)^{-1} = b * a$, produce a contradiction $x = x^{-1}$. Now it is clear that once we succeed to satisfy all the above requirements, the partial IP loop Q_{k+1}, to be obtained as the extension of Q_k through the operation $*$ constructed from the bijection η as described, will satisfy all the conditions (6), (7), (8) (with $k+1$ in place of k). Thus it is enough to show that there is indeed a "crossing avoiding" bijection $\eta\colon L_x \to R_x$ such that

$$(a, \eta(a)) \in \Gamma(Q_k)$$

for each $a \in L_x$. To this end we denote the common value of $\#L_x = \#R_x$ by m, enumerate the sets

$$L_x = \{a_1, \dots, a_m\}, \qquad R_x = \{b_1, \dots, b_m\}$$

in such a way that $i = j$ whenever $a_i = b_j \in L_x \cap R_x$, and introduce the graph G_x on the vertex set $V = \{1, \dots, m\}$, joining two vertices i, j by an edge if and only if $i \neq j$ and both $(a_i, b_j), (a_j, b_i) \in \Gamma(Q_k)$.

Claim 2. *The graph G_x has a Hamiltonian cycle.*

Proof. According to Lemma 2.3, it suffices to show that $m \geq 3$ and that the minimal degree of vertices in G_x is at least $m/2$. We keep in mind that both the right side and the left side multiplication in Q_k by a fixed element are injective maps.

Since $ax \in P \setminus \{1\}$ for every $a \in A$ such that $(a, x) \in D(Q_k)$, there are at most $\#P - 1$ pairs (a, x) in $D(Q_k)$. Hence

$$m = \#L_x \geq \#A - \#P + 1 \geq 4(\#P) > 3.$$

Let i be any vertex in G_x. Then i is not adjacent to a vertex j if and only if at least one of the pairs (a_i, b_j), (a_j, b_i) belongs to $D(Q_k)$. However, for fixed a_i or b_i, all such products $a_i b_j$ or $a_j b_i$ belong to P and, in both cases, every element of P occurs as a result at most once. Thus there are at most $2(\#P)$ vertices in G_x not adjacent to i. Therefore,

$$\deg(i) \geq m - 2(\#P) \geq m - \frac{m}{2} = \frac{m}{2}. \qquad \square$$

Let π be a cyclic permutation of the set V such that $\big(1, \pi(1), \dots, \pi^{m-1}(1)\big)$ is a Hamiltonian cycle in G_x. We define $\eta\colon L_x \to R_x$ by

$$\eta(a_i) = b_{\pi(i)}$$

for any $i \in V$. Obviously, η is bijective, $(a_i, \eta(a_i)) \in \Gamma(Q_k)$ for each $i \in V$, and, since $m \geq 3$, it avoids any crossing.

It follows that in the extension Q_{k+1} of the partial IP loop Q_k through the operation $*$ all the gaps from the set T_x are filled in, and the conditions (6), (7), (8) are preserved. The last partial IP loop $Q = Q_n$ satisfies already all the requirements of Proposition 1.5. $\square$

5 Steiner triples and the proof of Proposition 1.6

In the proof of Proposition 1.6 we will make use of Steiner loops and Steiner triple systems. A *Steiner loop* is an IP loop satisfying the identity $xx = 1$, i.e., an IP loop in which every element $x \neq 1$ has order two. Steiner loops are closely related to *Steiner triple systems*, which are systems $\mathcal{S}$ of three element subsets of a given base set X such that each two element subset $\{x, y\}$ of X is contained in exactly one set $\{x, y, z\} \in \mathcal{S}$. Namely, if L is a Steiner loop L then $X = L \setminus \{1\}$ becomes a base set of the Steiner triple system

$$\mathcal{S}_L = \big\{\{x, y, xy\} : x, y \in X\big\}.$$

Conversely, if $\mathcal{S}$ is a Steiner triple system with the base set X then, adjoining to X a new element $1 \notin X$, we obtain a Steiner loop with the base set $X^+ = X \cup \{1\}$, the unit 1 and the operation given by the casework

$$xy = \begin{cases} 1, & \text{if } x = y, \\ z, & \text{where } \{x, y, z\} \in \mathcal{S}, \text{ if } x \neq y, \end{cases}$$

for $x, y \in X$. Based on this definition, we will call a *Steiner triple* any three-element set $\{x, y, z\} \subseteq O_2(P)$ in any partial IP loop P, such that the product of any two of its elements equals the third one.

It is well known that there exists a Steiner triple system $\mathcal{S}$ on an n-element set X if and only if $n \equiv 1$ or $n \equiv 3 \pmod 6$ (see, e.g., Hwang [18]).

The construction reducing eventually the number of gaps in a given partial IP loop P, satisfying certain conditions which will be emerging gradually, is composed of several simpler steps, we are going to describe, now. At the same time, it depends on a six term progression $\mathbf{a} = (a_0, a_1, a_2, a_3, a_4, a_5)$ of pairwise distinct order two elements of P chosen in advance; the criteria for its choice will be clarified later on.

The first step is the *triplication construction*, which uses Steiner loops heavily. Given an arbitrary finite partial IP loop P such that $\#P \equiv 2$ or $\#P \equiv 4 \pmod 6$ we denote $n = \#P - 1$. Then Steiner triple systems on n-element sets, as well as Steiner loops on $(n + 1)$-element sets exist; assume that Y, Z are two n-element sets, such P, Y, Z are pairwise disjoint, and that both the sets $Y^+ = Y \cup \{1\}$, $Z^+ = Z \cup \{1\}$ are equipped with binary operations turning them into Steiner loops. In rather an ambiguous way, we denote by

$$3P = P \sqcup Y^+ \sqcup Z^+$$

the direct sum of the partial IP loop P with the Steiner loops Y^+ and Z^+ (see Section 4). It is a partial IP loop with the base set $P \cup Y \cup Z$, consisting of $3n + 1$ elements, and the domain

$$D(3P) = D(P) \cup (Y \times Y) \cup (Z \times Z) \cup \big(\{1\} \times (Y \cup Z)\big) \cup \big((Y \cup Z) \times \{1\}\big).$$

We will extend the partial operation on $3P$ by filling in all the gaps consisting of pairs of elements of different sets P, Y, Z. That way we'll obtain an extension $3P^*$ of $3P$ with the same base set $P \cup Y \cup Z$, such that $\Gamma(3P^*) = \Gamma(P)$. The extending operation $*$ is defined on the set

$$T = (P_0 \times (Y \cup Z)) \cup ((Y \cup Z) \times P_0) \cup (Y \times Z) \cup (Z \times Y) \subseteq \Gamma\big(P \sqcup Y^+ \sqcup Z^+\big),$$

where $P_0 = P \setminus \{1\}$. It depends on some arbitrary fixed enumerations

$$P_0 = \{x_0, \dots, x_{n-1}\}, \qquad Y = \{y_0, \dots, y_{n-1}\}, \qquad Z = \{z_0, \dots, z_{n-1}\}$$

of the sets P_0, Y, Z, respectively. Once having them we put

$$y_i * z_{i+k} = x_{i+2k}$$

for $0 \leq i, k < n$, with the addition of subscripts modulo n. Then, in order to satisfy the conditions of Lemma 2.1, we define

$$x_{i+2k} * z_{i+k} = y_i, \qquad y_i * x_{i+2k} = z_{i+k}, \qquad x_{i+2k}^{-1} * y_i = z_{i+k},$$
$$z_{i+k} * x_{i+2k}^{-1} = y_i, \qquad z_{i+k} * y_i = x_{i+2k}^{-1},$$

using the fact that all the elements of Y and Z are self-inverse. As all the pairs (x, y), (y, x), (x, z), (z, x), (y, z), (z, y), where $x \in P_0$, $y \in Y$, $z \in Z$, are gaps in $3P$, Lemma 3.1 guarantees that the extension $3P^*$ of the partial IP loop $3P$ through the operation $*$ is a partial IP loop, again. For lack of better terminology we will call it a *Steiner triplication* of the partial IP loop P and suppress the Steiner loops Y^+, Z^+ and the particular enumerations in its notation.

The Steiner triplication $3P^*$ of P satisfies $\Gamma(3P^*) = \Gamma(P)$, hence it still has the same number of gaps as P. However, Proposition 1.6 requires us to decrease this number. This will be achieved in a roundabout way. First we cancel some pairs in the domain $D(3P^*)$, creating that way the potential to fill in more gaps than we have added. In order to allow for this next step, P has to satisfy some additional conditions, namely, $\#P \geq 10$ (i.e., $n \geq 9$) and $o_2(P) \geq 6$. Though the enumerations of the sets P_0, Y, Z, used in the definition of the extending operation $*$, could have been arbitrary, we now assume that these sets were enumerated in such a way that the six term progression $\mathbf{a} = (a_0, a_1, a_2, a_3, a_4, a_5)$ chosen in advance coincides with the sixtuple $(x_0, x_2, x_1, x_5, x_3, x_{n-3})$ and that $\{y_0, y_1, y_3\}$ is a Steiner triple in Y^+. This artificial trick will facilitate us the description of the next step of our construction.

As special cases of the above defining relations for the operation $*$ in $3P^*$ we get $z_0 = y_0x_0 = y_3x_{n-3}$, $z_1 = y_0x_2 = y_1x_1$ and $z_3 = y_1x_5 = y_3x_3$. In other words, we have the following seven Steiner triples in $3P^*$:

$$\begin{array}{llll} \{x_0, y_0, z_0\}, & \{x_2, y_0, z_1\}, & \{x_1, y_1, z_1\}, & \{x_5, y_1, z_3\}, \\ \{x_3, y_3, z_3\}, & \{x_{n-3}, y_3, z_0\}, & \{y_0, y_1, y_3\}. & \end{array}$$

We delete these triples from the domain of $3P^*$. More precisely, for any one of these three-element sets we delete from $D(3P^*)$ all the six pairs consisting of its distinct elements. That way we obtain a partial IP loop $3P^- \leq 3P^*$, called the *reduction* of $3P^*$, which still is an extension of P, however, it has 42 more gaps than $3P^*$ (6 for each Steiner triple), and, since $\Gamma(P) = \Gamma(3P^*)$, than P, as well.

Instead we introduce some new triples consisting of the same elements, namely

$$\begin{array}{lll} \{x_0, x_2, y_0\}, & \{x_2, x_1, z_1\}, & \{x_1, x_5, y_1\}, \\ \{x_5, x_3, z_3\}, & \{x_3, x_{n-3}, y_3\}, & \{x_{n-3}, x_0, z_0\}, \\ \{y_0, y_1, z_1\}, & \{y_1, y_3, z_3\}, & \{y_3, y_0, z_0\}, \end{array}$$

which are intended to become Steiner triples, after we define a partial operation $\circ$ on the set $\{x_0, x_2, x_1, x_5, x_3, x_{n-3}, y_0, y_1, y_3, z_0, z_1, z_3\}$ by putting the product of any pair of distinct elements of a given three-element set from this list equal to the third one. That way we obtain an extending operation of the partial IP loop $3P^-$ if and only if all the pairs entering this new operation are gaps in $3P^-$. This is obviously true for the 18 pairs arising from the three triples in the last row above. However, this need not be the case for the pairs arising from the six triples in the first two rows. The problem can be reduced to the question which of the pairs (x_0, x_2), (x_2, x_1), (x_1, x_5), (x_5, x_3), (x_3, x_{n-3}), (x_{n-3}, x_0) belong to $\Gamma(P)$. If, e.g., $(x_0, x_2) \notin \Gamma(P)$ then we cannot put $x_0 \circ x_2 = y_0$, so that $\{x_0, x_2, y_0\}$ cannot become a Steiner triple.

Therefore, we include just those triples $\{x_i, x_j, y_k\}$ or $\{x_i, x_j, z_k\}$ for which the pair (x_i, x_j) is a gap in P. Every such "good" triple results in filling in six gaps. We already have 18 gaps filled in thanks to the last row. Thus we need at least five "good" triples in the first two rows in order to fill in additional 30 gaps; this would give $18+30 = 48 > 42$ gaps, while having just four "good" triples results in refilling back 42 gaps, only. In general, we can fill in $6(3+g)$ gaps, where $0 \le g \le 6$ is the number of gaps (x_i, x_j) in the list.

We refer to this last step of the construction as to "filling in the gaps along the path" $\mathbf{a}$ and denote the final resulting extension of the reduction $3P^-$ by $3P\langle\mathbf{a}\rangle$. Obviously, $3P\langle\mathbf{a}\rangle$ is an extension of the original IP loop P, as well, having $6(3+g) - 42 = 6(g-4)$ less gaps than P. This number can be negative, 0 or positive, depending on whether $g < 4$, $g = 4$, or $g > 4$. That's why we are interested just in the case when $g \ge 4$.

After all these preparatory accounts we can finally approach the proof of Proposition 1.6.

Proof of Proposition 1.6. Let P be a finite partial IP loop satisfying the conditions (4), i.e.,

$$3 \mid o_3(P),\ \#P \ge 10,\ \#P \equiv 4 \pmod 6 \text{ and } \varGamma(P) \subseteq O_2(P) \times O_2(P),$$

such that $\varGamma(P) \ne \emptyset$. We are to show that there is a finite partial IP loop $Q \ge P$ satisfying these conditions, as well, with less gaps than P.

Since $\varGamma(P) \subseteq O_2(P) \times O_2(P)$, it is an antireflexive and symmetric relation on the set $O_2(P)$. Thus we can form the *gap graph* $G(P) = (V, E)$ with the set of vertices

$$V = \{x \in O_2(P) : (x, y) \in \varGamma(P) \text{ for some } y \in O_2(P)\}$$

and the set of edges

$$E = \big\{\{x, y\} : (x, y) \in \varGamma(P)\big\}.$$

From the definition of the set of vertices V it follows there are no isolated vertices in $G(P)$. Let's record some less obvious useful facts about this graph.

Claim 3.

(a) The degree of each vertex in $G(P)$ is even.

(b) The number of edges in $G(P)$ is divisible by three.

Proof. (a): Let $x \in O_2(P)$. Then the conditions $xy = z$ and $xz = y$ are equivalent for any $y, z \in P$. Additionally, as $x \ne 1$, from $xy = z$ it follows that $y \ne z$. Thus the elements $y \in P$ such that $(x, y) \in D(P)$ can be grouped into pairs, hence their number is even. As $\#P$ is even, too, so is the degree

$$\deg(x) = \#\{y \in O_2(P) : (x, y) \in \varGamma(P)\} = \#P - \#\{y \in P : (x, y) \in D(P)\}.$$

(b): By Lemma 2.2 we have

$$\#\varGamma(P) \equiv (\#P - 1)(\#P - 2) - o_3(P) \equiv 0 \pmod 6.$$

On the other hand, $\#P \equiv 4 \pmod 6$ and $3 \mid o_3(P)$, yielding $3 \mid \#\varGamma(P)$. Obviously, the number of edges in $G(P)$ is half of the number of gaps $\#\varGamma(P)$, hence the number of edges in $G(P)$ must be divisible by three. □

The structure of connected components in $G(P)$ obeys the following alternative.

Claim 4. *Let C be a connected component of the graph $G(P)$. Then either C contains a triangle or a path of length five, or, otherwise, C is isomorphic to one of the following graphs: the cycle C_4 of length four, the cycle C_5 of length five or the complete bipartite graph $K_{2,m}$ where $m \geq 4$ is even.*

Proof. Let C be any connected component in $G(P)$. As $G(P)$ has no isolated vertices and the degree of every vertex in C is even (and therefore at least two), there is a cycle in C. Assume that C contains no triangle and no path of length five. Then the length of this cycle must be bigger than three and less than six. Thus there are just the following two options:

(a) There is a cycle of length five in C. Then there cannot be any other edge coming out from its vertices since then there would be a path of length five contained in C. Thus C coincides with this cycle.

(b) There is a cycle of length four in C; let us denote it by (v_0, v_1, v_2, v_3). Then, as $G(P)$ contains no triangle, neither $\{v_0, v_2\}$ nor $\{v_1, v_3\}$ is an edge in $G(P)$. If there are no more vertices in C then C is a cycle of length four.

Otherwise, we can assume, without loss of generality, that there is a fifth vertex $u_0 \in C$ adjacent to v_0. As u_0 has an even degree, it must be adjacent to some other vertex, too. If it were adjacent to some vertex u_1, distinct from all the vertices v_0, v_1, v_2, v_3, there would be a path $(u_1, u_0, v_0, v_1, v_2, v_3)$ of length five in C. If u_0 were adjacent to v_1 or to v_3, there would be a triangle (u_0, v_0, v_1) or (u_0, v_0, v_3) in C. That means that $\{u_0, v_2\}$ must be an edge in $G(P)$ and $\deg(u_0) = 2$.

It follows that every other vertex in C must have degree two and it must be adjacent either to v_0 and v_2 or to v_1 and v_3. However, the second option is impossible, since in that case $(u_0, v_0, v_1, u_1, v_3, v_2)$ would be a path of length five. This means that C is isomorphic to the complete bipartite graph $K_{2,m}$, where one term of this partition is formed by the set $\{v_0, v_2\}$ and the second one by the rest of the vertices in C. Since every vertex has an even degree, m must be even. At the same time, $m \geq 4$, as $K_{2,2}$ has just four vertices (and it is isomorphic to the cycle C_4). □

Thus the proof of Proposition 1.6 will be complete once we show how to construct the extension Q in each of the cases listed in Claim 4.

(a) $G(P)$ *contains a triangle,* i.e., a three-element set of vertices $\{x, y, z\}$ such that all its two-element subsets are edges. Then we can extend P through the operation $*$ turning $\{x, y, z\}$ into a Steiner triple. The corresponding extension Q of P has all the properties required and six less gaps than P.

(b) $G(P)$ *contains a path* $\mathbf{a} = (a_0, a_1, a_2, a_3, a_4, a_5)$ *of length five.* Then we can form the Steiner triplication $3P^*$ of P and, filling in the gaps along the path $\mathbf{a}$ in its reduction $3P^-$, we obtain the final extension $Q = 3P\langle\mathbf{a}\rangle$ satisfying the condition (4), again. If (a_5, a_0) is a gap in P (i.e., if $\mathbf{a}$ is a cycle of length five in $G(P)$) then Q has twelve gaps less than P, otherwise it still has six gaps less than P.

We still have to prove Proposition 1.6 in the case there is neither any triangle nor any path of length five in $G(P)$. To this end it is enough to construct, in each of the remaining cases listed in Claim 4, an extension Q of P such that the graph $G(Q)$ has the same number of edges as $G(P)$, however, there is a path $\mathbf{b}$ of length five in $G(Q)$. From such a Q we can construct another extension $3Q\langle\mathbf{b}\rangle \geq Q \geq P$ with a smaller number of gaps and still satisfying the condition (4), similarly as we did in the case (b). So let us have a closer look at the remaining cases.

(c) *$G(P)$ contains a connected component isomorphic to $K_{2,m}$, where $m \geq 4$.* Let $\{u_0, u_1\}$ be the two-element partition set and $\{v_0, v_1, v_2, v_3\}$ be any four-element subset of the m-element partition set in that component of $G(P)$. We denote by $\mathbf{a}$ the six-term progression $(v_0, u_0, v_1, v_2, u_1, v_3)$ and construct the extension $Q = 3P\langle\mathbf{a}\rangle$ of the partial IP loop P with the gap graph $G(Q)$. Then $\{v_0, u_0\}$, $\{u_0, v_1\}$, $\{v_2, u_1\}$, and $\{u_1, v_3\}$ are edges in $G(P)$, while $\{v_1, v_2\}$ and $\{v_3, v_0\}$ are not. Hence the new graph $G(Q)$ has the same number of edges as $G(P)$ and Q has the same number of gaps as P. At the same time, there are two distinct new vertices y_1, z_3 in $G(Q)$, occurring in the enumerations of the sets Y, Z, respectively. Now, one can easily verify that $\mathbf{b} = (v_0, u_1, v_1, y_1, v_2, u_0)$ is a path of length five in $G(Q)$.

(d) *There are two distinct connected components C and D in $G(P)$, each of them isomorphic to the cycle C_4 or C_5.* Let m and l denote any of the numbers 4 or 5. We assume that $(u_0, u_1, \ldots, u_{m-1})$ and $(v_0, v_1, \ldots, v_{l-1})$ are the cycles forming the components $C \cong C_m$ and $D \cong C_l$, respectively. Now we take the six term progression $\mathbf{a} = (u_0, u_1, u_2, v_0, v_1, v_2)$ and form the extension $Q = 3P\langle\mathbf{a}\rangle$. Once again, $\{u_0, u_1\}$, $\{u_1, u_2\}$, $\{v_0, v_1\}$ and $\{v_1, v_2\}$ are edges in $G(P)$, while $\{u_2, v_0\}$ and $\{v_2, u_0\}$ are not. Hence $G(Q)$ has the same number of edges as $G(P)$ and Q has the same number of gaps as P. Now, picking the new distinct vertices $y_1 \in Y$, $z_3 \in Z$, we obtain the path $\mathbf{b} = (u_3, u_2, y_1, v_0, v_{l-1}, v_{l-2})$ of length five in $G(Q)$.

(e) *$G(P)$ consists of a single connected component isomorphic either to C_4 or to C_5.* However, this is impossible, since the number of edges in $G(P)$ is divisible by three.

This concludes the proof of Proposition 1.6, as well as of Theorems 1.1, 1.2 and 1.3. □

6 Final remarks

The discussion from the introduction together with Theorem 1.3 naturally lead to the following question.

Problem 6.1. Is there some minimal (ore even the least) axiomatic class $\mathbf{K}$ of IP loops such that every group is locally embeddable into $\mathbf{K}_{\mathrm{fin}}$? Does this class (if it exists) satisfy the Finite Embeddability Property?

The first candidate which should be examined in this connection seems to be the class of all Moufang loops introduced in [25]: A *Moufang loop* is a loop satisfying one (hence all) of the following four equivalent identities

$$x(y(xz)) = ((xy)x)z, \qquad (xy)(zx) = (x(yz))x,$$
$$x(y(zy)) = ((xy)z)y, \qquad (xy)(zx) = x((yz)x),$$

cf. Pflugfelder [26], Kunnen [19]. It is well known that every Moufang loop is an IP loop.

The following is not the usual definition of the concept of a sofic group (see Gromov [15], Weiss [29], Ceccherini-Silberstein, Coornaert [2]), however, as proved by Gordon and Glebsky [12], it is equivalent to it. A group $(G, \cdot, 1)$ is *sofic* if for every nonempty finite set $X \subseteq G$ and every $\varepsilon > 0$ there exists a finite quasigroup $(Q, *)$ such that $X \subseteq Q$, for all $x, y \in X$ satisfying $xy \in X$ we have $xy = x * y$, as well as

$$\frac{\#\{q \in Q : (x * y) * q \neq x * (y * q)\}}{\#Q} < \varepsilon,$$

and, finally,

$$\frac{\#\{q \in Q : 1 * q \neq q\}}{\#Q} < \varepsilon.$$

No example of a non-sofic group is known up today, however, there is a general belief that not every group is sofic. Theorem 1.3 together with the above description of sofic groups indicate that the sofic groups could perhaps be characterized as groups locally embeddable into some "nice" subclass of the class of finite IP loops, fulfilling some "reasonable amount of associativity". A natural candidate is the class of all finite Moufang loops, once again. For some additional reasons in favor of this choice see [11].

As already indicated, one should start with trying to clarify the following question.

Problem 6.2. Does the class of all Moufang loops have the FEP?

If the answer is negative then it would make sense to elaborate on the following problem.

Problem 6.3. Characterize those groups which are locally embeddable into finite Moufang loops.

Finally, let us formulate two possible responses to Problem 6.3.

Conjecture 6.4. *Every group is locally embeddable into finite Moufang loops.*

Conjecture 6.5. *A group G is sofic if and only if it is locally embeddable into finite Moufang loops.*

Unless every group is sofic, these two conjectures contradict each other. Let us remark that we find the first one more probable to be true than the second one. This would follow from the affirmative answer to Problem 6.2, however it might be true even if the class of Moufang loops failed to have the FEP.

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On graphs with the smallest eigenvalue at least $-1-\sqrt{2}$, part III

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Abstract

There are many results on graphs with the smallest eigenvalue at least -2. In order to study graphs with the eigenvalues at least $-1-\sqrt{2}$, R. Woo and A. Neumaier introduced Hoffman graphs and $\mathcal{H}$-line graphs. They proved that a graph with the sufficiently large minimum degree and the smallest eigenvalue at least $-1-\sqrt{2}$ is a slim $\{[\mathfrak{h}_2],[\mathfrak{h}_5],[\mathfrak{h}_7],[\mathfrak{h}_9]\}$-line graph. After that, T. Taniguchi researched on slim $\{[\mathfrak{h}_2],[\mathfrak{h}_5]\}$-line graphs. As an analogue, we reveal the condition under which a strict $\{[\mathfrak{h}_1],[\mathfrak{h}_4],[\mathfrak{h}_7]\}$-cover of a slim $\{[\mathfrak{h}_7]\}$-line graph is unique, and completely determine the minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graphs.

Keywords: Hoffman graph, line graph, smallest eigenvalue.

Math. Subj. Class.: 05C50, 05C75

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1 Introduction

Throughout this paper, we will consider only undirected graphs without loops or multiple edges, and denote by $\lambda_{\min}(\Gamma)$ and $\delta(\Gamma)$ the minimum eigenvalue and the minimum degree of a graph Γ, respectively.

P. J. Cameron, J. M. Goethals, J. J. Seidel and E. E. Shult have characterized generalized line graphs as the graphs with the smallest eigenvalue at least -2 except for finitely many graphs with at most 36 vertices in [3]. After that, A. Hoffman proved the following theorem in [6].

Theorem 1.1. *There exists an integer valued function f defined on the intersection of the half-open interval $(-1-\sqrt{2},-2]$ and the set of the smallest eigenvalues of graphs, such that if Γ is a connected graph with $\delta(\Gamma) \geq f(\lambda_{\min}(\Gamma))$ then*

(i) if $-1 \geq \lambda_{\min}(\Gamma) > -2$ then Γ is a complete graph and $\lambda_{\min}(\Gamma) = -1$.

(ii) if $-2 \geq \lambda_{\min}(\Gamma) > -1-\sqrt{2}$ then Γ is a generalized line graph and $\lambda_{\min}(\Gamma) = -2$.

In [12], R. Woo and A. Neumaier introduced Hoffman graphs and $\mathcal{H}$-line graphs, where $\mathcal{H}$ is a family of isomorphism classes of Hoffman graphs, to extend the result of A. Hoffman, and proved Theorem 1.2. Moreover, they raised the problem [12, Open problem 3] to reveal the list of minimal forbidden graphs for the slim $\{[\mathfrak{h}_2],[\mathfrak{h}_5],[\mathfrak{h}_7],[\mathfrak{h}_9]\}$-line graphs. These Hoffman graphs and some ones that appear in this paper and [12] are listed in Figure 1 (here, the names $\mathfrak{h}_1,\mathfrak{h}_2,\ldots$ depend on [12]).

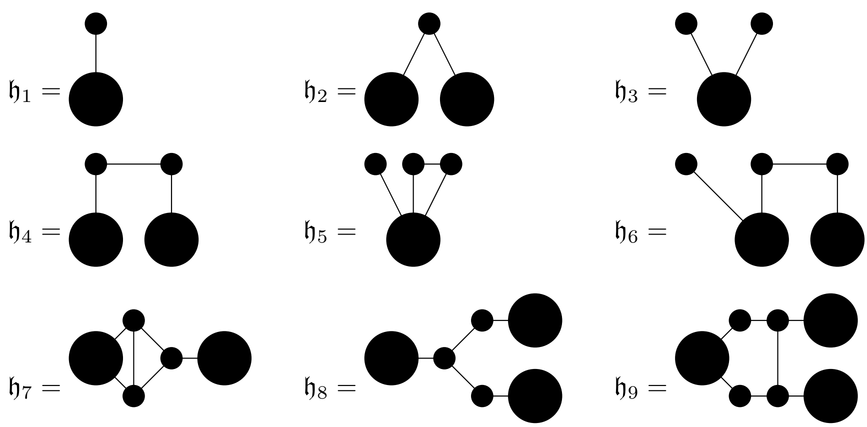


Figure 1: Hoffman graphs with slim (resp. fat) vertices depicted as small (resp. large) black dots.

Theorem 1.2. *Let α_4 (≈ -2.4812) be the smallest root of the polynomial x^3+2x^2-2x-2. There exists an integer valued function f defined on the intersection of the half-open interval $(\alpha_4,-1-\sqrt{2}]$ and the set of the smallest eigenvalues of graphs, such that if Γ is a graph with $\lambda_{\min}(\Gamma) \in (\alpha_4,-1-\sqrt{2}]$ and $\delta(\Gamma) \geq f(\lambda_{\min}(\Gamma))$, then Γ is an $\{[\mathfrak{h}_2],[\mathfrak{h}_5],[\mathfrak{h}_7],[\mathfrak{h}_9]\}$-line graph.*

Since it is difficult to solve the open problem, T. Taniguchi considered a partial problem. In [11], he completely determined the 38 minimal forbidden graphs for the slim $\{[\mathfrak{h}_2], [\mathfrak{h}_5]\}$-line graphs by using Theorem 1.3 [10].

Theorem 1.3. *A slim $\{[\mathfrak{h}_2], [\mathfrak{h}_5]\}$-line graph with at least 8 vertices has a unique strict $\{[\mathfrak{h}_2], [\mathfrak{h}_3], [\mathfrak{h}_5]\}$-cover up to equivalence.*

As an analogue of his result, we reveal the minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graph. In Section 2, we introduce a part of the basic theory of Hoffman graphs summarized in detail in [7]. In Section 3, we introduce minimal forbidden graphs. In Section 4, we aim to prove Theorem 4.11 which reveals the necessary and sufficient condition that a strict $\{[\mathfrak{h}_1], [\mathfrak{h}_4], [\mathfrak{h}_7]\}$-cover of a slim $\{[\mathfrak{h}_7]\}$-line graph becomes unique up to equivalence. Furthermore, when the condition is not satisfied, the theorem shows the shape of the slim $\{[\mathfrak{h}_7]\}$-line graph and indicates its strict $\{[\mathfrak{h}_1], [\mathfrak{h}_4], [\mathfrak{h}_7]\}$-covers are exactly two up to equivalence. This helps us to examine the minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graphs. In order to prove our main result Theorem 5.1, in which we determine the minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graphs, we computed the minimal forbidden graphs with at most 9 vertices by the software MAMGA [2]. In Section 5, we determine the minimal forbidden graphs apart from those with at most 9 vertices.

2 Hoffman graphs

We introduce definitions related to Hoffman graphs. Details are in [7].

Definition 2.1. A *Hoffman graph* $\mathfrak{h}$ is a pair (H, μ) of a graph $H = (V, E)$ and a labelling map $\mu\colon V \to \{f, s\}$, satisfying the following conditions:

(i) every vertex with label f is adjacent to at least one vertex with label s;

(ii) vertices with label f are pairwise non-adjacent.

We call a vertex with label s a *slim vertex*, and one with label f a *fat vertex*. We denote by $V_s(\mathfrak{h})$ (resp. $V_f(\mathfrak{h})$) the set of slim (resp. fat) vertices of $\mathfrak{h}$.

For a vertex x of a Hoffman graph $\mathfrak{h}$, we denote by $N^f_{\mathfrak{h}}(x)$ (resp. $N^s_{\mathfrak{h}}(x)$) the set of neighbors labelled f (resp. s) of x, and set $N_{\mathfrak{h}}(x) = N^f_{\mathfrak{h}}(x) \cup N^s_{\mathfrak{h}}(x)$. For a set X of vertices of $\mathfrak{h}$, we let $N^f_{\mathfrak{h}}(X) := \bigcup_{x\in X} N^f_{\mathfrak{h}}(x)$ and $N^s_{\mathfrak{h}}(X) := \bigcup_{x\in X} N^s_{\mathfrak{h}}(x)$. We regard an ordinary graph H without labelling as a Hoffman graph (H, μ) without fat vertices, that is, $\mu(x) = s$ for any vertex x of H. Such a graph is called a *slim graph*.

Definition 2.2. A Hoffman graph $\mathfrak{h}' = (H', \mu')$ is called an *induced Hoffman subgraph* of a Hoffman graph $\mathfrak{h} = (H, \mu)$, if H' is an induced subgraph of H and $\mu|_{V(H')} = \mu'$. For a subset S of $V_s(\mathfrak{h})$, we denote by $\langle\langle S\rangle\rangle_{\mathfrak{h}}$ the induced Hoffman subgraph of $\mathfrak{h}$ by $S \cup N^f_{\mathfrak{h}}(S)$.

We denote by $\langle S\rangle_\Gamma$ the ordinary induced subgraph by S of a graph Γ for a subset S of $V(\Gamma)$. For a Hoffman graph $\mathfrak{h}$, $\langle V_s(\mathfrak{h})\rangle_{\mathfrak{h}}$ is called the *slim subgraph* of $\mathfrak{h}$. The *diameter* of a graph is the maximum distance between two distinct vertices. Let Γ be a graph and C be a subset of $V(\Gamma)$. Then, C is a *clique* in Γ if the induced subgraph $\langle C\rangle_\Gamma$ is a complete graph. The size of the largest clique in Γ is called the *clique number*. A partition $\pi = \{C_1, C_2, \ldots, C_t\}$ of $V(\Gamma)$ is called a *clique partition* if all cells C_i are cliques. Focusing on cliques is useful for discovering the structure of line graphs. Also in this paper, we may focus on clique numbers and clique partitions.

Definition 2.3. Let $\mathfrak{h}$ be a Hoffman graph, and let $\mathfrak{h}^1$ and $\mathfrak{h}^2$ be two induced Hoffman subgraphs of $\mathfrak{h}$. The Hoffman graph $\mathfrak{h}$ is said to be the *sum* of $\mathfrak{h}^1$ and $\mathfrak{h}^2$, written as $\mathfrak{h} = \mathfrak{h}^1 \oplus \mathfrak{h}^2$, if the following conditions are satisfied:

(i) $V(\mathfrak{h}) = V(\mathfrak{h}^1) \cup V(\mathfrak{h}^2)$;

(ii) $V_s(\mathfrak{h}) = V_s(\mathfrak{h}^1) \cup V_s(\mathfrak{h}^2)$ and $V_s(\mathfrak{h}^1) \cap V_s(\mathfrak{h}^2) = \emptyset$;

(iii) if $x \in V_s(\mathfrak{h}^i)$, $y \in V_f(\mathfrak{h})$ for $i = 1$ or 2, and $x \sim y$, then $y \in V_f(\mathfrak{h}^i)$;

(iv) if $x \in V_s(\mathfrak{h}^1)$ and $y \in V_s(\mathfrak{h}^2)$, then x and y have at most one common fat neighbor, and they have one if and only if they are adjacent.

If $\mathfrak{h}$ is the sum of some two nonempty Hoffman graphs, then it is said to be *decomposable*. Otherwise, $\mathfrak{h}$ is said to be *indecomposable*.

Remark that the sum of Hoffman graphs satisfies commutative and associative laws.

Definition 2.4. Let $\mathfrak{h}$ and $\mathfrak{m}$ be Hoffman graphs, and let ϕ be a graph morphism from the underlying graph of $\mathfrak{h}$ to that of $\mathfrak{m}$. Mapping $\phi\colon \mathfrak{h} \to \mathfrak{m}$ is called a *morphism* if it preserves the labelling, that is, $\phi(V_s(\mathfrak{h})) \subset V_s(\mathfrak{m})$ and $\phi(V_f(\mathfrak{h})) \subset V_f(\mathfrak{m})$. If ϕ is a morphism and a graph isomorphism, then it is called an *isomorphism*, and $\mathfrak{h}$ and $\mathfrak{m}$ are said to be *isomorphic*, written as $\mathfrak{h} \simeq \mathfrak{m}$. Let $[\mathfrak{h}]$ denote the isomorphism class of $\mathfrak{h}$.

Definition 2.5. Let $\mathcal{H}$ be a family of isomorphism classes of Hoffman graphs. A Hoffman graph $\mathfrak{m}$ is called a *$\mathcal{H}$-line graph* if it is an induced subgraph of some Hoffman graph $\mathfrak{h} = \bigoplus_{i=1}^n \mathfrak{h}^i$, where $[\mathfrak{h}^i] \in \mathcal{H}$ for every i. In this case, $\mathfrak{m}$ is called a *slim $\mathcal{H}$-line graph* if $\mathfrak{m}$ is a slim graph, and $\mathfrak{h}$ is called a *strict $\mathcal{H}$-cover* of a graph Γ if $V_s(\mathfrak{h}) = V(\Gamma)$. Two strict $\mathcal{H}$-covers $\mathfrak{h}$ and $\mathfrak{h}'$ of a graph Γ are said to be *equivalent*, if there exists an isomorphism $\phi\colon \mathfrak{h} \to \mathfrak{h}'$ such that $\phi|_\Gamma$ is the identity automorphism of Γ.

Lemma 2.6. *Let $\mathcal{H}$ be a family of isomorphism classes of Hoffman graphs, and let $\mathcal{H}'$ be the family of the isomorphism classes of indecomposable induced Hoffman subgraphs by a nonempty set of slim vertices in a member of $\mathcal{H}$. Then, every slim $\mathcal{H}$-line graph has a strict $\mathcal{H}'$-cover.*

Proof. Let $\mathfrak{h} = \bigoplus_{i=1}^n \mathfrak{h}^i$, where $[\mathfrak{h}^i] \in \mathcal{H}$ for every i. Then, it holds that

$$\langle\langle S \rangle\rangle_\mathfrak{h} = \bigoplus_{i=1}^{n} \langle\langle S \cap V_s(\mathfrak{h}^i) \rangle\rangle_\mathfrak{h}$$

for a subset S of $V_s(\mathfrak{h})$. Therefore $\langle\langle S \rangle\rangle_\mathfrak{h}$ is a strict $\mathcal{H}'$-cover of the induced subgraph by S since every addend is the sum of some indecomposable induced Hoffman graphs of $\mathfrak{h}$. □

3 Minimal forbidden graphs

In graph theory, various important families of graphs can be described by a set of graphs that do not belong to that family. This is the concept of so-called minimal forbidden graphs. First, we give the definition. Suppose that a family $\mathcal{G}$ of graphs is closed under the operation to take induced subgraphs, that is, $\mathcal{G}$ satisfies the condition that for a graph G in $\mathcal{G}$, any induced subgraph of G is also in $\mathcal{G}$. Then, we say that a graph F is a *minimal forbidden graph* for $\mathcal{G}$ if both of the following are satisfied:

(i) F is not in $\mathcal{G}$;

(ii) Every proper induced subgraph of F is in $\mathcal{G}$.

On the family of ordinary line graphs [1] and the family of slim $\{[\mathfrak{h}_2],[\mathfrak{h}_5]\}$-line graphs [11], their minimal forbidden graphs are revealed. Besides this, characterizations of forests, perfect graphs [4] and Threshold graphs [5] by minimal forbidden graphs are also known. In addition, Sumner [9] claimed that if Γ is a connected $K_{1,3}$-free graph of even order, then Γ has a 1-factor. As such, there are also known results that properties of a family of graphs in the case that "forbidden graphs" are specified in advance. As we can see from these results, revealing the minimal forbidden graphs is one way to understand families of graphs. Unfortunately not being finite, but we are able to reveal the minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graphs.

4 The condition that an $\{[\mathfrak{h}_1],[\mathfrak{h}_4],[\mathfrak{h}_7]\}$-strict cover of a slim $\{[\mathfrak{h}_7]\}$-line graph is unique up to equivalence

In this section, set $\mathcal{H}=\{[\mathfrak{h}_1],[\mathfrak{h}_4],[\mathfrak{h}_7]\}$. Note that every slim $\{[\mathfrak{h}_7]\}$-line graph has a strict $\mathcal{H}$-cover by Lemma 2.6. For exmaple, the graph Γ in Figure 2 is a slim $\{[\mathfrak{h}_7]\}$-line graph. Indeed, considering the sum $\mathfrak{h}=\mathfrak{h}_7\oplus\mathfrak{h}_1\oplus\mathfrak{h}_4$ of Hoffman graphs in Figure 9, we see that

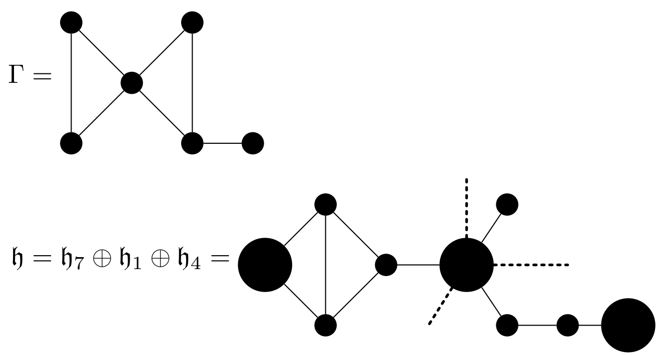


Figure 2: A slim $\{[\mathfrak{h}_7]\}$-line graph and its strict $\{[\mathfrak{h}_1],[\mathfrak{h}_4],[\mathfrak{h}_7]\}$-cover.

the slim subgraph of $\mathfrak{h}$ is the graph Γ. (In Figure 2, the dotted lines are used for convenience to show what kind of small Hoffman graphs the graph Γ is decomposed by. In addition, for two vertices x and y which belong to distinct addends, we omit the edge between x and y if they have a common fat neighbor since the existence of edge between x and y depends only on that of their common fat neighbor by Definition 2.3 (iv).) In addition, $\mathfrak{h}_1$ and $\mathfrak{h}_4$ are induced subgraphs of $\mathfrak{h}_7$, so the graph Γ is certainly a slim $\{[\mathfrak{h}_7]\}$-line graph. On the other hand, since $V_s(\mathfrak{h})=V(\Gamma)$ holds, the Hoffman graph $\mathfrak{h}$ is a strict $\{[\mathfrak{h}_1],[\mathfrak{h}_4],[\mathfrak{h}_7]\}$-cover of Γ.

Let $\mathfrak{h}=\bigoplus_{i=1}^{n}\mathfrak{h}^i$ where $[\mathfrak{h}^i]\in\mathcal{H}$ for every i. Then, we can regard $N_{\mathfrak{h}}^{f}$ as a mapping from $V_s(\mathfrak{h})$ to $V_f(\mathfrak{h})$ since every slim vertex is adjacent to exactly one fat vertex. For a slim vertex x of $\mathfrak{h}$, let $\mathfrak{h}(x)$ denote the addend $\mathfrak{h}^i$ containing x, and let $C_{\mathfrak{h}}(x)=N_{\mathfrak{h}}^{s}(N_{\mathfrak{h}}^{f}(x))$

and

$$\operatorname{cov}\mathfrak{h} := \{N^s_{\mathfrak{h}}(u) \mid u \in V_f(\mathfrak{h})\} = \{C_{\mathfrak{h}}(x) \mid x \in V_s(\mathfrak{h})\}.$$

Let x be a slim vertex of $\mathfrak{h}$. We show that $C_{\mathfrak{h}}(x)$ is a clique. First, we take $u \in V_f(\mathfrak{h})$ such that $N^f_{\mathfrak{h}}(x) = \{u\}$. We arbitrarily take two slim vertices y and z in $N^s_{\mathfrak{h}}(u)$ $(= C_{\mathfrak{h}}(x))$. It suffices to show that y and z are adjacent. If y and z are contained in the same indecomposable addend of $\mathfrak{h}$, then they are adjacent. Otherwise, so are they by Definition 2.3 (iv). Hence, the desired result follows. Note that $\operatorname{cov}\mathfrak{h}$ is a clique partition of $V_s(\mathfrak{h})$. Moreover, it holds clearly that $N^f_{\mathfrak{h}}|_\Delta = N^f_{\langle\langle\Delta\rangle\rangle_{\mathfrak{h}}}$ for any subset $\Delta \subset V_s(\mathfrak{h})$.

Lemma 4.1. *Let $\mathfrak{h} = \bigoplus_{i=1}^n \mathfrak{h}^i$, where $[\mathfrak{h}^i] \in \mathcal{H}$ for every i, and let C be a clique of the slim subgraph of $\mathfrak{h}$. Then, the following hold:*

(i) two distinct slim vertices x and y are adjacent if and only if $\mathfrak{h}(x) = \mathfrak{h}(y)$ or $N^f_{\mathfrak{h}}(x) = N^f_{\mathfrak{h}}(y)$;

(ii) $C \subset C_{\mathfrak{h}}(x)$ for any $x \in C$, or $C \subset V_s(\mathfrak{h}(y))$ for any $y \in C$;

(iii) If $C \subset C_{\mathfrak{h}}(x) \cap V_s(\mathfrak{h}(y))$ for some $x, y \in C$, then $|C| \leq 2$.

Proof. Statements (i) and (iii) hold clearly. Assume that $C \not\subset C_{\mathfrak{h}}(x)$ for some $x \in C$. There exists $y \in C$ such that $N^f_{\mathfrak{h}}(x) \neq N^f_{\mathfrak{h}}(y)$. Thus, $\mathfrak{h}(x) = \mathfrak{h}(y)$ holds by (i). Statement (ii) follows since $N^f_{\mathfrak{h}}(x) \neq N^f_{\mathfrak{h}}(z)$ or $N^f_{\mathfrak{h}}(x) \neq N^f_{\mathfrak{h}}(z)$ for each $z \in C$. □

We introduce some definitions to determine the strict $\mathcal{H}$-covers of a graph.

Definition 4.2. Let Γ be a graph, and let $\{C_i\}_{i\in I}$ be a partition of the vertex set of Γ. Then, define $n(x) = N_\Gamma(x) - C_i$ for $x \in C_i$. In addition, define $n^0(x) = \{x\}$ and

$$n^k(x) = n^{k-1}(x) \cup \bigcup_{y \in n^{k-1}(x)} n(y)$$

for a positive integer k. A vertex x of Γ is said to be *good* for the given partition $\{C_i\}_{i\in I}$ if x satisfies one of the following conditions:

(i) $n(x) = \emptyset$;

(ii) $n(x) = \{y\}$ for some y, and $n(y) = \{x\}$;

(iii) $n(x) = \{y, z\}$ for some y and z, $n(y) = \{x\}$, $n(z) = \{x\}$, and $y \sim z$;

(iv) $n(x) = \{y\}$ for some y, $n(y) = \{x, z\}$ for some z, $n(z) = \{y\}$, and $x \sim z$.

Furthermore, a set of vertices is said to be *good* if every element is good. Let $\mathcal{O}_\Gamma$ be the set of clique partitions for which every vertex is good.

We can regard cov as a mapping from the set of equivalent classes of strict $\mathcal{H}$-covers of Γ to $\mathcal{O}_\Gamma$, and Proposition 4.3 holds. It is clear that if $n(u)$ has a good vertex then u is good, and if u is good then $n(u)$ is good.

Proposition 4.3. *The mapping* cov *for a graph is bijective.*

Proof. We construct the inverse mapping of cov. Let $\{C_i\}_{i\in I} \in \mathcal{O}_\Gamma$. A Hoffman graph $\mathfrak{m}$ is defined as $V_s(\mathfrak{m}) := V(\Gamma)$, $V_f(\mathfrak{m}) := \{C_i\}_{i\in I}$ and

$$E(\mathfrak{m}) := E(\Gamma) \cup \{\{x, C\} \mid x \in V(\Gamma), \text{ and } C \in \{C_i\}_{i\in I} \text{ and } x \in C\}.$$

For $x \in V(\Gamma)$, define the induced Hoffman graph $\mathfrak{m}_x := \langle\langle n^2(x)\rangle\rangle_{\mathfrak{m}}$. It holds that

$$\mathfrak{m} = \bigoplus\{\mathfrak{m}_x \mid x \in V(\Gamma)\}, \qquad \text{and}$$
$$[\mathfrak{m}_x] = [\mathfrak{h}_1], [\mathfrak{h}_4] \text{ or } [\mathfrak{h}_7] \text{ for each vertex } x.$$

Hence, $\mathfrak{m}$ is a strict $\mathcal{H}$-cover of Γ. The mapping

$$\phi\colon \mathcal{O}_\Gamma \ni \{C_i\}_{i\in I} \mapsto \mathfrak{m} \in \text{ the set of strict } \mathcal{H}\text{-covers of } \Gamma$$

is the inverse mapping of the mapping cov. □

We have the following lemma:

Lemma 4.4. *Let* Γ *be a graph with a partition* $\{C_i\}_{i\in I}$ *of the vertex set. Then, a vertex* x *is good for* $\{C_i\}_{i\in I}$ *if and only if* x *is good for* $\{n^3(u) \cap C_i\}_{i\in I}$ *in* $\langle n^3(u)\rangle_\Gamma$.

Let Γ be a connected graph, and let K be a nonempty set of vertices. Then, let

$$\partial_{K,\Gamma}(x) = \partial_K(x) = \partial(x) := \min_{k\in K} d(x, k)$$

for $x \in V(\Gamma)$, where $d(x, y)$ is the distance between x and y. Define

$$\partial_{\max} = \max_{y\in V(\Gamma)} \partial_{K,\Gamma}(y),$$

and let $\Psi_\Gamma(K)$ denote the family

$$\big\{\{y \in \{x\} \cup N(x) \mid \partial_{K,\Gamma}(y) \geq \partial_{K,\Gamma}(x)\} \mid x \in V(\Gamma) \text{ and } \partial_{K,\Gamma}(x) \in 2\mathbb{N}+1\big\} \cup \{K\}$$

of sets of vertices. If $K \in \operatorname{cov}\mathfrak{h}$ then $\Psi_\Gamma(K) = \operatorname{cov}\mathfrak{h}$ for every strict $\mathcal{H}$-cover $\mathfrak{h}$ of Γ. This means that we can restore the clique partition if we find a member of a partition in $\mathcal{O}_\Gamma$. We have the following lemmas:

Lemma 4.5. *Let* Γ *be a graph with a clique* K*. If* $\Psi_\Gamma(K)$ *is a partition of* $V(\Gamma)$ *and* Γ *has no induced subgraph isomorphic to* $K_{1,3}$*, then* $\Psi_\Gamma(K)$ *is a clique partition.*

Lemma 4.6. *Let* Γ *be a connected slim* $\{[\mathfrak{h}_7]\}$*-line graph with a clique* C *of size* c*. Let* $\mathfrak{h}$ *be a strict* $\mathcal{H}$*-cover of* Γ*. If the following (i) or (ii) holds, then* $C_{\mathfrak{h}}(x)$ *is the maximal clique containing* C *for any* $x \in C$*, and a strict* $\mathcal{H}$*-cover of* Γ *is unique up to equivalence.*

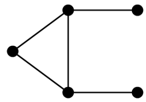
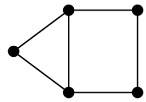
(i) $c \geq 4$,

(ii) $c = 3$*, and* $|N^f_{\mathfrak{m}}(C)| = 1$ *for any strict* $\mathcal{H}$*-cover* $\mathfrak{m}$ *of* Γ.

Proof. In the case of (i), for each clique D which contains C and any $x \in C$, $D \subset C_{\mathfrak{h}}(x)$ holds by Lemma 4.1 (ii) and $|D| \geq 4$. Hence, $C_{\mathfrak{h}}(x)$ is a unique maximal clique containing C for any $x \in C$. In the case of (ii), we can prove as well by Lemma 4.1 (ii) and (iii).

Next, we show the uniqueness of a strict $\mathcal{H}$-cover of Γ. The maximal clique D containing C is defined independently of a choice of a strict $\mathcal{H}$-cover. Hence, $\Psi_\Gamma(D)$ is also defined independently of one. By Proposition 4.3, a strict $\mathcal{H}$-cover of Γ is unique. □

We define

$S_1 =$  and $S_2 =$  .

Lemma 4.7. *Let $\mathfrak{h}$ be a strict $\mathcal{H}$-cover of a graph Γ. If Γ has an induced subgraph isomorphic to S_1 or S_2, then the vertices of the triangle of the induced subgraph are adjacent to the same fat vertex in $\mathfrak{h}$.*

Proof. Let Δ be the triangle in the induced subgraph $S \simeq S_1$ or S_2. Let $\mathfrak{m}$ be a strict $\mathcal{H}$-cover of Γ. We suppose that $|N^f_{\mathfrak{m}}(\Delta)| \geq 2$ to prove $|N^f_{\mathfrak{m}}(\Delta)| = 1$ by contradiction. Then, we have Δ is not contained in $C_{\mathfrak{m}}(x)$ for every $x \in V(\Gamma)$ since every slim vertex in $C_{\mathfrak{m}}(x)$ are adjacent to the same fat vertex. This together with Lemma 4.1 (ii) implies that

$$\Delta \subset V_s(\mathfrak{m}(y)) \text{ for any } y \in \Delta.$$

We take a vertex $y \in V(\Delta)$. Then, $\Delta \subset V_s(\mathfrak{m}(y))$, and hence $[\mathfrak{m}(y)] = [\mathfrak{h}_7]$. Moreover, $\langle\langle V(S)\rangle\rangle_{\mathfrak{m}}$ is a strict $\mathcal{H}$-cover of S. Hence, we have

$$\begin{aligned}\langle\langle V(S)\rangle\rangle_{\mathfrak{m}} &= \langle\langle \Delta\rangle\rangle_{\mathfrak{m}(y)} \oplus \langle\langle V(S)\setminus\Delta\rangle\rangle_{\mathfrak{m}'}\\ &\simeq \mathfrak{h}_7 \oplus \langle\langle V(S)\setminus\Delta\rangle\rangle_{\mathfrak{m}'},\end{aligned}$$

where $\mathfrak{m}'$ denotes the Hoffman graph so that $\mathfrak{m} = \mathfrak{m}(y) \oplus \mathfrak{m}'$. It is easy to verify that the slim subgraph of $\mathfrak{h}_7 \oplus \langle\langle V(S)\rangle\rangle_{\mathfrak{m}'}$ is distinct from S_1 and S_2. This is a contradiction to $S \simeq S_1$ or S_2. Therefore the desired result follows. □

The Lemma 4.6 gives conditions that a strict $\mathcal{H}$-cover is unique, and Lemma 4.7 gives a concrete situation satisfying one of the conditions.

Lemma 4.8. *If the slim subgraph of a Hoffman graph $\mathfrak{h} = \bigoplus_{i=1}^{N} \mathfrak{h}^i$ with $[\mathfrak{h}^i] \in \mathcal{H}$ for every i is connected, then that of $\bigoplus_{i=1, i\neq k}^{N} \mathfrak{h}^i$ is connected for some k.*

Proof. Note that an $\{[\mathfrak{h}_7]\}$-line graph is connected if and only if the slim subgraph is connected. Let Γ be the graph with the vertices $\{1, \dots, N\}$ whose two distinct vertices x and y are adjacent if and only if $V(\mathfrak{h}^i) \cap V(\mathfrak{h}^j) \neq \emptyset$. Since Γ is connected, there exists integer k such that $\Gamma - k$ is also connected. Hence, $\bigoplus_{i=1, i\neq k}^{N} \mathfrak{h}^i$ is connected, and the slim subgraph is connected. □

Let $t = (t_i)_{i=1}^{n}$ be a finite sequence of positive integers. Then, define the graphs P_t and C_t by

$$\begin{aligned} V(P_t) = V(C_t) &:= \{(i,j) \mid 1 \leq i \leq n, 1 \leq j \leq t_i\},\\ E(P_t) &:= \{\{(i,j),(i',j')\} \mid i - i' = 1, \text{ or } i = i' \text{ and } j \neq j'\},\\ E(C_t) &:= \{\{(i,j),(i',j')\} \mid i - i' \equiv 1 \pmod{n}, \text{ or } i = i' \text{ and } j \neq j'\},\end{aligned}$$

respectively (see Example 4.10). Let

$$[a_1, \dots, a_k] := \{(a_i, j) \in V(\Gamma) \mid 1 \leq i \leq k, 1 \leq j \leq t_{a_i}\}$$

for $\{a_1, \ldots, a_k\} \subset \{1, \ldots, n\}$, where $\Gamma = P_t$ or C_t. In addition, let

$$TP := \{(t_i)_{i=1}^n \in \{1,2\}^n \mid n \in \mathbb{Z}_{\geq 2}, t_i + t_{i+1} \leq 3\ (1 \leq \forall i \leq n-1)\} \quad \text{and} \tag{4.1}$$

$$TC := \{(t_i)_{i=1}^n \in \{1,2\}^n \mid n \in (2\mathbb{Z})_{\geq 4}, t_i + t_{(i+1 \bmod n)} \leq 3\ (1 \leq \forall i \leq n)\}. \tag{4.2}$$

Furthermore, a vertex u of a graph is said to be *end* if the graph is isomorphic to P_t for some $t \in TP$ with the length n, and $u \in [1]$ or $[n]$. In the following lemma, we see that P_t and C_t are slim $\{[\mathfrak{h}_7]\}$-line graphs, and reveal their strict $\mathcal{H}$-covers.

Lemma 4.9. *For* $t \in TP$ *of length* n*, we have* $\mathcal{O}_{P_t}$ *is the set of*

$$\{[1], [2,3], [4,5], [6,7], \ldots\} \qquad \textit{and} \qquad \{[1,2], [3,4], [5,6], \ldots\}. \tag{4.3}$$

For $t \in CP$ *of length* n*, we have* $\mathcal{O}_{C_t}$ *is the set of*

$$\{[1,2], [3,4], \ldots, [n-1,n]\} \qquad \textit{and} \qquad \{[n,1], [2,3], \ldots, [n-2,n-1]\}. \tag{4.4}$$

Namely, for $t \in TP$ *(resp.* TC*), the graph* P_t *(resp.* C_t*) has precisely two strict* $\mathcal{H}$*-covers up to equivalence.*

Proof. Recall that $\Psi_\Gamma(C) = \pi$ holds for every $C \in \pi$ where Γ is a slim $\{[\mathfrak{h}_7]\}$-line graph and $\pi \in \mathcal{O}_\Gamma$. In order to reveal $\mathcal{O}_\Gamma$, it suffices to verify whether $\Psi_\Gamma(K)$ is in $\mathcal{O}_\Gamma$ for every clique K of Γ.

We fix a sequence $t \in TP$ of length n, and determine $\mathcal{O}_{P_t}$. Since if $n = 2$ then desired result holds, we may assume that $n \geq 3$. On the other hand, every clique of P_t is contained in $[i, i+1]$ for an integer $i \in \{1, \ldots, n-1\}$. The clique partitions in (4.3) are obtained from cliques $[1]$, $[n]$ and $[i, i+1]$ for $i \in \{1, \ldots, n-1\}$. Moreover we can verify that $\Psi_\Gamma(K)$ is not in $\mathcal{O}_{P_t}$ for one clique K of the other following cliques:

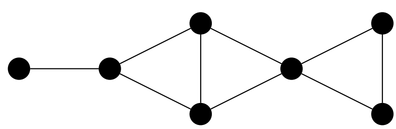
(i) non-empty subsets of $[i]$ for $i \in \{2, \ldots, n-1\}$;

(ii) $\{(i,1), (i+1,1)\}$ and $\{(i,2), (i+1,1)\}$ for $i \in \{1, \ldots, n-1\}$ with $t_i = 2$;

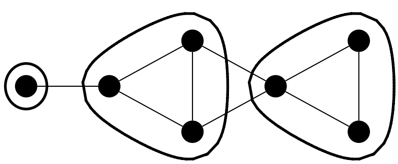
(iii) $\{(i,1), (i+1,1)\}$ and $\{(i,1), (i+1,2)\}$ for $i \in \{1, \ldots, n-1\}$ with $t_{i+1} = 2$.

Similarly, we can determine $\mathcal{O}_{C_t}$ for every $t \in TC$. Finally, by Proposition 4.3, which claims that cov is a bijection from the set of strict $\mathcal{H}$-cover of a slim $\{[\mathfrak{h}_7]\}$-line graph Γ to $\mathcal{O}_\Gamma$, P_t and C_t have precisely two strict $\mathcal{H}$-covers up to equivalence. □

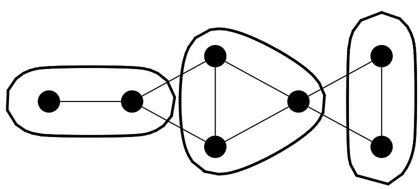
Example 4.10. We give examples of strict $\mathcal{H}$-covers of P_t and C_t. In the case of $t = (1,1,2,1,2)$, the graph P_t is



and $\mathcal{O}_{P_t}$ is the set consisting of the partitions corresponding to



and



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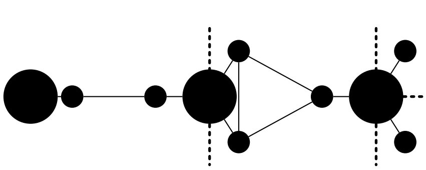
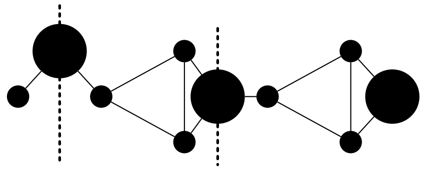
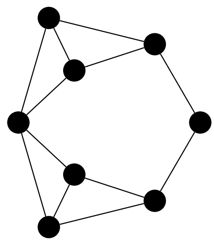
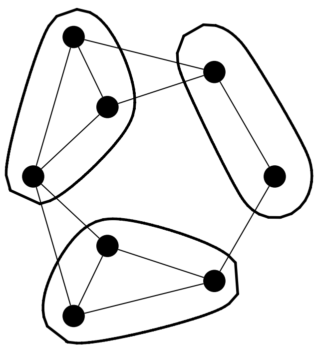
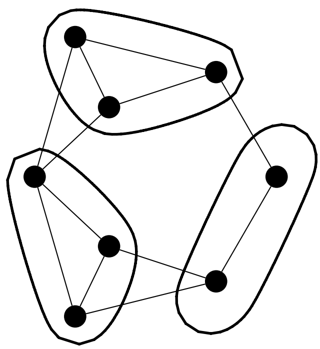
By Proposition 4.3, these clique partitions give strict $\mathcal{H}$-covers. They are

 and  ,

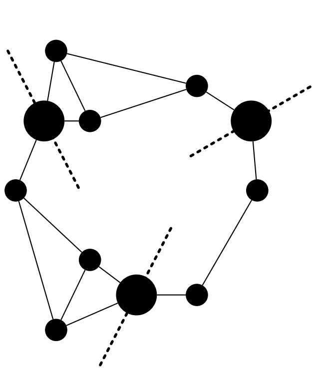
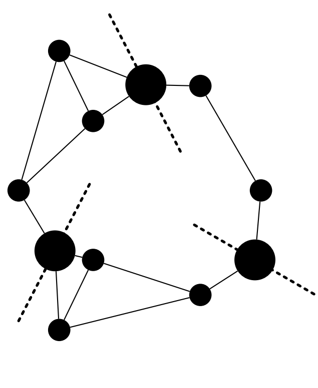
respectively. The similar consideration is applied to C_t for $t \in TC$. For example, we consider $t = (1, 1, 2, 1, 2, 1)$. Then the graph C_t is



and $\mathcal{O}_{C_t}$ is the set consisting of the partitions corresponding to

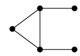
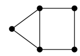
 and  .

By Proposition 4.3, we have the following two strict $\mathcal{H}$-covers:

 and  .

Theorem 4.11. *If a connected slim $\{[\mathfrak{h}_7]\}$-line graph Γ with the clique number c satisfies one of the following conditions:*

(a) $c = 1$ *or* $c \geq 4$*;*

(b) Γ *has an induced subgraph isomorphic to* $S_1 =$  *or* $S_2 =$ ,

then it has a unique strict $\{[\mathfrak{h}_1], [\mathfrak{h}_4], [\mathfrak{h}_7]\}$*-cover up to equivalence. Otherwise,* Γ *is isomorphic to* P_t *for some* $t \in TP$ *or* C_t *for some* $t \in TC$*, and it has precisely two strict* $\{[\mathfrak{h}_1], [\mathfrak{h}_4], [\mathfrak{h}_7]\}$*-covers up to equivalence.*

Proof. If (a) or (b) holds then a strict $\mathcal{H}$-cover is unique by Lemma 4.6 and Lemma 4.7 (see Example 4.12). Otherwise, it is proved that Γ is isomorphic to either P_t for some $t \in TP$ or C_t for some $t \in TC$ by induction on the number of addends of a strict $\mathcal{H}$-covers of Γ. Fix a strict $\mathcal{H}$-cover $\mathfrak{h} = \bigoplus_{i=1}^{N} \mathfrak{h}^i$, where $[\mathfrak{h}^i] \in \mathcal{H}$ for every i. If $N = 1$ then $\Gamma \simeq P_{\{1,1\}}$ or $P_{\{1,2\}}$. Otherwise, we can take an integer k such that the subgraph Γ' induced by the slim vertices of $\mathfrak{h}' = \bigoplus_{i=1, i\neq k}^{N} \mathfrak{h}^i$ is connected by Lemma 4.8. Each of S_1 and S_2 is not isomorphic to any induced subgraph in Γ'. Note that the clique number c' of Γ' is at most 3. Suppose $c' = 2$ or 3 since the result follows if $c' = 1$.

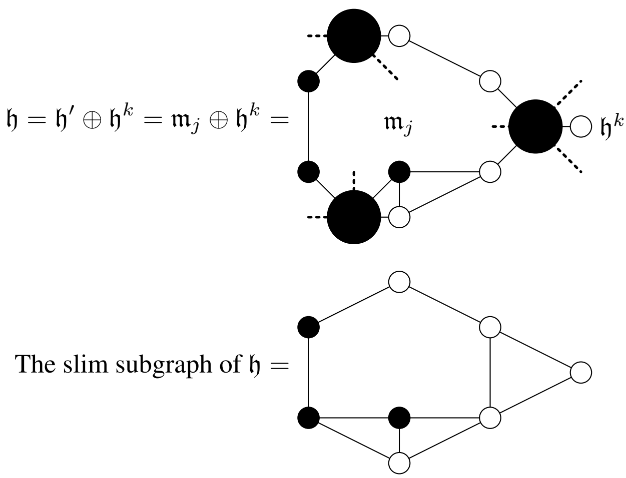


Figure 3: An example of the case that the slim subgraph of $\mathfrak{h}'$ is isomorphic to C_t for $t \in TC$.

If $\Gamma' \simeq C_{t'}$ for some $t' \in TC$, then $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}^k$ must have an induced subgraph isomorphic to either S_1 or S_2 (see Figure 3). Otherwise, $\Gamma' \simeq P_{t'}$ for some $t' \in TP$. Let

$$\mathfrak{m}_1 := \mathrm{cov}^{-1}(\{[1], [2,3], [4,5], [6,7], \ldots\}),$$
$$\mathfrak{m}_2 := \mathrm{cov}^{-1}(\{[1,2], [3,4], [5,6], \ldots\}),$$

and let n denote the length of t. By the induction hypothesis, we can take $j \in \{1,2\}$ so that $\mathfrak{h}'$ and $\mathfrak{m}_j$ are equivalent. Then, we show that the following two conditions hold:

(A) $|N^s_{\mathfrak{m}_j}(u)| + |N^s_{\mathfrak{h}^k}(u)| \leq 3$ holds for every $u \in V_f(\mathfrak{m}_j) \cap V_f(\mathfrak{h}^k)$;

(B) $N^s_{\mathfrak{m}_j}(u) = [1], [1,2], [n]$ or $[n-1, n]$ holds for every $u \in V_f(\mathfrak{m}_j) \cap V_f(\mathfrak{h}^k)$.

First, if $|N^s_{\mathfrak{m}_j}(u)| + |N^s_{\mathfrak{h}^k}(u)| \geq 4$ holds for $u \in V_f(\mathfrak{m}_j) \cap V_f(\mathfrak{h}^k)$, then $N^s_{\mathfrak{m}_j}(u) \cup N^s_{\mathfrak{h}^k}(u)$ is a clique of size greater than 4 in Γ, a contradiction to the assumption that Γ does not satisfy the condition (a). Second, we suppose that the condition (B) does not hold. Then we can take a fat vertex $u \in V_f(\mathfrak{m}_j) \cap V_f(\mathfrak{h}^k)$ so that

$$N^s_{\mathfrak{m}_j}(u) = [i, i+1]$$

for $i \in \{2, \ldots, n-2\}$. Thus, Γ has an induced subgraph isomorphic to S_1 (see Figure 4), a contradiction to the assumption that Γ does not satisfy the condition (b). Therefore the two conditions (A) and (B) are proved.

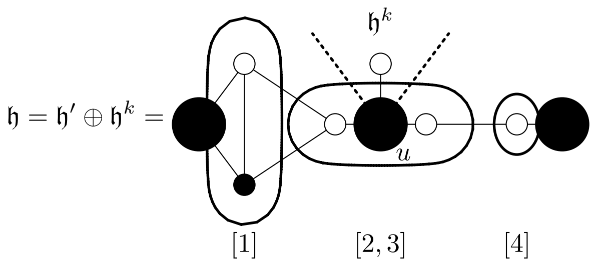


Figure 4: An example of the case that $\Gamma' \simeq P_{t'}$ and the condition (B) does not hold.

In the case of $[\mathfrak{h}^k] = [\mathfrak{h}_1]$, by the condition (B), the fat vertex u in $\mathfrak{h}^k$ equals

$$N^f_{\mathfrak{m}_j}([1]), \quad N^f_{\mathfrak{m}_j}([1,2]), \quad N^f_{\mathfrak{m}_j}([n]) \quad \text{or} \quad N^f_{\mathfrak{m}_j}([n-1,n])$$

for some $j = 1$ or 2. If $u = N^f_{\mathfrak{m}_j}([1])$ or $N^f_{\mathfrak{m}_j}([n])$ then Γ is isomorphic to P_y for some $y \in TP$. Otherwise, without loss of generality we can assume that

$$u = N^f_{\mathfrak{m}_j}([1,2]).$$

Then $t'_1 = t'_2 = 1$ by the condition (A). Hence, Γ is isomorphic to P_y for some $y \in TP$.

We consider the case of $[\mathfrak{h}^k] = [\mathfrak{h}_4]$ or $[\mathfrak{h}_7]$. If $n = 2$ then the desired result holds. Thus, we may assume that $n \geq 3$. Then

$$u \neq N^f_{\mathfrak{m}_j}([i, i+1]) \text{ for every fat vertex } u \in V_f(\mathfrak{h}^k) \text{ and } 1 \leq i \leq n-1 \tag{4.5}$$

since if (4.5) does not hold then $\mathfrak{h}$ has an induced subgraph isomorphic to either S_1 or S_2 (see Figure 5), a contradiction. Let u and v are distinct fat vertices of $\mathfrak{h}^k$. Then one of the following holds:

(i) $u = N^f_{\mathfrak{m}_j}([1])$ and $v \notin V_f(\mathfrak{h}')$, or $u = N^f_{\mathfrak{m}_j}([n])$ and $v \notin V_f(\mathfrak{h}')$;

(ii) $u = N^f_{\mathfrak{m}_j}([1]), v = N^f_{\mathfrak{m}_j}([n])$

by exchanging u and v if necessary. Hence, $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}^k$ is isomorphic to either P_y for some $y \in TP$ or C_y for some $y \in TC$. □

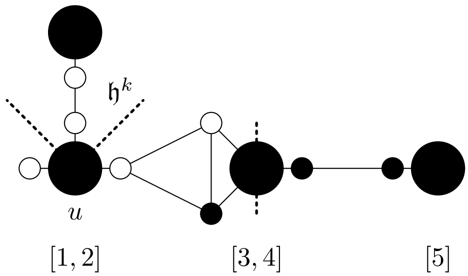


Figure 5: An example of the case that $\Gamma' = P_{t'}$, $\mathfrak{h}^k \simeq \mathfrak{h}_4$ and $u = N^f_{\mathfrak{m}_j}([1,2])$.

Example 4.12. In Theorem 4.11, there are the two conditions that a slim $\{[\mathfrak{h}_7]\}$-line graph has a unique strict $\mathcal{H}$-cover up to equivalence. For each condition, we give an example.

We let G and $\mathfrak{h}$ denote the slim $\{[\mathfrak{h}_7]\}$-line graph and its strict $\mathcal{H}$-cover in Figure 6, respectively. Then, the clique number c of G is equal to 4, and the set K of small circles of G is a maximal clique. Namely, G satisfies the condition (a) in Theorem 4.11. Take a vertex x in K. As shown in Lemma 4.6, $K = C_{\mathfrak{h}}(x)$ holds. Since $K = C_{\mathfrak{h}}(x) \in \operatorname{cov}\mathfrak{h}$, we have

$$\Psi_G(K) = \Psi_G(C_{\mathfrak{h}}(x)) = \operatorname{cov}\mathfrak{h}.$$

As with the proof of Proposition 4.3, we derive the Hoffman graph $\mathfrak{h}$ by adding fat vertices according to $\Psi_G(K)$.

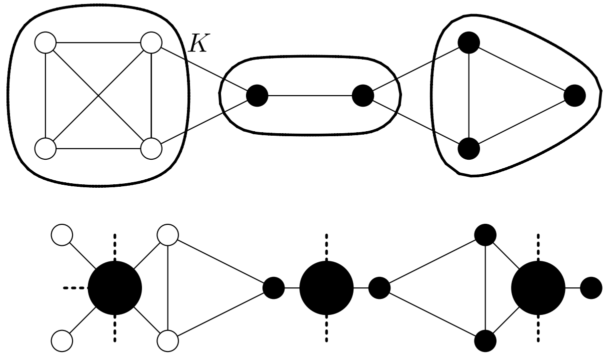


Figure 6: A slim $\{[\mathfrak{h}_7]\}$-line graph whose clique number c is 4 and its strict $\mathcal{H}$-cover corresponding to $\Psi_G(K)$.

Next, we let H and $\mathfrak{m}$ denote the slim $\{[\mathfrak{h}_7]\}$-line graph and its strict $\mathcal{H}$-cover in Figure 7, respectively. Let H' be the subgraph induced by the small circles in H. Then, the clique number c of H is equal to 3, and H' is isomorphic to S_1. Namely, H satisfies the condition (b) in Theorem 4.11. Let K be the triangle of H'. Take a vertex x in K. As shown in Lemma 4.7, the vertices in K are adjacent to the same fat vertex of $\mathfrak{m}$. In particular, $K = C_{\mathfrak{m}}(x)$ holds. Since $K = C_{\mathfrak{m}}(x) \in \operatorname{cov}\mathfrak{m}$, we have

$$\Psi_H(K) = \Psi_H(C_{\mathfrak{m}}(x)) = \operatorname{cov}\mathfrak{m}.$$

As with the proof of Proposition 4.3, we derive the Hoffman graph $\mathfrak{h}$ by adding fat vertices according to $\Psi_H(K)$. Similarly, for a slim $\{[\mathfrak{h}_7]\}$-line graph in Figure 8, we can construct its strict $\mathcal{H}$-cover.

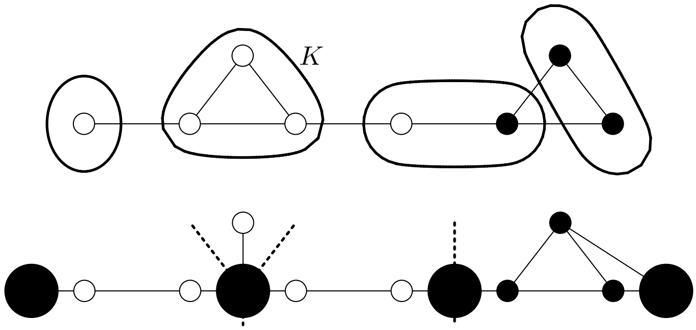


Figure 7: A slim $\{[\mathfrak{h}_7]\}$-line graph containing S_1 induced by the small circles and its strict $\mathcal{H}$-cover corresponding to $\Psi_G(K)$.

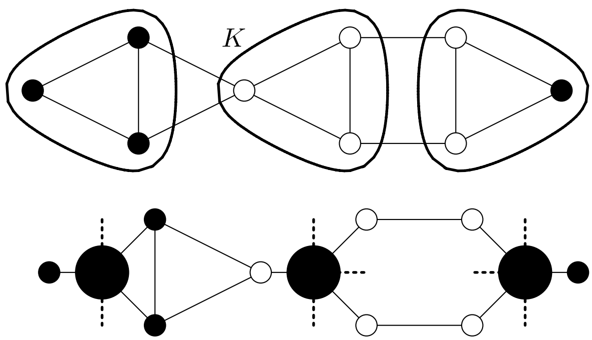


Figure 8: A $\{[\mathfrak{h}_7]\}$-line graph containing S_2 induced by the small circles and its strict $\mathcal{H}$-cover corresponding to $\Psi_H(K)$.

Remark 4.13. Let Γ be a connected slim $\{[\mathfrak{h}_7]\}$-line graph with the clique number at least 2. Let K be a maximal clique of Γ. We suppose that $|K| \geq 4$ when Γ satisfies the condition (a) in Theorem 4.11, and that K contains the triangle of S_1 or that of S_2 when Γ satisfies the condition (b). Then, as shown in Lemma 4.6, the strict $\{[\mathfrak{h}_1], [\mathfrak{h}_4], [\mathfrak{h}_7]\}$-cover is $\operatorname{cov}^{-1}(\Psi_\Gamma(K))$.

5 The minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graphs

The following theorem is the main result in this paper.

Theorem 5.1. *A graph is a minimal forbidden graph for the slim* $\{[\mathfrak{h}_7]\}$*-line graphs if and only if it is one of the following graphs:*

(i) M_i *(*$i = 1, 2, 3, 4, 6, 7, 11, 12, 19$*) in Figure 9;*

(ii) odd cycles with at least 5 *vertices;*

(iii) graphs in Figures 11 and 13.

We explain the reason that the graphs in Figure 9 are minimal forbidden graphs for the slim $\{\mathfrak{h}_7\}$-line graphs. They are obtained by enumeration by MAGMA. The following briefly describes the program.

The MAGMA program is available at [8]. It is also available at `https://doi.org/10.26493/1855-3974.1581.b47`.

Hoffman graphs can construct large new graphs little by little from small graphs by using the concept of sum. With this method, all possible $\{[\mathfrak{h}_7]\}$-line graphs with a small number of slim vertices can be obtained by considering all cases where fat vertices can be stuck together. Therefore, we can obtain all slim $\{[\mathfrak{h}_7]\}$-line graphs with a small number of vertices. On the other hand, the graphs up to 10 vertices have databases in MAGMA [2]. Using this, the list $\mathcal{F}$ of graphs with at most 10 vertices that are not slim $\{[\mathfrak{h}_7]\}$-line graphs is completely revealed. After that, the set of minimal elements of $\mathcal{F}$ can be calculated.

We will prove Theorem 5.1 separately.

(C1) Γ has an induced subgraph isomorphic to S_1, S_2 or the complete graph K_4;

(C2) For any maximal clique K containing the largest clique of some induced subgraph isomorphic to S_1, S_2 or K_4, $\Psi_\Gamma(K)$ is a partition of $V(\Gamma)$.

Proposition 5.2. *Let* Γ *be a minimal forbidden graph for the slim* $\{[\mathfrak{h}_7]\}$*-line graphs with at least* 10 *vertices. Then,* Γ *does not satisfy the condition (C1) if and only if* Γ *is an odd cycle.*

Proof. It is easy to verify that every odd cycle with at least 5 vertices is a minimal forbidden graph. Thus, the necessity is proved. Next, we prove the sufficiency. Pick two vertices x and y to determine the diameter of Γ. Then, $\Gamma - x$ and $\Gamma - y$ are connected and slim $\{[\mathfrak{h}_7]\}$-line graphs. By Theorem 4.11, $\Gamma - x$ is isomorphic to either P_t for some $t \in TP$ or C_t for some $t \in TC$. We have

$$|N_\Gamma(x)| \leq 4 \tag{5.1}$$

since $\Gamma - y$ is also isomorphic to either P_t for some $t \in TP$ or C_t for some $t \in TC$. By (4.1) and (4.2), which are the definition of TP and TC, we have the length l of t is at least 6 since

$$\sum_{i=1}^{l} t_i = |\Gamma| - 1 \geq 9.$$

In the rest of this proof, we will consider the decision on whether the vertex is end or non-end in $\Gamma - x$.

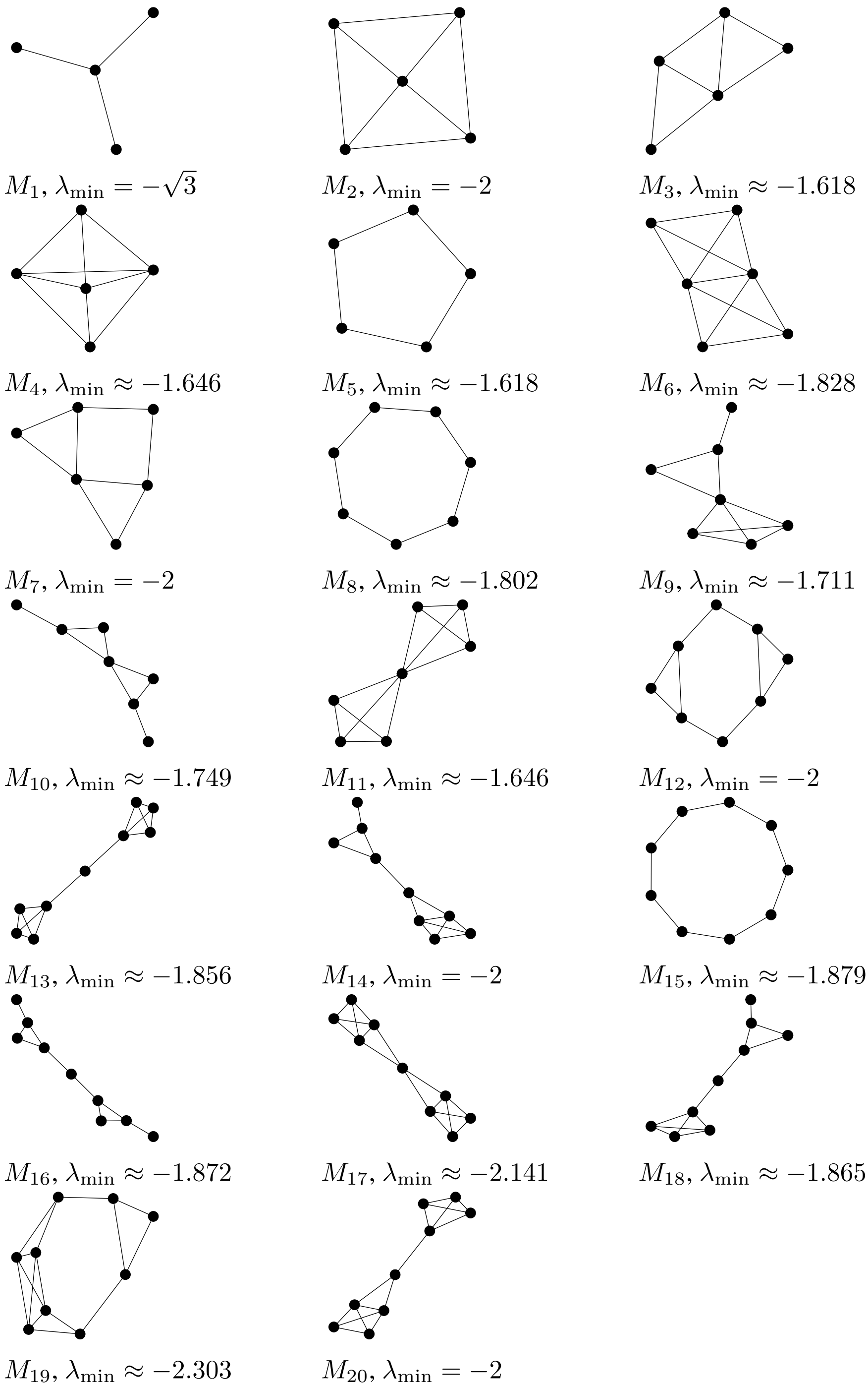


Figure 9: The minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graphs, with at most 9 vertices.

Step 1: Show that $N_\Gamma(x) \cap N_{\Gamma-x}(z)$ is non-empty if z is a non-end vertex in $N_\Gamma(x)$. Suppose that $N_\Gamma(x) \cap N_{\Gamma-x}(z)$ is empty. Then, there exists two distinct vertices u and v of $\Gamma - x$ such that $u \sim z \sim v$ and $u \not\sim v$. It is a contradiction that $M_1 \simeq \langle\{x, z, u, v\}\rangle_\Gamma$.

Step 2: Show that $|N_\Gamma(x) \cap N_{\Gamma-x}(z)| \leq 1$ for $z \in N_\Gamma(x)$. Suppose $|N_\Gamma(x) \cap N_{\Gamma-x}(z)| \geq 2$, and let z_1 and z_2 be two distinct vertices in $N_\Gamma(x) \cap N_{\Gamma-x}(z)$. We have $z_1 \not\sim z_2$ since $K_4 \simeq \langle\{x, z, z_1, z_2\}\rangle_\Gamma$ if $z_1 \sim z_2$. Thus, z is non-end. Let i be the integer such that $z \in [i]$. Then, the following hold:

(i) if $(t_{i-1}, t_i, t_{i+1}) = (1, 1, 2)$, then $M_3 \simeq \langle\{x\} \cup [i-1, i, i+1]\rangle_\Gamma$;

(ii) if $(t_{i-1}, t_i, t_{i+1}) = (1, 2, 1)$, then $M_2 \simeq \langle\{x\} \cup [i-1, i, i+1]\rangle_\Gamma$;

(iii) if $(t_{i-1}, t_i, t_{i+1}) = (2, 1, 2)$, then M_3 is isomorphic to an induced subgraph in $\{x\} \cup [i-1, i, i+1]$.

Then, Γ is isomorphic to either M_2 or M_3 by the minimality of Γ. This is a contradiction to $|V(\Gamma)| \geq 10$. Consider the case of $(t_{i-1}, t_i, t_{i+1}) = (1, 1, 1)$. If $|N_\Gamma(x)| = 3$ then $\Gamma \simeq P_{t'}$ or $C_{t'}$ where $t' = (t_1, \ldots, t_{i-1}, 2, t_{i+1}, \ldots, t_l)$. Otherwise, $|N_\Gamma(x)| = 4$ holds by (5.1), and hence we let $\{z, z_1, z_2, z_3\} = N_\Gamma(x)$. Then, the following hold:

(i) if $z_3 \in [i-2, i+2]$ then $M_3 \simeq \langle\{x\} \cup N_\Gamma(x)\rangle_\Gamma$;

(ii) if $z_3 \notin [i-2, i+2]$ then $M_1 \simeq \langle\{x, z_1, z_2, z_3\}\rangle_\Gamma$

(see Figure 10). These are contradictions to $|V(\Gamma)| \geq 10$.

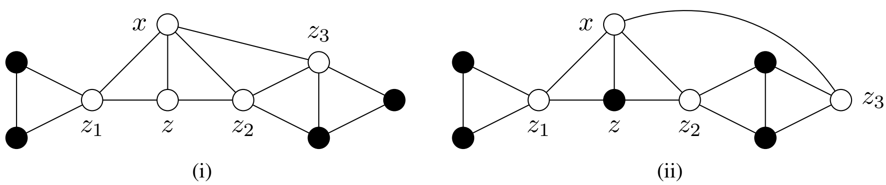


Figure 10: Examples of the case that $(t_{i-1}, t_i, t_{i+1}) = (1, 1, 1)$ and $|N_\Gamma(x)| = 4$ in Step 2.

Step 3: Show that the vertices in $N_\Gamma(x)$ are end. Suppose that a vertex z is non-end in $N_\Gamma(x)$. By Step 1 and 2, $|N_\Gamma(x) \cap N_{\Gamma-x}(z)| = 1$ holds. Thus, we can take a vertex z_1 so that

$$\{z_1\} = N_\Gamma(x) \cap N_{\Gamma-x}(z).$$

There are i and j such that $z \in [i]$ and $z_1 \in [j]$. Let $I = [i, j, i \pm 1, j \pm 1]$. It follows that $N_\Gamma(x) \cap I = \{z, z_1\}$ by Step 2. If z_1 is non-end, then some induced subgraph of Γ is isomorphic to M_1 if $i = j$, S_1 otherwise, a contradiction. Otherwise, we may assume that $i = 2$ and $j = 1$ without loss of generality. Then, the following hold:

(i) if $(t_1, t_2) = (2, 1)$, then M_1 is isomorphic to some induced subgraph of Γ;

(ii) if $(t_1, t_2) = (1, 2)$, then M_3 is isomorphic to some induced subgraph of Γ;

(iii) if $(t_1, t_2) = (1, 1)$ and $|N_\Gamma(x)| \geq 3$, then Γ has an induced subgraph isomorphic to S_1 or S_2;

(iv) if $(t_1, t_2) = (1, 1)$ and $|N_\Gamma(x)| = 2$, then $\Gamma \simeq P_{(t_1+1, t_2, \dots, t_l)}$.

The result follows. Moreover, $\Gamma - x$ is isomorphic to P_t.

Step 4: For $i = 1$ or l, if $N_\Gamma(x) \cap [i] \neq \emptyset$ then $N_\Gamma(x) \cap [i] = [i]$ since $N_\Gamma(x) \cap [i] \neq [i]$ implies that Γ has an induced subgraph isomorphic to S_1 by Step 3.

Step 5: If $N_\Gamma(x) = [1]$ then $\Gamma \simeq P_{(1,t_1,\dots,t_l)}$ is an $\{[\mathfrak{h}_7]\}$-line graph, a contradiction. Hence, $\Gamma \simeq C_{(1,t_1,\dots,t_l)}$. If l is odd then $C_{(1,t_1,\dots,t_l)}$ is an $\{[\mathfrak{h}_7]\}$-line graph, a contradiction. Thus, Γ is an odd cycle. □

Proposition 5.3. *Let Γ be a minimal forbidden graph for the slim $\{[\mathfrak{h}_7]\}$-line graphs, with at least 10 vertices and the condition (C1). Then, Γ is isomorphic to one of the graphs in Figure 11 if and only if Γ does not satisfy the condition (C2).*

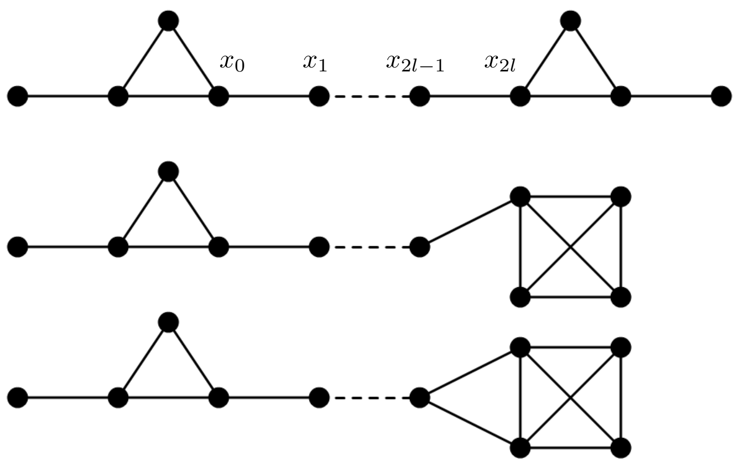


Figure 11: Minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graphs, with at least 10 vertices and the condition (C1), without (C2).

Proof. It is easy to verify that $\Psi_\Gamma(K)$ is not a partition, where Γ is one of the graphs of Figure 11, and K is the rightmost clique of size at least 3.

Next, we prove the necessity. By the condition (C1), Γ has an induced subgraph isomorphic to S_1, S_2 or K_4. Hence, we can take a maximal clique K containing the largest clique of some induced subgraph S isomorphic to S_1, S_2 or K_4. Since the condition (C2) is not satisfied, we may suppose that $\Psi_\Gamma(K)$ is not a partition of $V(\Gamma)$. If $|K| > 3$ then we replace S by $\langle K' \rangle_\Gamma$, where K' is a set of 4 vertices in K. Let $l = \lfloor (\partial_{\text{max}} - 1)/2 \rfloor$ and $V' = \{x \in V(\Gamma) \mid \partial_K(x) \leq 2l\}$.

Step 1: There is a vertex g not in $V(S) \cup K$ with $\partial_K(g) = \partial_{\text{max}}$. Then, $\{C \in \Psi_\Gamma(K) \mid \partial_K(c) \leq 2l$ for any $c \in C\} = \{C \in \Psi_{\Gamma - g}(K) \mid \partial_K(c) \leq 2l$ for any $c \in C\}$ is a clique partition of V' since $\Gamma - g$ is a slim $\{[\mathfrak{h}_7]\}$-line graph satisfying the condition (a) or (b) in

Theorem 4.11. Therefore, $\Psi_\Gamma(K)$ is not a partition if and only if there exists vertices x, y, p and q such that $\partial_K(x) = \partial_K(y) = 2l+1$,

$$q \in ((N(x) \cup \{x\}) \cap (N(y) \cup \{y\})) - V',$$

and

$$p \in ((N(x) \cup \{x\}) - (N(y) \cup \{y\})) - V'.$$

If $z \in V(\Gamma) - (\{x,y,p,q\} \cup S)$ with $\partial_K(z) \geq 2l+1$, then $\Gamma - z$ is an $\{[\mathfrak{h}_7]\}$-line graph and $p \sim y$ by Remark 4.13. This is a contradiction to $p \not\sim y$. It follows that

$$\{u \in V(\Gamma) \mid \partial_K(u) \geq 2l+1\} - V(S) = \{x,y,p,q\}. \tag{5.2}$$

Step 2: Assume that $l=0$. In the case of $|K| \geq 4$ (i.e., $S \simeq K_4$),

$$\{K - \{k\}, \{x,y,p,q\}\} \subset \Psi_{\Gamma-k}(K - \{k\})$$

is a clique partition for some $k \in K$ by Remark 4.13 and (5.2), i.e., $|K| \geq 6$. This is a contradiction to $p \not\sim y$. In the case of $|K| = 3$ (i.e., $S \simeq S_1$ or S_2), we obtain $|V(\Gamma)| \leq 9$ and a contradiction. Hence, l is a positive integer.

Step 3: Show that $x \sim y$. In addition, it holds that $x \neq p$ and $x = q$. Suppose that x and y are not adjacent. Then, $p = x$ and $q \notin \{x,y\}$ by Remark 4.13. If x and y are adjacent to a vertex r with $\partial_K(r) = 2l$, then Γ has an induced subgraph isomorphic to $K_{1,3}$, a contradiction. When $\partial_K(q) = 2l+1$, there is a neighbor q' of q with $\partial_K(q') = 2l$ such that $q' \notin N(x) \cap N(y)$. Without loss of generality we assume that q' and y are not adjacent. Then, the induced subgraph Γ' obtained by deleting the neighbors of y expect q has a clique partition $\Psi_{\Gamma'}(K)$ which contains a set $\{x,y,q\}$. This is a contradiction, and $\partial_K(q) = 2l+2$ follows. Then, Γ has an induced subgraph isomorphic to an odd cycle with at least 5 vertices. This is a contradiction since $\Psi_{\Gamma-q}(K)$ is a clique partition in $\mathcal{O}_{\Gamma-q}$.

Step 4: Let P denote a path $(x_0, \ldots, x_{2l+1})$ such that $x_0 \in K$ and $x = x_{2l+1}$. If $S \simeq K_4$, then replace S by $\langle K' \rangle_\Gamma$, where K' is a set of 4 vertices in K containing x_0 and a vertex not adjacent to x_1. The graph Γ' by deleting vertices other than $V(P) \cup V(S) \cup \{x,y,p\}$ is a slim $\{[\mathfrak{h}_7]\}$-line graph with a clique partition $\Psi_{\Gamma'}(K)$. Thus, it holds that $V(\Gamma) = V(P) \cup V(S) \cup \{x,y,p\}$ and $y \sim x_{2l}$.

Step 5: If $\partial_K(p) = 2l+1$, then $p \sim x_{2l}$. Thus, $\langle \{y,p,x_{2l},x_{2l-1}\} \rangle_\Gamma \simeq K_{1,3}$ holds, a contradiction. Thus, $\partial_K(p) = 2l+2$.

Step 6: In the case of $|K| \geq 4$ (i.e., $S \simeq K_4$), $\deg(x_1) \leq 3$ since $\Gamma - p$ is an $\{[\mathfrak{h}_7]\}$-line graph. Obtain the second and third graphs in Figure 11.

Step 7: In the case of $|K| = 3$ (i.e., $S \simeq S_1$ or S_2), if $x_1 \notin S$ then $|N(x_1) \cap K| = 1$ since M_1 and M_3 are minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graphs. Then, we can replace S by the induced subgraph by $K \cup \{x_1, w\}$, where w is a vertex of S not adjacent to x_0. Hence, $x_1 \in S$ holds. We can draw Γ as Figure 12. The edge e exists if and only if the edge e' does in Figure 12. If the edges e and e' exist, then $\Gamma - x_0$ is not an $\{[\mathfrak{h}_7]\}$-line graph. Otherwise, we obtain the first graphs in Figure 11. □

Let Γ be a connected graph, and let K and D be nonempty subsets of $V(\Gamma)$. D is said to be *deletable* for K if $K - D \neq \emptyset$, $\Gamma - D$ is connected, and $\Psi_{\Gamma-D}(K-D) = \{C - D \mid$

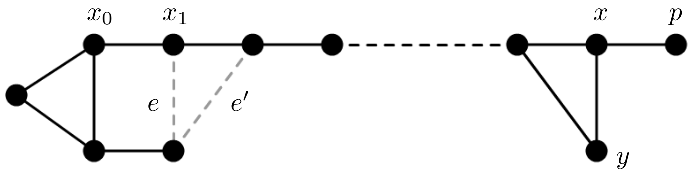


Figure 12: In the proof of Proposition 5.3.

$C \in \Psi_\Gamma(K)\} - \{\emptyset\}$. In addition, a vertex v is said to be *deletable* for K if $\{v\}$ is deletable for K.

Lemma 5.4. *Let Γ be a connected graph, and let K and D be nonempty subsets of $V(\Gamma)$. If $\Psi_\Gamma(K)$ is a partition of $V(\Gamma)$ and $\partial_{K,\Gamma}|_{\Gamma-D} = \partial_{K-D,\Gamma-D}$, then D is deletable for K.*

Proof. Write $\partial = \partial_{K,\Gamma}$ for short. Then, it holds that by $\partial_{K,\Gamma}|_{\Gamma-D} = \partial_{K-D,\Gamma-D}$,

$$\begin{aligned}&\Psi_{\Gamma-D}(K-D) - \{K-D\}\\ &\quad = \{\{y \in (\{x\} \cup N(x)) - D \mid \partial(y) \geq \partial(x)\} \mid x \in V(\Gamma - D), \partial(x) \in 2\mathbb{N}+1\},\end{aligned}$$

and

$$\begin{aligned}&\{C - D \mid C \in \Psi_\Gamma(K)\} - \{K-D\}\\ &\quad = \{\{y \in (\{x\} \cup N(x)) - D \mid \partial(y) \geq \partial(x)\} \mid x \in V(\Gamma), \partial(x) \in 2\mathbb{N}+1\}.\end{aligned}$$

Let $x \in D$ with odd $\partial(x)$, and take $C_x \in \Psi_\Gamma(K)$ containing x. Assuming that $C_x - D \neq \emptyset$, there exists $z \in \tilde{C}$ such that $\partial(z) = \partial(x)$ by the assumptions. It holds that

$$\begin{aligned}C_x - D &= \{y \in (\{x\} \cup N(x)) - D \mid \partial(y) \geq \partial(x)\}\\ &= \{y \in (\{z\} \cup N(z)) - D \mid \partial(y) \geq \partial(z)\} \in \Psi_{\Gamma-D}(K-D).\end{aligned}$$ □

Lemma 5.5. *Let Γ be a minimal forbidden graph for the slim $\{[\mathfrak{h}_7]\}$-line graphs, with the condition (C1) and (C2). Let S be an induced subgraph isomorphic to S_1, S_2 or K_4. Let K be a maximal clique of Γ contains the largest clique of S. Then, the following hold:*

(i) $\Psi_\Gamma(K)$ is a clique partition;

(ii) if u is a non good vertex for $\Psi_\Gamma(K)$, and $v \notin V(S) \cup K$ is a deletable vertex for K, then $v \in n^3(u)$ and v is non good for $\Psi_\Gamma(K)$, where $n^k(\cdot)$ is defined for $\Psi_\Gamma(K)$ and a non negative integer k.

Proof. By the condition (C2), $\Psi_\Gamma(K)$ is a partition of $V(\Gamma)$. Moreover, it is a clique partition of $V(\Gamma)$ by Lemma 4.5 since Γ has no induced subgraph isomorphic to $M_1 \simeq K_{1,3}$. Next, suppose that the vertex v is not in $n^3(u)$. Then,

$$\begin{aligned}\{C \cap n^3(u) \mid C \in \Psi_\Gamma(K)\} &= \{C \cap n^3(u) - \{v\} \mid C \in \Psi_\Gamma(K)\}\\ &= \{C \cap n^3(u) \mid C \in \Psi_{\Gamma-v}(K)\}\end{aligned}$$

holds. Thus, the vertex u is good for $\Psi_{\Gamma-v}(K)$ if and only if it is good for $\Psi_{\Gamma-v}(K)$ in $\Gamma - v$ by Lemma 4.4. On other hand, $v \notin V(S) \cup K$ and $\Gamma - v$ is connected. Hence, $\Psi_{\Gamma-v}(K) \in \mathcal{O}_{\Gamma-v}$, that is, every vertex of $\Gamma - v$ is good for $\Psi_\Gamma(K)$ by Remark 4.13 since Γ is a minimal forbidden graph for the slim $\{[\mathfrak{h}_7]\}$-line graphs. Therefore, u is good for $\Psi_{\Gamma-v}(K)$ and $\Psi_\Gamma(K)$. This is a contradiction that u is non good for $\Psi_\Gamma(K)$. □

Proposition 5.6. *Let* Γ *be a minimal forbidden graph for the slim* $\{[\mathfrak{h}_7]\}$*-line graphs, with at least* 10 *vertices and the condition (C1). Then,* Γ *is one of the graphs in Figure 13 if and only if* Γ *satisfies the condition (C2).*

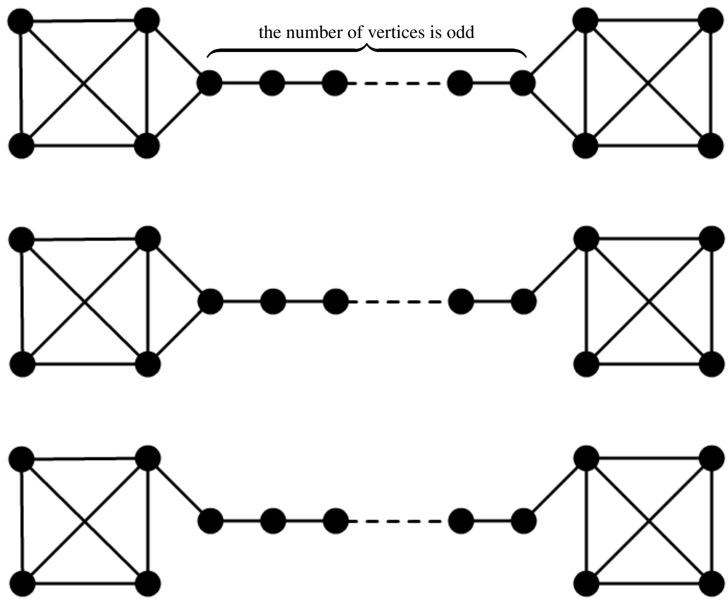


Figure 13: The minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graphs, with at least 10 vertices and the conditions (C1) and (C2).

Proof. The sufficiency obviously holds. Prove the necessity. Fix an induced subgraph S isomorphic to S_1, S_2 or K_4, and let K be a maximal clique containing the largest clique of S. By Lemma 5.5 (i), $\Psi_\Gamma(K)$ is a clique partition. Then, $n^k(\cdot)$ is defined for $\Psi_\Gamma(K)$ and a non negative integer k.

If Γ has an induced subgraph isomorphic to K_4, then replace S by it. Let $l = \lfloor(\partial_{\max} - 1)/2\rfloor$. By the definition of $\Psi_\Gamma(K)$, we can pick the subset

$$\big\{\{y \in N(x) \cup \{x\} \mid \partial_K(y) \geq \partial_K(x)\} \mid x \in V(\Gamma), \partial_K(x) = 2l+1\big\}$$

of $\Psi_\Gamma(K)$. We denote by $\{C_i\}_{i=1}^{n}$ the subset. Note that C_i are pairwise distinct. If $l = 0$ then let $D_1 = K$ and $m = 1$. Otherwise we pick the subset

$$\big\{\{y \in N(x) \cup \{x\} \mid \partial_K(y) \geq \partial_K(x)\} \mid x \in V(\Gamma), \partial_K(x) = 2l-1\big\}$$

of $\Psi_\Gamma(K)$. We denote by $\{D_i\}_{i=1}^{n'}$ the subset. Note that D_i are pairwise distinct. Without loss of generality, we can take an integer m such that $D_i - V(S)$ is empty if and only if $i > m$.

In the case of $\partial_{\max} = 2l+1$, we show that Γ is isomorphic to one of the graphs in Figure 13.

Step 1: Show that $l \neq 0$. Suppose that $l = 0$ to prove by contradiction. Set $B = V(\Gamma) - (V(S) \cup K)$. Then, every vertex in B is deletable by Lemma 5.4. Moreover, the deletable vertex is non good for $\Psi_\Gamma(K)$ by applying Lemma 5.5 (ii) to a non good vertex for $\Psi_\Gamma(K)$ and the deletable vertex. We obtain a contradiction by checking the following:

(i) $|B| \leq 3$;

(ii) if $7 \leq |V(\Gamma) - K|$, then $5 \leq |V(\Gamma) - K| - 2 \leq |B|$;

(iii) if $4 \leq |V(\Gamma) - K| \leq 6$, then $S \simeq K_4$ and $4 \leq |B|$;

(iv) if $|V(\Gamma) - K| \leq 3$, then Γ is an $\{[\mathfrak{h}_7]\}$-line graph.

Assume that $|B| \geq 4$. If we find a vertex $k \in K$ with $|n(k)| \geq 3$, then we can take a vertex $b \in B$ such that $|n(k) - \{b\}| \geq 3$. This is a contradiction since the vertex b is deletable and $\Psi_{\Gamma-b}(K) \in \mathcal{O}_{\Gamma-b}$ by Remark 4.13. Thus,

$$|n(k)| \leq 2 \tag{5.3}$$

holds for every $k \in K$. Fix a vertex $b \in B$. Then, $|n(b)| \geq 3$ holds by (5.3) and applying Lemma 5.5 (ii) to the non good vertex b and each vertex in $B - \{b\}$. We obtain a contradiction as well and $|B| \leq 3$. Next, In the case of $|V(\Gamma) - K| \leq 3$, we have $|B| \leq 3$, $|K| \geq 7$ and hence $S \simeq K_4$. If $2 \leq |B| \leq 3$, then $|n(b)| \leq 2$ for every vertex $b \in B$. Hence, we can pick a vertex k in

$$K - \bigcup_{b \in B} n(b).$$

It is clear that k is deletable and $\Psi_{\Gamma-k}(K - \{k\}) \in \mathcal{O}_{\Gamma-k}$. Thus every vertex of Γ is good for $\Psi_\Gamma(K)$, a contradiction to Γ being a non $\{[\mathfrak{h}_7]\}$-line graph.

Step 2: Every vertex x with $\partial_K(x) = \partial_{\max}$ is not in $V(S) \cup K$ and deletable by $l \geq 1$ and Lemma 5.4. Moreover, such a vertex x is non good for $\Psi_\Gamma(K)$ by applying Lemma 5.5 (ii) to a non good vertex for $\Psi_\Gamma(K)$ and the deletable vertex x. Fix a vertex u with $\partial_K(u) = \partial_{\max}$. By applying Lemma 5.5 (ii) to u and each vertex x with $\partial_K(x) = \partial_{\max}$, we have

$$\{x \in V(\Gamma) \mid \partial_K(x) = \partial_{\max}\} \subset n^2(u)$$

since Γ has no induced subgraph isomorphic to $M_1 \simeq K_{1,3}$. For a vertex x with $\partial_K(x) = \partial_{\max} - 1 = 2l$, it holds that $x \in n^3(u)$ since if $n(x) = \emptyset$ then x is deletable by Lemma 5.4 and $x \in n^3(u)$ holds by applying Lemma 5.5 (ii) to the vertices u and x. Thus,

$$n^3(u) = \{x \in V(\Gamma) \mid \partial_K(x) \geq 2l\}. \tag{5.4}$$

Furthermore, D_i contains a vertex v_i with $\partial_K(v_i) = \partial_{\max} - 1$ for every $1 \leq i \leq m$, since every deletable vertex not in $V(S) \cup K$ is contained in $n^3(u)$ by Lemma 5.5 (ii).

Step 3: Show that $n = m = 1$. If $n \geq 2$ then Γ has an induced subgraph isomorphic to M_1 by (5.4), a contradiction. Thus, $n = 1$. Suppose that $m \geq 2$ to prove by contradiction. Without loss of generality, we can assume that $v_1 \sim u \sim v_2$ by (5.4). In the case of $m \geq 3$, the vertex v_3 is deletable clearly, and the vertex u is non good for $\Psi_{\Gamma-v_3}(K)$ in $\Gamma - v_3$. This is a contradiction to $\Psi_{\Gamma-d_3}(K) \in \mathcal{O}_{\Gamma-d_3}$ by Remark 4.13.

In the case of $m = 2$, we have $C_1 = \{u\}$ since if we find an vertex u' in C_1, then the vertex u' is deletable by Lemma 5.4 and u is non good for $\Psi_{\Gamma-u'}(K)$ in $\Gamma - u'$, a contradiction to $\Psi_{\Gamma-u'}(K) \in \mathcal{O}_{\Gamma-u'}$. Fix a vertex $v_i' \in D_i$ with $\partial_K(v_i') = 2l-1$ for $i = 1$ and 2, respectively. Then, the set

$$D_1 \cup D_2 - (V(S) \cup \{v_1, v_2, v_1', v_2'\})$$

is deletable and u is good. Hence, the set is empty. If v_1 and v_2 are not adjacent in Γ, then Γ has an induced subgraph isomorphic to an odd cycle with at least 5 vertices since u is deletable and $\Psi_\Gamma(K) - \{C_1\} = \Psi_{\Gamma-u}(K) \in \mathcal{O}_{\Gamma-u}$. This is a contradiction to the minimality of Γ. Hence, v_1 and v_2 are adjacent in Γ. First, consider the case of $l \geq 2$. Let d be the vertex adjacent to v_2' with $\partial_K(d) = 2l-2$. Note that d is not in S. Then, $\Gamma - d$

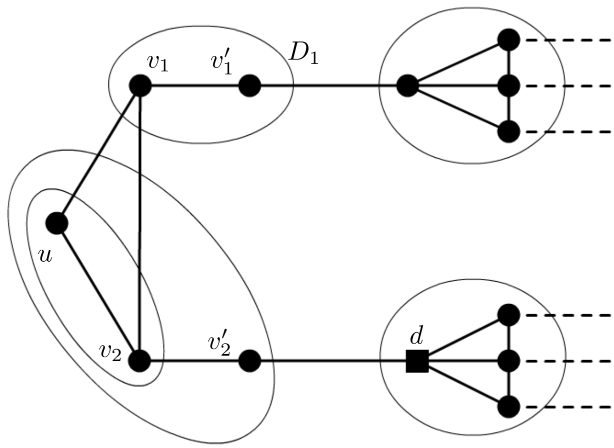


Figure 14: The cases of $n = 1, m = 2$ and $l \geq 2$ in the proof of Proposition 5.6.

is not an $\{[\mathfrak{h}_7]\}$-line graph since $\Psi_{\Gamma-d}(K) \notin \mathcal{O}_{\Gamma-d}$ (cf. Figure 14). Second, consider the case of $l = 1$. Suppose that $S \simeq K_4$. Note that $n' = m = 2$ holds. If $|K| \geq 5$, then we can take a deletable vertex k in K. By Remark 4.13, $\Psi_{\Gamma-k}(K - \{k\}) \in \mathcal{O}_{\Gamma-k}$ holds since $K - \{k\}$ is a maximal clique with at least 4 vertices. However, u is non good for $\Psi_{\Gamma-k}(K - \{k\})$, a contradiction to the minimality of Γ. We have $|V(\Gamma)| = |K|+|D_1|+|D_2|+|C_1| = 4+2+2+1 = 9$, a contradiction to $|V(\Gamma)| \geq 10$. Thus, Γ has no induced subgraph isomorphic to K_4. Suppose that $S \simeq S_1$ or S_2. We define the vertices w_i of S as Figure 15. Note that the vertex v_i' is not in $V(S)$ for $i = 1, 2$ since $|V(\Gamma)| \geq 10$. If both v_1' and v_2' are adjacent only to w_1, then Γ has an induced subgraph isomorphic to M_1, a contradiction. Hence, it is not so. Without loss of generality, we can assume that v_2' and w_2 are adjacent. Since Γ has no induced subgraph isomorphic to K_4, v_2' is not adjacent to some vertex $w \in \{w_1, w_3\}$. The vertices v_2' and w_5 are adjacent since $\langle v_2', w, w_2, w_5\rangle_\Gamma$ is not isomorphic to M_1. Furthermore, v_2 is also adjacent to w_5 since $\Psi_\Gamma(K)$ is a clique partition. Hence, v_2' is deletable for K, a contradiction to Ψ. We have $n = m = 1$.

Step 4: Let $p = |C_1|$ and $q = |\{x \in V(\Gamma) \mid \partial_K(x) = 2l\}|$. The induced subgraph $\langle C_1 \cup D_1\rangle_\Gamma$ is an $\{[\mathfrak{h}_7]\}$-line graph by the minimality of Γ. Hence, $\Psi_{\langle C_1 \cup D_1\rangle_\Gamma}(D_1) =$

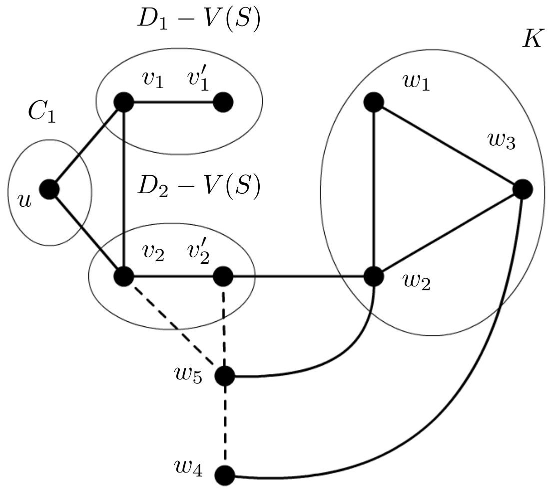


Figure 15: The cases of $n = 1, m = 2$ and $l = 1$ in the proof of Proposition 5.6.

$\{C_1, D_1\}$ holds by $|D_1| \geq 4$ and Remark 4.13. This is a contradiction since non good vertices for $\Psi_\Gamma(K)$ in $C_1 \cup D_1$ are also non good for $\Psi_{\langle C_1 \cup D_1 \rangle_\Gamma}(D_1)$. Thus, $q \leq 2$. When $q = 1$, $p = 3$ holds and we obtain the second and third graph in Figure 13 in the same way as the Step 4 in the proof of Proposition 5.3. Consider the case of $q = 2$. If $p = 1$ then Γ is an $\{[\mathfrak{h}_7]\}$-line graph. When $p = 2$, we obtain the first and second graph in Figure 13 since Γ has no induced subgraph isomorphic to M_3. If $p \geq 3$ then we can assume that $n(u) = \{x \in V(\Gamma) \mid \partial_K(x) = 2l\}$ by (5.4). Then, $\Gamma - w$ is not an $\{[\mathfrak{h}_7]\}$-line graph for some $w \in C_1 - \{v\}$, a contradiction.

Suppose that $\partial_{\max} = 2l + 2$. We have $l \geq 0$ and every vertex u with $\partial_K(u) = 2l + 2$ is deletable for K. By Lemma 5.5 (ii), the vertex u is non good for $\Psi_\Gamma(K)$. Moreover, every vertex v with $\partial_K(v) \leq 2l + 1$ is good for $\Psi_\Gamma(K)$ since $\langle n^3(v) \rangle_\Gamma$ is the induced subgraph of $\Gamma - u$, where $\partial_K(u) = 2l + 2$. Hence, a vertex u is good if and only if $\partial_K(u) \leq 2l + 1$. We have $n \neq 1$ since a vertex v with $\partial_K(v) = 2l + 2$ is non good. Let v be a vertex with $\partial_K(v) = 2l + 1$. If v is not a vertex of S, then we have a contradiction to Lemma 5.5 (ii) that v is deletable for K. Hence, v is a vertex of S. Thus, $S \simeq S_1$, $l = 0$ and $n = 2$. Then, $|V(\Gamma)| \leq 9$ since $|K| = 3$, $|C_1| \leq 3$ and $|C_2| \leq 3$, a contradiction. □

Proof of Theorem 5.1. The minimal forbidden graphs with at most 9 vertices are revealed in Figure 9. This theorem follows by Proposition 5.2, 5.3 and 5.6. □

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Lobe, edge, and arc transitivity of graphs of connectivity 1

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Abstract

We give necessary and sufficient conditions for lobe-transitivity of locally finite and locally countable graphs whose connectivity equals 1. We show further that, given any biconnected graph Λ and a "code" assigned to each orbit of $\mathrm{Aut}(\Lambda)$, there exists a unique lobe-transitive graph Γ of connectivity 1 whose lobes are copies of Λ and is consistent with the given code at every vertex of Γ. These results lead to necessary and sufficient conditions for a graph of connectivity 1 to be edge-transitive and to be arc-transitive. Countable graphs of connectivity 1 the action of whose automorphism groups is, respectively, vertex-transitive, primitive, regular, Cayley, and Frobenius had been previously characterized in the literature.

Keywords: Lobe, lobe-transitive, edge-transitive, orbit, connectivity.

Math. Subj. Class.: 05C25, 05C63, 05C38, 20B27

1 Introduction

Throughout this article, Γ denotes a connected simple graph. Consider the equivalence relation $\cong$ on the edge-set $E\Gamma$ of Γ whereby $e_1 \cong e_2$ whenever the edges e_1 and e_2 lie on a common cycle of Γ. A *lobe* is a subgraph of Γ induced by an equivalence class with respect to $\cong$. Equivalently, a *lobe* is a subgraph that either consists of a cut-edge with its two incident vertices or is a maximal biconnected subgraph[1]. A vertex is a *cut-vertex* if it belongs to at least two different lobes. Connected graphs other than K_2 have connectivity 1 if and only if they have a cut-vertex. Clearly no *finite* vertex-transitive graph admits a cut-vertex.

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[1]The term "lobe" is due to O. Ore [6]. We eschew the term "block" for this purpose, as it leads to ambiguity when discussing imprimitivity.

Graphs of connectivity 1 whose automorphism groups have certain given properties have been characterized. Those whose automorphism groups are, respectively, vertex-transitive, primitive, and regular were characterized in [5]. In particular, primitive *planar* graphs of connectivity 1 were characterized in [11][2]. Cayley graphs of connectivity 1 were characterized in [9]. Graphs of connectivity 1 with Frobenius automorphism groups were characterized in [10]. In the present work, we complete this investigation; we characterize graphs of connectivity 1 whose automorphism groups act transitively on their set of lobes. As a consequence, we obtain characterizations of edge-transitive graphs and arc-transitive graphs of connectivity 1.

The conditions for a graph of connectivity 1 to be lobe-transitive or to be vertex-transitive are independent; such a graph may have either property or neither one or both. Such is not the case for edge- and arc-transitivity. In Section 3 we give necessary and sufficient conditions for a graph to be lobe-transitive. We further show that, given any biconnected graph Λ and a "list" of orbit-multiplicities of copies of $\mathrm{Aut}(\Lambda)$, one can construct a lobe-transitive graph of connectivity 1 all of whose lobes are isomorphic to Λ and locally respects the given list. We give necessary and sufficient conditions for a countable graph of connectivity 1 to be edge-transitive in Section 4 and to be arc-transitive in Section 5. As the sets of conditions for these latter two properties are more intertwined with lobe-transitivity than the characterization of vertex-transitivity (for graphs of connectivity 1), scattered throughout are examples that illustrate some algebraic distinctions among these various properties.

2 Preliminaries

Throughout this article, the symbol $\mathbb{N}$ denotes the set of positive integers. The symbols $\mathbb{I}, \mathbb{J}$, and $\mathbb{K}$, often subscripted, denote subsets of $\mathbb{N}$ of the form $\{1, 2, \ldots, n\}$ or the set $\mathbb{N}$ itself; they appear as sets of indices. All graphs (and their valences) in this article are finite or countably infinite. The symbol $\delta_{i,j}$ (the so-called "Kronecker delta") assumes the value 1 if $i = j$ and 0 if $i \neq j$. For a graph Λ and any subgroup $H \leq \mathrm{Aut}(\Lambda)$, the set of orbits of H acting on $V\Lambda$ is denoted by $\mathscr{O}(H)$.

The set of lobes of a graph Γ is denoted by $\mathscr{L}(\Gamma)$. We let $\{\mathscr{L}_k : k \in \mathbb{K}\}$ denote the partition of $\mathscr{L}(\Gamma)$ into isomorphism classes of lobes. For given $k \in \mathbb{K}$ and a lobe $\Lambda \in \mathscr{L}_k$, we let $\mathscr{O}(\mathrm{Aut}(\Lambda)) = \{(V\Lambda)_j : j \in \mathbb{J}_k\}$, and we understand that if $\sigma \colon \Lambda \to \Theta$ is an isomorphism between lobes in $\mathscr{L}_k$, then $\sigma((V\Lambda)_j) = (V\Theta)_j$ for all $j \in \mathbb{J}_k$. Finally, for each $k \in \mathbb{K}$ and $j \in \mathbb{J}_k$, we define the function $\tau_j^{(k)} \colon V\Gamma \to \mathbb{N}$ by

$$\tau_j^{(k)}(v) = |\{\Lambda \in \mathscr{L}_k : v \in (V\Lambda)_j\}|. \tag{2.1}$$

For $\Lambda_0 \in \mathscr{L}(\Gamma)$ and $n \in \mathbb{N}$, we recursively define the subgraphs

$$\Gamma_0(\Lambda_0) = \Lambda_0,$$
$$\Gamma_{n+1}(\Lambda_0) = \bigcup\{\Lambda \in \mathscr{L}(\Gamma) : V\Lambda \cap V\Gamma_n(\Lambda_0) \neq \emptyset\}.$$

Lemma 2.1 ([5, Lemma 3.1]). *Let $\Lambda, \Theta \in \mathscr{L}(\Gamma)$ and let $n \in \mathbb{N}$. If for each $k \in \mathbb{K}$ and $j \in \mathbb{J}_k$, the function $\tau_j^{(k)}$ is constant on $V\Gamma$, then any isomorphism $\sigma_n \colon \Gamma_n(\Lambda) \to \Gamma_n(\Theta)$ admits an extension to an automorphism $\sigma \in \mathrm{Aut}(\Gamma)$.*

[2] For a short algebraic proof that all 1-ended planar graphs with primitive automorphism group are biconnected, see [8].

This lemma was used in [5] to prove the following characterization of vertex-transitive graphs of connectivity 1.

Theorem 2.2 ([5, Theorem 3.2])**.** *Let* Γ *be a graph of connectivity* 1*. A necessary and sufficient condition for* Γ *to be vertex-transitive is that all the functions* $\tau_j^{(k)}$ *be constant on* $V\Gamma$.

Notation. When all the lobes of the graph Γ are pairwise isomorphic, that is, the index set $\mathbb{K}$ has but one element, then in Equation (2.1) the index k is suppressed; we simply replace $\mathbb{J}_k$ by $\mathbb{J}$ and $\tau_j^{(k)}$ by τ_j.

3 Lobe-transitivity

Let Γ be a graph of connectivity 1. It is immediate from the above definitions that the edge-sets of the lobes of Γ are blocks of imprimitivity of the group $\mathrm{Aut}(\Gamma)$ acting on $E\Gamma$. Hence any automorphism of Γ must map lobes onto lobes, and therefore, if $\mathrm{Aut}(\Gamma)$ is to act transitively on $\mathscr{L}(\Gamma)$, then all the lobes of Γ must be pairwise isomorphic. However, pairwise-isomorphism of the lobes alone is not sufficient for lobe-transitivity, even when every vertex of Γ lies in the same number of lobes.

Let us first dispense with trees; the proof is elementary and hence omitted.

Proposition 3.1. *A finite or countable tree is lobe-transitive (and simultaneously, edge-transitive) if and only if there exist* $n_1, n_2 \in \mathbb{N} \cup \{\aleph_0\}$ *such that every edge has one incident vertex of valence* n_1 *and the other of valence* n_2*. If* $n_1 = n_2$*, the tree is also arc-transitive.*

For graphs of connectivity 1 other than trees, we have the following characterization of lobe-transitivity.

Theorem 3.2. *Let* Γ *be a graph of connectivity* 1*, and let* Λ_0 *be an arbitrary lobe of* Γ*. Let* $\{P_i : i \in \mathbb{I}\}$ *be the set of orbits of* $\mathrm{Aut}(\Gamma)$*, and let* $\mathscr{Q} = \{Q_j : j \in \mathbb{J}\}$ *be the set of those orbits of the stabilizer in* $\mathrm{Aut}(\Gamma)$ *of* Λ_0 *that are contained in* Λ_0*. Then necessary and sufficient conditions for the graph* Γ *to be lobe-transitive are:*

(1) For each lobe $\Lambda \in \mathscr{L}(\Gamma)$*, there exists an isomorphism* $\sigma_\Lambda : \Lambda_0 \to \Lambda$*.*

(2) For each $j \in \mathbb{J}$*, there exists a function* $\tau_j : V\Gamma \to \mathbb{N} \cup \{0, \aleph_0\}$ *such that*

(a) for all $v \in V\Gamma$*,*

$$\tau_j(v) = |\{\Lambda \in \mathscr{L}(\Gamma) : v \in \sigma_\Lambda(Q_j)\}| \tag{3.1}$$

and

(b) for each $i \in \mathbb{I}$*,* τ_j *is constant on* P_i *and is nonzero if and only if* $Q_j \subset P_i$*.*

Proof. (Necessity) Suppose that Γ is lobe-transitive. For each lobe $\Lambda \in \mathscr{L}(\Gamma)$, there is an automorphism $\overline{\sigma}_\Lambda \in \mathrm{Aut}(\Gamma)$ that maps the fixed lobe Λ_0 onto Λ. The restriction to Λ_0 of $\overline{\sigma}_\Lambda$ is an isomorphism $\sigma_\Lambda : \Lambda_0 \to \Lambda$ that satisfies condition (1).

For any lobe Λ, an automorphism $\alpha \in \mathrm{Aut}(\Gamma)$ is in the stabilizer of Λ if and only if $\overline{\sigma}_\Lambda^{-1} \alpha \overline{\sigma}_\Lambda$ is in the stabilizer of Λ_0. It follows that the partition $\{\sigma_\Lambda(Q_j) : j \in \mathbb{J}\}$ of $V\Lambda$ is the set of orbits of the stabilizer of Λ that are contained in Λ. Furthermore, since the stabilizer of Λ_0 is a subgroup of $\mathrm{Aut}(\Gamma)$, the partition $\{\sigma_\Lambda(Q_j) : j \in \mathbb{J}\}$ of $V\Lambda$

refines the partition $\{P_i \cap V\Lambda : i \in \mathbb{I}\}$. If for some indices i and j, the vertex v satisfies $v \in \sigma_\Lambda(Q_j) \subset P_i$, then for any lobe Θ, the vertex $\sigma_\Theta\sigma_\Lambda^{-1}(v)$ lies in $P_i \cap \sigma_\Theta(Q_j)$. This implies that, for all $j \in \mathbb{J}$, the function τ_j as given in Equation (3.1) is well-defined and constant on P_i.

Suppose that for an arbitrary index $i \in \mathbb{I}$, the vertex v lies in P_i. Since by Equation (3.1), $\tau_j(v)$ counts for each $j \in \mathbb{J}$ the number of lobes Λ such that v lies in $\sigma_\Lambda(Q_j)$, it follows that $\tau_j(v)$ is positive exactly when $\sigma_\Lambda^{-1}(v) \in Q_j \subset P_i$ holds, concluding the proof of condition (2.b).

(Sufficiency) Assume conditions (1) and (2). To prove that Γ is lobe-transitive, it suffices to prove that every isomorphism $\sigma_\Theta \colon \Lambda_0 \to \Theta$ is extendable to an automorphism of Γ. (Note that in this direction of the proof, σ_Θ is not presumed to be the restriction to Λ_0 of an automorphism $\overline{\sigma}_\Theta \in \mathrm{Aut}(\Gamma)$ but, in fact, it is.)

Fix a lobe Θ_0 and a vertex $v \in V\Lambda_0$ and let $w = \sigma_{\Theta_0}(v)$. For some $j \in \mathbb{J}$, the vertex v lies in Q_j, and so $w \in \sigma_{\Theta_0}(Q_j)$. Since both $\tau_j(v)$ and $\tau_j(w)$ are therefore positive, both v and w lie in the same orbit P_i of $\mathrm{Aut}(\Gamma)$ by condition (2.b). Furthermore, since τ_j is constant on P_i for each $j \in \mathbb{J}$, there exists a bijection β_j from the set of lobes Λ such that $v \in \sigma_\Lambda(Q_j)$ onto the set of lobes Θ such that $w \in \sigma_\Theta(Q_j)$. Let Λ_1 be a lobe in the former set, and let $\Theta_1 = \beta_j(\Lambda_1)$.

Although v and w lie in the images of the same orbit Q_j in lobes Λ_1 and Θ_1, respectively, the vertices $\sigma_{\Lambda_1}^{-1}(v)$ and $\sigma_{\Theta_1}^{-1}(w)$ need not be the same vertex of Λ_0. However, since both vertices lie in the same orbit Q_j of Λ_0, there exists an automorphism $\alpha \in \mathrm{Aut}(\Lambda_0)$ such that $\alpha\sigma_{\Lambda_1}^{-1}(v) = \sigma_{\Theta_1}^{-1}(w)$. Then $\sigma_{\Theta_1}\alpha\sigma_{\Lambda_1}^{-1}$ is an isomorphism from Λ_1 onto Θ_1 that maps v onto w and therefore agrees with σ_{Θ_0} at the vertex v common to Λ_0 and Λ_1.

The amalgamation of $\sigma_{\Theta_1}\alpha\sigma_{\Lambda_1}^{-1}$ with σ_{Θ_0} is an isomorphism from $\Lambda_0 \cup \Lambda_1$ to $\Theta_0 \cup \Theta_1$. By repeating this same technique, we can extend σ_{Θ_0} to all lobes adjacent to Λ_0 and then to all of their adjacent lobes and inductively to all of Γ. □

Example 3.3. Suppose that the lobes of Γ are copies of some biconnected, vertex-transitive graph and that every vertex of Γ is incident with exactly m lobes where $m \geq 2$. By Theorem 2.2, Γ is vertex-transitive. By Theorem 3.2, Γ is lobe-transitive, with $\mathscr{O}(\mathrm{Aut}(\Gamma))$ being the trivial partition (with just one big cell P_1). Also $|\mathbb{J}| = 1$ and $\tau_1(v) = m$ for all $v \in V\Gamma$.

Remark 3.4. There exists a "degenerate" family of lobe-transitive graphs Γ of connectivity 1 that have but a single cut-vertex. For some cardinal $\mathfrak{K} \geq 2$, consider a collection of $\mathfrak{K}$ copies of a biconnected graph Λ_0, and let $v_0 \in V\Lambda_0$. Let $\sigma_\Lambda \colon \Lambda_0 \to \Lambda$ be an isomorphism as in Theorem 3.2, and let $\sigma_\Lambda(v_0) = v_\Lambda$ for each copy Λ of Λ_0 in the collection. We obtain Γ by identifying v_0 and all the vertices v_Λ and naming the new amalgamated vertex w, which forms a singleton orbit $\{w\}$ of $\mathrm{Aut}(\Gamma)$. Clearly Γ is lobe-transitive and w is its unique cut-vertex. If Λ_0 has finite diameter, then Γ has zero ends (see [3]) when Λ_0 is finite and has $\mathfrak{K}$ ends when Λ_0 is infinite; if Λ_0 has infinite diameter, then Γ has at least $\mathfrak{K}$ ends. Other than the graphs just described, all *countable* lobe-transitive graphs of connectivity 1 are "tree-like" with $\aleph_0$ cut vertices and either 2 or $2^{\aleph_0}$ ends.

Theorem 3.5. *Let Λ_0 be any biconnected graph. Let $H \leq \mathrm{Aut}(\Lambda_0)$, let $\mathscr{Q} = \mathscr{O}(H) = \{Q_j : j \in \mathbb{J}\}$, and let $\mathscr{R} = \{R_k : k \in \mathbb{K}\}$ be a partition of $V\Lambda_0$ refined by $\mathscr{Q}$. For each $k \in \mathbb{K}$, let the function $\mu_k \colon \mathbb{J} \to \mathbb{N} \cup \{0, \aleph_0\}$ satisfy $\mu_k(j) > 0$ if and only if $Q_j \subseteq R_k$ and (to avoid the triviality of a single lobe) $\sum_{j \in \mathbb{J}} \mu_k(j) \geq 2$ for at least one $k \in \mathbb{K}$. Then there exists (up to isomorphism) a unique lobe-transitive graph Γ of connectivity* 1 *such that*

(1) for each lobe $\Lambda \in \mathscr{L}(\Gamma)$, there exists an isomorphism $\sigma_\Lambda \colon \Lambda_0 \to \Lambda$;

(2) for each vertex $v \in V\Gamma$ and each $j \in \mathbb{J}$, we have

$$\mu_k(j) = |\{\Lambda \in \mathscr{L}(\Gamma) : \sigma_\Lambda^{-1}(v) \in Q_j \subseteq R_k\}|.$$

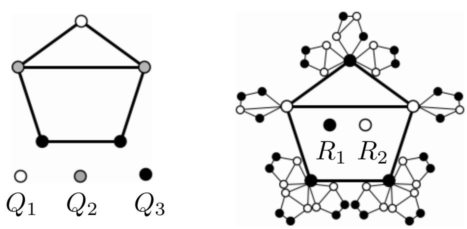
Proof. Let $\Lambda_0,\ H,\ \mathscr{Q}$, and μ_k be as postulated. Let $\Gamma_0 = \Lambda_0$ from which we construct Γ_1 as follows.

Let v be any vertex of Λ_0. For some j, k, it must hold that $v \in Q_j \subseteq R_k$, and so $\mu_k(j) > 0$. For each ℓ such that $Q_\ell \subseteq R_k$, we postulate the existence of $\mu_k(\ell)$ copies Λ of Λ_0 (including Λ_0 itself when $\ell = j$) such that, if $\sigma_\Lambda \colon \Lambda_0 \to \Lambda$ is an isomorphism, then some vertex in $\sigma_\Lambda(Q_\ell)$ is identified with the vertex v. The graph Γ_1 is produced by repeating this process for each vertex of Λ_0. We repeat this process starting at each vertex $w \in V\Gamma_1 \setminus V\Gamma_0$, the only notational change being that, if specifically $w \in \sigma_\Lambda(Q_{j'})$ for some $j' \in \mathbb{J}$, then we consider the subset $\sigma_\Lambda(Q_{j'})$ of $V\Lambda$ (instead of Q_j in Λ_0) to which w belongs. Thus we construct Γ_2.

Inductively, suppose that Γ_n has been constructed for some $n \geq 2$. Let $w \in V\Gamma_n \setminus V\Gamma_{n-1}$, and so $w \in \sigma_\Lambda(Q_m)$ holds for some $m \in \mathbb{J}$ and a unique lobe $\Lambda \in \mathscr{L}(\Gamma_n) \setminus \mathscr{L}(\Gamma_{n-1})$. Supposing that $Q_m \subseteq R_k$, we postulate the existence of $\mu_k(m)$ new copies of Λ_0 that share only the vertex w with Γ_n according to the above identification. In this way we construct Γ_{n+1}. Finally, let $\Gamma = \bigcup_{n=0}^{\infty} \Gamma_n$.

It remains only to prove that Γ so-constructed is lobe-transitive. Let Θ be any lobe of Γ. By the above construction, all lobes of Γ are pairwise isomorphic, and so there exists an isomorphism $\sigma_\Theta \colon \Lambda_0 \to \Theta$. Starting with $\Gamma'_0 = \Theta$ and by using the technique in the proof of Sufficiency in Theorem 3.2, one constructs a sequence $\Gamma'_0, \Gamma'_1, \ldots$ so that for all $n \in \mathbb{N}$, we have $\Gamma'_n \cong \Gamma_n$, and σ_Θ is extendable to an isomorphism from Γ_n to Γ'_n. Thus $\Gamma \cong \bigcup_{n=0}^{\infty} \Gamma'_n$, and σ_Θ can be extended to an automorphism of Γ. □

Example 3.6. In the notation of Theorem 3.5, let Λ_0 be the 5-cycle with one chord as shown in Figure 1(a), and let $H = \mathrm{Aut}(\Lambda)$, yielding the orbit partition $\{Q_1, Q_2, Q_3\}$ as indicated. Let $R_1 = Q_1 \cup Q_3$ and $R_2 = Q_2$, giving $\mathbb{J} = \{1, 2, 3\}$ and $\mathbb{K} = \{1, 2\}$. Define $\mu_1(1) = 3,\ \mu_1(3) = 1$, and $\mu_2(2) = 2$. Note that all other values of μ_1 and μ_2 must equal 0. Then Γ_1 is as seen in Figure 1(b).



(a) Λ_0 (b) Γ_1

Figure 1: Λ_0 and Γ_1 from Example 3.6.

The pairs of conditions in Theorems 3.2 and 3.5 may appear alike, but there is a notable difference between them. This occurs when the arbitrarily chosen subgroup $H \leq \mathrm{Aut}(\Lambda_0)$

of Theorem 3.5 is a *proper* subgroup of the stabilizer of Λ_0 in $\mathrm{Aut}(\Gamma)$, where Γ is the graph constructed from Λ_0 and the functions μ_k of Theorem 3.5. We illustrate this distinction with following example.

Example 3.7. Our initial lobe Λ_0 is a copy of K_4, with vertices labeled as in Figure 2(a), and so $\mathrm{Aut}(\Lambda_0) \cong \mathrm{Sym}(4)$ of order 24. We use Λ_0 to "build" the lobe-transitive graph Γ shown four times in Figure 2(b). The action on Λ_0 by the stabilizer of Λ_0 in $\mathrm{Aut}(\Gamma)$ is the 4-element group $\langle g_1 \rangle \times \langle g_2 \rangle$ whose generators have cycle representation $g_1 = (v_1, v_2)$ and $g_2 = (v_3, v_4)$. The shadings of the vertices in the four depictions of Γ in Figure 2(b) correspond respectively to the four different subgroups of $\langle g_1 \rangle \times \langle g_2 \rangle$ described below. For the sake of simplicity, we assume $\mathscr{R} = \mathscr{Q}$.

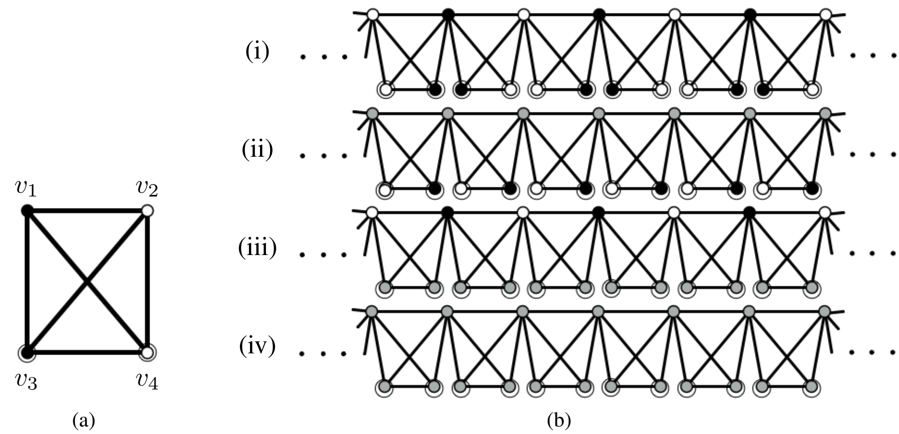


Figure 2: The clothesline graph.

(i) H is the trivial group $\{\iota\}$. Thus H induces four orbits $Q_j = \{v_j\}$ for $j \in \{1, 2, 3, 4\}$. The functions μ_k are then given by $\mu_k(j) = 2\delta_{j,k}$ for $k \in \{1, 2\}$ and $\mu_k(j) = \delta_{j,k}$ for $k \in \{3, 4\}$.

(ii) $H = \langle g_1 \rangle$. There are three orbits of H: $Q_1 = \{v_1, v_2\}$, $Q_2 = \{v_3\}$, and $Q_3 = \{v_4\}$, which give $\mu_1(1) = 2, \mu_2(2) = \mu_3(3) = 1$. All other functional values are zero.

(iii) $H = \langle g_2 \rangle$. Again there are three orbits of H but not the same ones: $Q_1 = \{v_1\}$, $Q_2 = \{v_2\}$, and $Q_3 = \{v_3, v_4\}$. This gives $\mu_k(j) = 2\delta_{j,k}$ for $k \in \{1, 2\}$ and $\mu_3(j) = \delta_{j,3}$.

(iv) $H = \langle g_1 \rangle \times \langle g_2 \rangle$. Now there are just two orbits: $Q_1 = \{v_1, v_2\}$ and $Q_2 = \{v_3, v_4\}$. Finally, $\mu_1(1) = 2$, $\mu_2(2) = 1$, and all other functional values are zero.

All four choices for H, the partition $\mathscr{Q}$, and the functions μ_k clearly yield the same lobe-transitive graph Γ of connectivity 1 by the construction of Theorem 3.5

4 Edge-transitivity

Lemma 4.1. *If Γ is an edge-transitive (respectively, arc-transitive) graph, then Γ is lobe-transitive and its lobes are also edge-transitive (respectively, arc-transitive).*

Proof. For $i = 1, 2$, let $\Theta_i \in \mathscr{L}(\Gamma)$, and let e_i be an edge (respectively, arc) of Θ_i. There exists an an automorphism $\varphi \in \mathrm{Aut}(\Gamma)$ such that $\varphi(e_1) = e_2$. Since φ maps cycles through e_1 onto cycles through e_2, φ must map Θ_1 onto Θ_2. If e_1 and e_2 lie in the same lobe Θ, then φ leaves Θ invariant, and so its restriction to Θ is an automorphism of Θ. □

Theorem 4.2. *Let Γ be a graph of connectivity 1 with more than one lobe, and let $\Lambda \in \mathscr{L}(\Gamma)$. Necessary and sufficient conditions for Γ to be edge-transitive are the following:*

(1) The lobes of Γ are edge-transitive.

(2) For each lobe $\Theta \in \mathscr{L}(\Gamma)$, there exists an isomorphism $\sigma_\Theta \colon \Lambda \to \Theta$.

(3) Exactly one of the following descriptions of Γ holds:

(a) Both Γ and Λ are vertex-transitive, in which case every vertex is incident with the same number ≥ 2 of lobes.

(b) The graph Γ is vertex-transitive but Λ is not vertex-transitive, in which case Λ is bipartite with bipartition $\{Q_1, Q_2\}$, and there exist constants $m_1, m_2 \in \mathbb{N} \cup \{\aleph_0\}$ such that for $j = 1, 2$ and all $v \in V\Gamma$, it holds that

$$m_j = |\{\Theta \in \mathscr{L}(\Gamma) : v \in \sigma_\Theta(Q_j)\}|.$$

(c) The graph Γ is not vertex-transitive, in which case Γ is bipartite with bipartition $\{P_1, P_2\}$ and there exist constants $m_1, m_2 \in \mathbb{N} \cup \{\aleph_0\}$, at least one of which is at least 2, such that for $i = 1, 2$, if $v \in P_i$, then

$$|\{\Theta \in \mathscr{L}(\Gamma) : v \in \sigma_\Theta(P_j \cap V\Lambda)\}| = m_j \delta_{i,j}.$$

Proof. (Sufficiency) Assume all the conditions in the hypothesis and let $e_1, e_2 \in E\Gamma$ be arbitrary edges in lobes Λ_1 and Λ_2, respectively. By condition (2), there exists an isomorphism $\sigma \colon \Lambda_1 \to \Lambda_2$. By condition (1), there exists an automorphism $\alpha \in \mathrm{Aut}(\Lambda_2)$ such that $e_2 = \alpha\sigma(e_1)$. Each of the three cases of condition (3) is seen to satisfy the hypothesis of Lemma 2.1, implying that $\alpha\sigma \colon \Lambda_1 \to \Lambda_2$ is extendable to an isomorphism of Γ mapping e_1 to e_2.

(Necessity) Suppose that Γ is an edge-transitive graph of connectivity 1. By Lemma 4.1, Γ is also lobe-transitive and its lobes are edge-transitive, proving condition (1). Condition (2), which establishes notation for the remainder of this proof, also follows from Lemma 4.1.

To prove (3), we continue the notation of Theorem 3.2 with $\mathbb{I}$ being the index set for the set of orbits of $\mathrm{Aut}(\Gamma)$ and $\mathbb{J}$ being the index set for the orbits of the stabilizer of Λ that are contained in Λ. Since both Γ and all of its lobes are edge-transitive, $|\mathbb{I}|$ and $|\mathbb{J}|$ equal either 1 or 2.

If both Γ and Λ are vertex-transitive, then $|\mathbb{I}| = |\mathbb{J}| = 1$, and for every vertex $v \in V\Gamma$, $\tau_1(v) = m$ holds for some $m \geq 2$. This is case (3.a).

Since any odd cycle in Γ would be contained in a lobe of Γ, it holds that Γ is bipartite if and only if every lobe is bipartite. If either Γ or Λ is not vertex-transitive, then each – and hence both – are bipartite, and the sides of the bipartitions (whether or not they are entire orbits of the appropriate automorphism group) are blocks of imprimitivity systems. Let $\{P_1, P_2\}$ be the bipartition of $V\Gamma$, and so $\{P_1 \cap V\Theta, P_2 \cap V\Theta\}$ is the bipartition of any lobe Θ. Equivalently, letting $\{Q_1, Q_2\}$ denote the bipartition of Λ, we have $P_i = \bigcup_{\Theta \in \mathscr{L}(\Gamma)} \sigma_\Theta(Q_i)$ for $i = 1, 2$.

Suppose that Γ is vertex-transitive but Λ is not, and so $|\mathbb{I}| = 1$ and $|\mathbb{J}| = 2$. By Theorem 2.2, there exist constants $m_1, m_2 \in \mathbb{N} \cup \{\aleph_0\}$ such that for all $v \in V\Gamma$ and $j \in \mathbb{J}$, we have $m_j = \tau_j(v) = |\{\Theta \in \mathscr{L}(\Gamma) : v \in \sigma_\Theta(Q_j)\}|$.

Finally, suppose that Γ is not vertex-transitive, and so P_1 and P_2 are the orbits of $\mathrm{Aut}(\Gamma)$. Also, Λ is bipartite with bipartition $\{Q_1, Q_2\}$, where $Q_i = P_i \cap V\Lambda$. As no automorphism of Γ swaps P_1 with P_2, no automorphism of Γ swaps Q_1 with Q_2 (even though Λ may be vertex-transitive!). Hence $|\mathbb{I}| = |\mathbb{J}| = 2$. Since Γ is lobe-transitive, it follows now from the "necessity" argument of Theorem 3.2 that, for $j = 1, 2$, the function τ_j satisfies the condition $\tau_j(v) > 0$ if and only if $v \in P_j$. That means that there exist constants $m_1, m_2 \in \mathbb{N} \cup \{\aleph_0\}$, at least one of which is greater than 1, such that, if $v \in P_i$, then $\tau_j(v) = m_j \delta_{i,j}$. □

Example 4.3. Suppose in the notation of Theorem 4.2 that Γ is edge-transitive and Λ is the complete bipartite graph $K_{s,t}$ with $|Q_1| = s$ and $|Q_2| = t$. Suppose that every vertex of Γ is incident with exactly two lobes isomorphic to Λ. If $s = t$, then both Γ and Λ are vertex-transitive, and we have case (3.a) of the theorem. If $s \neq t$ and every vertex lies in one image of Q_1 and one image of Q_2, then we have the situation of case (3.b). If again $s \neq t$ but each vertex lies in either two images of Q_1 or two images of Q_2, then we have the situation described in case (3.c).

Remark 4.4. With regard to Example 4.3, we note that having $s = t$ does not assure vertex-transitivity of edge-transitive bipartite graphs. There exist edge-transitive, non-vertex-transitive, finite bipartite graphs where the two sides of the bipartition have the same size. Such graphs are called *semisymmetric*. The smallest such graph, on 20 vertices with valence 4, was found by J. Folkman [2], who also found several infinite families of semisymmetric graphs. Many more such families as well as forbidden values for s were determined by A. V. Ivanov [4].

Example 4.5. This simple example illustrates how the converse of Lemma 4.1 is false, even though the lobes themselves may be highly symmetric. Let Γ be a graph of connectivity 1 whose lobes are copies of the Petersen graph (which is 3-arc-transitive!). For each lobe Λ, let $(V\Lambda)_1$ and $(V\Lambda)_2$ denote the vertex sets of disjoint 5-cycles indexed "consistently," i.e., if Λ and Θ share a vertex v, then $v \in (V\Lambda)_i \cap (V\Theta)_i$ for $i = 1$ or $i = 2$. (Observe that Γ is not bipartite.) For $i = 1, 2$ define $P_i = \bigcup_{\Theta \in \mathscr{L}(\Gamma)} (V\Theta)_i$, and suppose that each vertex in P_i belongs to exactly m_i lobes. The graph Γ is lobe-transitive by Theorem 3.2, and Γ is both vertex- and edge-transitive when $m_1 = m_2$, but Γ is neither vertex- nor edge-transitive when $m_1 \neq m_2$.

5 Arc-transitivity

Theorem 5.1. *Let* Γ *be a graph of connectivity* 1*. Necessary and sufficient conditions for* Γ *to be arc-transitive are the following:*

(1) The lobes of Γ *are arc-transitive.*

(2) The lobes of Γ *are pairwise isomorphic.*

(3) All vertices of Γ *are incident with the same number of lobes.*

Proof. (Necessity) Suppose that Γ is arc-transitive. Conditions (1) and (2) follow from Lemma 4.1. Since arc-transitivity implies vertex-transitivity, condition (3) holds.

(Sufficiency) Assume that the three conditions hold. For $k = 1, 2$, let a_k be an arc of Γ, and let Θ_k be the lobe containing a_k. By condition (2), there exists an isomorphism $\sigma\colon \Theta_1 \to \Theta_2$. By condition (1), there exists an automorphism $\alpha \in \mathrm{Aut}(\Theta_2)$ such that $\alpha\sigma(a_1) = a_2$. By condition (3), the functions τ_j of Equation (2.1) are constant on $V\Gamma$. (In fact, since the lobes are vertex-transitive, there is only one such function.) It now follows from Lemma 2.1 that $\alpha\sigma$ is extendable to all of Γ. □

Remark 5.2. If conditions (1) and (3) of Theorem 5.1 were replaced by *the lobes are edge-transitive and* Γ *is vertex-transitive*, the *sufficiency* argument would fail. There exist finite graphs [1] and countably infinite graphs with polynomial growth rate [7] that are vertex- and edge-transitive but not arc-transitive. Let Λ denote such a graph, and consider a graph Γ whose lobes are isomorphic to Λ with the same number of lobes incident with every vertex. Then Γ itself is vertex- and edge-transitive but not arc-transitive.

The following proposition is elementary.

Proposition 5.3. *For all $k \geq 2$, the only k-arc-transitive graphs of connectivity* 1 *are trees of constant valence.*

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Bipartite edge-transitive bi-p-metacirculants*

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Abstract

A graph is a bi-Cayley graph over a group if the group acts semiregularly on the vertex set of the graph with two orbits. Let G be a non-abelian metacyclic p-group for an odd prime p. In this paper, we prove that if G is a Sylow p-subgroup in the full automorphism group $\mathrm{Aut}(\Gamma)$ of a graph Γ, then G is normal in $\mathrm{Aut}(\Gamma)$. As an application, we classify the half-arc-transitive bipartite bi-Cayley graphs over G of valency less than $2p$, while the case for valency 4 was given by Zhang and Zhou in 2019. It is further shown that there are no semisymmetric or arc-transitive bipartite bi-Cayley graphs over G of valency less than p.

Keywords: Bi-Cayley graph, half-arc-transitive graph, metacyclic group.

Math. Subj. Class.: 05C10, 05C25, 20B25

1 Introduction

All graphs considered in this paper are finite, connected, simple and undirected. For a graph Γ, we use $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. A graph Γ is said to be *vertex-transitive*, *edge-transitive* or *arc-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$, $E(\Gamma)$ or $A(\Gamma)$ respectively, *semisymmetric* if it is edge-transitive but not vertex-transitive, and *half-arc-transitive* if it is vertex-transitive, edge-transitive, but not arc-transitive.

Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular. A group G is *metacyclic* if it has a normal subgroup N such that both N and G/N are cyclic.

Let Γ be a graph with $G \leq \mathrm{Aut}(\Gamma)$. Then Γ is called a *Cayley graph* over G if G is regular on $V(\Gamma)$ and a *bi-Cayley graph* over G if G is semiregular on $V(\Gamma)$ with two orbits.

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In particular, if G is normal in $\mathrm{Aut}(\Gamma)$, the Cayley graph or the bi-Cayley graph Γ is called *a normal Cayley graph* or *a normal bi-Cayley graph* over G, respectively.

Determining the automorphism group of a graph is fundamental in algebraic graph theory, but very difficult in general. If Γ is a connected normal Cayley graph over a group G, then $\mathrm{Aut}(\Gamma)$ is determined by Godsil [27], and if Γ is a connected normal bi-Cayley graph over G, then $\mathrm{Aut}(\Gamma)$ is also determined by Zhou and Feng [55]. Thus a natural problem is to determine normality of Cayley graphs or bi-Cayley graphs over groups.

The normality of Cayley graphs over cyclic group of order a prime and over group of order twice a prime was solved by Alspach [1] and Du et al. [19], respectively. Dobson [14] determined all non-normal Cayley graphs over group of order a product of two distinct primes, and Dobson and Witte [16] determined all non-normal Cayley graphs over group of order a prime square. Dobson and Kovács [15] determined the full automorphism groups of Cayley graphs over elementary abelian group of rank 3. However, it seems still very difficult to obtain normality of Cayley graphs for general valencies. On the other hand, many results on the normality of Cayley graphs with small valencies were obtained, and for example, one may refer to [20, 21, 22] for finite non-abelian simple groups and to [4, 23, 26, 51, 54] for solvable groups. Due to nice properties on automorphism groups of non-abelian p-groups, the normality of Cayley graphs with general valencies over certain non-abelian p-groups was obtained. A connected Cayley graph or bi-Cayley graph over a non-abelian metacyclic p-group, for an odd prime p, is called a *p-metacirculant* or a *bi-p-metacirculant*, respectively. Li and Sim [34] proved that a p-metacirculant Γ is normal except a special case when the non-abelian metacyclic p-group is a Sylow p-subgroup of $\mathrm{Aut}(\Gamma)$, and Wang and Feng [50] proved that this special case cannot occur. In this paper we prove the following theorem.

Theorem 1.1. *Let Γ be a connected bipartite bi-p-metacirculant over a non-abelian metacyclic p-group G. If G is a Sylow p-subgroup of $\mathrm{Aut}(\Gamma)$, then G is normal in $\mathrm{Aut}(\Gamma)$.*

It is well-known that Cayley graphs play an important role in the study of symmetry of graphs. However, graphs with various symmetries can be constructed by bi-Cayley graphs. The smallest trivalent semisymmetric graph is the Gray graph [6], which is a bi-Cayley graph over a non-abelian metacyclic group of order 27, and infinite semisymmetric graphs were constructed in [17, 18, 37]. Boben et al. [5] studied properties of cubic bi-Cayley graphs over cyclic groups and the configurations arising from these graphs. Kovács et al. [31] gave a description of arc-transitive one-matching bi-Cayley graphs over abelian groups. All cubic vertex-transitive bi-Cayley graphs over cyclic groups, abelian groups or dihedral groups were determined in [39, 52, 54]. Recently, Conder et al. [11] investigated bi-Cayley graphs over abelian groups, dihedral groups and metacyclic p-groups, and using these results, a complete classification of connected trivalent edge-transitive graphs of girth at most 6 was obtained. Furthermore, Qin et al. [41] classified connected edge-transitive bi-p-metacirculants of valency p, and as an application of Theorem 1.1, we prove that there are no such graphs with valency less than p.

Theorem 1.2. *For any odd prime p, there are no connected arc-transitive or semisymmetric bipartite bi-p-metacirculants of valency less than p.*

In 1966, Tutte [46] initiated an investigation of half-arc-transitive graphs by showing that a vertex- and edge-transitive graph with odd valency must be arc-transitive. A few years later, in order to answer Tutte's question on the existence of half-arc-transitive graphs of even valency, Bouwer [7] constructed a $2k$-valent half-arc-transitive graph for every $k \geq 2$. One of the standard problems in the study of half-arc-transitive graphs is to classify such graphs for certain orders. Let p be a prime. It is well known that there are no half-arc-transitive graphs of order p or p^2, and no such graphs of order $2p$ by Cheng and Oxley [8]. Alspach and Xu [2] classified half-arc-transitive graphs of order $3p$ and Kutnar et al. [33] classified such graphs of order $4p$. Despite all of these efforts, however, further classifications of half-arc-transitive graphs with general valencies seem to be very difficult, and special attention has been paid to the study of half-arc-transitive graphs with small valencies, which were extensively studied from different perspectives over decades by many authors; see [3, 9, 10, 24, 25, 29, 32, 35, 38, 40, 43, 47, 48, 49] for example.

The smallest half-arc-transitive graph constructed in Bouwer [7] is a bi-Cayley graph over the non-abelian metacyclic group of order 27 with exponent 9. Zhang and Zhou [56] proved that a half-arc-transitive bi-Cayley graph over cyclic group has valency at least 6, and this was extended to abelian groups by Conder et al. [11]. In fact, half-arc-regular bi-Cayley graphs of valency 6, over cyclic groups, were classified in [56], and two infinite families of bipartite tetravalent half-arc-transitive bi-p-metacirculants of order p^3 were constructed in [11], of which one is Cayley and the other is not Cayley. Furthermore, Zhang and Zhou [53] classified tetravalent half-arc-transitive bi-p-metacirculants, and all these graphs are bipartite. This was the main motivation for the research leading to Theorem 1.3, namely the classification of bipartite half-arc-transitive bi-p-metacirculants of valency less than $2p$. It was also motivated in part by the classification of half-arc-transitive p-metacirculants of valency less than $2p$, given by Li and Sim [35].

For a positive integer n, denote by $\mathbb{Z}_n$ the cyclic group of order n, as well as the ring of integers modulo n, and by $\mathbb{Z}_n^*$ the multiplicative group of the ring $\mathbb{Z}_n$ consisting of numbers coprime to n.

Theorem 1.3. *Let p be an odd prime and let Γ be a connected bipartite bi-p-metacirculant of valency $2k$ with $k < p$ over a non-abelian metacyclic p-group G. Then Γ is half-arc-transitive if and only if*

$$k \geq 2, \quad k \mid (p-1), \quad G \cong G_{\alpha,\beta,\gamma} \quad \textit{and} \quad \Gamma \cong \Gamma^{\pm}_{m,k,\ell},$$

where $0 < \gamma < \alpha \leq \beta + \gamma$, $m \in \mathbb{Z}^*_{p^{\alpha-\gamma}}$, $0 \leq \ell < k$ with $\frac{k}{(k,\ell)} \mid \frac{(p-1)}{2}$, and $\mathrm{Aut}(\Gamma^{\pm}_{m,k,\ell}) \cong (G_{\alpha,\beta,\gamma} \rtimes \mathbb{Z}_k).\mathbb{Z}_2$.

The groups $G_{\alpha,\beta,\gamma}$ and graphs $\Gamma^{\pm}_{m,k,\ell}$ above are defined in Equation (2.1) and Equation (4.3). By Zhang and Zhou [53], the graphs $\Gamma^{\pm}_{m,2,\ell}$ can be Cayley or non-Cayley for certain values m and ℓ, and this implies that the extensions $(G_{\alpha,\beta,\gamma} \rtimes \mathbb{Z}_2).\mathbb{Z}_2$ above can be split or non-split.

2 Background results

Let G be a finite metacyclic p-group. Lindenberg [36] proved that the automorphism group of G is a p-group when G is non-split. The following proposition describes the automorphism group of the remaining case when G is split. It is easy to show that every non-abelian split metacyclic p-group G for an odd prime p has the following presentation:

$$G_{\alpha,\beta,\gamma} = \langle a, b \mid a^{p^\alpha} = 1,\ b^{p^\beta} = 1,\ b^{-1}ab = a^{1+p^\gamma} \rangle, \tag{2.1}$$

where α, β, γ are positive integers such that $0 < \gamma < \alpha \leq \beta + \gamma$. Li and Sim characterized the automorphism group $\mathrm{Aut}(G_{\alpha,\beta,\gamma})$ of the group $G_{\alpha,\beta,\gamma}$.

Proposition 2.1 ([35, Theorem 2.8]). *For an odd prime p, we have*

$$|\mathrm{Aut}(G_{\alpha,\beta,\gamma})| = (p-1)p^{\min(\alpha,\beta)+\min(\beta,\gamma)+\beta+\gamma-1}.$$

Moreover, all Hall p'-subgroups of $\mathrm{Aut}(G_{\alpha,\beta,\gamma})$ are conjugate and isomorphic to $\mathbb{Z}_{p-1}$. In particular, the map $\theta\colon a \mapsto a^{\varepsilon},\ b \mapsto b$ induces an automorphism of $G_{\alpha,\beta,\gamma}$ of order $p-1$, where ε is an element of order $p-1$ in $\mathbb{Z}_{p^\alpha}^*$.

A p-group G is said to be *regular* if for any $x, y \in G$ there exist $d_i \in \langle x, y\rangle'$, $1 \leq i \leq n$, for some positive integer n such that $x^p y^p = (xy)^p \prod_{i=1}^{n} d_i^p$. If G is metacyclic, then the derived subgroup G' is cyclic, and hence G is regular by [30, Kapitel III, 10.2 Satz]. For regular p-groups, the following proposition holds by [30, Kapitel III, 10.8 Satz].

Proposition 2.2. *Let G be a metacyclic p-group for an odd prime p. If $|G'| = p^n$, then for any $m \geq n$, we have*

$$(xy)^{p^m} = x^{p^m} y^{p^m},$$

for any $x, y \in G$.

Remark 2.3. For the non-abelian split metacyclic group $G_{\alpha,\beta,\gamma}$ given in Equation (2.1), we have $|G'_{\alpha,\beta,\gamma}| = p^{\alpha-\gamma}$ and $\alpha - \gamma \leq \beta$, and by Proposition 2.2, if $(p, m) = 1$ then $o(b^m a^n) = \max\{o(a^n), p^\beta\}$, and if $\beta < \alpha$ and $p \mid n$ then $o(b^m a^n) \leq p^{\alpha-1}$.

For a finite group G, $N \leq G$ means that N is a subgroup of G, and $N < G$ means that N is a proper subgroup of G. The following proposition lists non-abelian simple groups having a proper subgroup of index prime-power order.

Proposition 2.4 ([28, Theorem 1]). *Let T be a non-abelian simple group with $H < T$, and let $|T : H| = p^a$ for a prime p. Then one of the following holds.*

(1) $T = \mathrm{PSL}(n, q)$ and H is the stabilizer of a line or hyperplane. Furthermore, $|T : H| = (q^n - 1)/(q - 1) = p^a$ and n must be a prime.

(2) $T = \mathrm{A}_n$ and $H \cong \mathrm{A}_{n-1}$ with $n = p^a$.

(3) $T = \mathrm{PSL}(2, 11)$ and $H \cong \mathrm{A}_5$.

(4) $T = \mathrm{M}_{23}$ and $H \cong \mathrm{M}_{22}$ or $T = \mathrm{M}_{11}$ and $H \cong \mathrm{M}_{10}$.

(5) $T = \mathrm{PSU}(4, 2) \cong \mathrm{PSp}(4, 3)$ and H is the parabolic subgroup of index 27.

For a group G and a prime p, denote by $\mathrm{O}_p(G)$ the largest normal p-subgroup of G, and by $\mathrm{O}_{p'}(G)$ the largest normal subgroup of G whose order is not divisible by p. The next proposition is about transitive permutation groups of prime-power degree.

Proposition 2.5 ([34, Lemma 2.5]). *Let p be a prime, and let A be a transitive permutation group with p-power degree. Let B be a nontrivial subnormal subgroup of A. Then B has a proper subgroup of p-power index, and $\mathrm{O}_{p'}(B) = 1$. In particular, $\mathrm{O}_{p'}(A) = 1$.*

It is well-known that $\mathrm{GL}(d, q)$ has a cyclic group of order $q^d - 1$, the so called Singer-cycle subgroup, which also induces a cyclic subgroup of $\mathrm{PSL}(d, q)$.

Proposition 2.6 ([30, Kapitel II, 7.3 Satz]). *The group $G = \mathrm{GL}(d, q)$ contains a cyclic subgroup of order $q^d - 1$, and it induces a cyclic subgroup of order $\frac{q^d-1}{(q-1)(q-1,d)}$ of $\mathrm{PSL}(d, q)$.*

Let G and E be two groups. We call an extension E of G by N a *central extension* if N lies in the center of E and $E/N \cong G$, and if E is further perfect, that is, the derived group $E' = E$, we call E a *covering group* of G. Schur [42] proved that for every non-abelian simple group G there is a unique maximal covering group M such that every covering group of G is a factor group of M (also see [30, Chapter 5, Section 23]). This group M is called the *full covering group* of G, and the center of M is the *Schur multiplier* of G, denoted by $\mathrm{M}(G)$. For a group G, we denote by $\mathrm{Out}(G)$ the outer automorphism group of G, that is, $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$, where $\mathrm{Inn}(G)$ is the inner automorphism group of G induced by conjugation.

The following proposition is about outer automorphism group and Schur multiplier of a non-abelian simple group having a proper subgroup of prime-power index.

Proposition 2.7 ([34, Lemma 2.3]). *Let p be an odd prime and let T be a non-abelian simple group that has a subgroup H of index $p^\ell > 1$. Then*

(1) $p \nmid |\mathrm{M}(T)|$;

(2) either $p \nmid |\mathrm{Out}(T)|$, or $T \cong \mathrm{PSL}(2, 8)$ and $p^\ell = 3^2$.

A group G is said to be *a central product* of its subgroups $H_1, \ldots, H_n$ $(n \geq 2)$ if $G = H_1 \cdots H_n$ and for any $i \neq j$, H_i and H_j commute elementwise. A group G is called *quasisimple* if $G' = G$ and $G/Z(G)$ is a non-abelian simple group, where $Z(G)$ is the centralizer of G. A group G is called *semisimple* if $G' = G$ and $G/Z(G)$ is a direct product of non-abelian simple groups. Clearly, a quasisimple group is semisimple.

Proposition 2.8 ([45, Theorem 6.4]). *A central product of two semisimple groups is also semisimple. Any semisimple group can be decomposed into a central product of quasisimple groups, and this set of quasisimple groups is uniquely determined.*

A subnormal quasisimple subgroup of a group G is called a *component* of G. By [45, 6.9(iv), p. 450], any two distinct components of G commute elementwise, and by Proposition 2.8, the product of all components of G is semisimple, denoted by $E(G)$, which is characteristic in G. We use $F(G)$ to denote the *Fitting subgroup* of G, that is, $F(G) = \mathrm{O}_{p_1}(G) \times \mathrm{O}_{p_2}(G) \times \cdots \times \mathrm{O}_{p_t}(G)$, where $p_1, p_2, \ldots, p_t$ are the distinct prime factors of $|G|$. Set $F^*(G) = F(G)E(G)$ and call $F^*(G)$ the *generalized Fitting subgroup* of G. The following is one of the most significant properties of $F^*(G)$. For a group G and a subgroup H of G, denote by $\mathrm{C}_G(H)$ the centralizer of H in G.

Proposition 2.9 ([45, Theorem 6.11]). *For any finite group G, we have*

$$\mathrm{C}_G(F^*(G)) \leq F^*(G).$$

An *action* of a group G on a set Ω is a homomorphism from G to the symmetric group S_Ω on Ω. We denote by $\Phi(G)$ the *Frattini subgroup* of G, that is, the intersection of all maximal subgroups of G. Note that for a prime p, $\mathrm{O}_p(G)$ is a p-group and $\mathrm{O}_p(G)/\Phi(\mathrm{O}_p(G))$ is an elementary abelian p-group. Thus, $\mathrm{O}_p(G)/\Phi(\mathrm{O}_p(G))$ can be viewed as a vector space over the field $\mathbb{Z}_p$. The following lemma considers a natural action of a group G on the vector space $\mathrm{O}_p(G)/\Phi(\mathrm{O}_p(G))$.

Proposition 2.10 ([50, Lemma 2.9]). *For a finite group G and a prime p, let $H = \mathrm{O}_p(G)$ and $V = H/\Phi(H)$. Then G has a natural action on V, induced by conjugation via elements of G on H. If $\mathrm{C}_G(H) \leq H$, then H is the kernel of this action of G on V.*

Let a group T act on two sets Ω and Σ, and these two actions are *equivalent* if there is a bijection $\lambda\colon \Omega \mapsto \Sigma$ such that

$$(\alpha^t)^\lambda = (\alpha^\lambda)^t \text{ for all } \alpha \in \Omega \text{ and } t \in T.$$

When the two actions above are transitive, there is a simple criterion on whether or not they are equivalent.

Proposition 2.11 ([13, Lemma 1.6B]). *Assume that a group T acts transitively on the two sets Ω and Σ, and let W be a stabilizer of a point in the first action. Then the actions are equivalent if and only if W is the stabilizer of some point in the second action.*

For a group G and two subgroups H and K of G, we consider the actions of G on the right cosets of H and K by right multiplication. The stabilizers of Hx and Ky are H^x and K^y, respectively. By Proposition 2.11, these two right multiplication actions are equivalent if and only if H and K are conjugate in G.

3 Automorphisms of bipartite bi-p-metacirculants

Let Γ_N be the *quotient graph* of a graph Γ with respect to $N \leq \mathrm{Aut}(\Gamma)$, that is, the graph having the orbits of N as vertices with two orbits O_1, O_2 adjacent in Γ_N if and only if there exist some $u \in O_1$ and $v \in O_2$ such that $\{u, v\}$ is an edge in Γ. Denote by $[O_1]$ the induced subgraph of Γ by O_1, and by $[O_1, O_2]$ the subgraph of $[O_1 \cup O_2]$ with edge set $\{\{u, v\} \in E(\Gamma) \mid u \in O_1, v \in O_2\}$.

Proof of Theorem 1.1. Let G a non-abelian metacyclic p-group of order p^n for an odd prime p and a positive integer n, and let Γ be a connected bipartite bi-p-metacirculant over G. Set $A = \mathrm{Aut}(\Gamma)$, and let $G \in \mathrm{Syl}_p(A)$, where $\mathrm{Syl}_p(A)$ is the set of Sylow p-subgroups of A. To finish the proof, it suffices to show that $G \trianglelefteq A$.

Let W_0 and W_1 be the two parts of the bipartite graph Γ. Then $\{W_0, W_1\}$ is a complete block system of A on $V(\Gamma)$ with $|W_0| = |W_1| = |G| = p^n$. Let A^* be the kernel of A on $\{W_0, W_1\}$, that is, the subgroup of A fixing W_0 and W_1 setwise. Then $A^* \trianglelefteq A$, $A/A^* \leq \mathbb{Z}_2$ and $\mathrm{Syl}_p(A) = \mathrm{Syl}_p(A^*)$. It follows that $G \in \mathrm{Syl}_p(A^*)$. Noting that $|G| = p^n$, we have $p^n \bigm| |A|$ and $p^{n+1} \nmid |A|$, that is, $p^n \parallel |A|$. The group G has exactly two orbits, that is, W_0 and W_1, and G is regular on both W_0 and W_1. By Frattini argument [30, Kapitel I, 7.8 Satz], $A^* = GA^*_u$ for any $u \in V(\Gamma)$, implying that A^*_u is a p'-group. Clearly, $A_u = A^*_u$, and so A_u is also a p'-group.

Let K be the kernel of A^* acting on W_0. Then $K \leq A^*_v$ for any $v \in W_0$, and $K \trianglelefteq A^*$. The orbits of K on W_1 have the same length, and so it is a divisor of p^n. It follows that if $K \neq 1$ then $p \bigm| |K|$, which is impossible because A^*_v is a p'-group. Thus, A^* acts faithfully on W_0 (resp. W_1). Since Sylow p-subgroups of A are conjugate, every p-subgroup of A is semiregular on both W_0 and W_1.

Claim 1. Any minimal normal subgroup N of A^* is abelian.

We argue by contradiction and we suppose that N is non-abelian. Then $N \cong T_1 \times \cdots \times T_k$ with $k \geq 1$, where $T_i \cong T$ is a non-abelian simple group. By Proposition 2.5, $p \bigm| |N|$ and so $p \bigm| |T_i|$ for each $1 \leq i \leq k$. Since $G \in \mathrm{Syl}_p(A^*)$, we have $G \cap N \in \mathrm{Syl}_p(N)$, and hence $G \cap N = P_1 \times \cdots \times P_k$ for some $P_i \in \mathrm{Syl}_p(T_i)$, where $P_i \neq 1$ for each $1 \leq i \leq k$. Since G is metacyclic and $G \cap N \trianglelefteq G$, $G \cap N$ is metacyclic and this implies $k \leq 2$.

Set $\Omega = \{T_1, \ldots, T_k\}$ and write $B = \mathrm{N}_{A^*}(T_1)$. Considering the conjugation action of A^* on Ω, we have $B \trianglelefteq A^*$ as $k \leq 2$, and hence $A^*/B \lesssim \mathrm{S}_2$, forcing $B \trianglelefteq A^*$. Thus, $\mathrm{Syl}_p(B) = \mathrm{Syl}_p(A^*)$ and so B is transitive on both W_0 and W_1.

Let Γ_{T_1} be the quotient graph of Γ with respect to T_1. Since $T_1 \trianglelefteq B$, all orbits of T_1 on W_0 have the same length, and the length must be a p-power as $|W_0| = p^n$. Since each p-subgroup is semiregular, this length is the order of a Sylow p-subgroup of T_1. Similarly, all orbits of T_1 on W_1 have the same length and it is also the order of a Sylow p-subgroup of T_1. Thus, $V(\Gamma_{T_1}) = \{\Delta_1, \ldots, \Delta_s, \Delta_1', \ldots, \Delta_s'\}$, the set of all orbits of T_1, with $W_0 = \Delta_1 \cup \cdots \cup \Delta_s$ and $W_1 = \Delta_1' \cup \cdots \cup \Delta_s'$. Furthermore, for any $1 \leq i, j \leq s$ we have $|\Delta_i| = |\Delta_j'| = p^m$ for some $1 \leq m \leq n$, and hence $s = p^{n-m}$. Since $T_1 \trianglelefteq B$, B has a natural action on $V(\Gamma_{T_1})$ and let K be the kernel of this action. Clearly, $T_1 \leq K$. Recall that $p \nmid |A_u|$ for any $u \in V(\Gamma)$. Then $p \nmid |(T_1)_u|$, and by Guralnick [28, Corollary 2], T_1 is 2-transitive on each Δ_i or Δ_i'. Since $p^n \parallel |A^*|$, we have $p^n \parallel |B|$, implying that $p^m \parallel |K|$. Since $(T_1)_u$ is a proper subgroup of T_1 of index p-power, Proposition 2.7 implies that either $T_1 = \mathrm{PSL}(2, 8)$ with $p^m = 3^2$, or $p \nmid |\mathrm{Out}(T_1)|$. To finish the proof of Claim 1, we will obtain a contradiction for both cases.

Case 1. $T_1 = \mathrm{PSL}(2, 8)$ with $p^m = 3^2$.

In this case, $|\Delta_i| = |\Delta_j'| = 9$. If $s = 1$ then $|G| = p^m = 3^2$, contradicting that G is non-abelian. Thus $s \geq 2$ and $s = 3^{n-2}$. By Atlas [12], $\mathrm{PSL}(2, 8)$ has only one conjugate class of subgroups of index 9, and by Proposition 2.11, T_1 acts equivalently on Δ_i and Δ_j'.

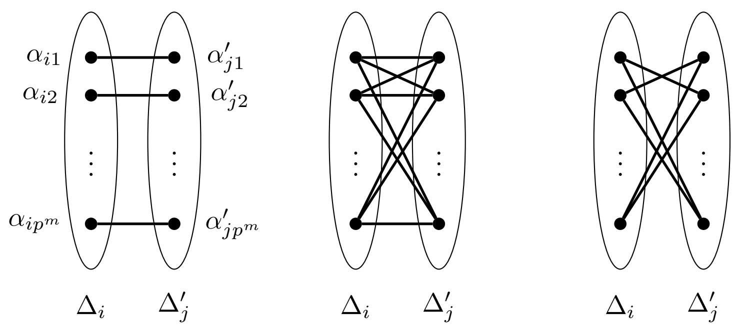

Figure 1: The subgraphs $[\Delta_i, \Delta_j']$.

Set $\Delta_i = \{\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{i9}\}$ and $\Delta_j' = \{\alpha_{j1}', \alpha_{j2}', \ldots, \alpha_{j9}'\}$ for $1 \leq i, j \leq 3^{n-2}$. Recall that T_1 is 2-transitive on Δ_i and Δ_j'. Since T_1 acts equivalently on Δ_i and Δ_j', by Proposition 2.11, we may assume that $(T_1)_{\alpha_{i\ell}} = (T_1)_{\alpha_{j\ell}'}$ for any $1 \leq i, j \leq 3^{n-2}$ and $1 \leq \ell \leq 3^2$. The subgraph $[\Delta_i, \Delta_j']$ is either a null graph, or one of the three graphs in Figure 3 because $(T_1)_{\alpha_{i\ell}} = (T_1)_{\alpha_{j\ell}'}$ acts transitively on both $\Delta_i \setminus \{\alpha_{i\ell}\}$ and $\Delta_j' \setminus \{\alpha_{j\ell}'\}$.

The three graphs have edge sets $\{\{\alpha_{i\ell}, \alpha'_{j\ell}\} \mid 1 \leq \ell \leq 3^2\}$, $\{\{\alpha_{ik}, \alpha'_{j\ell}\} \mid 1 \leq k, \ell \leq 3^2\}$ or $\{\{\alpha_{ik}, \alpha'_{j\ell}\} \mid 1 \leq k, \ell \leq 3^2, k \neq \ell\}$, respectively.

For any $g \in \mathrm{S}_9$, define a permutation σ_g on $V(\Gamma)$ by $(\alpha_{i\ell})^{\sigma_g} = \alpha_{i\ell^g}$ and $(\alpha'_{j\ell})^{\sigma_g} = \alpha'_{j\ell^g}$ for any $1 \leq i, j \leq 3^{n-2}$ and $1 \leq \ell \leq 3^2$. Then σ_g fixes each Δ_i and Δ'_j, and permutes the elements of Δ_i and Δ'_j in the 'same way' for each $1 \leq i, j \leq 3^{n-2}$. Since $[\Delta_i, \Delta'_j]$ is either a null graph, or one graph in Figure 3, σ_g induces an automorphism of $[\Delta_i, \Delta'_j]$, for all $1 \leq i, j \leq 3^{n-2}$. Also σ_g induces automorphisms of $[\Delta_i]$ and $[\Delta'_j]$ for all $1 \leq i, j \leq 3^{n-2}$ because $[\Delta_i]$ and $[\Delta'_j]$ have no edges (Γ is bipartite). It follows that $\sigma_g \in \mathrm{Aut}(\Gamma)$. Thus, $L := \{\sigma_g \mid g \in \mathrm{S}_9\} \leq \mathrm{Aut}(\Gamma)$ and $L \cong \mathrm{S}_9$.

Clearly, $L \leq A^*$. If $L \nleq B$, there exists $x \in L$ such that $T_1^x \neq T_1$, and hence $N \cong T_1 \times T_2$ with $k = 2$ and $T_1^x = T_2$, which implies that T_1 and T_2 have the same orbits because x fixes each orbit of T_1, contradicting that Sylow p-subgroups of N are semiregular. Thus $L \leq B$. Recall that K is the kernel of B acting on $V(\Gamma_{T_1})$ and $3^2 = p^m \parallel |K|$. Since L fixes each orbit of T_1 and $3^3 \bigm| |L|$, we have $L \leq K$ and $3^3 \bigm| |K|$, a contradiction.

Case 2. $p \nmid |\mathrm{Out}(T_1)|$.

Since $B/T_1\mathrm{C}_B(T_1) \lesssim \mathrm{Out}(T_1)$, we have $p^n \parallel |T_1\mathrm{C}_B(T_1)|$. Since T_1 is non-abelian simple, $T_1 \cap \mathrm{C}_B(T_1) = 1$ and hence $T_1\mathrm{C}_B(T_1) = T_1 \times \mathrm{C}_B(T_1)$. If $p \bigm| |\mathrm{C}_B(T_1)|$, then G is conjugate to $Q_1 \times Q_2$, where $Q_1 \in \mathrm{Syl}_p(T_1)$ and $Q_2 \in \mathrm{Syl}_p(\mathrm{C}_B(T_1))$. Since G is metacyclic, G can be generated by two elements, and since G is a p-group, any minimal generating set of G has cardinality 2. It follows that both Q_1 and Q_2 are cyclic, and so G is abelian, a contradiction. Thus, $p \nmid |\mathrm{C}_B(T_1)|$ and hence $p^n \parallel |T_1|$, forcing $s = 1$. Furthermore, $W_0 = \Delta_1$, $W_1 = \Delta'_1$ and T_1 is 2-transitive on both W_0 and W_1. Note that $(T_1)_u$ is proper subgroup of T_1 of index p^n. Since G is a Sylow p-subgroup of A of order p^n, all Sylow p-subgroups of T_1 are also Sylow p-subgroups of A, and so they are isomorphic to G. Without loss of generality, we may assume $G \leq T_1$. By Proposition 2.4, $T_1 = \mathrm{PSL}(2, 11)$, M_{11}, M_{23}, $\mathrm{PSU}(4, 2)$, A_{p^n}, or $\mathrm{PSL}(d, q)$ with $\frac{q^d-1}{q-1} = p^n$ and d a prime.

Suppose $T_1 = \mathrm{PSL}(2, 11)$, M_{11} or M_{23}. By Proposition 2.4, $|W_0| = |W_1| = 11$, 11 or 23 respectively, and hence $|G| = 11$, 11 or 23, contradicting that G is non-abelian.

Suppose $T_1 = \mathrm{PSU}(4, 2)$ or A_{p^n}. For the former, T_1 has one conjugate class of subgroups of index 27 by Atlas [12], and for the latter, T_1 has one conjugate class of subgroups of index p^n. By Proposition 2.11, T_1 acts equivalently on W_0 and W_1, and since Γ is connected, the 2-transitivity of T_1 on W_0 and W_1 implies that $\Gamma \cong K_{p^n,p^n}$ or $K_{p^n,p^n} - p^n K_2$. Then $A = \mathrm{S}_{p^n} \wr \mathrm{S}_2$ or $\mathrm{S}_{p^n} \times \mathbb{Z}_2$ respectively. Since G is non-abelian, we have $n \geq 3$, and so $p^{n+1} \bigm| |A|$, a contradiction.

Suppose $T_1 = \mathrm{PSL}(d, q)$ with $\frac{q^d-1}{q-1} = p^n$ and d a prime. By Proposition 2.6, T_1 has a cyclic subgroup of order $\frac{q^d-1}{(q-1)(q-1,d)}$. Since d is a prime, either $(q-1, d) = 1$ or $(q-1, d) = d$. Note that $(q-1, d) \bigm| \frac{q^d-1}{q-1}$. If $(q-1, d) = d$ then $d = p$ and $p \mid (q-1)$. Since $p \geq 3$ and $p^2 \mid (q^2-1)(q-1)$, we have $p^{n+1} \bigm| \frac{(q^p-1)(q^p-q)\cdots(q^p-q^{p-1})}{(q-1)(q,d)}$, that is, $p^{n+1} \bigm| |T_1|$, a contradiction. If $(q-1, d) = 1$ then T_1 has a cyclic subgroup of order $\frac{q^d-1}{q-1} = p^n$, contradicting that G is non-abelian. This completes the proof of Claim 1.

Claim 2. $\mathrm{C}_{A^*}(\mathrm{O}_p(A^*)) \leq \mathrm{O}_p(A^*)$.

Suppose that D is a component of A^*, that is, a subnormal quasisimple subgroup of A^*. Then $D = D'$ and $D/Z(D) \cong T$, a non-abelian simple group. By Proposition 2.5, D has a proper subgroup C of p-power index and $Z(D)$ is a p-group. Since $|D : C| = |D : CZ(D)| \cdot |CZ(D) : C|$, we have that $|D : CZ(D)|$ is a p-power. If $D = CZ(D)$ then $D = D' = C'$, contradicting that C is a proper subgroup of D. Thus, $CZ(D) \neq D$. Since $|D/Z(D) : CZ(D)/Z(D)| = |D : CZ(D)|$, we have that $D/Z(D)$ has a proper subgroup $CZ(D)/Z(D)$ of p-power index. By Proposition 2.7(1), $p \nmid |M(D/Z(D))|$ and hence $p \nmid |Z(D)|$. Since $Z(D)$ is a p-group, we have $Z(D) = 1$ and so $D \cong T$. Recall that $E(A^*)$ is the product of all components of A^*. Then $D \leq E(A^*)$ and since $D \cong T$, D is a direct factor of $E(A^*)$. Clearly, D^a is also a direct factor of $E(A^*)$ for any $a \in A^*$. It follows that A^* contains a minimal normal subgroup which is isomorphic to T^ϵ with $\epsilon \geq 1$, contradicting Claim 1. Thus, A^* has no component and $E(A^*) = 1$. It follows that the generalized Fitting subgroup $F^*(A^*) = F(A^*)$. By Proposition 2.5, $\mathrm{O}_{p'}(A^*) = 1$ and hence $F^*(A^*) = \mathrm{O}_p(A^*)$. By Proposition 2.9, $\mathrm{C}_{A^*}(\mathrm{O}_p(A^*)) \leq \mathrm{O}_p(A^*)$, as claimed.

Now we are ready to finish the proof. Since $|A : A^*| \leq 2$ and G has no subgroups of index 2, we only need to show $G \trianglelefteq A^*$.

Let $H = \mathrm{O}_p(A^*)$. By Claim 1, $H \neq 1$. Write $\overline{H} = H/\Phi(H)$ and $\overline{A^*} = A^*/\Phi(H)$. Then $\mathrm{O}_p(A^*/H) = 1$ and $H \leq G$ as $G \in \mathrm{Syl}_p(A^*)$. By Claim 2 and Proposition 2.10, $A^*/H \leq \mathrm{Aut}(\overline{H})$. Since G is metacyclic, $\overline{H} = \mathbb{Z}_p$ or $\mathbb{Z}_p \times \mathbb{Z}_p$.

Assume $\overline{H} = \mathbb{Z}_p$. Then $A^*/H \leq \mathbb{Z}_{p-1}$, and $G = H \trianglelefteq A^*$, as required.

Assume $\overline{H} = \mathbb{Z}_p \times \mathbb{Z}_p$. Then $A^*/H \leq \mathrm{GL}(2,p)$. If $p \nmid |A^*/H|$ then $G = H \trianglelefteq A^*$, as required. To finish the proof, we suppose $p \mid |A^*/H|$ and will obtain a contradiction.

Since $p \parallel |\mathrm{GL}(2,p)|$, we have $p \parallel |A^*/H|$, and since $\mathrm{Syl}_p(\mathrm{SL}(2,p)) = \mathrm{Syl}_p(\mathrm{GL}(2,p))$, we have $\mathrm{Syl}_p(A^*/H) \subseteq \mathrm{Syl}_p(\mathrm{SL}(2,p))$. Note that $A^*/H \cdot \mathrm{SL}(2,p) \leq \mathrm{GL}(2,p)$. Then $p \parallel |A^*/H \cdot \mathrm{SL}(2,p)|$ and so $p \mid |(A^*/H) \cap \mathrm{SL}(2,p)|$. Since $\mathrm{O}_p(A^*/H) = 1$, A^*/H has at least two Sylow p-subgroups, and hence $(A^*/H) \cap \mathrm{SL}(2,p)$ has at least two Sylow p-subgroups, that is, $(A^*/H) \cap \mathrm{SL}(2,p)$ has no normal Sylow p-subgroups. By [44, Theorem 6.17], $(A^*/H) \cap \mathrm{SL}(2,p)$ contains $\mathrm{SL}(2,p)$, that is, $\mathrm{SL}(2,p) \leq A^*/H \leq \mathrm{GL}(2,p)$. In particular, the induced faithful representation of A^*/H on the linear space $\overline{H}$ is irreducible, and hence $\overline{H}$ is a minimal normal subgroup of $\overline{A^*}$.

Recall that $A^* \trianglelefteq A$ and $H = \mathrm{O}_p(A^*)$, which is characteristic in A^*. Then $H \trianglelefteq A$, and since $\Phi(H)$ is characteristic in H, we have $\Phi(H) \trianglelefteq A$. Let $\Gamma_{\Phi(H)}$ be the quotient digraph of Γ relative to $\Phi(H)$, and let L be the kernel of A acting on $V(\Gamma_{\Phi(H)})$. Clearly, $\Gamma_{\Phi(H)}$ is bipartite. Furthermore, $L \trianglelefteq A$, $L \leq A^*$, $\Phi(H) \leq L$ and $L = \Phi(H)L_u$ for any $u \in V(\Gamma)$ because both $\Phi(H)$ and L are transitive on the orbit of $\Phi(H)$ containing u. Since $\Phi(H) \leq G$, $\Phi(H)$ is semiregular on $V(\Gamma)$, and hence $\Phi(H) \cap L_u = 1$. Since $p \nmid |A_u|$, L_u is a Hall p'-subgroup of L. Since $\Phi(H) \trianglelefteq L$ and $\Phi(H) \in \mathrm{Syl}_p(L)$, the Schur-Zassenhaus Theorem implies that all Hall p'-subgroup of L are conjugate. By Frattini argument [30, Kapitel I, 7.8 Satz], $A = L\mathrm{N}_A(L_u) = \Phi(H)L_u\mathrm{N}_A(L_u) = \Phi(H)\mathrm{N}_A(L_u)$ and $H = H \cap A = H \cap (\Phi(H)\mathrm{N}_A(L_u)) = \Phi(H)(H \cap \mathrm{N}_A(L_u))$. Since Frattini subgroup is generated by nongenerators (see [30, Kapitel III, 3.2 Satz]), $H = \Phi(H)(H \cap N_A(L_u))$ if and only if $H = H \cap N_A(L_u)$, that is, $H \leq N_A(L_u)$. It follows that $A = \mathrm{N}_A(L_u)$, that is, $L_u \trianglelefteq A$. By taking $u \in W_0$, we have $L_u = L_v$ for any $v \in W_0$ because A is transitive on W_0, and since A^* acts faithfully on W_0, we have $L_u = 1$. It follows that $L = \Phi(H)$, that is, the kernel of A acting on $V(\Gamma_{\Phi(H)})$ is $\Phi(H)$. Thus $\overline{A} = A/\Phi(H)$ is faithful on $V(\Gamma_{\Phi(H)})$, and then $\overline{A^*}$ is faithful on each of the parts of $V(\Gamma_{\Phi(H)})$, that is,

$\overline{A^*}$ is a transitive permutation group with p-power degree (the cardinality of each part of $V(\Gamma_{\Phi(H)})$).

Since $\overline{A^*}/\overline{H} \cong A^*/H$, we have $\mathrm{SL}(2,p) \leq \overline{A^*}/\overline{H} \leq \mathrm{GL}(2,p)$. Write $\overline{R}/\overline{H} = Z(\overline{A^*}/\overline{H})$. Then $\overline{R} \trianglelefteq \overline{A^*}$ and $1 \neq \overline{R}/\overline{H}$ is a p'-group. Since $\overline{H} \trianglelefteq \overline{R}$ and $\overline{H} \in \mathrm{Syl}_p(\overline{R})$, the Schur-Zassenhaus Theorem [44, Theorem 8.10] implies that there is a p'-group $\overline{V} \leq \overline{R}$ such that $\overline{R} = \overline{HV}$ and all Hall p'-subgroup of $\overline{R}$ are conjugate. Note that $\overline{V} \neq 1$. By Frattini argument [30, Kapitel I, 7.8 Satz], $\overline{A^*} = \overline{R}N_{\overline{A^*}}(\overline{V}) = \overline{H}N_{\overline{A^*}}(\overline{V})$. Since $\overline{H}$ is abelian, $\overline{H} \cap N_{\overline{A^*}}(\overline{V}) \trianglelefteq \overline{A^*}$, and by the minimality of $\overline{H}$, we have $\overline{H} \cap N_{\overline{A^*}}(\overline{V}) = \overline{H}$ or 1. If $\overline{H} \cap N_{\overline{A^*}}(\overline{V}) = \overline{H}$ then $\overline{H} \leq N_{\overline{A^*}}(\overline{V})$ and $\overline{A^*} = \overline{H}N_{\overline{A^*}}(\overline{V}) = N_{\overline{A^*}}(\overline{V})$, that is, $\overline{V} \trianglelefteq \overline{A^*}$. This implies that $\mathrm{O}_{p'}(\overline{A^*}) \neq 1$, contradicting Proposition 2.5. If $\overline{H} \cap N_{\overline{A^*}}(\overline{V}) = 1$ then $\overline{A^*} = \overline{H}N_{\overline{A^*}}(\overline{V})$ implies $\mathrm{ASL}(2,p) \leq \overline{A^*} \leq \mathrm{AGL}(2,p)$ as $\mathrm{SL}(2,p) \leq \overline{A^*}/\overline{H} \leq \mathrm{GL}(2,p)$. It follows that a Sylow p-subgroup of $\overline{A^*}$ is not metacyclic. On the other hand, since both normal subgroups and quotient groups of a metacyclic group are metacyclic, any Sylow p-subgroup of $\overline{A^*}$ is metacyclic because each Sylow p-subgroup of A^* is metacyclic, a contradiction. This completes the proof. □

4 Edge-transitive bipartite bi-p-metacirculants

A connected edge-transitive graph should be semisymmetric, arc-transitive or half-arc-transitive. In this section, as an application of Theorem 1.1, we prove that there are no connected arc-transitive or semisymmetric bipartite bi-p-metacirculants with valency less than p. Furthermore, we classify the connected half-arc-transitive bipartite bi-p-metacirculants with valency less than $2p$.

Let G be a group and let R, L and S be subsets of G such that $R = R^{-1}$, $L = L^{-1}$, $1 \notin R \cup L$ and $1 \in S$, where 1 is the identity of G. Let $\mathrm{BiCay}(G,R,L,S)$ be the graph having vertex set the union of the *right part* $W_0 = \{g_0 \mid g \in G\}$ and the *left part* $W_1 = \{g_1 \mid g \in G\}$, and edge set the union of the *right edges* $\{\{h_0, g_0\} \mid gh^{-1} \in R\}$, the *left edges* $\{\{h_1, g_1\} \mid gh^{-1} \in L\}$ and the *spokes* $\{\{h_0, g_1\} \mid gh^{-1} \in S\}$. For $g \in G$, define a permutation $\hat{g}$ on $V(\Gamma) = W_0 \cup W_1$ by the rule

$$h_i^{\hat{g}} = (hg)_i, \ \forall i \in \mathbb{Z}_2, \ h, g \in G.$$

It is easy to check that $\hat{g}$ is an automorphism of $\mathrm{BiCay}(G,R,L,S)$ and $\hat{G} = \{\hat{g} \mid g \in G\}$ is a semiregular group of automorphisms of $\mathrm{BiCay}(G,R,L,S)$ with two orbits W_0 and W_1. Thus, $\mathrm{BiCay}(G,R,L,S)$ is a bi-Cayley graph over $\hat{G}$, and $\mathrm{BiCay}(G,R,L,S)$ is also called a *bi-Cayley graph* over G relative to R, L and S. Furthermore, $\mathrm{BiCay}(G,R,L,S)$ is connected if and only if $G = \langle R \cup L \cup S \rangle$, and $\mathrm{BiCay}(G,R,L,S) \cong \mathrm{BiCay}(G, R^\theta, L^\theta, S^\theta)$ for any $\theta \in \mathrm{Aut}(G)$.

On the other hand, if Γ is a Bi-Cayley graph over G then $\Gamma \cong \mathrm{BiCay}(G,R,L,S)$ for some subsets R, L and S of G satisfying $R = R^{-1}$, $L = L^{-1}$, $1 \notin R \cup L$ and $1 \in S$.

For $\theta \in \mathrm{Aut}(G)$ and $x, y, g \in G$, define two permutations on $V(\mathrm{BiCay}(G,R,L,S)) = W_0 \cup W_1$ as following:

$$\delta_{\theta,x,y} \colon h_0 \mapsto (xh^\theta)_1, \ h_1 \mapsto (yh^\theta)_0, \ \forall h \in G,$$
$$\sigma_{\theta,g} \colon h_0 \mapsto (h^\theta)_0, \ h_1 \mapsto (gh^\theta)_1, \ \forall h \in G.$$

Set

$$I := \{\delta_{\theta,x,y} \mid \theta \in \mathrm{Aut}(G) \text{ s.t. } R^\theta = x^{-1}Lx,\ L^\theta = y^{-1}Ry,\ S^\theta = y^{-1}S^{-1}x\},$$
$$F := \{\sigma_{\theta,g} \mid \theta \in \mathrm{Aut}(G) \text{ s.t. } R^\theta = R,\ L^\theta = g^{-1}Lg,\ S^\theta = g^{-1}S\}.$$

The following proposition characterizes the normalizer of $\hat{G}$ in $\mathrm{Aut}(\Gamma)$.

Proposition 4.1 ([55, Theorem 1.1])**.** *Let $\Gamma = \mathrm{BiCay}(G, R, L, S)$ be a connected bi-Cayley graph over a group G, where R, L and S are subsets of G with $R = R^{-1}$, $L = L^{-1}$, $1 \notin R \cup L$ and $1 \in S$. If $I = \emptyset$ then $\mathrm{N}_{\mathrm{Aut}(\Gamma)}(\hat{G}) = \hat{G} \rtimes F$, and if $I \neq \emptyset$, then $\mathrm{N}_{\mathrm{Aut}(\Gamma)}(\hat{G}) = \hat{G}\langle F, \delta_{\theta,x,y}\rangle$ for some $\delta_{\theta,x,y} \in I$.*

Write $N = \mathrm{N}_{\mathrm{Aut}(\Gamma)}(\hat{G})$. By Proposition 4.1, $N_{1_0} = F$ and $N_{1_0 1_1} = \{\sigma_{\theta,1} \mid \theta \in \mathrm{Aut}(G) \text{ s.t. } R^\theta = R,\ L^\theta = L,\ S^\theta = S\}$. In particular, F is a group. For the special case $R = L = \emptyset$, it is easy to see that $F = \{\sigma_{\theta,s} \mid \theta \in \mathrm{Aut}(G), s \in S, S^\theta = s^{-1}S\}$ as $1 \in S$.

Lemma 4.2. *Let $\Gamma = \mathrm{BiCay}(G, \emptyset, \emptyset, S)$ be a connected bipartite bi-Cayley graph over G relative to S with $1 \in S$. Then $F = \{\sigma_{\theta,s} \mid \theta \in \mathrm{Aut}(G), s \in S, S^\theta = s^{-1}S\}$ is faithful on $S_1 = \{s_1 \mid s \in S\}$. If G is a p-group and F is a p'-group, then $F \cong \{\theta \mid \sigma_{\theta,s} \in F\}$.*

Proof. Set $L = \{\theta \mid \sigma_{\theta,s}\}$. Since Γ is connected, $G = \langle S\rangle$, and since $F_{1_1} = \{\sigma_{\theta,1} \mid \theta \in \mathrm{Aut}(G) \text{ s.t. } S^\theta = S\}$, F is faithful on S_1. The group F has operation $\sigma_{\theta,x}\sigma_{\delta,y} = \sigma_{\theta\delta,yx^\delta}$ for any $\sigma_{\theta,x}, \sigma_{\delta,y} \in F$, and so the map $\varphi\colon \sigma_{\theta,s} \mapsto \theta$ is an epimorphism from F to L. Let K be the kernel of φ. Then $\sigma_{\theta,s} \in K$ if and only if $\theta = 1$.

Let G be a p-group and F a p'-group. If $\sigma_{1,s} \in K$ for some $1 \neq s \in S$, then $s_1^{\langle\sigma_{1,s}\rangle} = \{s_1, s_1^2, \ldots, s^{o(s)-1}, 1_1\}$ because $1_1^{\sigma_{1,s}} = s_1$ and $(s_1^{l-1})^{\sigma_{1,s}} = s_1^l$ for any positive integer l. Since G is a p-group, $o(s)$ is a p-power and hence $p \mid o(\sigma_{1,s})$, which is impossible because F is a p'-group. Thus, $s = 1$ and hence $K = 1$. Since φ is an epimorphism from F to L, we have $F \cong L$. □

By Equation (2.1), $G_{\alpha,\beta,\gamma} = \langle a, b \mid a^{p^\alpha} = 1,\ b^{p^\beta} = 1,\ b^{-1}ab = a^{1+p^\gamma}\rangle$ with $0 < \gamma < \alpha \leq \beta + \gamma$.

Lemma 4.3. *In $G_{\alpha,\beta,\gamma}$, the following properties hold:*

(1) For any non-negative integers i, j, we have $a^i b^j = b^j a^{i(1+p^\gamma)^j}$;

(2) Let $\theta \in \mathrm{Aut}(G_{\alpha,\beta,\gamma})$ such that $a^\theta = b^m a^n$ with $(m, p) = 1$. Then $\beta < \alpha$.

Proof. From $b^{-1}ab = a^{1+p^\gamma}$, we have $b^{-1}a^ib = a^{i(1+p^\gamma)}$ and hence $b^{-j}a^ib^j = a^{i(1+p^\gamma)^j}$. Part (1) follows. Since $a^\theta = b^m a^n$, we have $o(b^m a^n) = o(a) = p^\alpha$, and since $\langle a\rangle \trianglelefteq G_{\alpha,\beta,\gamma}$, we have $\langle b^m a^n\rangle \trianglelefteq G_{\alpha,\beta,\gamma}$. Then $(p, m) = 1$ implies $G_{\alpha,\beta,\gamma} = \langle a, b^m a^n\rangle = \langle a\rangle\langle b^m a^n\rangle$, and hence

$$p^{\alpha+\beta} = |G_{\alpha,\beta,\gamma}| = \frac{|\langle a\rangle| \cdot |\langle b^m a^n\rangle|}{|\langle a\rangle \cap \langle b^m a^n\rangle|} = \frac{p^\alpha \cdot p^\alpha}{|\langle a\rangle \cap \langle b^m a^n\rangle|} \leq p^\alpha \cdot p^\alpha,$$

that is, $\beta \leq \alpha$. If $\beta = \alpha$, then $|\langle a\rangle \cap \langle b^m a^n\rangle| = 1$ and hence $G_{\alpha,\beta,\gamma} = \langle a\rangle \times \langle b^m a^n\rangle$, contradicting that $G_{\alpha,\beta,\gamma}$ is non-abelian. Thus, $\beta < \alpha$ and part (2) follows. □

A graph Γ is called *locally-transitive* if the stabilizer $\mathrm{Aut}(\Gamma)_u$, for any $u \in V(\Gamma)$, is transitive on the neighborhood of u in $V(\Gamma)$.

Theorem 4.4. *There are no connected locally-transitive bipartite bi-p-metacirculants of valency less than p for any odd prime p.*

Proof. Suppose to the contrary that Γ is a connected locally-transitive bipartite bi-Cayley graph of valency less than p over a non-abelian metacirculant p-group G. Since p is odd, the two orbits of G are exactly the parts of Γ, and we may assume that $\Gamma = \mathrm{BiCay}(G, \emptyset, \emptyset, S)$, where $1 \in S$, $|S| < p$ and $G = \langle S\rangle$. Let $A = \mathrm{Aut}(\Gamma)$. Since Γ has valency less than p, A_{1_0} is a p'-group, and by Theorem 1.1, $\hat{G} \trianglelefteq A$. Write $F = \{\sigma_{\theta,g} \mid \theta \in \mathrm{Aut}(G), S^\theta = g^{-1}S\}$ and $L = \{\theta \mid \sigma_{\theta,s} \in F\}$. By Proposition 4.1, $A_{1_0} = F$, and by Lemma 4.2, $F \cong L$.

Assume that G is non-split. By Lindenberg [36], the automorphism group of G is a p-group. Thus, $p \mid |L|$ and hence $p \mid |A_{1_0}|$, a contradiction.

Assume that G is split. Then $G = G_{\alpha,\beta,\gamma}$, as defined in Equation (2.1). Since F is a p'-group and $F \cong L$, Proposition 2.1 implies that F is cyclic and $|F| \mid (p-1)$. Let $|S_1| = k$ and $F = \langle \sigma_{\theta,s}\rangle$, where $\theta \in \mathrm{Aut}(G)$, $s \in S$ and $S^\theta = s^{-1}S$. Since F is transitive on S_1, $\sigma_{\theta,s}$ permutes all elements in S_1 cyclically, and so $\sigma_{\theta,s}^k$ fixes all elements in S_1. By Proposition 4.1, F is faithful on S_1, implying that $\sigma_{\theta,s}^k = 1$. It follows that $\sigma_{\theta,s}$ has order k and is regular on S_1. Since $F \cong L$, θ also has order k. Furthermore, $S_1 = 1_1^{\langle \sigma_{\theta,s}\rangle} = \{1_1, s_1, (ss^\theta)_1, \ldots, (ss^\theta \cdots s^{\theta^{k-2}})_1\}$ and $1_1 = (ss^\theta \cdots s^{\theta^{k-1}})_1$, that is, $S = \{1, s, ss^\theta, \ldots, ss^\theta \cdots s^{\theta^{k-2}}\}$ and

$$ss^\theta \cdots s^{\theta^{k-1}} = 1. \tag{4.1}$$

Note that for any $\tau \in \mathrm{Aut}(G)$, we have $\Gamma = \mathrm{BiCay}(G, \emptyset, \emptyset, S) \cong \mathrm{BiCay}(G, \emptyset, \emptyset, S^\tau)$, where $S^\tau = \{1, t, tt^{\theta^\tau}, \ldots, tt^{\theta^\tau} \cdots t^{(\theta^\tau)^{k-2}}\}$ and $tt^{\theta^\tau} \cdots t^{(\theta^\tau)^{k-1}} = 1$ with $t = s^\tau$. By Proposition 2.1, all cyclic groups of order k in $\mathrm{Aut}(G)$ are conjugate, and so we may assume that θ is the automorphism induced by $a \mapsto a^e$, $b \mapsto b$, where $e \in \mathbb{Z}_{p^\alpha}^*$ has order k.

Let $s = b^i a^j \in G_{\alpha,\beta,\gamma}$. By Lemma 4.3, $a^i b^j = b^j a^{i(1+p^\gamma)^j}$, and since $a^\theta = a^e$ and $b^\theta = b$, we have $ss^\theta \cdots s^{\theta^{k-1}} = b^{ki} a^\epsilon$ for some $\epsilon \in \mathbb{Z}_{p^\alpha}$. By Equation (4.1), $b^{ki} = 1$, that is, $ki \equiv 0 \pmod{p^\beta}$. Since $k < p$, we have $i \equiv 0 \pmod{p^\beta}$, and hence $G = \langle S\rangle = \langle 1, a^j, a^j a^{je}, \ldots, a^j a^{je} \cdots a^{je^{k-2}}\rangle \leq \langle a\rangle$, a contradiction. This completes the proof. □

Theorem 1.2 is a direct corollary of Theorem 4.4.

To prove Theorem 1.3, we need two technical lemmas on integer numbers.

Lemma 4.5. *Let p be an odd prime and α a positive integer. Let e be an element of order k ($k \geq 2$) in $\mathbb{Z}_{p^\alpha}^*$ with $k \mid (p-1)$. Then $e^i - 1 \in \mathbb{Z}_{p^\alpha}^*$ for any $1 \leq i < k$, and $1 + e + \cdots + e^{k-1} \equiv 0 \pmod{p^\alpha}$.*

For $i \in \mathbb{Z}_k$, let $t_i = (e-1)^{-1}(e^i - 1)$ and $T = \{t_i \mid i \in \mathbb{Z}_k\}$, where $(e-1)^{-1}$ is the inverse of $e-1$ in $\mathbb{Z}_{p^\alpha}^*$. For $x, y \in \mathbb{Z}_{p^\alpha}$, let $Tx + y = \{tx + y \mid t \in T\}$. Then $Tx + y = T$ in $\mathbb{Z}_{p^\alpha}$ if and only if $x \equiv e^l \pmod{p^\alpha}$ and $y \equiv (e-1)^{-1}(e^l - 1) \pmod{p^\alpha}$ for some $l \in \mathbb{Z}_k$. In particular, $Tx = T$ in $\mathbb{Z}_{p^\alpha}$ if and only if $x \equiv 1 \pmod{p^\alpha}$.

Proof. Suppose $e^i - 1 \notin \mathbb{Z}_{p^\alpha}^*$ for some $1 \leq i < k$. Then $p \mid (e^i - 1)$, and since e has order k, we have $e^i \not\equiv 1 \pmod{p^\alpha}$ and $(e^i)^k \equiv 1 \pmod{p^\alpha}$. Furthermore, $p \mid (e^i - 1)$ implies that there exist $l \in \mathbb{Z}_{p^\alpha}^*$ ($p \nmid l$) and $1 \leq s < \alpha$ such that $e^i = 1 + lp^s$. Note that

$$(e^i)^k - 1 = (1 + lp^s)^k - 1 = klp^s + C_k^2(lp^s)^2 + \cdots + C_k^{k-1}(lp^s)^{k-1} + (lp^s)^k.$$

Since $(e^i)^k \equiv 1 \pmod{p^\alpha}$, we have $p \mid kl$, and since $2 \leq k < p$, we have $p \mid l$, a contradiction. Thus, $p \nmid (e^i - 1)$, that is, $e^i - 1 \in \mathbb{Z}_{p^\alpha}^*$. The equation $1+e+\cdots+e^{k-1} \equiv 0 \pmod{p^\alpha}$ follows from $(e-1)(1+e+\cdots+e^{k-1}) = e^k - 1 \equiv 0 \pmod{p^\alpha}$ and $e-1 \in \mathbb{Z}_{p^\alpha}^*$.

Note that $T \subseteq \mathbb{Z}_{p^\alpha}$ and $Tx + y \subseteq \mathbb{Z}_{p^\alpha}$. Since $1 + e + \cdots + e^{k-1} \equiv 0 \pmod{p^\alpha}$, we have

$$\begin{aligned}\sum_{t\in T} t &= (e-1)^{-1} \sum_{i\in\mathbb{Z}_k} (e^i - 1)\\ &= (e-1)^{-1}[(e-1) + \cdots + (e^{k-1} - 1)]\\ &= -k(e-1)^{-1} \in \mathbb{Z}_{p^\alpha}^*.\end{aligned}$$

Assume $Tx + y = T$ in $\mathbb{Z}_{p^\alpha}$. Then $\sum_{t\in T}(tx + y) = \sum_{t\in T} t$, and hence $ky = (1-x)\sum_{t\in T} t = -(1-x)k(e-1)^{-1}$ in $\mathbb{Z}_{p^\alpha}$. It follows $y = (e-1)^{-1}(x-1)$ because $k \in \mathbb{Z}_{p^\alpha}^*$. Then $Tx + (e-1)^{-1}(x-1) = T$ implies $x[T(e-1)+1] = T(e-1)+1$. Since $T(e-1)+1 = \{e^i \mid i \in \mathbb{Z}_k\} = \langle e\rangle$, we have $x\langle e\rangle = \langle e\rangle$ in $\mathbb{Z}_{p^\alpha}^*$, that is, $x \equiv e^l \pmod{p^\alpha}$ for some $l \in \mathbb{Z}_k$. Furthermore, $y \equiv (e-1)^{-1}(e^l - 1) \pmod{p^\alpha}$.

On the other hand, let $x \equiv e^l \pmod{p^\alpha}$ and $y \equiv (e-1)^{-1}(e^l - 1) \pmod{p^\alpha}$ for some $l \in \mathbb{Z}_k$. Then in $\mathbb{Z}_{p^\alpha}$, we have

$$\begin{aligned}Tx + y &= \{e^l(e-1)^{-1}(e^i - 1) + (e-1)^{-1}(e^l - 1) \mid i \in \mathbb{Z}_k\}\\ &= (e-1)^{-1}\{e^l(e^i - 1) + (e^l - 1) \mid i \in \mathbb{Z}_k\}\\ &= (e-1)^{-1}\{e^{i+l} - 1 \mid i \in \mathbb{Z}_k\} = \{(e-1)^{-1}(e^i - 1) \mid i \in \mathbb{Z}_k\} = T.\end{aligned}$$

Thus $Tx + y = T$ in $\mathbb{Z}_{p^\alpha}$ if and only if $x \equiv e^l \pmod{p^\alpha}$ and $y \equiv (e-1)^{-1}(e^l - 1) \pmod{p^\alpha}$ for some $l \in \mathbb{Z}_k$. Applying this with $y = 0$, we obtain that $Tx = T$ in $\mathbb{Z}_{p^\alpha}$ if and only if $x \equiv 1 \pmod{p^\alpha}$. □

Lemma 4.6. *Let p be an odd prime and let α, γ be positive integers with $0 < \gamma < \alpha$. Let e be an element of order k ($k \geq 2$) in $\mathbb{Z}_{p^\alpha}^*$ with $k \mid (p-1)$. Then for any $m \in \mathbb{Z}_{p^{\alpha-\gamma}}^*$ and any $0 \leq \ell \leq k-1$, the following equation in $\mathbb{Z}_{p^\alpha}$*

$$e^\ell(1+p^\gamma)^m = [(1+p^\gamma)^m - x(1-e)]^2 \tag{4.2}$$

has a solution if and only if $\frac{k}{(k,\ell)} \mid \frac{(p-1)}{2}$, and in this case, there are exactly two solutions.

Proof. Since $e^\ell(1+p^\gamma)^m \in \mathbb{Z}_{p^\alpha}^*$, Equation (4.2) has a solution if and only if $e^\ell(1+p^\gamma)^m$ is a square in $\mathbb{Z}_{p^\alpha}^*$. Since $\mathbb{Z}_{p^\alpha}^* \cong \mathbb{Z}_{p^{\alpha-1}(p-1)}$, squares in $\mathbb{Z}_{p^\alpha}^*$ consists of the unique subgroup of order $\frac{(p-1)}{2}p^{\alpha-1}$ in $\mathbb{Z}_{p^\alpha}^*$, and so Equation (4.2) has a solution if and only if the order of $e^\ell(1+p^\gamma)^m$ in $\mathbb{Z}_{p^\alpha}^*$ is a divisor of $\frac{(p-1)}{2}p^{\alpha-1}$. Clearly, $(1+p^\gamma)^m$ has order $p^{\alpha-\gamma}$, and e^ℓ has order $\frac{k}{(k,\ell)}$. Thus, Equation (4.2) has a solution if and only if $\frac{k}{(k,\ell)} \mid \frac{(p-1)}{2}$. If $e^\ell(1+p^\gamma)^m = u^2$ for some $u \in \mathbb{Z}_{p^\alpha}^*$ then $(1-e)^{-1}[(1+p^\gamma)^m \pm u]$ are the only two solutions of Equation (4.2) in $\mathbb{Z}_{p^\alpha}$. □

Now we construct the half-arc-transitive graphs in Theorem 1.3. Let p be an odd prime, and let α, β, γ be positive integers such that $0 < \gamma < \alpha \leq \beta + \gamma$. Let e be an element of order k ($k \geq 2$) in $\mathbb{Z}_{p^\alpha}^*$ with $k \mid (p-1)$. Choose $0 \leq \ell < k$ such that $\frac{k}{(k,\ell)} \mid \frac{(p-1)}{2}$. Recall that

$$G_{\alpha,\beta,\gamma} = \langle a, b \mid a^{p^\alpha} = 1,\ b^{p^\beta} = 1,\ b^{-1}ab = a^{1+p^\gamma} \rangle.$$

Let

$$U = \{a^t \mid t \in \{(e-1)^{-1}(e^i - 1) \mid i \in \mathbb{Z}_k\}\} \text{ and}$$
$$V = \{b^m a^i \mid i \in \{(e-1)^{-1}(e^i - 1)(1+p^\gamma)^m + e^i n \mid i \in \mathbb{Z}_k\}\},$$

where $m \in \mathbb{Z}_{p^{\alpha-\gamma}}^*$ and n is a solution of $e^\ell(1+p^\gamma)^m = [(1+p^\gamma)^m - x(1-e)]^2$. Define

$$\Gamma_{m,k,\ell}^n = \mathrm{BiCay}(G_{\alpha,\beta,\gamma}, \emptyset, \emptyset, U \cup V). \tag{4.3}$$

By Lemma 4.6, there are exactly two solutions n of equation $e^\ell(1+p^\gamma)^m = [(1+p^\gamma)^m - x(1-e)]^2$ in $\mathbb{Z}_{p^\alpha}$, and so the notation $\Gamma_{m,k,\ell}^n$ is also written as $\Gamma_{m,k,\ell}^\pm$, as used in Theorem 1.3. We first prove the sufficiency of Theorem 1.3.

Lemma 4.7. *The graphs $\Gamma_{m,k,\ell}^\pm$ are independent from the choice of element e of order k in $\mathbb{Z}_{p^\alpha}^*$ and half-arc-transitive, and $\mathrm{Aut}(\Gamma_{m,k,\ell}^\pm) \cong (G_{\alpha,\beta,\gamma} \rtimes \mathbb{Z}_k).\mathbb{Z}_2$.*

Proof. Write $\Gamma = \Gamma_{m,k,\ell}^n$ and $A = \mathrm{Aut}(\Gamma)$. Let $T = \{(e-1)^{-1}(e^i - 1) \mid i \in \mathbb{Z}_k\}$ and $T' = \{(e-1)^{-1}(e^i-1)(1+p^\gamma)^m + e^i n \mid i \in \mathbb{Z}_k\}$. Then $U = \{a^\eta \mid \eta \in T\}$ and $V = \{b^m a^\eta \mid \eta \in T'\}$. Furthermore, $\Gamma = \mathrm{BiCay}(G, \emptyset, \emptyset, S)$ with $G = G_{\alpha,\beta,\gamma}$ and $S = U \cup V$. Clearly, $1 \in U$ and $G_{\alpha,\beta,\gamma} = \langle S \rangle$, implying that Γ is connected. Note that $T' = T[(1+p^\gamma)^m + n(e-1)] + n$.

Since $e \in \mathbb{Z}_{p^\alpha}^*$, any element of order k in $\mathbb{Z}_{p^\alpha}^*$ can be written as e^q with $(q,k) = 1$ and hence $\{e^i \mid i \in \mathbb{Z}_k\} = \langle e \rangle = \langle e^q \rangle = \{(e^q)^i \mid i \in \mathbb{Z}_k\}$. By Lemma 4.5, $e - 1 \in \mathbb{Z}_{p^\alpha}^*$ and $e^q - 1 \in \mathbb{Z}_{p^\alpha}^*$. Let

$$\begin{aligned}
\overline{T} &= \{(e^q-1)^{-1}((e^q)^i - 1) \mid i \in \mathbb{Z}_k\},\\
\overline{T'} &= \{(e^q-1)^{-1}((e^q)^i - 1)(1+p^\gamma)^m + (e^q)^i n \mid i \in \mathbb{Z}_k\},\\
\overline{U} &= \{a^\eta \mid \eta \in \overline{T}\} \text{ and}\\
\overline{V} &= \{b^m a^\eta \mid \eta \in \overline{T'}\}.
\end{aligned}$$

It is easy to see that $a \mapsto a^{(e-1)(e^q-1)^{-1}}$ and $b \mapsto b$ induce an automorphism of G, say ρ. Then

$$\begin{aligned}
U^\rho &= \{a^{(e-1)(e^q-1)^{-1}(e-1)^{-1}(e^i-1)} \mid i \in \mathbb{Z}_k\}\\
&= \{a^{(e^q-1)^{-1}((e^q)^i-1)} \mid i \in \mathbb{Z}_k\} = \{a^\eta \mid \eta \in \overline{T}\} = \overline{U},
\end{aligned}$$

and similarly, $V^\rho = \overline{V}$. Thus, $\mathrm{BiCay}(G, \emptyset, \emptyset, U \cup V) \cong \mathrm{BiCay}(G, \emptyset, \emptyset, \overline{U} \cup \overline{V})$, that is, Γ is independent from the choice of element e of order k in $\mathbb{Z}_{p^\alpha}^*$. To finish the proof, it suffices to prove that Γ is half-arc-transitive with $\mathrm{Aut}(\Gamma) \cong (G_{\alpha,\beta,\gamma} \rtimes \mathbb{Z}_k).\mathbb{Z}_2$.

Claim 1. $p \nmid |A_{1_0}|$.

We argue by contradiction and we suppose $p \mid |A_{1_0}|$. Let P is a Sylow p-subgroup of A containing $\hat{G}$ and let $X = \mathrm{N}_A(\hat{G})$. Then $\hat{G} < P$, and hence $\hat{G} < \mathrm{N}_P(\hat{G}) \leq X$. In particular, $p \mid |X : \hat{G}|$, and so $p \mid |X_{1_0}|$. Let τ be the automorphism of G induced by $a \mapsto a^e$ and $b \mapsto b$.

First we prove $\sigma_{\tau,a} \in X_{1_0}$. By Proposition 4.1, it is enough to show $S^\tau = a^{-1}S$. Clearly,

$$U^\tau = \{a^{e\eta} \mid \eta \in T\} = \{a^\eta \mid \eta \in Te\} \text{ and}$$
$$a^{-1}U = \{a^{\eta-1} \mid \eta \in T\} = \{a^\eta \mid \eta \in T-1\}.$$

By taking $\ell = 1$ in Lemma 4.5, we have $Te = T - 1$ and hence $U^\tau = a^{-1}U$. Similarly,

$$V^\tau = \{b^m a^{e\eta} \mid \eta \in T'\} = \{b^m a^\eta \mid \eta \in T'e\} \text{ and}$$
$$a^{-1}V = \{a^{-1}b^m a^\eta \mid \eta \in T'\} = \{b^m a^{-(1+p^\gamma)^m} a^\eta \mid \eta \in T'\}$$
$$= \{b^m a^\eta \mid \eta \in T' - (1+p^\gamma)^m\}.$$

By Equation (4.2), $(1+p^\gamma)^m + n(e-1) \in \mathbb{Z}_{p^\alpha}^*$, and hence $Te = T - 1$ implies

$$T[(1+p^\gamma)^m + n(e-1)]e + ne = T[(1+p^\gamma)^m + n(e-1)] + n - (1+p^\gamma)^m.$$

Since $T' = T[(1+p^\gamma)^m + n(e-1)] + n$, we have $T'e = T' - (1+p^\gamma)^m$, that is, $V^\tau = a^{-1}V$. It follows that $S^\tau = a^{-1}S$, as required.

Set $U_1 = \{u_1 \mid u \in U\}$, $V_1 = \{v_1 \mid v \in V\}$ and $S_1 = \{s_1 \mid s \in S\}$. Then $U_1 = 1_1^{\langle\sigma_{\tau,a}\rangle}$ and $V_1 = (b^m a^n)_1^{\langle\sigma_{\tau,a}\rangle}$. Since $\sigma_{\tau,a} \in X_{1_0}$, either X_{1_0} has two orbits of length k on S_1, or is transitive on S_1. By Lemma 4.2, X_{1_0} acts faithfully on S_1, and since $p \mid |X_{1_0}|$, any element of order p of X_{1_0} has an orbit of length p on S_1, implying that X_{1_0} is transitive on S_1 as $k < p$. From $|X_{1_0}| = |X_{1_0 1_1}| \cdot |1_1^{X_{1_0}}| = |X_{1_0 1_1}| \cdot 2k$, we have $p \mid |X_{1_0 1_1}|$. By Proposition 4.1, $X_{1_0 1_1} = \{\sigma_{\theta,1} \mid \theta \in \mathrm{Aut}(G) \text{ s.t. } S^\theta = S\}$. Let $\sigma_{\theta,1} \in X_{1_0 1_1}$ be of order p with $\theta \in \mathrm{Aut}(G)$. Then θ has order p and $S^\theta = S$. Recall that $k \geq 2$.

Assume $k > 2$. Since $a \in S$, we have $a^\theta \in S^\theta = S = U \cup V$. If $a^\theta \in V$ then $a^\theta = b^m a^i$ for some $i \in T'$. Note that $a^{1+e} \in S$ as $k > 2$. Since $m \in \mathbb{Z}_{p^{\alpha-\gamma}}^*$, we have $(m,p) = 1$, and by Lemma 4.5, $(p, 1+e) = 1$. Then $(a^{1+e})^\theta = (b^m a^i)^{1+e} \in V$, and considering the powers of b, we have $m(1+e) \equiv m \pmod{p^\beta}$, that is, $e \equiv 0 \pmod{p^\beta}$. It follows that $p \mid e$, contradicting that $e \in \mathbb{Z}_{p^\alpha}^*$. Thus, $a^\theta \in U$, and hence, $a^\theta = a^j$ for some $j \in T$. If $a^\theta \neq a$ then $a_1^{\sigma_{\theta,1}} = \{a_1, a_1^\theta, \ldots, a_1^{\theta^{p-1}}\}$ is an orbit of length p of $\sigma_{\theta,1}$ on S_1, which is impossible because there are exactly $k < p$ elements of type a^j in S. Thus, $a^\theta = a$ and θ fixes U pointwise. Furthermore, θ also fixes V pointwise because $|V| = k < p$. It follows that $\theta = 1$ as $G = \langle S\rangle$, and so $\sigma_{\theta,1} = 1$, a contradiction.

Assume $k = 2$. Then $e \equiv -1 \pmod{p^\alpha}$ and

$$S_1 = \{1_1, a_1, (b^m a^n)_1, (b^m a^{(1+p^\gamma)^m - n})_1\}.$$

Since $1_1^{\sigma_{\theta,1}} = 1_1$ and $p \geq 3$, $\sigma_{\theta,1}$ has order 3 and we may assume that $a_1^{\sigma_{\theta,1}} = (b^m a^n)_1$, $(b^m a^n)_1^{\sigma_{\theta,1}} = (b^m a^{(1+p^\gamma)^m - n})_1$ and $(b^m a^{(1+p^\gamma)^m - n})_1^{\sigma_{\theta,1}} = a_1$ (replace $\sigma_{\theta,1}$ by $\sigma_{\theta,1}^2$ if necessary), that is, $a^\theta = b^m a^n$, $(b^m a^n)^\theta = b^m a^{(1+p^\gamma)^m - n}$ and $(b^m a^{(1+p^\gamma)^m - n})^\theta = a$.

By Lemma 4.3, $\beta < \alpha$. It follows that

$$a = (b^m a^{(1+p^\gamma)^m - n})^\theta = [(b^m a^n) a^{(1+p^\gamma)^m - 2n}]^\theta$$
$$= b^m a^{(1+p^\gamma)^m - n} (b^m a^n)^{(1+p^\gamma)^m - 2n},$$

and so $0 \equiv m + m[(1+p^\gamma)^m - 2n] \pmod{p^\beta}$. Thus, $p \mid (1-n)$, which is impossible because otherwise $p^\alpha = o(a^{\theta^2}) = o(b^m a^{(1+p^\gamma)^m - n}) < p^\alpha$.

Summing up, we have proved $p \nmid |A_{1_0}|$, and this completes the proof of Claim 1.

By Claim 1, A_{1_0} is a p'-group, and by Theorem 1.1, Γ is normal. By Proposition 4.1, $A_{1_0} = X_{1_0} = F = \{\sigma_{\theta,s} \mid \sigma \in \mathrm{Aut}(G), s \in S, S^\theta = s^{-1}S\}$, and by Lemma 4.2, $F \cong L \leq \mathrm{Aut}(G)$, where $L = \{\theta \mid \sigma_{\theta,s} \in F\}$. By Proposition 2.1, F is cyclic, and since $\sigma_{\tau,a} \in A_{1_0}$, F is transitive on S_1 or has two orbits. By Lemma 4.2, F is faithful on S_1, and since F is cyclic, either F is regular on S_1, or $F = \langle \sigma_{\tau,a} \rangle$.

We suppose that F is regular on S_1 and will obtain a contradiction. Note that $F \cong \mathbb{Z}_{2k}$ and $|F : \langle \sigma_{\tau,a} \rangle| = 2$. Then $\langle \sigma_{\tau,a} \rangle \trianglelefteq F$, and the two orbits U_1 and V_1 of $\langle \sigma_{\tau,a} \rangle$ consist of an imprimitive block system of F on S_1. By the regularity of F, there exists $\sigma_{\theta,s} \in F$ such that $1_1^{\sigma_{\theta,s}} = (b^m a^n)_1$, implying that $s = b^m a^n$ and $S^\theta = s^{-1}S = (b^m a^n)^{-1}S$, and hence $U_1^{\sigma_{\theta,s}} = V_1$ because $1_1 \in U_1$ and $(b^m a^n)_1 \in V_1$. It follows that $a^\theta \in (b^m a^n)^{-1}V$. It is easy to see that

$$(b^m a^n)^{-1}S = (b^m a^n)^{-1}U \cup (b^m a^n)^{-1}V,$$

where

$$(b^m a^n)^{-1}U = \{(b^m a^n)^{-1}a^\eta \mid \eta \in T\} = \{b^{-m} a^{-n(1+p^\gamma)^{-m}+\eta} \mid \eta \in T\}$$
$$= \{b^{-m} a^\eta \mid \eta \in T - n(1+p^\gamma)^{-m}\} \text{ and}$$
$$(b^m a^n)^{-1}V = \{(b^m a^n)^{-1} b^m a^\eta \mid \eta \in T'\}$$
$$= \{a^{-n+\eta} \mid \eta \in T'\} = \{a^\eta \mid \eta \in T' - n\}.$$

Since $T' = T[(1+p^\gamma)^m + n(e-1)] + n$, we have

$$(b^m a^n)^{-1}V = \{a^\eta \mid \eta \in T[(1+p^\gamma)^m + n(e-1)]\}.$$

Let $a^\theta = a^r \in (b^m a^n)^{-1}V$ for some $r \in T[(1+p^\gamma)^m + n(e-1)]$. Since $p^\alpha = o(a^\theta) = o(a^r)$, we have $r \in \mathbb{Z}_{p^\alpha}^*$. Note that

$$U^\theta = \{a^{\eta r} \mid \eta \in T\} = \{a^\eta \mid \eta \in Tr\} \subseteq (b^m a^n)^{-1}V.$$

Then

$$U^\theta = (b^m a^n)^{-1}V = \{a^\eta \mid \eta \in T[(1+p^\gamma)^m + n(e-1)]\},$$

and so $Tr = T[(1+p^\gamma)^m + n(e-1)]$ in $\mathbb{Z}_{p^\alpha}$. By Lemma 4.5, $r = (1+p^\gamma)^m + n(e-1)$.

Since $S^\theta = (b^m a^n)^{-1}S = (b^m a^n)^{-1}U \cup (b^m a^n)^{-1}V$, we have $V^\theta = (b^m a^n)^{-1}U$. In particular, $(b^m a^n)^\theta = b^{-m} a^t$ for some $t \in T - n(1+p^\gamma)^{-m}$. For $\eta \in T'$, since

$$(b^m a^\eta)^\theta = [(b^m a^n) a^{\eta - n}]^\theta = b^{-m} a^t a^{r(\eta - n)} = b^{-m} a^{r\eta - rn + t},$$

we have

$$\begin{aligned}\{b^{-m}a^{\eta} \mid \eta \in T - n(1+p^{\gamma})^{-m}\}\\ &= (b^m a^n)^{-1}U = V^{\theta} = \{(b^m a^{\eta})^{\theta} \mid \eta \in T'\}\\ &= \{b^{-m}a^{r\eta - rn + t} \mid \eta \in T'\} = \{b^{-m}a^{\eta} \mid \eta \in T'r - rn + t\}.\end{aligned}$$

This implies that

$$\begin{aligned}T - n(1+p^{\gamma})^{-m}\\ &= T'r - rn + t = Tr[(1+p^{\gamma})^m + n(e-1)] + rn - rn + t\\ &= Tr[(1+p^{\gamma})^m + n(e-1)] + t = T[(1+p^{\gamma})^m + n(e-1)]^2 + t\end{aligned}$$

in $\mathbb{Z}_{p^{\alpha}}$. By Equation (4.2), $e^{\ell}(1+p^{\gamma})^m = [(1+p^{\gamma})^m + n(e-1)]^2$. It follows that

$$T - n(1+p^{\gamma})^{-m} = Te^{\ell}(1+p^{\gamma})^m + t,$$

and hence

$$T = Te^{\ell}(1+p^{\gamma})^m + t + n(1+p^{\gamma})^{-m}.$$

By Lemma 4.5, there exists $\ell' \in \mathbb{Z}_k$ such that $e^{\ell'} \equiv e^{\ell}(1+p^{\gamma})^m \pmod{p^{\alpha}}$, that is, $e^{\ell'-\ell} = (1+p^{\gamma})^m \pmod{p^{\alpha}}$. Since e is an element of order k, we have $(1+p^{\gamma})^{mk} \equiv 1 \pmod{p^{\alpha}}$ and since $(mk, p) = 1$, we have $p^{\gamma} \equiv 0 \pmod{p^{\alpha}}$, implying that $\gamma \geq \alpha$, which is impossible because $0 < \gamma < \alpha$.

Thus, $A_{1_0} = F = \langle \sigma_{\tau,a} \rangle \cong \mathbb{Z}_k$. Since A_{1_0} has two orbits on S_1, that is U_1 and V_1, Γ is not arc-transitive. To prove the half-arc-transitivity of Γ, we only need to show that A is transitive on $V(\Gamma)$ and $E(\Gamma)$. Note that $1_1 \in U_1$ and $(b^m a^n)_1 \in V_1$. By Proposition 4.1, it suffices to construct a $\lambda \in \mathrm{Aut}(G)$ such that

$$\delta_{\lambda, b^m a^n, 1} \in I = \{\delta_{\lambda,x,y} \mid \lambda \in \mathrm{Aut}(G), S^{\lambda} = y^{-1}S^{-1}x\},$$

that is $S^{\lambda} = S^{-1}b^m a^n$, because $(1_0, 1_1)^{\delta_{\lambda, b^m a^n, 1}} = ((b^m a^n)_1, 1_0)$.

Let $\mu = -(1+p^{\gamma})^m - n(e-1)$ and $\nu = -(e-1)^{-1}\mu^2 - (e-1)^{-1}\mu$. Then $\mu + 1 + n(e-1) \equiv 0 \pmod{p^{\gamma}}$ and hence

$$\begin{aligned}\nu - \mu n &= -(e-1)^{-1}\mu^2 - (e-1)^{-1}\mu - \mu n\\ &\equiv -(e-1)^{-1}\mu[\mu + 1 + n(e-1)] \equiv 0 \pmod{p^{\gamma}}.\end{aligned}$$

By Proposition 2.2, $o(b^m a^{\nu-\mu n}) = p^{\beta}$. Denote by m^{-1} the inverse of m in $\mathbb{Z}_{p^{\beta}}$. Then $(b^m a^{\nu-\mu n})^{m^{-1}} = ba^{\epsilon}$ for some ϵ in $\mathbb{Z}_{p^{\alpha}}$, and it is easy to check that a^{μ} and $(b^m a^{\nu-\mu n})^{m^{-1}}$ have the same relations as do a and b. Define λ as the automorphism of G induced by $a \mapsto a^{\mu}$, $b \mapsto (b^m a^{\nu-\mu n})^{m^{-1}}$. Clearly, $(b^m)^{\lambda} = b^m a^{\nu-\mu n}$.

Note that $S = U \cup V$. First we have

$$\begin{aligned}U^{\lambda} &= \{a^{\eta\mu} \mid \eta \in T\} = \{a^{\eta} \mid \eta \in T\mu\} \text{ and}\\ V^{-1}b^m a^n &= \{(b^m a^{\eta})^{-1}b^m a^n \mid \eta \in T'\}\\ &= \{a^{-\eta+n} \mid \eta \in T'\} = \{a^{\eta} \mid \eta \in -T' + n\}.\end{aligned}$$

Recall that $T' = T[(1+p^\gamma)^m + n(e-1)] + n = -T\mu + n$. Then

$$-T' + n = T\mu - n + n = T\mu,$$

and so $U^\lambda = V^{-1}b^m a^n$.

On the other hand,

$$\begin{aligned} V^\lambda &= \{(b^m a^\eta)^\lambda \mid \eta \in T'\} = \{b^m a^{\nu-\mu n} a^{\eta\mu} \mid \eta \in T'\} \\ &= \{b^m a^\eta \mid \eta \in T'\mu - \mu n + \nu\} \text{ and} \\ U^{-1}b^m a^n &= \{(a^\eta)^{-1} b^m a^n \mid \eta \in T\} = \{b^m a^{-\eta(1+p^\gamma)^m + n} \mid \eta \in T\} \\ &= \{b^m a^\eta \mid \eta \in -T(1+p^\gamma)^m + n\}. \end{aligned}$$

To prove $V^\lambda = U^{-1}b^m a^n$, we only need to show $T'\mu - \mu n + \nu \equiv -T(1+p^\gamma)^m + n$ in $\mathbb{Z}_{p^\alpha}$, which is equivalent to show that $T(1+p^\gamma)^m = T\mu^2 - \nu + n$ because $T' = -T\mu + n$. By Equation (4.2),

$$e^\ell(1+p^\gamma)^m = [(1+p^\gamma)^m - n(1-e)]^2 = \mu^2,$$

and by Lemma 4.5, $T = Te^\ell + (e-1)^{-1}(e^\ell - 1)$. It follows

$$\begin{aligned} T(1+p^\gamma)^m &= Te^\ell(1+p^\gamma)^m + (e-1)^{-1}(e^\ell-1)(1+p^\gamma)^m \\ &= T\mu^2 + (e-1)^{-1}[\mu^2 - (1+p^\gamma)^m]. \end{aligned}$$

Note that

$$\begin{aligned} -\nu + n &= (e-1)^{-1}\mu^2 + (e-1)^{-1}\mu + n \\ &= (e-1)^{-1}[\mu^2 + \mu + n(e-1)] = (e-1)^{-1}[\mu^2 - (1+p^\gamma)^m]. \end{aligned}$$

Then $T(1+p^\gamma)^m = T\mu^2 - \nu + n$, and hence $V^\lambda = U^{-1}b^m a^n$.

Thus, $S^\lambda = U^\lambda \cup V^\lambda = V^{-1}b^m a^n \cup U^{-1}b^m a^n = S^{-1}b^m a^n$, and so Γ is half-arc-transitive.

Let A^* be the subgroup of A fixing the two parts of Γ setwise. Then $A = A^*.\mathbb{Z}_2$. Since $A_{1_0} \cong \mathbb{Z}_k$ and Γ is normal, we have $A^* \cong G \rtimes \mathbb{Z}_k$ and hence $A \cong (G \rtimes \mathbb{Z}_k).\mathbb{Z}_2$. □

Now we prove the necessity of Theorem 1.3.

Lemma 4.8. *For an odd prime p, let Γ be a connected bipartite half-arc-transitive bi-p-metacirculant of valency $2k$ ($k < p$) over G. Then $k \geq 2$, $k \mid (p-1)$, $G \cong G_{\alpha,\beta,\gamma}$ and $\Gamma \cong \Gamma^{\pm}_{m,k,\ell}$, where $m \in \mathbb{Z}^*_{p^{\alpha-\gamma}}$ and $0 \leq \ell < k$ with $\frac{k}{(k,\ell)} \mid \frac{(p-1)}{2}$.*

Proof. Clearly, the two orbits of G are exactly the two parts of Γ. Then we may assume that $\Gamma = \mathrm{BiCay}(G, \emptyset, \emptyset, S)$, where $1 \in S$, $|S| < 2p$ and $G = \langle S \rangle$. Let $A = \mathrm{Aut}(\Gamma)$.

Since Γ is half-arc-transitive, Γ has valency at least 4, that is, $k \geq 2$, and A_{1_0} has exactly two orbits on $S_1 = \{s_1 \mid s \in S\}$, say U_1 and V_1 with $1_1 \in U_1$, where U and V are subsets of G with $1 \in U$. Then $S = U \cup V$, $|U| = |V| = k \geq 2$ and $|S| = 2k$. Since $k < p$, the Orbit-Stabilizer theorem implies that A_{1_0} is a p'-group. By Theorem 1.1, $\hat{G} \trianglelefteq A$, and by Proposition 4.1, $A_{1_0} = F = \{\sigma_{\theta,s} \mid \theta \in \mathrm{Aut}(G), s \in S, S^\theta = s^{-1}S\}$. By Lemma 4.2, A_{1_0} is faithful on S_1, and $F \cong L := \{\sigma \mid \sigma_{\theta,s} \in F\}$.

Suppose that G is non-split. By Lindenberg [36], the automorphism group of G is a p-group. Thus, $p \bigm| |L|$ and hence $p \bigm| |A_{1_0}|$, a contradiction.

Thus, G is split, namely $G = G_{\alpha,\beta,\gamma}$, as defined in Equation (2.1). By Proposition 2.1, F is a cyclic subgroup of $\mathbb{Z}_{p-1}$, and hence $F = \langle \sigma_{\theta,s} \rangle$ for some $\theta \in \mathrm{Aut}(G)$ and $s \in S$ with $S^\theta = s^{-1}S$. Then $\sigma_{\theta,s}$ has order k and $\langle \sigma_{\theta,s} \rangle$ is regular on both U_1 and V_1. Furthermore, $A_{1_0} = F = \langle \sigma_{\theta,s} \rangle \cong \mathbb{Z}_k$,

$$U_1 = 1_1^{\langle \sigma_{\theta,s} \rangle} = \{1_1, s_1, (ss^\theta)_1, \ldots, (ss^\theta \cdots s^{\theta^{k-2}})_1\}, \text{ and}$$
$$V_1 = t_1^{\langle \sigma_{\theta,s} \rangle} = \{t_1, (st^\theta)_1, (ss^\theta t^{\theta^2})_1, \ldots, (ss^\theta \cdots s^{\theta^{k-2}} t^{\theta^{k-1}})_1\}$$

with $(ss^\theta \cdots s^{\theta^{k-1}})_1 = 1_1$ for any $t \in V$. It follows

$$U = \{1, s, ss^\theta, \ldots, ss^\theta \cdots s^{\theta^{k-2}}\} \text{ and}$$
$$V = \{t, st^\theta, ss^\theta t^{\theta^2}, \ldots, ss^\theta \cdots s^{\theta^{k-2}} t^{\theta^{k-1}}\}.$$

In particular, $k \mid (p-1)$, θ has order k, and

$$ss^\theta \cdots s^{\theta^{k-1}} = 1. \tag{4.4}$$

By Proposition 2.1, we may assume that θ is the automorphism induced by $a \mapsto a^e$, $b \mapsto b$, where $e \in \mathbb{Z}_{p^\alpha}^*$ has order k.

Let $s = b^i a^j$ and $t = b^m a^n$ with $i, m \in \mathbb{Z}_{p^\beta}$ and $j, n \in \mathbb{Z}_{p^\alpha}$. Since $s^\theta = b^i a^{ej}$, we have $ss^\theta \cdots s^{\theta^{k-1}} = b^{ki} a^\epsilon$ for some $\epsilon \in \mathbb{Z}_{p^\alpha}$. By Equation (4.4), $b^{ki} = 1$, that is, $ki \equiv 0 \pmod{p^\beta}$. Since $k < p$, we have $i \equiv 0 \pmod{p^\beta}$, and hence $s = a^j$. Since

$$ss^\theta \cdots s^{\theta^{i-1}} = a^j a^{je} \cdots a^{je^{i-1}} = a^{j(e-1)^{-1}(e^i-1)},$$

we have

$$U = \{1, a^j, a^j a^{je}, \ldots, a^j a^{je} \cdots a^{je^{k-2}}\} = \{a^{j(e-1)^{-1}(e^i-1)} \mid i \in \mathbb{Z}_k\}.$$

By Lemma 4.3,

$$a^{j(e-1)^{-1}(e^i-1)} b^m = b^m a^{j(e-1)^{-1}(e^i-1)(1+p^\gamma)^m},$$

and since $(b^m a^n)^{\theta^i} = b^m a^{e^i n}$, we have

$$\begin{aligned} V &= \{a^{j(e-1)^{-1}(e^i-1)} (b^m a^n)^{\theta^i} \mid i \in \mathbb{Z}_k\} \\ &= \{b^m a^{j(e-1)^{-1}(e^i-1)(1+p^\gamma)^m + e^i n} \mid i \in \mathbb{Z}_k\}. \end{aligned}$$

By the connectedness of Γ, $G = \langle S \rangle = \langle U \cup V \rangle \leq \langle a^j, a^n, b^m \rangle$, forcing $G = \langle a^j, a^n, b^m \rangle$. It follows that $p \nmid m$ and so $m \in \mathbb{Z}_{p^\beta}^*$.

Since Γ is half-arc-transitive, Proposition 4.1 implies that there exists $\delta_{\lambda,x,y} \in I$ such that $(1_0, 1_1)^{\delta_{\lambda,x,y}} = ((b^m a^n)_1, 1_0)$ with $\lambda \in \mathrm{Aut}(G)$ and $S^\lambda = y^{-1} S^{-1} x$. In particular, $(b^m a^n)_1 = 1_0^{\delta_{\lambda,x,y}} = x_1$ and $1_0 = 1_1^{\delta_{\lambda,x,y}} = y_0$. It follows that $x = b^m a^n$, $y = 1$ and $S^\lambda = S^{-1} b^m a^n = U^{-1} b^m a^n \cup V^{-1} b^m a^n$. Furthermore,

$$\begin{aligned} U^{-1} b^m a^n &= \{a^{-j(e-1)^{-1}(e^i-1)} b^m a^n \mid i \in \mathbb{Z}_k\} \\ &= \{b^m a^{-j(e-1)^{-1}(e^i-1)(1+p^\gamma)^m + n} \mid i \in \mathbb{Z}_k\}, \end{aligned}$$

and since

$$(b^m a^{j(e-1)^{-1}(e^i-1)(1+p^\gamma)^m+e^i n})^{-1} b^m a^n = a^{-j(e-1)^{-1}(e^i-1)(1+p^\gamma)^m+n(1-e^i)},$$

we have

$$V^{-1} b^m a^n = \{a^{-j(e-1)^{-1}(e^i-1)(1+p^\gamma)^m+n(1-e^i)} \mid i \in \mathbb{Z}_k\}.$$

Suppose $p \mid j$. Since $G = \langle a^j, a^n, b^m \rangle$, we have $p \nmid n$ and $p \nmid m$. By Proposition 2.2, every element in both V and $U^{-1}b^m a^n$ has order $\max\{p^\alpha, p^\beta\}$. Clearly, every element in U has order less than p^α, but the element $a^{-j(1+p^\gamma)^m+n(1-e)} \in V^{-1}b^m a^n$ has order p^α because $p \nmid (1-e)$ by Lemma 4.5. This is impossible as $\lambda \in \mathrm{Aut}(G)$ and $(U \cup V)^\lambda = S^\lambda = S^{-1}b^m a^n = U^{-1}b^m a^n \cup V^{-1}b^m a^n$. Thus, $p \nmid j$.

Now, there is an automorphism of G mapping a^j to a and b to b, and so we may assume $j = 1$ and $s = a$. It follows that

$$U = \{a^\eta \mid \eta \in T\}, \tag{4.5}$$

where $T = \{(e-1)^{-1}(e^i - 1) \mid i \in \mathbb{Z}_k\}$;

$$V = \{b^m a^\eta \mid \eta \in T'\}, \tag{4.6}$$

where $T' = \{(e-1)^{-1}(e^i-1)(1+p^\gamma)^m + e^i n \mid i \in \mathbb{Z}_k\}$.

As

$$\begin{aligned}(e-1)^{-1}&(e^i-1)(1+p^\gamma)^m + e^i n \\ &= [(e-1)^{-1}(e^i-1)][(1+p^\gamma)^m + n(e-1)] + n,\end{aligned}$$

we have

$$T' = T[(1+p^\gamma)^m + n(e-1)] + n. \tag{4.7}$$

Since

$$-(e-1)^{-1}(e^i-1)(1+p^\gamma)^m + n = [(e-1)^{-1}(e^i-1)][-(1+p^\gamma)^m] + n$$

and

$$\begin{aligned}-(e-1)^{-1}&(e^i-1)(1+p^\gamma)^m + n(1-e^i) \\ &= [(e-1)^{-1}(e^i-1)][-(1+p^\gamma)^m + n(1-e)],\end{aligned}$$

we have

$$U^{-1} b^m a^n = \{b^m a^\eta \mid \eta \in T_1\}, \tag{4.8}$$

where $T_1 = T[-(1+p^\gamma)^m] + n$;

$$V^{-1} b^m a^n = \{a^\eta \mid \eta \in T_1'\}, \tag{4.9}$$

where $T_1' = T[-(1+p^\gamma)^m + n(1-e)]$.

Noting that $T, T', T_1, T_1' \subseteq \mathbb{Z}_{p^\alpha}$, we have $U, V, U^{-1}b^m a^n, V^{-1}b^m a^n \subseteq G$.

Claim 1. $a^\lambda \in V^{-1} b^m a^n$.

Suppose to the contrary that $a^\lambda \notin V^{-1}b^m a^n$. Since $a^\lambda \in S^\lambda = U^{-1}b^m a^n \cup V^{-1}b^m a^n$, we have $a^\lambda \in U^{-1}b^m a^n$, that is, $a^\lambda = b^m a^\mu$ for $\mu \in T_1$. By Lemma 4.3, $\beta < \alpha$. Recall $k \geq 2$.

Let $k > 2$. Then $a^{1+e} \in U$ and $(a^{1+e})^\lambda = (b^m a^\mu)^{1+e} \in U^{-1}b^m a^n$. Note that $p \nmid m$ and by Lemma 4.5, $p \nmid (1+e)$. Considering the power of b of $(b^m a^\mu)^{1+e}$ and elements in $U^{-1}b^m a^n$, we have $m(1+e) \equiv m \pmod{p^\beta}$ and so $e \equiv 0 \pmod{p^\beta}$, contradicting $e \in \mathbb{Z}_{p^\alpha}^*$.

Let $k = 2$. Then $T = \{0, 1\}$ and $e \equiv -1 \pmod{p^\alpha}$. By Equations (4.5) and (4.6),

$$S = \{1, a, b^m a^n, b^m a^{(1+p^\gamma)^m - n}\},$$

and by Equations (4.8) and (4.9),

$$S^{-1}b^m a^n = \{1, a^{-(1+p^\gamma)^m+2n}, b^m a^n, b^m a^{-(1+p^\gamma)^m+n}\}.$$

Note that $a^\lambda \in U^{-1}b^m a^n = \{b^m a^n, b^m a^{-(1+p^\gamma)^m+n}\}$.

Case 1. $a^\lambda = b^m a^n$.

As $S^\lambda = S^{-1}b^m a^n$, it is easy to see that

$$\begin{aligned}&((b^m a^n)^\lambda, (b^m a^{(1+p^\gamma)^m-n})^\lambda)\\ &\quad = (a^{-(1+p^\gamma)^m+2n}, b^m a^{-(1+p^\gamma)^m+n}) \text{ or } (b^m a^{-(1+p^\gamma)^m+n}, a^{-(1+p^\gamma)^m+2n}).\end{aligned}$$

For the former,

$$\begin{aligned}b^m a^{-(1+p^\gamma)^m+n} &= (b^m a^{(1+p^\gamma)^m-n})^\lambda = [(b^m a^n) a^{(1+p^\gamma)^m-2n}]^\lambda\\ &= a^{-(1+p^\gamma)^m+2n}(b^m a^n)^{(1+p^\gamma)^m-2n},\end{aligned}$$

implying that $m \equiv m[(1+p^\gamma)^m - 2n] \pmod{p^\beta}$, and since $p \nmid m$, we have $p \mid n$. This is impossible because otherwise $p^\alpha = o(a^\lambda) = o(b^m a^n) < p^\alpha$ ($\beta < \alpha$). For the latter, we can verify that

$$\begin{aligned}a^{-(1+p^\gamma)^m+2n} &= (b^m a^{(1+p^\gamma)^m-n})^\lambda = [(b^m a^n) a^{(1+p^\gamma)^m-2n}]^\lambda\\ &= b^m a^{-(1+p^\gamma)^m+n}(b^m a^n)^{(1+p^\gamma)^m-2n}.\end{aligned}$$

Thus, $0 \equiv m + m[(1+p^\gamma)^m - 2n] \pmod{p^\beta}$, and hence $p \mid (1-n)$, but it is also impossible because otherwise $p^\alpha = o(a^\lambda) = o(b^m a^n) = o((b^m a^n)^\lambda) = o(b^m a^{-(1+p^\gamma)^m+n}) < p^\alpha$.

Case 2. $a^\lambda = b^m a^{-(1+p^\gamma)^m+n}$.

In this case, we have

$$\begin{aligned}&((b^m a^n)^\lambda, (b^m a^{(1+p^\gamma)^m-n})^\lambda)\\ &\quad = (a^{-(1+p^\gamma)^m+2n}, b^m a^n) \text{ or } (b^m a^n, a^{-(1+p^\gamma)^m+2n}).\end{aligned}$$

For the former,

$$\begin{aligned}b^m a^n &= (b^m a^{(1+p^\gamma)^m-n})^\lambda = [(b^m a^n) a^{(1+p^\gamma)^m-2n}]^\lambda\\ &= a^{-(1+p^\gamma)^m+2n}(b^m a^{-(1+p^\gamma)^m+n})^{(1+p^\gamma)^m-2n},\end{aligned}$$

implying $m \equiv m[(1+p^\gamma)^m - 2n] \pmod{p^\beta}$, and since $p \nmid m$, we have $p \mid n$. By Proposition 2.2,

$$\begin{aligned} o(b^m a^{-(1+p^\gamma)^m+n}) &= o(b^m a^{(1+p^\gamma)^m-n}) \\ &= \max\{o(a^{(1+p^\gamma)^m-n}) = o(a^{-(1+p^\gamma)^m+n}), o(b^m)\}, \end{aligned}$$

and it follows that

$$\begin{aligned} p^\alpha = o(a^\lambda) &= o(b^m a^{-(1+p^\gamma)^m+n}) = o(b^m a^{(1+p^\gamma)^m-n}) \\ &= o((b^m a^{(1+p^\gamma)^m-n})^\lambda) = o(b^m a^n) < p^\alpha, \end{aligned}$$

a contradiction. For the latter,

$$\begin{aligned} a^{-(1+p^\gamma)^m+2n} &= (b^m a^{(1+p^\gamma)^m-n})^\lambda = [(b^m a^n) a^{(1+p^\gamma)^m-2n}]^\lambda \\ &= b^m a^n (b^m a^{-(1+p^\gamma)^m+n})^{(1+p^\gamma)^m-2n}. \end{aligned}$$

Thus, $0 \equiv m + m[(1+p^\gamma)^m - 2n] \pmod{p^\beta}$, and hence $p \mid (1-n)$, but it is also impossible because otherwise $p^\alpha = o(a^\lambda) = o(b^m a^{-(1+p^\gamma)^m+n}) < p^\alpha$. This completes the proof of Claim 1.

By Claim 1, $a^\lambda = a^\mu \in V^{-1} b^m a^n$ for some $\mu \in T_1'$. Since $p^\alpha = o(a^\lambda) = o(a^\mu)$, we have $\mu \in \mathbb{Z}_{p^\alpha}^*$. By Equations (4.5) and (4.9),

$$U^\lambda = \{a^{\eta\mu} \mid \eta \in T\} = \{a^\eta \mid \eta \in T\mu\} \subseteq V^{-1} b^m a^n.$$

Then $U^\lambda = V^{-1} b^m a^n = \{a^\eta \mid \eta \in T_1'\}$, and so $T\mu = T_1'$ in $\mathbb{Z}_{p^\alpha}$. By Equation (4.9), $T\mu = T[-(1+p^\gamma)^m + n(1-e)]$. Since $p \nmid \mu$, we have $T = T[-(1+p^\gamma)^m + n(1-e)]\mu^{-1}$. By Lemma 4.5, $\mu = -(1+p^\gamma)^m + n(1-e)$.

Since $S^\lambda = S^{-1} b^m a^n = U^{-1} b^m a^n \cup V^{-1} b^m a^n$, we have $V^\lambda = U^{-1} b^m a^n$. In particular, $(b^m a^n)^\lambda = b^m a^\nu$ for some $\nu \in T_1$. For $\eta \in T'$, since

$$(b^m a^\eta)^\lambda = [(b^m a^n) a^{\eta-n}]^\lambda = b^m a^\nu a^{\eta\mu - \mu n} = b^m a^{\eta\mu - \mu n + \nu},$$

we have that

$$\begin{aligned} \{b^m a^\eta \mid \eta \in T_1\} &= U^{-1} b^m a^n = V^\lambda = \{(b^m a^\eta)^\lambda \mid \eta \in T'\} \\ &= \{b^m a^{\eta\mu - \mu n + \nu} \mid \eta \in T'\} = \{b^m a^\eta \mid \eta \in T'\mu - \mu n + \nu\}. \end{aligned}$$

By Equations (4.7) and (4.8),

$$\begin{aligned} T[-(1+p^\gamma)^m] + n = T_1 &= T'\mu - \mu n + \nu \\ &= T[(1+p^\gamma)^m + n(e-1)]\mu + \mu n - \mu n + \nu \end{aligned}$$

in $\mathbb{Z}_{p^\alpha}$. Thus, $T[(1+p^\gamma)^m - n(1-e)]^2 (1+p^\gamma)^{-m} - (\nu - n)(1+p^\gamma)^{-m} = T$. By Lemma 4.5, there exists $\ell \in \mathbb{Z}_k$ such that $e^\ell = [(1+p^\gamma)^m - n(1-e)]^2 (1+p^\gamma)^{-m}$, that is, n satisfies Equation (4.2).

Recall that $\alpha - \gamma \leq \beta$ and $m \in \mathbb{Z}_{p^\beta}^*$. Let $m = m_1 + lp^{\alpha-\gamma}$ with $m_1 \in \mathbb{Z}_{p^{\alpha-\gamma}}^*$. Since $(1+p^\gamma)$ has order $p^{\alpha-\gamma}$ in $\mathbb{Z}_{p^\alpha}^*$, we have

$$(1+p^\gamma)^m = (1+p^\gamma)^{m_1 + lp^{\alpha-\gamma}} = (1+p^\gamma)^{m_1}.$$

This implies that replacing m by m_1, Equation (4.2) has the same solutions, and

$$T' = \{(e-1)^{-1}(e^i - 1)(1+p^\gamma)^{m_1} + e^i n \mid i \in \mathbb{Z}_k\} \subseteq \mathbb{Z}_{p^\alpha}.$$

The automorphism of G induced by $a \mapsto a$ and $b \mapsto b^{m_1 m^{-1}}$, maps U to U, and $V = \{b^m a^\eta \mid \eta \in T'\}$ to $\{b^{m_1} a^\eta \mid \eta \in T'\}$. Thus, we may assume that $m \in \mathbb{Z}^*_{p^{\alpha-\gamma}}$, and therefore, $\Gamma \cong \Gamma^n_{m,k,l}$. □

Proof of Theorem 1.3. This is a consequence of Lemmas 4.7 and 4.8. □

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On flag-transitive automorphism groups of symmetric designs*

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Abstract

In this article, we study flag-transitive automorphism groups of non-trivial symmetric (v, k, λ) designs, where λ divides k and $k \geqslant \lambda^2$. We show that such an automorphism group is either point-primitive of affine or almost simple type, or point-imprimitive with parameters $v = \lambda^2(\lambda + 2)$ and $k = \lambda(\lambda + 1)$, for some positive integer λ. We also provide some examples in both possibilities.

Keywords: Symmetric design, flag-transitive, point-primitive, point-imprimitive, automorphism group.

Math. Subj. Class.: 05B05, 05B25, 20B25

1 Introduction

A *t-design* $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ with parameters (v, k, λ) is an incidence structure consisting of a set $\mathcal{P}$ of v *points*, and a set $\mathcal{B}$ of k-element subsets of $\mathcal{P}$, called *blocks*, such that every t-element subset of points lies in exactly λ blocks. The design $\mathcal{D}$ is *non-trivial* if $t < k < v - t$, and is *symmetric* if $|\mathcal{B}| = v$. By [7, Theorem 1.1], if $\mathcal{D}$ is symmetric and non-trivial, then $t \leqslant 2$, see also [12, Theorem 1.27]. Thus we study non-trivial symmetric 2-designs with parameters (v, k, λ) which we simply call non-trivial *symmetric* (v, k, λ) *designs*. A *flag* of $\mathcal{D}$ is an incident pair (α, B), where α and B are a point and a block of $\mathcal{D}$, respectively. An *automorphism* of a symmetric design $\mathcal{D}$ is a permutation of the points permuting the blocks and preserving the incidence relation. An automorphism group G of $\mathcal{D}$ is called *flag-transitive* if it is transitive on the set of flags of $\mathcal{D}$. If G leaves invariant a non-trivial partition of $\mathcal{P}$, then G is said to be *point-imprimitive*; otherwise G is called

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point-primitive. We here adopt the standard notation as in [8, 23] for finite simple groups of Lie type. For example, we use $\mathrm{PSL}_n(q)$, $\mathrm{PSp}_n(q)$, $\mathrm{PSU}_n(q)$, $\mathrm{P\Omega}_{2n+1}(q)$ and $\mathrm{P\Omega}^{\pm}_{2n}(q)$ to denote the finite classical simple groups. Symmetric and alternating groups on n letters are denoted by S_n and A_n, respectively. Further notation and definitions in both design theory and group theory are standard and can be found, for example in [10, 12, 14]. We also use the software GAP [21] for computational arguments.

Flag-transitive incidence structures have been of most interest. In 1961, Higman and McLaughlin [11] proved that a flag-transitive automorphism group of a linear space must act primitively on its points set, and then Buekenhout, Delandtsheer and Doyen [5] studied this action in details and proved that a linear space admitting a flag-transitive automorphism group (which is in fact point-primitive) is either of affine, or almost simple type. Thereafter, a deep result [6], namely the classification of flag-transitive finite linear spaces relying on the Classification of Finite Simple Groups (CFSG) was announced. Although, flag-transitive symmetric designs are not necessarily point-primitive, Regueiro [18] proved that a flag-transitive and point-primitive automorphism group of such designs for $\lambda \leqslant 4$ is of affine or almost simple type, and so using CFSG, she determined all flag-transitive and point-primitive biplanes ($\lambda = 2$). In conclusion, she gave a classification of flag-transitive biplanes except for the 1-dimensional affine case [17]. Tian and Zhou [22] proved that a flag-transitive and point-primitive automorphism group of a symmetric design with $\lambda \leqslant 100$ must be of affine or almost simple type. Generally, Zieschang [25] proved in 1988 that a flag-transitive automorphism group of a 2-design with $\gcd(r, \lambda) = 1$ is (point-primitive) of affine or almost simple type, and this result has been generalised by Zhuo and Zhan [24] for $\lambda \geqslant \gcd(r, \lambda)^2$.

1.1 Main result

In this paper, we study flag-transitive automorphism groups of symmetric (v, k, λ) designs, where λ divides k and $k \geqslant \lambda^2$, and we show that such an automorphism group is not necessarily point-primitive:

Theorem 1.1. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a non-trivial symmetric (v, k, λ) design with $\lambda \geqslant 1$, and let G be a flag-transitive automorphism group of $\mathcal{D}$. If λ divides k and $k \geqslant \lambda^2$, then one of the following holds:*

(a) G is point-primitive of affine or almost simple type;

(b) G is point-imprimitive and $v = \lambda^2(\lambda + 2)$ and $k = \lambda(\lambda + 1)$, for some positive integer λ. In particular, if G has d classes of imprimitivity of size c, then there is a constant l such that, for each block B and each class Δ, the size $|B \cap \Delta|$ is either 0, or l, and $(c, d, l) = (\lambda^2, \lambda + 2, \lambda)$ or $(\lambda + 2, \lambda^2, 2)$.

We highlight here that if λ divides k, then $\gcd(k, \lambda)^2 = \lambda^2 > \lambda$ which does not satisfy the conditions which have been studied in [24, 25]. Moreover, in Section 1.2, we provide some examples to show that both possibilities in Theorem 1.1 can actually occur.

In order to prove Theorem 1.1(a), we apply O'Nan-Scott Theorem [15] and discuss possible types of primitive groups in Section 3. We further note that our proof for part (a) relies on CFSG. To prove part (b), we use an important result by Praeger and Zhou [20, Theorem 1.1] on characterisation of imprimitive flag-transitive symmetric designs.

1.2 Examples and comments on Theorem 1.1

Here, we give some examples of symmetric (v, k, λ) designs admitting flag-transitive automorphism groups, where λ divides k and $k \geqslant \lambda^2$. In Table 1, we list some small examples of such designs with $\lambda \leqslant 3$. To our knowledge the design in Line 2 is the only point-primitive example of symmetric designs with $v \leqslant 2500$ satisfying the conditions of Theorem 1.1 and this motivates the authors to investigate symmetric designs admitting symplectic automorphism groups [3]. More examples of symmetric designs admitting flag-transitive and point-imprimitive automorphism groups can be found in [20] and references therein.

Line 1. Hussain [13] showed that there are exactly three symmetric $(16, 6, 2)$ designs, and Regueiro proved that exactly two of such designs are flag-transitive and point-imprimitive [18, p. 139].

Line 2. The symmetric design in this line arises from the study of primitive permutation groups with small degrees. This design belongs to a class of symmetric designs with parameters $(3^m(3^m+1)/2, 3^{m-1}(3^m-1)/2, 3^{m-1}(3^{m-1}-1)/2)$, for some positive integer $m > 1$, see [4, 9]. If $m = 2$, then we obtain the symmetric $(45, 12, 3)$ design admitting $\mathrm{PSp}_4(3)$ or $\mathrm{PSp}_4(3) : 2$ as flag-transitive automorphism group of rank 3, see [4].

Lines 3 – 4. Mathon and Spence [16] constructed 2616 pairwise non-isomorphic symmetric $(45, 12, 3)$ designs with non-trivial automorphism groups. Praeger [19] proved that there are exactly two flag-transitive symmetric $(45, 12, 3)$ designs, exactly one of which admits a point-imprimitive group, and this example satisfies Line 4, but not Line 3.

Table 1: Some symmetric designs satisfying the conditions in Theorem 1.1.

Line	v	k	λ	c	d	l	Case	Examples	Reference	Comments
1	16	6	2	4	4	2	(b)	2	[13], [18]	imprimitive
2	45	12	3	–	–	–	(a)	1	[4]	primitive
3	45	12	3	5	9	2	(b)	None	[19]	imprimitive
4	45	12	3	9	5	3	(b)	1	[19]	imprimitive

2 Preliminaries

In this section, we state some useful facts in both design theory and group theory.

Lemma 2.1 ([1, Lemma 2.1]). *Let $\mathcal{D}$ be a symmetric (v, k, λ) design, and let G be a flag-transitive automorphism group of $\mathcal{D}$. If α is a point in $\mathcal{P}$ and $H := G_\alpha$, then*

(a) $k(k-1) = \lambda(v-1)$;

(b) k *divides* $|H|$ *and* $\lambda v < k^2$.

Lemma 2.2 ([2, Corollary 4.3]). *Let T be a finite simple classical group of dimension n over a finite field $\mathbb{F}_q$ of size q. Then* *(a) If* $T = \mathrm{PSL}_n(q)$ *with* $n \geqslant 2$*, then* $|T| > q^{n^2-2}$*;*

(b) If $T = \mathrm{PSU}_n(q)$ *with* $n \geqslant 3$*, then* $|T| > (1-q^{-1})q^{n^2-2}$*;*

(c) If $T = \mathrm{PSp}_n(q)$ *with* $n \geqslant 4$*, then* $|T| > q^{\frac{1}{2}n(n+1)}/(2\alpha)$*, where* $\alpha = \gcd(2, q-1)$*;*

(d) If $T = \mathrm{P}\Omega_n^{\epsilon}(q)$ *with* $n \geqslant 7$*, then* $|T| > q^{\frac{1}{2}n(n-1)}/(4\beta)$*, where* $\beta = \gcd(2, n)$*.*

Lemma 2.3. *Let* T *be a non-abelian finite simple group satisfying*

$$|T| < 8 \cdot |\operatorname{Out}(T)|^3. \tag{2.1}$$

Then T *is isomorphic to* A_5 *or* A_6*.*

Proof. If T is a sporadic simple group or an alternating group A_n with $n \geqslant 7$, then $|\operatorname{Out}(T)| \in \{1, 2\}$, and so by (2.1), we must have $|T| < 64$, which is a contradiction. Note that the alternating groups A_5 and A_6 satisfy (2.1) as claimed. Therefore, we only need to consider the case where T is a finite simple group of Lie type. In what follows, we discuss each case separately.

Let $T = \mathrm{PSL}_n(q)$ with $q = p^a$ and $n \geqslant 2$. If $n = 2$, then $q \geqslant 4$ and $|\operatorname{Out}(T)| = a \cdot \gcd(2, q-1)$, and so by Lemma 2.2(a) and (2.1), we have that $q^2 < |\mathrm{PSL}_2(q)| < 8a^3 \cdot \gcd(2, q-1)^3 \leqslant 64a^3$. Thus, $q^2 < 64a^3$. This inequality holds only for $(p, a) \in \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (3, 1), (3, 2), (3, 3), (5, 1), (7, 1)\}$. Note in this case that $q \geqslant 4$, and hence by (2.1), we conclude that T is either

$$\mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5) \cong A_5, \quad \text{or} \quad \mathrm{PSL}_2(9) \cong A_6,$$

as claimed. If $n = 3$, then by Lemma 2.2(a), we have that $q^7 < 64a^3 \cdot \gcd(3, q-1)^3 < 64a^3q^3$, and so $q^4 < 64a^3$. If q would be odd, then we would have $3^{4a} < 64a^3$, which is impossible. If $q = 2^a$, then $2^a < 64a^3$ would hold only for $a = 1, 2$. Therefore, T is isomorphic to $\mathrm{PSL}_3(2)$ or $\mathrm{PSL}_3(4)$. These simple groups do not satisfy (2.1). If $n \geqslant 4$, then (2.1) implies that $q^{11} < 64a^3$, but this inequality has no possible solution.

Let $T = \mathrm{PSU}_n(q)$ with $q = p^a$ and $n \geqslant 3$. By Lemma 2.2(b), we have that $|T| > (1-q^{-1})q^{n^2-2}$, and so (2.1) implies that $(1-q^{-1})q^{n^2-2} < 64a^3 \cdot \gcd(n, q+1)^3$. If $n = 3$, then $(1-q^{-1})q^7 < 64a^3 \cdot \gcd(n, q+1)^3$, and so $q^6 < 27 \cdot 64a^3$. This inequality holds only for $(p, a) \in \{(2, 1), (2, 2), (3, 1)\}$. Note that $\mathrm{PSU}_3(2)$ is not simple. Therefore, T is isomorphic to $\mathrm{PSU}_3(3)$ or $\mathrm{PSU}_3(4)$. These simple groups do not satisfy (2.1). If $n \geqslant 4$, then since $(q+1)^3 < 4 \cdot q^3(q-1)$, we would have $q^{n^2-3} < 64a^3 \cdot \gcd(n, q+1)^3/(q-1) < 4 \cdot 64a^3(q+1)^3/4(q-1) < 4 \cdot 64a^3q^3$, and so $q^{n^2-6} < 4 \cdot 64a^3$, and hence $q^{10} < 4 \cdot 64a^3$, which is impossible.

Let $T = \mathrm{PSp}_n(q)$ with $q = p^a$ and $n \geqslant 4$. By Lemma 2.2(c), we observe that $|T| > q^{\frac{1}{2}n(n+1)}/2\gcd(2, q-1) \geqslant q^{\frac{1}{2}n(n+1)}/4$. By (2.1), we have that $q^{10} \leqslant q^{\frac{1}{2}n(n+1)} < 4 \cdot 64a^3$, and so $q^{10} < 4 \cdot 64a^3$, which is impossible.

Let $T = \mathrm{P}\Omega_n(q)$ with $q = p^a$ odd and $n \geqslant 7$. Then we conclude by Lemma 2.2(d) that $|T| > q^{\frac{1}{2}n(n-1)}/8$. Since $|\operatorname{Out}(T)| = 2a$ and $n \geqslant 7$, it follows from (2.1) that $q^{21} < 8^3a^3$, which is impossible.

Let $T = \mathrm{P}\Omega_n^{\epsilon}(q)$ with $q = p^a$ and $n \geqslant 8$ and $\epsilon = \pm$. It follows from Lemma 2.2(d) that $|T| > q^{\frac{1}{2}n(n-1)}/8$. Note that $|\operatorname{Out}(T)| \leqslant 6a \cdot \gcd(4, q^{\frac{n}{2}} - \epsilon) \leqslant 24a$. Then (2.1) implies that $q^{28} < 8^2 \cdot 24^3a^3$, which is impossible.

Let T be one of the finite exceptional groups

$$F_4(q),\ E_6(q),\ E_7(q),\ E_8(q),\ {}^2F_4(q)\ (q = 2^{2m+1}),\ {}^3D_4(q) \text{ and } {}^2E_6(q).$$

Then $|T| > q^{20}$, and so (2.1) implies that $q^{20} < 8 \cdot 2^3 \cdot 3^3 a^3$, which is impossible. If $T = G_2(q)$ with $q = p^a \neq 2$. Then by (2.1), we have that $q^{12} < q^6(q^2-1)(q^6-1) < 8 \cdot 2^3 a^3$, and so $q^{12} < 8 \cdot 2^3 a^3$, which is impossible. Similarly, if T is one of the groups ${}^2B_2(q)$ with $q = 2^{2m+1}$ and ${}^2G_2(q)$ with $q = 3^{2m+1}$, then $|T| > q^4$, and so (2.1) implies that $q^4 < 8a^3$, which is impossible. □

3 Point-primitive designs

In what follows, we assume that $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a non-trivial symmetric (v, k, λ) design admitting a flag-transitive and point-primitive automorphism group G. Let also λ divide k and $k \geqslant \lambda^2$ and set $t := k/\lambda$. Notice that $\lambda < k$, and so $t \geqslant 2$. We moreover observe by Lemma 2.1(a) that

$$k = \frac{v+t-1}{t}; \tag{3.1}$$

$$\lambda = \frac{v+t-1}{t^2}. \tag{3.2}$$

Since also G is a primitive permutation group on $\mathcal{P}$, by O'Nan-Scott Theorem [15], G is of one of the following types:

(a) Affine;

(b) Almost simple;

(c) Simple diagonal;

(d) Product;

(e) Twisted wreath product.

3.1 Product and twisted wreath product type

In this section, we assume that G is a primitive group of product type on $\mathcal{P}$, that is to say, $G \leqslant H \wr S_\ell$, where H is of almost simple or diagonal type on the set Γ of size $m := |\Gamma| \geqslant 5$ and $\ell \geqslant 2$. In this case, $\mathcal{P} = \Gamma^\ell$.

Lemma 3.1. *Let G be a flag-transitive point-primitive automorphism group of product type. Then k divides $\lambda\ell(m-1)$.*

Proof. See the proof of Lemma 4 in [18]. □

Proposition 3.2. *If $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a non-trivial symmetric (v, k, λ) design admitting a flag-transitive and point-primitive automorphism group G, where λ divides k and $k \geqslant \lambda^2$, then G is not of product type.*

Proof. Assume the contrary. Suppose that G is of product type. Then $v = m^\ell$. Note by Lemma 3.1 that k divides $\lambda\ell(m-1)$, and so $t = k/\lambda$ divides $\ell(m-1)$. We also note by

Lemma 2.1(b) that $\lambda v < k^2$. Then $v < \lambda t^2$, and since $\lambda \leqslant t$, we have that $v < t^3$. Recall that t divides $\ell(m-1)$. Hence

$$m^\ell < \ell^3(m-1)^3. \tag{3.3}$$

Then $m^\ell < \ell^3 m^3$, or equivalently, $m^{\ell-3} < \ell^3$. Since $m \geqslant 5$, it follows that $5^{\ell-3} < \ell^3$, and this is true for $2 \leqslant \ell \leqslant 6$. If $\ell = 6$, then since $m^{6-3} < 6^3$, we conclude that $m = 5$, but $(m, \ell) = (5, 6)$ does not satisfy (3.3). Therefore, $2 \leqslant \ell \leqslant 5$.

Suppose first that $\ell = 5$. Then by (3.3), we have that $m^5 < 5^3(m-1)^3$, and so $5 \leqslant m \leqslant 9$. It follows from (3.1) that t divides $m^5 - 1$. For each $5 \leqslant m \leqslant 9$, we can obtain divisors t of $m^5 - 1$. Note by (3.2) that t^2 must divide $m^5 - t + 1$. This is true only for $m = 7$ when $t = 2$ or 6 for which

$$(v, k, \lambda) = (16807, 8404, 4202) \text{ or } (16807, 2802, 467),$$

respectively. Since $\lambda^2 \leqslant k$, these parameters can be ruled out.

Suppose that $\ell = 4$. Then by (3.3), we have that $m^5 < 4^3(m-1)^3$, and so $5 \leqslant m \leqslant 9$. By the same argument as in the case where $\ell = 5$, by (3.1) and (3.2), we obtain possible parameters (m, t, v, k, λ) as in Table 2. Note by Lemma 3.1 that k must divide $4\lambda(m-1)$, and this is not true, for all parameters in Table 2.

Table 2: Possible values for (m, t, v, k, λ) when $\ell = 4$.

m	t	v	k	λ
13	51	28561	561	11
31	555	923521	1665	3
47	345	4879681	14145	41
57	416	10556001	25376	61

Suppose now that $\ell = 3$. We again apply Lemma 3.1 and conclude that t divides $3(m-1)$. Then there exists a positive integer x such that $3(m-1) = tx$, and so $m = (tx+3)/3$. By (3.2), we have that

$$\lambda = \frac{m^2 + t - 1}{t^2} = \frac{t^2x^3 + 9tx^2 + 27x + 27}{27t}.$$

Then $27\lambda t = t^2x^3 + 9tx^2 + 27x + 27$. Therefore, t must divide $27x + 27$, and so $ty = 27x + 27$, for some positive integer y. Thus,

$$\lambda = \frac{t(ty-27)^3 + 9 \cdot 27(ty-27)^2 + 27^3 y}{27^4}, \tag{3.4}$$

for some positive integers t and y. Since $\lambda^2 \leqslant k$, we have that $\lambda \leqslant t$, and so

$$t(ty-27)^3 + 9 \cdot 27(ty-27)^2 + 27^3 y \leqslant 27^4 t. \tag{3.5}$$

If $y \geqslant 32$, then

$$\begin{aligned} &t(ty-27)^3 + 9 \cdot 27(ty-27)^2 + 27^3 y \\ &\qquad \geqslant t(32t-27)^3 + 9 \cdot 27(32t-27)^2 + 32 \cdot 27^3 > 27^4 t, \end{aligned}$$

for $t \geqslant 2$. Thus $1 \leqslant y \leqslant 31$, and so by (3.5), we conclude that $2 \leqslant t \leqslant 107$. For each such y and t, by straightforward calculation, we observe that λ as in (3.4) is not a positive integer.

Suppose finally that $\ell = 2$. Recall by Lemma 3.1 that t divides $2(m-1)$. Then $2(m-1) = tx$ for some positive integer x, and so $m = (tx+2)/2$. It follows from (3.2) that $\lambda = (tx^2+4x+4)/4t$, or equivalently, $4t\lambda = tx^2+4x+4$. This shows that t divides $4x+4$, and so $ty = 4x+4$, for some positive integer y. Therefore, $4^3\lambda = (ty-4)^2+16y$. Since $\lambda^2 \leqslant k$, we have that $\lambda \leqslant t$, and so $(ty-4)^2+16y \leqslant 4^3t$. If $y \geqslant 6$, then $(6t-4)^2+6\cdot 16 \leqslant 4^3t$, which has no possible solution for t. Thus $1 \leqslant y \leqslant 5$. Since also $(t-4)^2+16 \leqslant 4^3t$, we conclude that $2 \leqslant t \leqslant 71$, and so (3.1) and (3.2) imply that

$$k = \frac{t(t^2y^2 - 8ty + 16y + 16)}{64} \quad \text{and} \quad \lambda = \frac{(ty-4)^2+16y}{64},$$

where $2 \leqslant t \leqslant 71$ and $1 \leqslant y \leqslant 5$. For these values of t and y, considering the fact that $m \geqslant 5$, $k \geqslant \lambda^2$ and λ divides k, we obtain $(v,k,\lambda) = (121,25,5)$ or $(441,56,7)$ respectively when $(t,y) = (5,4)$ or $(8,3)$. These possibilities can be ruled out by [4] or [22, Theorem 1.1]. □

Proposition 3.3. *If $\mathcal{D} = (\mathcal{P},\mathcal{B})$ is a non-trivial symmetric (v,k,λ) design admitting a flag-transitive and point-primitive automorphism group G, where λ divides k and $k \geqslant \lambda^2$, then G is not of twisted wreath product type.*

Proof. If G would be of twisted wreath product type, then by [15, Remark 2(ii)], it would be contained in the wreath product $H \wr S_m$ with $H = T \times T$ of simple diagonal type, and so G would act on $\mathcal{P}$ by product action, and this contradicts Proposition 3.2. □

3.2 Simple diagonal type

In this section, we suppose that G is a primitive group of diagonal type. Let $M = \mathrm{Soc}(G) = T_1 \times \cdots \times T_m$, where $T_i \cong T$ is a non-abelian finite simple group, for $i = 1,\dots,m$. Then G may be viewed as a subgroup of $M \cdot (\mathrm{Out}(T) \times S_m)$. Here, G_α is isomorphic to a subgroup of $\mathrm{Aut}(T) \times S_m$ and $M_\alpha \cong T$ is a diagonal subgroup of M, and so $|\mathcal{P}| = |T|^{m-1}$.

Lemma 3.4. *Let G be a flag-transitive point-primitive automorphism group of simple diagonal type with socle T^m. Then k divides $\lambda m_1 h$, where $m_1 \leqslant m$ and h divides $|T|$.*

Proof. See the proof of Proposition 3.1 in [22]. □

Proposition 3.5. *If $\mathcal{D} = (\mathcal{P},\mathcal{B})$ is a non-trivial symmetric (v,k,λ) design admitting a flag-transitive and point-primitive automorphism group G, where λ divides k and $k \geqslant \lambda^2$, then G is not of simple diagonal type.*

Proof. Suppose by contradiction that G is a primitive group of simple diagonal type. Then $v = |T|^{m-1}$, and so by Lemma 2.1(b), $\lambda v < k^2$. This implies that $\lambda|T|^{m-1} < k^2 = \lambda^2 t^2$. Since $\lambda^2 \leqslant k$, we must have $\lambda \leqslant t$, and hence

$$|T|^{m-1} < t^3. \tag{3.6}$$

Note by Lemma 3.4 that k divides $\lambda m_1 h$ and $m_1 h \leqslant m|T|$. Then t divides $m_1 h$, and so $t \leqslant m|T|$. We now apply (3.6) and conclude that $|T|^{m-1} < m^3|T|^3$. Therefore,

$|T|^{m-4} < m^3$. Since $|T| \geqslant 60$, we must have $m < 6$. If $m = 5$, then $|T| < 5^3$, and it follows that $T \cong A_5$. Note that k divides $\lambda(v-1) = \lambda(|T|^{m-1}-1)$. Then t divides $|T|^{m-1}-1 = 60^4-1 = 13\cdot 59\cdot 61\cdot 277$. Since $t \leqslant m|T| = 300$ and $t \geqslant 2$, it follows that $t \in \{13, 59, 61, 277\}$. For each such t, we have that $\lambda \leqslant t$ and $k = t\lambda$, and so we easily observe that these parameters does not satisfy Lemma 2.1(a). Therefore $m \in \{2, 3, 4\}$. Note that G_α is isomorphic to a subgroup of $\mathrm{Aut}(T) \times S_m$. Then by Lemma 2.1(b), the parameter k divides $|G_\alpha|$, and so k divides $(m!)\cdot|T|\cdot|\mathrm{Out}(T)|$. On the other hand, Lemma 2.1(a) implies that k divides $\lambda(|T|^{m-1}-1)$, and so t divides $|T|^{m-1}-1$ implying that $\gcd(t, |T|) = 1$. Since k divides $(m!)\cdot|T|\cdot|\mathrm{Out}(T)|$ and t is a divisor of k, we conclude that t divides $(m!)\cdot|\mathrm{Out}(T)|$. Recall by (3.6) that $|T|^{m-1} < t^3$. Therefore,

$$|T|^{m-1} < (m!)^3 \cdot |\mathrm{Out}(T)|^3, \tag{3.7}$$

where $m \in \{2, 3, 4\}$.

If $m = 2$, then $|T| < 8\cdot|\mathrm{Out}(T)|^3$. If $m = 3$, then $|T|^2 < 6^3|\mathrm{Out}(T)|^3$, and so $|T| < 6^{\frac{3}{2}}|\mathrm{Out}(T)|$. If $m = 4$, then $|T|^3 < 24^3|\mathrm{Out}(T)|^3$, and $|T| < 24|\mathrm{Out}(T)|$. Thus for $m \leqslant 4$, we always have

$$|T| < 8\cdot|\mathrm{Out}(T)|^3,$$

where T is a non-abelian finite simple group. We now apply Lemma 2.3 and conclude that T is isomorphic to A_5 or A_6. If $m = 2$, then since t divides $|T|^{m-1}-1 = |T|-1$, we have that t divides 59 or 359 when T is isomorphic to A_5 or A_6, respectively. Thus $(v, k, \lambda) = (60, 59\lambda, \lambda)$ or $(v, k, \lambda) = (360, 359\lambda, \lambda)$. Since $\lambda > 1$, in each case, we conclude that $k > v$, which is a contradiction. For $m = 3, 4$, since $|\mathrm{Out}(A_5)| = 2$ and $|\mathrm{Out}(A_6)| = 4$, it follows from (3.7) that $|T| < 48$ or $|T| < 96$ when T is isomorphic to A_5 or A_6, respectively, which is a contradiction. □

4 Proof of the main result

In this section, we prove Theorem 1.1. Suppose that $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a non-trivial symmetric (v, k, λ) design with λ divides k and $k \geqslant \lambda^2$. Suppose also that G is a flag-transitive automorphism group of $\mathcal{D}$.

Proof of Theorem 1.1. If G is point-primitive, then by O'Nan-Scott Theorem [15] and Propositions 3.2, 3.3 and 3.5, we conclude that G is of affine or almost simple type. Suppose now that G is point-imprimitive. Then G leaves invariant a non-trivial partition $\mathcal{C}$ of $\mathcal{P}$ with d classes of size c. By [20, Theorem 1.1], there is a constant l such that, for each $B \in \mathcal{B}$ and $\Delta \in \mathcal{C}$, $|B \cap \Delta| \in \{0, l\}$ and one of the following holds:

(a) $k \leqslant \lambda(\lambda-3)/2$;

(b) $(v, k, \lambda) = (\lambda^2(\lambda+2), \lambda(\lambda+1), \lambda)$ with $(c, d, l) = (\lambda^2, \lambda+2, \lambda)$ or $(\lambda+2, \lambda^2, 2)$;

(c)

$$(v, k, \lambda, c, d, l) = \left(\frac{(\lambda+2)(\lambda^2-2\lambda+2)}{4}, \frac{\lambda^2}{2}, \lambda, \frac{\lambda+2}{2}, \frac{\lambda^2-2\lambda+2}{2}, 2\right),$$

and either $\lambda \equiv 0 \pmod 4$, or $\lambda = 2u^2$, where u is odd, $u \geqslant 3$, and $2(u^2-1)$ is a square;

(d)

$$(v, k, \lambda, c, d, l) = \left(\frac{(\lambda+6)(\lambda^2+4\lambda-1)}{4}, \frac{\lambda(\lambda+5)}{2}, \lambda, \lambda+6, \frac{\lambda^2+4\lambda-1}{4}, 3\right),$$

where $\lambda \equiv 1$ or $3 \pmod 6$.

We easily observe that the cases (a) and (c) can be ruled out as $k \geqslant \lambda^2$. If case (d) occurs, then $\lambda(\lambda+5)/2 = k \geqslant \lambda^2$ implying that $\lambda \leqslant 5$. Since $\lambda \equiv 1$ or $3 \pmod 6$, it follows that $\lambda = 3$ for which $(v, k, \lambda, c, d, l) = (45, 12, 3, 9, 5, 3)$ which satisfies the condition in Theorem 1.1(b). Therefore, the case (b) can occur as claimed. □

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The symmetric genus spectrum of abelian groups

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Abstract

Let $\mathcal{S}$ denote the set of positive integers that appear as the symmetric genus of a finite abelian group and let $\mathcal{S}_0$ denote the set of positive integers that appear as the strong symmetric genus of a finite abelian group. The main theorem of this paper is that $\mathcal{S} = \mathcal{S}_0$. As a result, we obtain a set of necessary and sufficient conditions for an integer g to belong to $\mathcal{S}$. This also shows that $\mathcal{S}$ has an asymptotic density and that it is approximately 0.3284.

Keywords: Symmetric genus, strong symmetric genus, Riemann surface, abelian groups, genus spectrum, density.

Math. Subj. Class.: 57M60, 30F35, 20F38

1 Introduction

Let G be a finite group. Among the various genus parameters associated with G, one of the most important is the *symmetric* genus $\sigma(G)$, the minimum genus of any Riemann surface on which G acts faithfully. The origins of this parameter can be traced back over a century to the work of Hurwitz, Poincare, Burnside and others. The modern terminology was introduced in the important article [10].

A natural problem is to determine the positive integers that occur as the symmetric genus of a group (or a particular type of group), that is, to determine the symmetric genus spectrum for the particular type of group. Important results about the symmetric genus spectrum of all finite groups were obtained by Conder and Tucker [1]. They showed that the symmetric genus spectrum of finite groups contains well over 88 percent of all positive integers. In particular, they showed that if g is any non-negative integer such that g is not congruent to 8 or $14 \pmod{18}$, then g is in the spectrum [1, Theorem 1.2]. However, there are no known gaps in the spectrum, and evidence suggests that there are none. Here see [1, Conjecture 1.3].

Our focus here is the symmetric genus spectrum of abelian groups. Let

$$\mathcal{S} = \{g \in \mathbb{N} : g = \sigma(A) \text{ for some abelian group } A\}$$

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denote the symmetric genus spectrum of abelian groups. Henceforth, we will refer to $\mathcal{S}$ simply as the "spectrum."

Closely related to the symmetric genus is the *strong symmetric* genus $\sigma^0(G)$, the minimum genus of any Riemann surface on which G acts faithfully and preserving orientation. Obviously $\sigma(G) \leq \sigma^0(G)$ always, but in some (important) cases, the two parameters agree. If the group G does not have a subgroup of index 2, then G cannot act on a surface reversing orientation and thus $\sigma(G) = \sigma^0(G)$. In particular, if G is a group of odd order, then $\sigma(G) = \sigma^0(G)$. Spectrum questions about this parameter have been considered, with some success. The basic problem was settled for the family of all finite groups in [6]: there is a group of strong symmetric genus g, for all $g \in \mathbb{N}$. The strong symmetric genus spectrum of abelian groups was studied in [3]. Let

$$\mathcal{S}_0 = \left\{g \in \mathbb{N} : g = \sigma^0(A) \text{ for some abelian group } A\right\}.$$

Necessary and sufficient conditions were developed in [3] for an integer g to belong to the spectrum of abelian groups for this parameter; further, this spectrum was shown to have an asymptotic density, approximately equal to 0.3284. In addition, the strong symmetric genus spectrum of nilpotent groups was shown to have lower asymptotic density greater than or equal to $\frac{8}{9}$ in [9].

Our original expectation was that the two spectra $\mathcal{S}$ and $\mathcal{S}_0$ would have a significant intersection, but that there would be integers that are in each spectrum but not the other. Interestingly, this is not the case. Our main result is the following.

Theorem 1.1. $\mathcal{S} = \mathcal{S}_0$.

Our fundamental tool here is the result that determines the symmetric genus $\sigma(A)$ of an abelian group A [5, Theorem 5.7]. An easy but important consequence of this result is that the spectrum $\mathcal{S}$ contains the entire congruence class $g \equiv 1 \pmod 4$.

To establish the containment $\mathcal{S} \subset \mathcal{S}_0$, we show that, given an abelian group A, either $\sigma(A) \equiv 1 \pmod 4$ or $\sigma(A) = \sigma^0(A)$ (or both), unless the Sylow 2-subgroup group of A has rank 2 and is isomorphic to $Z_2 \times Z_{2^\ell}$, for some $\ell \geq 1$. Our approach utilizes the strong constraints on the Sylow 2-subgroup of a group (not necessarily abelian) acting on a surface of even genus or a surface of genus congruent to $3 \pmod 4$; here see [7, Theorem 8] and [8, Theorem 5]. In the exceptional case (in which the Sylow 2-subgroup has a special form), we show that there exists an abelian group A_1 such that $\sigma(A) = \sigma^0(A_1)$.

To establish the reverse containment $\mathcal{S}_0 \subset \mathcal{S}$, we utilize the characterization of the integers in the spectrum $\mathcal{S}_0$ in [3, Theorem 1]. For each integer g satisfying one of the five conditions in that result, we exhibit an abelian group G such that $g = \sigma(G)$.

2 Background results

Let A be a non-trivial finite abelian group of rank r. Then A has the standard canonical representation

$$A \cong Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_r}, \tag{2.1}$$

with invariants $m_1, m_2, \ldots, m_r$ subject to $m_1 > 1$ and $m_i \mid m_{i+1}$ for $1 \leq i < r$.

The abelian group A also has another canonical form that is useful in calculating genus parameters of abelian groups. Define the *alternate canonical form* for A as the direct product of three subgroups T, B and D of A. First, the group D is the subgroup of A

generated by the factors Z_{m_s}, where m_s is divisible by 4. Then write $A = D \times \overline{D}$. Now let T be the Sylow 2-subgroup of $\overline{D}$, which is elementary abelian, and B be its direct summand of odd order. Therefore, $A = T \times B \times D$. Define $t = \text{rank}(T)$, $b = \text{rank}(B)$, $d = \text{rank}(D)$. It follows that $r = \text{rank}(A) = d + \max(b, t)$. We point out that this notation differs from that used in [5].

The groups of symmetric genus zero are the classical, well-known groups that act on the Riemann sphere (possibly reversing orientation) [2, §6.3.2]. The abelian group A has symmetric genus zero if and only if A is Z_n, $Z_2 \times Z_{2n}$, or $(Z_2)^3$; see [2, §6.3.2].

The groups of symmetric genus one have also been classified, at least in a sense. These groups act on the torus and fall into 17 classes, corresponding to quotients of the 17 Euclidean space groups [2, §6.3.3]. Each class is characterized by a presentation, typically a partial one. The abelian group A has symmetric genus one if and only if A is $Z_m \times Z_{mn}$ with $m \geq 3$, $Z_2 \times Z_2 \times Z_{2n}$ with $n \geq 2$, or $(Z_2)^4$; see [2, §6.3.3].

Let A be a finite abelian group. The strong symmetric genus of A has been completely determined by Maclachlan [4, Theorem 4], and if A has odd order, then $\sigma(A) = \sigma^0(A)$.

The focus of [5] was the determination of the symmetric genus of an abelian group of even order. The approach was to show that, among the various genus actions of A, there is one induced by an NEC group with a signature of one of three types. We established the following result [5, Theorem 3.10].

Theorem A. *Let A be an abelian group of even order. Among the NEC groups with minimal non-euclidean area that act on A, there is a group Γ whose signature has one of the following forms:*

(I) $(g, +, [\lambda_1, \ldots, \lambda_n], \{\ \})$;

(II) $(0, +, [\lambda_1, \ldots, \lambda_s], \{(\ )^k\})$ *for some* $k \geq 1$;

(III) $(0, +, [\,], \{(\ )^u, (2^v)\})$ *for some* $v \geq 2$.

Furthermore, in cases (I) and (II), λ_i divides λ_{i+1} for $1 \leq i \leq r-1$.

In (III) the notation (2^v) means, as usual, a period cycle with v link periods equal to 2.

We denote by $\tau(A)$ (here and in [5]) the minimum genus of any action of A induced by an NEC group of Type II. The size of the largest elementary abelian 2-group factor of A determines whether $\sigma(A)$ is given by an action induced by a group of Type I, II or III. The main result of [5] is the following [5, Theorem 5.7].

Theorem B. *Let A be an abelian group of even order with canonical form*

$$A \cong (Z_2)^a \times Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_q},$$

where $m_1 > 2$. If the symmetric genus $\sigma(A) \geq 2$, then

(i) $\sigma(A) = 1 + |A| \cdot (a + 3q - 4)/8$, *if* $a \geq q + 2$;

(ii) $\sigma(A) = \tau(A)$, *if* $1 \leq a \leq q + 1$;

(iii) $\sigma(A) = \min\{\sigma^0(A), \tau(A)\}$, *if* $a = 0$.

Thus Theorem B gives the symmetric genus of an abelian group A in terms of the invariants of A and the numbers $\sigma^0(A)$ and $\tau(A)$.

The main result in [3] is the characterization [3, Theorem 1] of the integers in the spectrum $\mathcal{S}_0$, and this will be important here.

Theorem C. *Let $g \geq 2$. Then $g \in \mathcal{S}_0$ if and only if g satisfies one of the following conditions:*

(i) $g \equiv 1 \pmod 4$ *or* $g \equiv 55 \pmod{81}$*;*

(ii) $g - 1$ *is divisible by* p^4 *for some odd prime* p*;*

(iii) $g - 1$ *is divisible by* a^2 *for some odd integer* a *with* $(a-1) \mid g$*;*

(iv) $g-1$ *is divisible by* $b^2a^2(a-1)$ *for some odd integers* $a, b > 1$*, with* $a \equiv 3 \pmod 4$*.*

3 $\mathcal{S}_0 \subset \mathcal{S}$

To establish the containment $\mathcal{S}_0 \subset \mathcal{S}$, we use the characterization of the integers in the spectrum $\mathcal{S}_0$ in Theorem C. For each integer g satisfying one of the five conditions in that result, we exhibit an abelian group G such that $g = \sigma(G)$. This is quite easy, as we shall see.

In this section, we will assume that A is always written in alternate canonical form, $A = T \times B \times D$.

We begin by noting some consequences of Theorem B. Directly from part (i) we have the following; this formula was also pointed out in [7, p. 4094].

Proposition 3.1. $\sigma(Z_2^3 \times Z_{2m}) = 1 + 4m$ *for any integer* $m \geq 2$*.*

Since $\sigma((Z_2)^4) = 1$ and $\sigma((Z_2)^5) = 5$ [7] (the general genus formula is $\sigma((Z_2)^n) = 1 + 2^{n-3}(n-4)$ [5, Corollary 5.4]), it follows that the spectrum $\mathcal{S}$ contains the entire congruence class $g \equiv 1 \pmod 4$. These odd integers are also in $\mathcal{S}_0$ [3, p. 342].

A special case of Theorem B [5, p. 423] will be important here.

Theorem 3.2. *Let the abelian group A have alternate canonical form $A = T \times B \times D$. If T is trivial, then $\sigma(A) = \sigma^0(A)$.*

Let A be a finite abelian group. Then $\sigma(A) = \sigma^0(A)$ in another important case.

Lemma 3.3. *Let A be an abelian group of rank three or more. If the Sylow 2-subgroup S_2 of A is cyclic, then $\sigma(A) = \sigma^0(A)$.*

Proof. By Theorem 3.2, we may assume that T is non-trivial. If S_2 is cyclic, then we must have $S_2 = Z_2$ and $D = 1$. Now write $A \cong Z_2 \times B \cong Z_2 \times Z_{\beta_1} \times Z_{\beta_2} \times \cdots \times Z_{\beta_b}$ for $b \geq 3$, where each β_i is odd. In this case, $t = 1$, $d = 0$ and $b \geq 3$. Since $t \leq d + 1$, [5, p. 416] gives

$$\tau(A) = 1 + \frac{1}{2}|A| \left(-1 + \sum_{i=1}^{b} \left(1 - \frac{1}{\beta_i} \right) \right)$$

(see also (4.1) in Section 4).

The group A has canonical form $A = Z_{\beta_1} \times Z_{\beta_2} \times \cdots \times Z_{2\beta_b}$, and is the image of a Fuchsian group Γ with signature $(0, +, [\beta_1, \ldots, 2\beta_b, 2\beta_b], \{\ \})$. When we calculate the genus arising from the action of this Fuchsian group on A, we get that it is equal to $\tau(A)$. Now applying Maclachlan's formula shows that $\sigma^0(A) \leq \tau(A)$ and hence $\sigma(A) = \sigma^0(A)$ by Theorem B. $\square$

Theorem 3.4. $\mathcal{S}_0 \subset \mathcal{S}$.

Proof. First suppose that $g \equiv 1 \pmod 4$. Then $g \in \mathcal{S}$ by Proposition 3.1 (and the comments after its statement).

Next suppose that $g \equiv 55 \pmod{81}$. Write $g = 55+81j$. Let $G = Z_3 \times Z_3 \times Z_3 \times Z_{3n}$, where $n = j + 1$. Then by Maclachlan's formula, $g = \sigma^0(G)$; here also see [3, p. 344]. But since the Sylow 2-subgroup of G is clearly cyclic, we also have $\sigma(G) = \sigma^0(G) = g$ by Lemma 3.3.

Now assume that $g = 1 + mp^4$ for an odd prime p. First let $p \geq 5$ and set $G = Z_p \times Z_p \times Z_p \times Z_{mp}$. Then $g = \sigma^0(G)$; see [3, p. 344]. Again the Sylow 2-subgroup of G is cyclic, and $\sigma(G) = \sigma^0(G) = g$. For the prime $p = 3$, see the calculations in [3, p. 344] and use Lemma 3.3.

Suppose that g satisfies condition (iii) of Theorem C for some odd a. Then, as shown in the proof of Proposition 5 of [3, p. 343], g is the strong symmetric genus of the group $G = Z_a \times Z_a \times Z_{an}$ for some n. Once again, $\sigma(G) = \sigma^0(G) = g$ by Lemma 3.3.

Finally, assume that g satisfies condition (iv) of Theorem C for some odd $a, b > 1$, with $a \equiv 3 \pmod 4$. Then $g = \sigma^0(G)$, where G is a group of the form $Z_a \times Z_{ab} \times Z_{abn}$ [3, p. 343], a group with a cyclic Sylow 2-subgroup, so that $\sigma(G) = \sigma^0(G) = g$ by Lemma 3.3. □

4 The Type II genus

The Type II genus $\tau(A)$ was considered in [5, §4]. The genus $\tau(A)$ is the minimum genus of any action of A induced by an NEC group with signature $(0, +, [\lambda_1, \ldots, \lambda_s], \{(\,)^k\})$ for some $k \geq 1$ (the integer k is the number of empty period cycles). Among the signatures of the NEC groups that induce the Type II genus $\tau(A)$, Lemmas 4.2 and 4.3 of [5] identify one value of k that can be used to calculate $\tau(A)$. These two lemmas are correct. Unfortunately, there is a mistake in [5, Formula (4.5)], which is used in the final determination of $\tau(A)$. We correct that here.

Let $A = T \times B \times D$ be the alternate canonical form for A. Remember that $t = \operatorname{rank}(T)$, $b = \operatorname{rank}(B)$, $d = \operatorname{rank}(D)$ and so $r = \operatorname{rank}(A) = d + \max(b, t)$. The odd order group B is generated by elements with orders $\beta_1, \ldots, \beta_b$, where β_i divides β_j for $i < j$.

In the case in which $t \leq d + 1$, the formula for $\tau(A)$ [5, p. 416] is correct. Note that $k = t$ gives minimal area by [5, Lemma 4.2]. In the new notation, this formula is

$$\tau(A) = 1 + \frac{1}{2}|A| \left(k - 2 + \sum_{i=1}^{b} \left(1 - \frac{1}{\beta_i}\right) + \sum_{i=1}^{d+1-t} \left(1 - \frac{1}{\delta_i}\right) \right), \tag{4.1}$$

where the group D is generated by elements with orders $\delta_1, \ldots, \delta_d$ satisfying δ_i divides δ_j for $i < j$.

Next, we consider the case $t > d + 1$.

Theorem 4.1. *Let A be an abelian group in alternate canonical form. Suppose that $t > d + 1$ and k is given by [5, Lemma 4.3]. There are two cases and in each case, define $\nu = b + d - k + 1$.*

(i) Suppose that $t + d$ is odd. Then $k = (t + d + 1)/2$;

(a) If $b \leq (k - 1) - d$, then $\tau(A) = 1 + \frac{1}{2}|A|(k - 2)$;

(b) If $b > (k-1) - d$*, then* $\nu \geq 1$ *and*

$$\tau(A) = 1 + \frac{1}{2}|A|\left(k - 2 + \sum_{i=1}^{\nu}\left(1 - \frac{1}{\beta_i}\right)\right);$$

(ii) Suppose that $t + d$ *is even. Then* $k = (t+d)/2$*;*

(a) If $b \leq (k-1) - d$*, then* $\tau(A) = 1 + \frac{1}{2}|A|(k - \frac{3}{2})$*;*

(b) If $b > (k-1) - d$*, then* $\nu \geq 1$ *and*

$$\tau(A) = 1 + \frac{1}{2}|A|\left(k - 2 + \sum_{i=1}^{\nu-1}\left(1 - \frac{1}{\beta_i}\right) + \left(1 - \frac{1}{2\beta_\nu}\right)\right).$$

Proof. Among the NEC groups that induce the Type II genus $\tau(A)$, let Γ be one with signature $(0; +; [\lambda_1, \dots; \lambda_s]; (\ )^k)$ in which the number k of empty period cycles is given by $k = [(t + d + 1)/2]$ [5, Lemma 4.3, p. 414]. It follows that $k \geq d + 1$. The group Γ has generators $x_1, \dots, x_s$, $e_1, \dots, e_k$, and involutions $c_1, \dots, c_k$. The defining relations for Γ consist of $x_1 \cdots x_s e_1 \cdots e_k = 1$, conditions on the order of the elements x_i and certain elements commuting. Clearly, one generator is redundant, and, since $\mu(\Gamma)$ is minimal, we may assume that e_k is that generator. Now let $a_1, \dots, a_r$ be a generating set for A in canonical form so that the orders of these elements satisfy the standard divisibility condition. With $\mu(\Gamma)$ minimal, the elements $e_1, \dots, e_{k-1}$ are mapped onto the subgroup generated by the $k-1$ elements $a_{r-k+2}, \dots, a_r$ of highest order. In particular, since $k-1 \geq d$, the subgroup D of A is contained in the image of $\langle e_1, \dots, e_{k-1}\rangle$.

If $t + d$ is odd, then $2k - 1 = t + d$. Since $t + d$ is the rank of the Sylow 2-subgroup S_2 of A, we have that S_2 is contained in the image of $\langle c_1, \dots, c_k, e_1, \dots, e_{k-1}\rangle$. It follows that $\langle T, D\rangle$ is contained in the image of $\langle c_1, \dots, c_k, e_1, \dots, e_{k-1}\rangle$. If $t + d$ is even, then $2k = t + d$. In this case, there is an additional generator x_ℓ so that $\langle T, D\rangle$ is contained in the image of $\langle x_\ell, c_1, \dots, c_k, e_1, \dots, e_{k-1}\rangle$.

If $b \leq (k-1) - d$, then the images of the generators e_i which are not mapped into D generate all of B. Therefore, if $t + d$ is odd, then A is the image of the NEC group with signature $(0; +; [\,]; \{(\ )^k\})$ and $\tau(A) = 1 + \frac{1}{2}|A|(k-2)$. If $t + d$ is even, then we need a generator x_1 in order to map onto A, and with $\mu(\Gamma)$ minimal, $|x_1| = 2$. Therefore, if $t + d$ is even, then A is the image of the NEC group with signature $(0; +; [2]; \{(\ )^k\})$ and $\tau(A) = 1 + \frac{1}{2}|A|(k - \frac{3}{2})$.

The last case is when $b > (k-1) - d$. Let E be the subgroup of A generated by the images of $e_1, \dots, e_{k-1}$. Then the subgroup B can be decomposed as $B = B_1 \times B_2$, where $B_2 = B \cap E$. Let $\nu = b + d - k + 1$ so that ν is the rank of B_1 and $\nu \geq 1$. We need generators $x_1, \dots, x_\nu$ to map onto B_1. If $t + d$ is odd, then A is the image of the NEC group with signature $(0; +; [\beta_1, \dots, \beta_\nu]; \{(\ )^k\})$ and

$$\tau(A) = 1 + \frac{1}{2}|A|\left(k - 2 + \sum_{i=1}^{\nu}\left(1 - \frac{1}{\beta_i}\right)\right).$$

Suppose that $t + d$ is even. As in the previous case, we need generators $x_1, \dots, x_\nu$ to map onto B_1. However, the generator x_ν must map onto an element of order $2\beta_\nu$ for the NEC group to map onto A. This is because there is the extra involution not in the image of $\langle c_1, \ldots, c_k \rangle$. If $t + d$ is even, then A is the image of the NEC group with signature $(0; +; [\beta_1, \ldots, \beta_{\nu-1}, 2\beta_\nu]; \{(\ )^k\})$ and

$$\tau(A) = 1 + \frac{1}{2}|A| \left(k - 2 + \sum_{i=1}^{\nu-1} \left(1 - \frac{1}{\beta_i}\right) + \left(1 - \frac{1}{2\beta_\nu}\right) \right). \qquad \square$$

5 General results

Again in this section, we assume that A is written in alternate canonical form, $A = T \times B \times D$.

Let A be a finite abelian group so that the integer $g = \sigma(A)$ is in the spectrum $\mathcal{S}$. We want to show that g is in $\mathcal{S}_0$ as well. This is clearly the case if A has rank one or two or A has a trivial factor T in its alternate canonical form. Thus we may assume that A has rank at least 3 and T is not trivial. In particular, A has even order.

Our approach utilizes the Sylow 2-subgroup S_2 of A.

Lemma 5.1. *Let A be an abelian group of rank three or more with $\sigma(A) \geq 2$. If the Sylow 2-subgroup of A is isomorphic to $(Z_2)^3$, then $\tau(A) \equiv 1 \pmod 4$.*

Proof. Since $S_2 \cong (Z_2)^3$, we have $T = S_2$, $D = 1$ and $A = T \times B$. Now $t = 3$, $d = 0$, and $k = 2$. Since $b = 1$ would imply $\sigma(A) = 1$, $b \geq 2$. We have $t > d + 1$ with $t + d$ odd. The Type II genus is $\tau(A) = 1 + |A| \cdot M/2$, where $M = (k - 2 + \sum_{i=1}^{\nu}(1 - \frac{1}{\beta_i}))$ by Theorem 4.1(i)(b). Since $M \cdot |B|$ is an integer and $|A| = 8|B|$, we clearly have $\tau(A) \equiv 1 \pmod 4$. $\square$

Lemma 5.2. *Let A be an abelian group of rank three or more with $\sigma(A) \geq 2$. If the Sylow 2-subgroup of A is isomorphic to $Z_2 \times Z_2 \times Z_{2^\ell}$ for some $\ell \geq 2$, then $\tau(A) \equiv 1 \pmod 4$.*

Proof. In this case we have $T = Z_2 \times Z_2$ and D is cyclic with order divisible by 2^ℓ. Therefore, $t = 2$ and $d = 1$. This implies that $k = t = 2$ by [5, Lemma 4.2]. Since $b = 1$ would imply $\sigma(A) = 1$, $b \geq 2$. We have $t = d + 1$ and by (4.1), the Type II genus $\tau(A) = 1 + |A| \cdot M/2$ where $M = (k - 2 + \sum_{i=1}^{b}(1 - \frac{1}{\beta_i}))$. Since $M \cdot |B|$ is an integer and $|A| = 4 \cdot |D| \cdot |B|$ with $|D|$ divisible by 2^ℓ, again $\tau(A) \equiv 1 \pmod 4$. $\square$

Lemma 5.3. *Let A be an abelian group. If the Sylow 2-subgroup of A is isomorphic to $(Z_2)^4$, then $\tau(A) \equiv 1 \pmod 4$.*

Proof. Since $S_2 \cong (Z_2)^4$, we have $T = S_2$, $D = 1$ and $A = T \times B$. Now $t = 4$, $d = 0$, $k = 2$ and $b \geq 1$. We have $t > d+1$ with $t+d$ even. We apply Theorem 4.1(ii) and get that $k = 2$. If $b = 1$, then $M = 1/2$ by Theorem 4.1(ii)(a). If $b \geq 2$, then by Theorem 4.1(ii)(b), $M \cdot 2|B|$ is an integer. In either case, $2M \cdot |B|$ is an integer. Since $|A| = 16|B|$, we again have $\tau(A) \equiv 1 \pmod 4$. $\square$

Theorem 5.4. *Let A be an abelian group of rank three or more with $\sigma(A) \geq 2$. Then either $\sigma(A) \equiv 1 \pmod 4$ or $\sigma(A) = \sigma^0(A)$ (or both), unless the Sylow 2-subgroup of A has rank 2 and is isomorphic to $Z_2 \times Z_{2^\ell}$, for some $\ell \geq 1$.*

Proof. First, if A has odd order, then $\sigma(A) = \sigma^0(A)$. Assume, then, that A has even order so that $\sigma(A)$ is given by Theorem B. Let A have canonical form

$$A \cong (Z_2)^a \times Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_q},$$

as in Theorem B. It is easy to see in case (i), we always have that $\sigma(A) \equiv 1 \pmod 4$.

Suppose $a \leq q+1$. By Theorem B, $\sigma(A)$ is either equal to $\sigma^0(A)$ or $\tau(A)$. Let A act on a surface X of genus $g = \sigma(A) \geq 2$, and write $|A| = 2^n \cdot m$, where m is odd.

Assume first that g is even. Then by [7, Theorem 9], A is not a 2-group so that $m \neq 1$. We consider the possibilities for the Sylow 2-subgroup S_2 of A. If S_2 is cyclic, then by $\sigma(A) = \sigma^0(A)$ by Lemma 3.3. Assume then that S_2 is not cyclic. If A contains an element of order 2^{n-1} with $|S_2| = 2^n$, then S_2 is isomorphic to $Z_2 \times Z_{2^{n-1}}$, the exceptional case.

Assume then that A has no elements of order 2^{n-1}, and apply [7, Theorem 8]. Since A and S_2 are abelian, the only possibility is that S_2 is isomorphic to $(Z_2)^3$. But in this case $\tau(A) \equiv 1 \pmod 4$ by Lemma 5.1.

Therefore, by Theorem B, if g is even, then either $\sigma(A) \equiv 1 \pmod 4$ or $\sigma(A) = \sigma^0(A)$, unless $S_2 \cong Z_2 \times Z_{2^\ell}$, for some $\ell \geq 1$.

Now suppose $g \equiv 3 \pmod 4$, and use [8, Theorem 5]. The Sylow 2-subgroup S_2 also acts of the surface X of genus $g \geq 2$. By [8, Theorem 5], S_2 contains an element of order 2^{n-3} or larger. Further, if $\mathrm{Exp}(S_2) = 2^{n-3}$, then S_2 contains a dihedral subgroup of index 4. We consider the possibilities for $\mathrm{Exp}(S_2)$.

If S_2 is cyclic, then by $\sigma(A) = \sigma^0(A)$ by Lemma 3.3.

If S_2 is not cyclic and contains an element of order 2^{n-1}, then S_2 is isomorphic to $Z_2 \times Z_{2^{n-1}}$, the exceptional case.

Suppose $\mathrm{Exp}(S_2) = 2^{n-2}$. Then S_2 is isomorphic to either $Z_2 \times Z_2 \times Z_{2^{n-2}}$ or $Z_4 \times Z_{2^{n-2}}$. If $S_2 \cong Z_2 \times Z_2 \times Z_{2^{n-2}}$, then by Lemma 5.2, $\tau(A) \equiv 1 \pmod 4$. If on the other hand, $S_2 \cong Z_4 \times Z_{2^{n-2}}$, then by Theorem 3.2, $\sigma(A) = \sigma^0(A)$.

Suppose $\mathrm{Exp}(S_2) = 2^{n-3}$ and S_2 has a dihedral subgroup of index 4. Since S_2 is abelian, this forces $n = 4$ and $S_2 \cong (Z_2)^4$. In this case, $\tau(A) \equiv 1 \pmod 4$ by Lemma 5.3.

Therefore, by Theorem B, if $g \equiv 3 \pmod 4$, then either $\sigma(A) = \tau(A) \equiv 1 \pmod 4$ or $\sigma(A) = \sigma^0(A)$, unless $S_2 \cong Z_2 \times Z_{2^\ell}$, for some $\ell \geq 1$. □

A consequence of the proof is perhaps worth noting, in connection with the well-known conjecture that "almost all" groups are 2-groups.

Theorem 5.5. *If A be an abelian 2-group of positive symmetric genus, then*

$$\sigma(A) \equiv 1 \pmod 4.$$

Proof. Assume A is an abelian 2-group with $\sigma(A) \geq 2$. Then $\sigma(A)$ is not even by [7, Theorem 9]. The proof of Theorem 5.4 shows that $\sigma(A)$ cannot be congruent to $3 \pmod 4$ either, since $A = S_2$ and A is not an abelian group of genus zero or one. □

Next we handle the exceptional case in Theorem 5.4.

Theorem 5.6. *Let A be an abelian group of rank three or more with $\sigma(A) \geq 2$. If the Sylow 2-subgroup of A is isomorphic to $Z_2 \times Z_{2^\ell}$, for some $\ell \geq 1$, then there exists an abelian group A_1 such that $\sigma(A) = \sigma^0(A_1)$.*

Proof. Let A have alternate canonical form $A = T \times B \times D$. We consider two cases, $\ell = 1$ and $\ell \geq 2$.

First assume that $S_2 \cong (Z_2)^2$. Now $T = S_2$, $D = 1$ and $A = T \times B$, with $t = 2$, $d = 0$, and $k = 1$. Write $A \cong Z_2 \times Z_2 \times B \cong Z_2 \times Z_2 \times Z_{\beta_1} \times Z_{\beta_2} \times \cdots \times Z_{\beta_b}$ for $b \geq 3$, where each β_i is odd. By Theorem 4.1(ii)(b), the Type II genus is

$$\tau(A) = 1 + \frac{1}{2}|A| \left(-1 + \sum_{i=1}^{b-1} \left(1 - \frac{1}{\beta_i}\right) + \left(1 - \frac{1}{2\beta_b}\right) \right).$$

By Theorem B, $\sigma(A) = \min\{\sigma^0(A), \tau(A)\}$. If $\sigma(A) = \sigma^0(A)$, then set $A_1 = A$ and we are done. So we assume that $\tau(A) < \sigma^0(A)$. Maclachlan's formula uses the non-euclidean areas of the groups Δ_0 with signature $(0, +, [\beta_1, \ldots, 2\beta_{b-1}, 2\beta_b, 2\beta_b], \{\ \})$ and Γ_p with signatures $(p, +, [\beta_1, \ldots, \beta_{b-2p}, \beta_{b-2p}], \{\ \})$ for $p \geq 1$ to obtain $\sigma^0(A)$.

Let $A_1 = Z_{\beta_1} \times \cdots \times Z_{\beta_{b-1}} \times Z_{4\beta_b}$ be an abelian group. Maclachlan's formula for $\sigma^0(A_1)$ uses the minimum non-euclidean areas of the Fuchsian groups Γ_0 with signature $(0, +, [\beta_1, \ldots, \beta_{b-1}, 4\beta_b, 4\beta_b], \{\ \})$ and Γ_p for $p \geq 1$ as in the calculation of $\sigma^0(A)$. By assumption the Fuchsian groups Γ_p for $p \geq 1$ give genus larger than $\tau(A)$. The Fuchsian group Γ_0 gives the same genus as $\tau(A)$. Therefore, $\tau(A) = \sigma^0(A_1)$ and so $\sigma(A) = \sigma^0(A_1)$.

Now assume that $S_2 \cong Z_2 \times Z_{2^\ell}$, for some $\ell \geq 2$. Now $T = Z_2$ and D is isomorphic to Z_{m2^ℓ}, where m is odd. We have alternate canonical form $A = Z_2 \times B \times Z_{m2^\ell} \cong Z_2 \times Z_{\beta_1} \times Z_{\beta_2} \times \cdots \times Z_{\beta_{b-1}} \times Z_{m2^\ell}$, where each β_i is odd, β_i divides β_j for $i < j$ and m is divisible by β_{b-1}. It follows that $t = 1$, $b \geq 3$, $d = 1$, and $k = 1$ by [5, Lemma 4.2]. By (4.1) the Type II genus is

$$\tau(A) = 1 + \frac{1}{2}|A| \left(-1 + \sum_{i=1}^{b} \left(1 - \frac{1}{\beta_i}\right) + \left(1 - \frac{1}{m2^\ell}\right) \right).$$

By Theorem B, $\sigma(A) = \min\{\sigma^0(A), \tau(A)\}$. If $\sigma(A) = \sigma^0(A)$, then let $A_1 = A$ and we are done. So we will assume that $\tau(A) < \sigma^0(A)$. Now let $A_1 = Z_{\beta_1} \times \cdots \times Z_{\beta_b} \times Z_{m2^{\ell+1}}$. The Fuchsian group with signature $(0, +, [\beta_1, \ldots, \beta_b, m2^{\ell+1}, m2^{\ell+1}], \{\ \})$ has genus action equal to $\tau(A)$. As in the previous case, $\tau(A) = \sigma^0(A_1)$ and so $\sigma(A) = \sigma^0(A_1)$. □

Combining Theorems 5.4 and 5.6 yields the following.

Theorem 5.7. $\mathcal{S} \subset \mathcal{S}_0$.

Of course, Theorems 5.7 and 3.4 provide the proof of Theorem 1.1. Theorem 1.1 and the results in [3] give some interesting results about the symmetric genus spectrum $\mathcal{S}$ of abelian groups. Now Theorem C gives a necessary and sufficient condition for a positive integer to be in the spectrum $\mathcal{S}$. Two other consequences are perhaps worth stating.

Corollary 5.8. *The spectrum $\mathcal{S}$ has an asymptotic density $\delta(\mathcal{S}) \approx 0.3284$.*

Corollary 5.9. *If $g - 1$ is a square-free integer, then $g \notin \mathcal{S}$.*

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Extremal embedded graphs*

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Abstract

Let G be a ribbon graph and $\mu(G)$ be the number of components of the virtual link formed from G as a cellularly embedded graph via the medial construction. In this paper we first prove that $\mu(G) \leq f(G)+\gamma(G)$, where $f(G)$ and $\gamma(G)$ are the number of boundary components and Euler genus of G, respectively. A ribbon graph is said to be extremal if $\mu(G) = f(G) + \gamma(G)$. We then obtain that a ribbon graph is extremal if and only if its Petrial is plane. We introduce a notion of extremal minor and provide an excluded extremal minor characterization for extremal ribbon graphs. We also point out that a related result in the monograph by Ellis-Monaghan and Moffatt is not correct and prove that two related conjectures raised by Huggett and Tawfik hold for more general ribbon graphs.

Keywords: Ribbon graph, medial graph, Petrie dual, extremal minor, orientation.

Math. Subj. Class.: 05C10, 05C83, 57M15

1 Introduction

A cellularly embedded graph is a graph G embedded in a surface Σ such that every connected component of $\Sigma - G$ is a 2-cell, called a face. If $G \subset \Sigma$, the homeomorphism class of the surface Σ generates an equivalence class of cellularly embedded graphs, and we say that cellularly embedded graphs are equal if they are in the same equivalence class. We assume familiarity with cellularly embedded graphs, referring the reader to [6, 13] for further details.

We use standard notations $V(G)$, $E(G)$ and $F(G)$ to denote the sets of vertices, edges, and faces, respectively, of a cellularly embedded graph G and $v(G) = |V(G)|$, $e(G) =$

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$|E(G)|$ and $f(G) = |F(G)|$, respectively. In general, a finite number of connected cellularly embedded graphs will form a disconnected cellulary embedded graph. Let $k(G)$ denote the number of connected components of G. A graph is said to be bipartite if it does not contain odd cycles. A face of a cellularly embedded graph is called even if its boundary has an even number of edges (it is possible that a single edge appears twice in the boundary of a face and under such circumstances the edge will be counted twice).

Let $\mu(G)$ be the number of components of the virtual link formed from a cellulary embedded graph G via the medial construction. It was originally called *left-right paths* of cellularly embedded graphs in [16]. Let $T_G(x, y)$ be the Tutte polynomial [17] of the graph G. It is shown in [10, 11] that

$$T_G(-1,-1) = (-1)^{e(G)}(-2)^{\mu(G)-1},$$

if G is embedded in the plane, the real projective plane or the torus. For many families of planar graphs, their link component numbers have been determined, see, for example, [5, 8, 14, 15].

Cellularly embedded graphs are equivalent to ribbon graphs. In this paper we mainly work in the language of ribbon graphs. In this paper we first prove that $\mu(G) \leq f(G) + \gamma(G)$ for any ribbon graph G, where $f(G)$ and $\gamma(G)$ are the number of boundary components (i.e. faces of the corresponding cellularly embedded graph) and Euler genus of G, respectively. A ribbon graph is said to be *extremal* if $\mu(G) = f(G) + \gamma(G)$ and a ribbon graph is plane if its Euler genus is zero. In [9], the authors studied extremal plane graphs and in [7], Huggett and Tawfik studied extremal cellularly embedded graphs on orientable surfaces of positive genus. In this paper we extend their results to any ribbon graph and also obtain that a ribbon graph is extremal if and only if its Petrial is plane. In [12], Moffatt introduced the notion of minors of ribbon graphs and gave an excluded minor characterization of the family of ribbon graphs that represent knot and link diagrams. In [3], Chudnovsky et al introduced a notion of bipartite minor and proved a bipartite analog of Wagner's theorem. In this paper we imitate them and introduce a notion of extremal minor and provide an excluded extremal minor characterization for extremal ribbon graphs.

In [7], Huggett and Tawfik also conjectured

Conjecture 1.1 ([7])**.** *If G is an extremal graph cellularly embedded on a torus then each face of G is even.*

Furthermore,

Conjecture 1.2 ([7])**.** *If G is an extremal graph cellularly embedded on a torus then G is bipartite.*

It is obvious Conjecture 1.2 implies Conjecture 1.1. In this paper, we shall show Conjecture 1.1 holds for any extremal cellularly embedded graphs and Conjecture 1.2 holds for any orientable extremal cellularly embeded graphs, but is not true for non-orientable extremal embedded graphs. We also point out that the second claim of Proposition 3.27 in [4] is not correct.

2 Preliminaries

There are several ways to represent cellularly embedded graphs, and it is often more convenient and natural to work in the language of one or the other of these representations.

Thus, we briefly describe ribbon graphs. We refer the readers to [1, 2] and the monograph [4] for details. Readers familiar with them can skip this section.

Definition 2.1 ([1]). A ribbon graph $G = (V(G), E(G))$ is a (possibly non-orientable) surface with boundary, represented as the union of two sets of topological discs, a set $V(G)$ of vertices, and a set $E(G)$ of edges such that

1. the vertices and edges intersect in disjoint line segments, we call them *common line segments*;
2. each such line segment lies on the boundary of precisely one vertex and precisely one edge;
3. every edge contains exactly two such line segments.

If we delete the two common line segments from the boundary of an edge in a ribbon graph then we obtain exactly two disjoint line segments, we call them *edge line segments* of the edge. Two ribbon graphs are said to be equivalent or equal if there is a homeomorphism between them that preserves the vertex-edge structure. A ribbon graph is said to be orientable if it is orientable when viewed as a surface with boundary. Similarly, the genus, $g(G)$, of a ribbon graph G is its genus when viewed as a punctured surface. The genus of a disconnected ribbon graph is the sum of the genera of its connected components. The genus of a surface is not additive under connected sums. For example the connected sum of a torus and a real projective plane, which, are both surfaces of genus 1, is homeomorphic to the connected sum of three real projective planes, a surface of genus 3. To get over this technical difficulty, the Euler genus, γ, which is additive under the connected sum, is defined. Let G be a connected ribbon graph. Then

$$\gamma(G) = \begin{cases} 2g(G), & \text{if } G \text{ is orientable,} \\ g(G), & \text{if } G \text{ is non-orientable.} \end{cases}$$

If G is not connected, then $\gamma(G)$ is defined as the sum of the Euler genus of each of its connected components. We say that a ribbon graph G is a plane graph if $\gamma(G) = 0$.

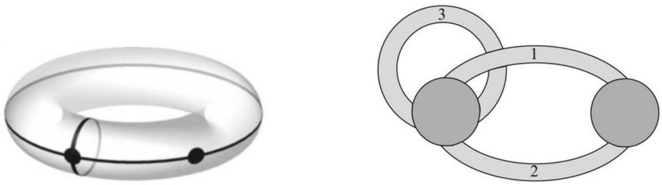
It is well known that ribbon graphs and cellularly embedded graphs are equivalent: if G is a cellularly embedded graph, a ribbon graph representation results from taking a small neighbourhood of the embedded graph G. Neighbourhoods of vertices of the graph G form the vertices of a ribbon graph, and neighbourhoods of the edges of the embedded graph form the edges of the ribbon graph. On the other hand, if G is a ribbon graph, we simply cap off the punctures to obtain a closed surface and then the core of the ribbon graph is exactly cellularly embedded in this surface. An example is given in Figure 1 [4]. Since cellularly embedded graphs and ribbon graphs are equivalent, we can, and will, move freely between these representations, choosing whichever is most convenient at the time for our purposes. In the context of ribbon graphs, $f(G)$ is the number of boundary components of the ribbon graph G.

The Euler characteristic, $\chi(G)$, of a ribbon graph G, is defined by

$$\chi(G) = v(G) - e(G) + f(G).$$

It is related to the Euler genus by the following formula:

$$\chi(G) = v(G) - e(G) + f(G) = 2k(G) - \gamma(G).$$



(a) A cellularly embedded graph G. (b) G as a ribbon graph.

Figure 1: Two presentations of the same embedded graph.

A loop (respectively, cycle) in a cellularly embedded graph is said to be non-orientable if its small neighbourhood is homeomorphic to a Möbius band. Otherwise it is said to be orientable. It is more obvious if we use the language of ribbon graphs.

We use the language of ribbon graphs to define the deletion and contraction of an edge. Let $G = (V(G), E(G))$ be a ribbon graph and $e \in E(G)$. We denote by $G - e$ the ribbon graph obtained from G by deleting the edge e. We denote by G/e the ribbon graph obtained from G by contracting the edge e. In the case that $e = (u, v)$ is not a loop, G/e is obtained from G by deleting e, u and v and adding a vertex disc along the boundary of e, u and v and in the case that $e = (v, v)$ is a loop, G/e is obtained from G by deleting e and v and adding a vertex disc or two vertex discs along the boundary of e and v (it is an annulus if the loop is orientable and a Möbius band otherwise). Alternatively, $G/e := (G^* - e)^*$, where * represents the geometrical dual.

3 Medial graphs and Petrials

Medial graphs as a tool will be used very often throughout this paper. If G is cellularly embedded in Σ, we construct its medial graph G_m in the embedded surface by placing a vertex on each of its edges, and for each face f with boundary $e_1, e_2, \ldots, e_{d(f)}$, drawing $d(f)$ edges $\{e_1, e_2\}, \ldots, \{e_{d(f)}, e_1\}$ inside the face f along the boundary of f. It is obvious that G_m is also cellularly embedded in Σ. We can form the medial graph of a ribbon graph inside the ribbon graph as shown in Figure 2. In particular, the medial graph of an isolated vertex is a free loop. Another example is given in Figure 3. In the language of medial graphs, $\mu(G)$ is exactly the number of "straight-ahead" walks in the medial graph G_m, as described in [18].

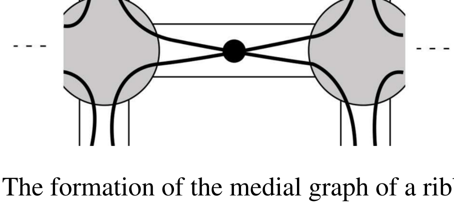


Figure 2: The formation of the medial graph of a ribbon graph.

An all-crossing direction of G_m is an assignment of a direction to each edge of G_m in such a way that at each vertex v of G_m, when we follow the cyclic order of the directed

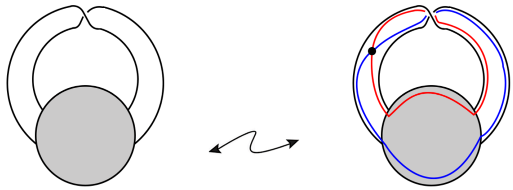


Figure 3: A non-orientable loop G and its medial graph G_m ($\mu(G) = 2$, red component and blue component).

edges incident to v, we find head, head, tail and tail. If G_m is equipped with an all-crossing direction, then we can partition the vertices of G_m into c-vertices and d-vertices according to the scheme shown in Figure 4. Accordingly edges of G are divided into c-edges and d-edges.

(a) c-vertex (b) d-vertex

Figure 4: c-vertex and d-vertex.

The Petrial of a cellularly embedded graph G, which we denote by $G^{\times}$, is formed with the same vertices and edges as G, but for the faces taking the Petrie polygons, which are the result of closed "straight-ahead" walks in G_m (see Wilson [18]). Since the "straight-ahead" pattern effectively means crossing over each edge, when the graph is viewed as a ribbon graph, this is simply giving each edge a half-twist, and hence $G^{\times}$ is simply the result of giving a half-twist to all of the edges as shown in Figure 5. An example of a ribbon graph and its Petrial is given in Figure 7.

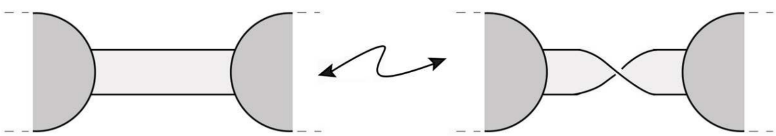


Figure 5: Add a half-twist to an edge of a ribbon graph.

Lemma 3.1. *Let G be a ribbon graph. Then*

$$\mu(G) = f(G^{\times}).$$

Proof. This follows immediately from Figure 6. □

Notice that rather than adding a half-twist to each of the edges of G, we may add half-twists to only some of the edges of G. The result is a partial Petrial of G. Let G be a

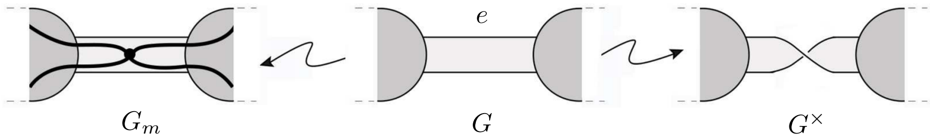


Figure 6: The relation between medial graph and Petrial.

ribbon graph and $A \subseteq E(G)$. Then the partial Petrial, $G^{\tau(A)}$, of G with respect to A is the ribbon graph obtained from G by adding a half-twist to each of the edges in A. Let $\mathrm{Orb}_{(\tau)}(G) = \{G^{\tau(A)} \mid A \subseteq E(G)\}$ denote the set of all partial Petrials of G. Ellis-Monaghan and Moffatt in [4] claimed:

Proposition 3.2 ([4])**.** *Let G be a ribbon graph. Then*

1. *$|\mathrm{Orb}_{(\tau)}(G)|$ is bounded above by two raised to the power of the number of cycles in G.*
2. *if G is bipartite, then $G^{\times} = G$.*

In fact the second claim of Proposition 3.2 is not true. A counterexample is given below.

Example 3.3. As shown in Figure 7, we have $\gamma(G_2) = 0$ and $\gamma(G_2^{\times}) = 2$, and hence $G_2 \neq G_2^{\times}$ but G is bipartite, contradicting the second claim of Proposition 3.2.

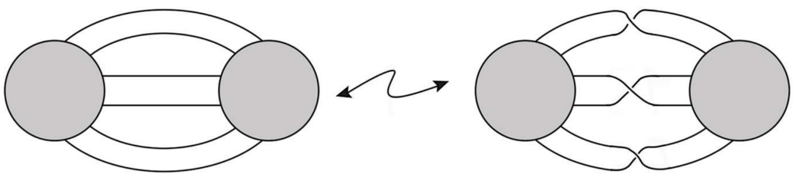


Figure 7: Counterexample G_2 (and its Petrial $G_2^{\times}$) of the second claim of Proposition 3.2.

4 Upper bound and extremal graphs

In this section, we give some basic properties of $\mu(G)$, extending results in [7] and [9] from orientable ribbon graphs to non-orientable ones, and obtaining several new results at the same time.

Lemma 4.1. *Let $G + e$ be the ribbon graph obtained from a ribbon graph G by adding a new edge e connecting two (not necessarily distinct) vertices of G. Then*

$$\mu(G) - 1 \leq \mu(G + e) \leq \mu(G) + 1.$$

Proof. If e is a loop and the common line segments of the loop are adjacent, then there are two cases:

Case 1.1: If e is an orientable loop, then $\mu(G + e) = \mu(G)$, as in Figure 8.

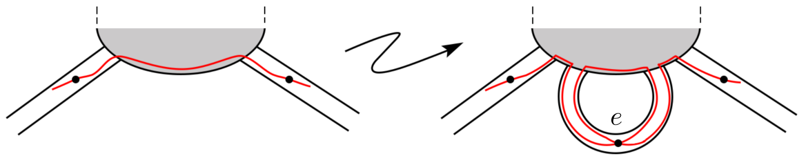


Figure 8: Case 1.1.

Case 1.2: If e is a non-orientable loop, then $\mu(G+e)=\mu(G)+1$, as in Figure 9.

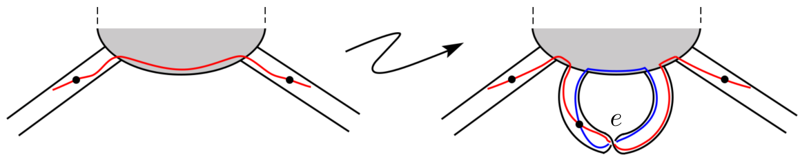


Figure 9: Case 1.2.

Otherwise, there are also two cases (see Figure 10).

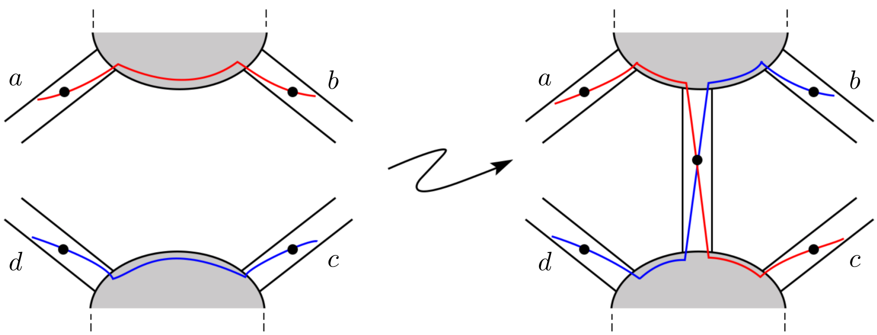


Figure 10: Case 2.1 and 2.2.

Case 2.1: If the arcs α (joining a and b) and β (joining c and d) are contained in different components of G (here "components of G" means components of the virtual link formed from G via the medial construction, in the following we take this convention and it causes no confusion in the context), then $\mu(G+e)=\mu(G)-1$.

Case 2.2: If the arcs α and β are contained in same component of G, then there are two subcases.

Case 2.2.1: Along this component, if the order of the four endpoints of the two arcs α and β is a,b,c,d, then $\mu(G+e)=\mu(G)$.

Case 2.2.2: If the order of the four endpoints of the two arcs α and β is a,b,d,c, then $\mu(G+e)=\mu(G)+1$. $\square$

Theorem 4.2. *Let G be a ribbon graph. Then*

$$k(G) \leq \mu(G) \leq f(G) + \gamma(G).$$

Proof. It is trivial that $k(G) \leq \mu(G)$. Take a maximal forest F of G, it is obvious that $\mu(F) = k(G)$. There are $e(G) - v(G) + k(G)$ edges of G outside F. By Lemma 4.1, we have

$$\mu(G) \leq k(G) + (e(G) - v(G) + k(G)) = f(G) + \gamma(G).$$ □

A ribbon graph G is called extremal if $\mu(G) = f(G) + \gamma(G)$. The following lemma expresses $\mu(G)$ in terms of G and its petrial $G^{\times}$.

Lemma 4.3. *Let G be a ribbon graph. Then*

$$\mu(G) = \gamma(G) + f(G) - \gamma(G^{\times}).$$

Proof. Note that

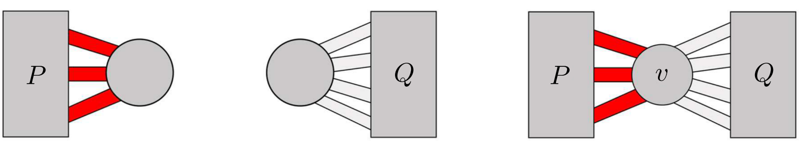
$$\chi(G) = v(G) - e(G) + f(G) = 2k(G) - \gamma(G),$$
$$\chi(G^{\times}) = v(G^{\times}) - e(G^{\times}) + f(G^{\times}) = 2k(G^{\times}) - \gamma(G^{\times}).$$

In addition, $v(G) = v(G^{\times}), e(G) = e(G^{\times})$, $k(G) = k(G^{\times})$ and $f(G^{\times}) = \mu(G)$ by Lemma 3.1. Hence $\mu(G) = \gamma(G) + f(G) - \gamma(G^{\times})$. □

The upper bound in Theorem 4.2 is also a direct consequence of Lemma 4.3 since $\gamma(G^{\times}) \geq 0$. The following theorem is simple, but critical in the next two sections.

Theorem 4.4. *A ribbon graph G is extremal if and only if $\gamma(G^{\times}) = 0$, i.e. $G^{\times}$ is plane.*

If G, P and Q are ribbon graphs, then we say that G is the *join* of P and Q, written $G = P \vee Q$, if G can be obtained by identifying an arc on the boundary of a vertex of P with an arc on the boundary of a vertex of Q as indicated in Figure 11. The two arcs that are identified do not intersect any common line segments.



(a) P (b) Q (c) $P \vee Q$

Figure 11: The join $P \vee Q$ of P and Q.

Lemma 4.5. *Let $G = B_1 \vee B_2 \vee B_3 \vee \cdots \vee B_k$. Then*

$$\mu(G) = \sum_{i=1}^{k} \mu(B_i) - k + 1.$$

Proof. It suffices to show that Lemma 4.5 holds for $k = 2$. Let v be the new vertex of G formed by merging a vertex of B_1 and a vertex of B_2. Since v is a cut vertex, the arcs c_1 and c_2 must belong to a single component, and c_1' and c_2' must belong to different components, that is, splitting $B_1 \vee B_2$ at v into B_1 and B_2 increases the number of components by one, as in Figure 12. Thus $\mu(G) = \mu(B_1) + \mu(B_2) - 1$. □

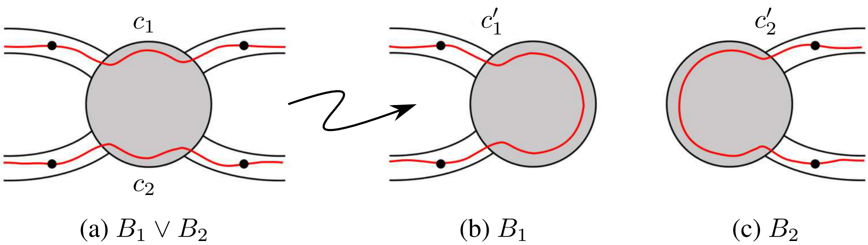


Figure 12: A component of $B_1 \vee B_2$ is split into a component of B_1 and a component of B_2.

Lemma 4.6. *Let G be a ribbon graph and e be a bridge of G. Then $\mu(G) = \mu(G/e)$.*

Proof. This follows immediately from Figure 13. Since e is a bridge, the arcs c_1 and c_2 must belong to a single component, and so do the arcs c_1' and c_2'. We have $\mu(G) = \mu(G/e)$. □

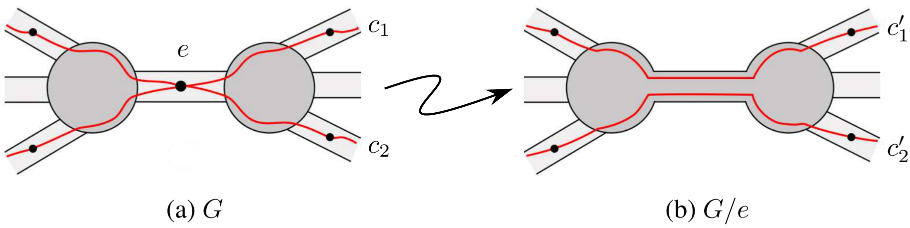


Figure 13: The medial graphs of G and G/e.

Lemma 4.7. *Let G be a ribbon graph and e_1 and e_2 be two distinct edges of G (see Figure 4.7).*

1. *If the 2-cycle given by $\{e_1, e_2\}$ is orientable and the common line segments of e_1 and e_2 are adjacent as in Case 1, then $\mu(G) = \mu(G - e_1 - e_2)$.*
2. *If the 2-cycle given by $\{e_1, e_2\}$ is non-orientable and the common line segments of e_1 and e_2 are adjacent as in Case 2, then $\mu(G) = \mu(G/e_1/e_2)$.*
3. *If e_1 and e_2 are not parallel edges, but are incident with a common vertex of degree 2 as in Case 3, then $\mu(G) = \mu(G/e_1/e_2)$.*

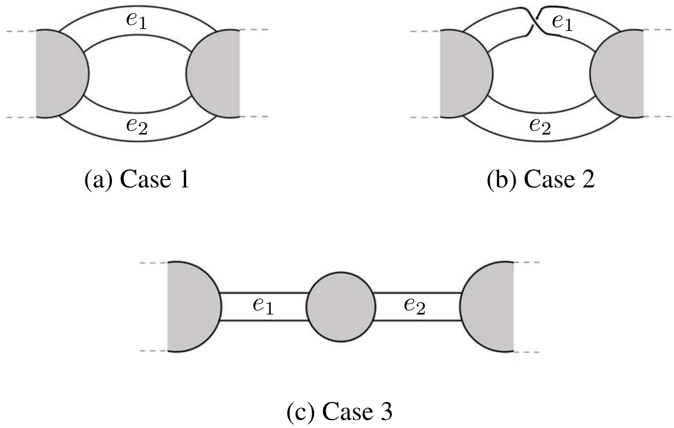
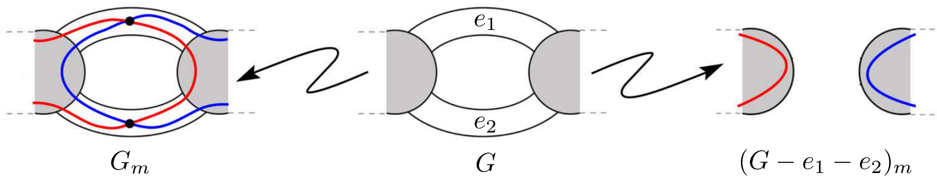


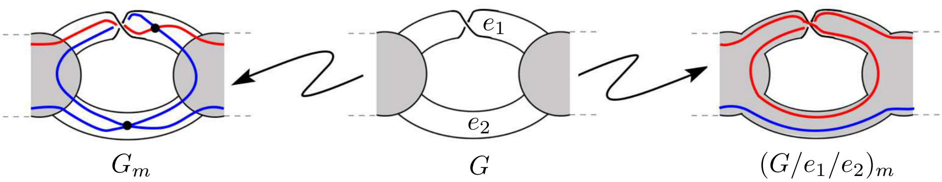
Figure 14: The three cases of Lemma 4.7.

Proof. We illustrate the proof with the following figures.

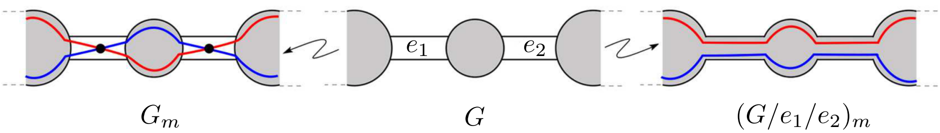
Case 1:



Case 2:



Case 3:



□

Theorem 4.8. *Let G be a ribbon graph.*

(1) If G is extremal and e is not a bridge of G, then $G - e$ is extremal.

(2) If $G = B_1 \vee B_2 \vee B_3 \vee \cdots \vee B_k$, then G is extremal if and only if each B_i is extremal.

(3) If e is a bridge of G, then G/e is extremal if and only if G is extremal.

(4) Let v be a vertex of degree 2 with exactly one adjacent vertex, the two edges joining these vertices are as in Case 1 of Lemma 4.7. Then $G - v$ is extremal if and only if G is extremal.

(5) Let v be a vertex of degree 2 with two different adjacent vertices x and y. Then $G/\{v,x\}/\{v,y\}$ is extremal if and only if G is extremal.

Proof. (1): Since G is extremal, $\mu(G) = f(G) + \gamma(G)$. By Lemma 4.1, we have

$$\mu(G - e) \geq f(G) + \gamma(G) - 1.$$

Moreover, $\mu(G - e) \leq f(G - e) + \gamma(G - e)$ by Theorem 4.2 and

$$f(G - e) + \gamma(G - e) = f(G) + \gamma(G) - 1.$$

Therefore $\mu(G - e) = f(G - e) + \gamma(G - e)$. Hence $G - e$ is extremal.

(2): Suppose that G is extremal. Then from Lemma 4.5 we have

$$\begin{aligned}\mu(G) &= \sum_{i=1}^{k} \mu(B_i) - k + 1 \\ &= f(G) + \gamma(G) \\ &= \sum_{i=1}^{k} f(B_i) - k + 1 + \sum_{i=1}^{k} \gamma(B_i) \\ &= \sum_{i=1}^{k} f(B_i) + \gamma(B_i) - k + 1.\end{aligned}$$

Therefore

$$\sum_{i=1}^{k} \mu(B_i) = \sum_{i=1}^{k} f(B_i) + \gamma(B_i),$$

and so each B_i is extremal. The converse is proved similarly.

(3): This follows from Lemma 4.6:

$$\mu(G/e) = \mu(G), \quad f(G/e) = f(G) \quad \text{and} \quad \gamma(G/e) = \gamma(G).$$

(4): This follows from Case 1 of Lemma 4.7:

$$\mu(G - v) = \mu(G) - 1, \quad f(G - v) = f(G) - 1 \quad \text{and} \quad \gamma(G - v) = \gamma(G).$$

(5): This follows from Case 3 of Lemma 4.7:

$$\mu(G/\{v,x\}/\{v,y\}) = \mu(G), \qquad f(G/\{v,x\}/\{v,y\}) = f(G) \quad \text{and}$$
$$\gamma(G/\{v,x\}/\{v,y\}) = \gamma(G).$$

□

5 Excluded extremal minor characterization

Definition 5.1. Let $G = (V, E)$ be a ribbon graph and $v \in V$ and $e \in E$. A deletion $G - e$ or $G - v$ of G is admissible if e is not a bridge of G or v is an isolated vertex of G; a contraction G/e or G/v is admissible if e is a bridge of G or v is a vertex of degree 2 with two different adjacent vertices u, w and $G/v = G/\{v, u\}/\{v, w\}$.

Definition 5.2. Let G be a ribbon graph. We say that a ribbon graph H is an extremal minor of G, denoted $H \prec G$, if there is a sequence of ribbon graphs $G = G_0, G_1, \ldots, G_t = H$ where for each i, G_{i+1} is obtained from G_i by either an admissible deletion or an admissible contraction.

Lemma 5.3. *Let G be an extremal ribbon graph and $H \prec G$. Then H is extremal.*

Proof. It suffices to prove that admissible deletions and admissible contractions preserve extremity. Cases of $G - e$, G/e and G/v follow from Statements (1), (3), and (5) of Theorem 4.8, respectively. Let v be an isolated vertex of G. Then $\mu(G - v) = \mu(G) - 1$, $f(G - v) = f(G) - 1$ and $\gamma(G - v) = \gamma(G)$. Thus if G is extremal, $G - v$ is also extremal. □

Lemma 5.4. *$H^\times \prec G^\times$ if and only if $H \prec G$.*

Proof. We only need to prove the sufficiency since $G = (G^\times)^\times$. Then by Definition 5.2, it suffices to prove that Lemma 5.4 holds for $H = G - e$ and $H = G - v$ (admissible deletions), and $H = G/e$ and $H = G/v$ (admissible contractions). For clarity, we denote by $e^\times$ (respectively, $v^\times$) the edge (respectively, the vertex) of $G^\times$ corresponding to the edge e (respectively, the vertex v) of G. Then e is a bridge if and only if $e^\times$ is a bridge of $G^\times$, v is an isolated vertex if and only if $v^\times$ is an isolated vertex of $G^\times$, and v is a vertex of degree 2 with two different adjacent vertices of G if and only if $v^\times$ is a vertex of degree 2 with two different adjacent vertices of $G^\times$. Hence $H \prec G$ implies $H^\times \prec G^\times$. □

Theorem 5.5. *Let G be a ribbon graph. Then G is extremal if and only if it contains no extremal minor equivalent to $B_1, B_{\overline{2}}, I_3, I_{\overline{2}}, T_1$ or $T_{\overline{2}}$ (see Figure 15).*

Proof. It is not difficult to obtain that

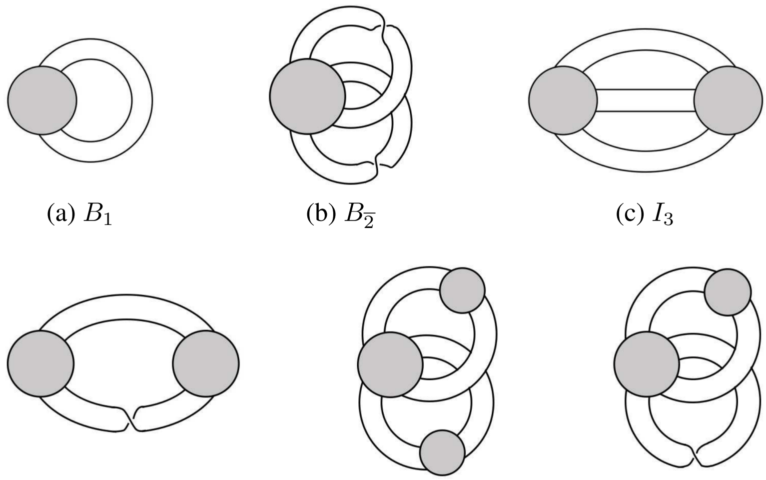
$$\gamma(B_1^\times) = \gamma(I_{\overline{2}}^\times) = 1 \quad \text{and} \quad \gamma(B_{\overline{2}}^\times) = \gamma(I_3^\times) = \gamma(T_1^\times) = \gamma(T_{\overline{2}}^\times) = 2.$$

By Theorem 4.4, $B_1, B_{\overline{2}}, I_3, I_{\overline{2}}, T_1$ and $T_{\overline{2}}$ are not extremal. It follows from Lemma 5.3 that G contains no extremal minor equivalent to $B_1, B_{\overline{2}}, I_3, I_{\overline{2}}, T_1$ or $T_{\overline{2}}$. To prove the converse we suppose that G is not extremal. Without loss of generality, we assume that G is connected. Then, by Theorem 4.4, $\gamma(G^\times) > 0$.

Case 1: If $G^\times$ is non-orientable, then $G^\times$ contains a non-orientable cycle C. If C is odd, then C in G is orientable. The edges of G not on C can be deleted or contracted admissibly, and thus $C \prec G$. Note that $B_1 \prec C$ and hence $B_1 \prec G$, a contradiction. If C is even, then C in G is non-orientable. It follows that $C \prec G$, $I_{\overline{2}} \prec C$ and thus $I_{\overline{2}} \prec G$, a contradiction.

Case 2: If $G^\times$ is orientable, then $g(G^\times) \geq 1$.

Step 1: If $f(G^\times) > 1$, we can take an edge e of $G^\times$ whose two edge line segments belong to different boundary components of $G^\times$. Such an edge e must be not a bridge, we can



(a) B_1 (b) $B_{\overline{2}}$ (c) I_3

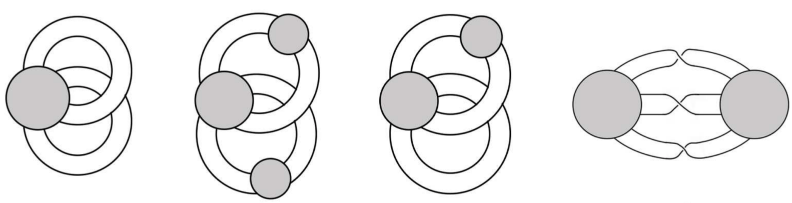
(d) $I_{\overline{2}}$ (e) T_1 (f) $T_{\overline{2}}$

Figure 15: The "forbidden" minors in Theorem 5.5.

delete it in $G^\times$. Deleting such an e will decrease boundary component number by 1 and preserve orientability and the genus. Repeat Step 1 we obtain an orientable ribbon graph $(G^\times)'$ with $f((G^\times)') = 1$ and $g((G^\times)') = g(G^\times)$.

Step 2: If $g((G^\times)') > 1$, $(G^\times)'$ must have non-bridges. Take a non-bridge e of $(G^\times)'$, deleting e will preserve orientability and connectedness, and must increase boundary components by 1 since $f((G^\times)') = 1$ and hence decrease the genus by 1.

Now go to Step 1. Then carry out Step 2 if the genus is greater than 1. Repeat above process, finally we obtain an orientable ribbon graph $(G^\times)''$ with $f((G^\times)'') = 1$ and $g((G^\times)'') = 1$. Note that $(G^\times)'' \prec G^\times$ and $(G^\times)''$ has the cyclomatic number 2. After contracting all bridges and vertices of degree 2 with two distinct adjacent vertices in $(G^\times)''$, we obtain four possible $(G^\times)'''$ as shown in Figure 16. Note that $a^\times, b^\times, c^\times$ and $d^\times$ in Figure 16 are $B_{\overline{2}}, T_1, T_{\overline{2}}$ and I_3, respectively. By Lemma 5.4, we have $B_{\overline{2}} \prec G$, $I_3 \prec G$, $T_1 \prec G$ or $T_{\overline{2}} \prec G$, a contradiction. □



(a) (b) (c) (d)

Figure 16: Four possibilities for $(G^\times)'''$.

As a corollary, we obtain an excluded extremal minor characterization of extremal plane graphs. We need the following lemma and leave its proof to readers.

Lemma 5.6. *If* $H \prec G$*, then* $\gamma(H) \leq \gamma(G)$.

Corollary 5.7. *A plane graph* G *is extremal if and only if it contains no extremal minor equivalent to* B_1 *or* I_3.

Proof. By Lemma 5.6, an extremal plane graph can not contain $B_{\overline{2}}$, $I_{\overline{2}}, T_1$ and $T_{\overline{2}}$ as extremal minors. It then follows from Theorem 5.5. □

6 Two conjectures and their generalizations

We first prove Conjecture 1.2 holds for any extremal cellularly embedded graphs on orientable surfaces.

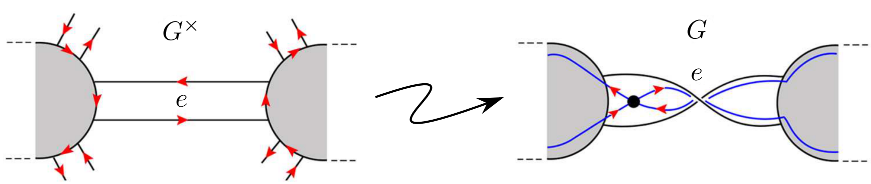
Theorem 6.1. *If* G *is an orientable extremal ribbon graph, then* G *is bipartite.*

Proof. By Theorem 4.4, $\gamma(G^{\times}) = 0$ which implies that $G^{\times}$ is a plane graph and thus orientable. If G is not bipartite, then it contains an odd cycle C. C, as a subgraph of the orientable ribbon graph G, is also orientable which will become a non-orientable cycle of $G^{\times}$. It follows that $G^{\times}$ is non-orientable, a contradiction. □

This theorem is not true for non-orientable extremal ribbon graphs. For example, the non-orientable loop, as in Figure 3, is extremal, but not bipartite. By Theorem 6.1, any orientable extremal ribbon graph with non-zero Euler genus (including the $G^{\times}$ in Example 3.3) is a counterexample of the second claim of Proposition 3.2. To show that Conjecture 1.1 holds for any extremal cellularly embedded graphs we need the following lemma.

Lemma 6.2. *Let* G *be a ribbon graph. If* $G^{\times}$ *is orientable, then* G_m *has an all-crossing direction such that each of the edges of* G *is a* d*-edge.*

Proof. Since $G^{\times}$ is orientable, it follows that $G^{\times}$ can be drawn on the plane such that each of its edge discs is untwisted. We orient each edge disc anticlockwise as illustrated in Figure 17(a). Note that straight-ahead walks of G_m correspond exactly to boundary components of $G^{\times}$. Then G_m is oriented in Figure 17(b) as boundary components of $G^{\times}$. This is an all-crossing direction such that each of the edges of G is a d-edge. □



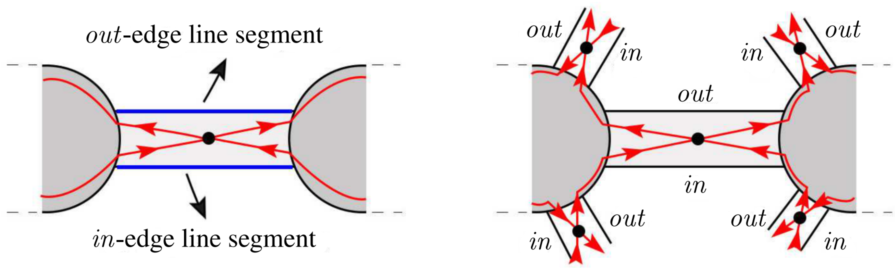
(a) Orient each edge disc anticlockwise.

(b) Get an all-crossing direction of G_m and e is a d-edge.

Figure 17: The all-crossing direction of G_m.

Theorem 6.3. *If G is an extremal ribbon graph, then each face of G is even.*

Proof. Since G is an extremal ribbon graph, it follows that $\gamma(G^{\times}) = 0$ by Theorem 4.4, and thus $G^{\times}$ is orientable. By Lemma 6.2, G_m has an all-crossing direction such that each of the edges of G is a d-edge. Since each edge is d-edge, we call one of edge line segments in-edge line segment if two marking arrows are all "in" arising from the all-crossing direction of G_m. Otherwise, we call the other out-edge line segment, as in Figure 18(a). It is immediate that the in-edge line segments and out-edge line segments are alternating in the every face boundary of G, as in Figure 18(b). It follows that each face of G is even. □



(a) (b)

Figure 18: in-edge line segment and out-edge line segment.

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Reconfiguring vertex colourings of 2-trees

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Abstract

Let H be a graph and let $k \geq \chi(H)$ be an integer. The *k-colouring graph of H*, denoted $G_k(H)$, is the graph whose vertex set consists of all proper k-vertex-colourings (or simply k-colourings) of H using colours $\{1, 2, \dots, k\}$; two vertices of $G_k(H)$ are adjacent if and only if the corresponding k-colourings differ in colour on exactly one vertex of H. If $G_k(H)$ has a Hamilton cycle, then H is said to have a *Gray code* of k-colourings, and the *Gray code number* of H is the least integer $k_0(H)$ such that $G_k(H)$ has a Gray code of k-colourings for all $k \geq k_0(H)$. Choo and MacGillivray determine the Gray code numbers of trees. We extend this result to 2-trees. A 2*-tree* is constructed recursively by starting with a complete graph on three vertices and connecting each new vertex to an existing clique on two vertices. We prove that if H is a 2-tree, then $k_0(H) = 4$ unless H is isomorphic to the join of a tree T and a vertex u, where T is a star on at least three vertices, or the bipartition of T has two even parts; in these cases, $k_0(H) = 5$.

Keywords: 2*-trees, graph colouring, Gray codes, Hamilton cycles, reconfiguration problems.*

Math. Subj. Class.: 05C15, 05C45

1 Introduction

Let H be a graph and k a positive integer. We define a *proper k-vertex-colouring* of H as a function $f\colon V(H) \to \{1, 2, \dots, k\}$ for which $f(x) \neq f(y)$ for any $xy \in E(H)$. Since we are concerned only with proper k-vertex-colourings, we use the simpler term *k-colouring*, and refer to $f(x)$ as the *colour* of x. For notation and terminology not defined here, the reader is referred to Bondy and Murty [1].

A graph H has a *Gray code of k-colourings* if it is possible to list all the k-colourings of H in such a way that consecutive colourings in the list (including the last and the first) differ on exactly one vertex of H, and the *Gray code number of H* is the least integer

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$k_0(H)$ for which H has a Gray code of its k-colourings for all $k \geq k_0(H)$. Equivalently, we may define the *k-colouring graph of H*, denoted $G_k(H)$, to be the graph whose vertices correspond to all k-colourings of H, and whose edges connect two k-colourings of H that differ on exactly one vertex of H. In this context, H has a Gray code of k-colourings if and only if $G_k(H)$ has a Hamilton cycle, and the *Gray code number of H* is the the least integer $k_0(H)$ for which $G_k(H)$ has a Hamilton cycle for all $k \geq k_0(H)$.

The k-colouring graph is an example of a *reconfiguration graph*. Such graphs are often used in the study of what are known as *reconfiguration problems*. Generally, a reconfiguration problem asks: given two (different) feasible solutions to a problem, can one solution be transformed to the other through a sequence of allowable moves, while maintaining feasibility at each stage? In the context of k-colourings, the k-colouring graph is connected if and only if any k-colouring can be reconfigured into any other k-colouring by re-colouring one vertex at a time in such a way that each intermediate colouring is a k-colouring. Recently, reconfiguration problems have been receiving wide attention, and have been studied for various graph colouring problems [2, 3, 4, 10, 15], for dominating sets [12, 13, 18], and for various other graph problems including vertex covers, cliques, and independent sets [14].

The k-colouring graph arises in the context of theoretical physics, where it is the graph of the Glauber dynamics Markov chain; the goal is to find efficient algorithms for almost uniform sampling of k-colourings of graphs [16]. The Glauber dynamics Markov chain converges to the uniform distribution provided that the k-colouring graph is connected, prompting Cereceda, van den Heuvel and Johnson [4] to ask the question: given a graph H and a positive integer k, is $G_k(H)$ connected? They prove that if H has chromatic number $k \in \{2, 3\}$, then $G_k(H)$ is never connected, whereas for $k \geq 4$, there are k-chromatic graphs H for which $G_k(H)$ is connected, and other k-chromatic graphs H for which $G_k(H)$ is not connected. In general, they prove that $G_k(H)$ is connected for all $k \geq \mathrm{col}(H) + 1$, where $\mathrm{col}(H)$, the *colouring number* of H, is defined as $\mathrm{col}(H) := \max\{\delta(G) \mid G \subseteq H\} + 1$. A slightly weaker version of this result is proven in [8].

Choo and MacGillivray [6] initiated the study of Hamilton cycles in k-colouring graphs by proving that the Gray code number of H is well defined, i.e., $G_k(H)$ has a Hamilton cycle for all $k \geq \mathrm{col}(H) + 2$. This gives the upper bound $k_0(H) \leq \mathrm{col}(H) + 2$. Note that if T is a tree, then $\mathrm{col}(T) = 2$ and $G_2(T)$ is disconnected; hence $3 \leq k_0(T) \leq 4$. Choo and MacGillivray [6] determine which trees have $k_0(T) = 3$ and which have $k_0(T) = 4$. In particular, they prove that if T is a tree, then $k_0(T) = 3$ unless $T \cong K_{1,2\ell}$, for $\ell \geq 1$, in which case $k_0(T) = 4$. They also determine the Gray code numbers of complete graphs and cycles. Celaya, Choo, MacGillivray and Seyffarth [3] establish the Gray code numbers of complete bipartite graphs. Haas [11] studies a variation of k-colouring graphs, namely, canonical colouring graphs, and uses techniques developed in [6] to show that canonical colouring graphs of trees have Hamilton cycles.

Given the results for trees and complete graphs, a natural question is to determine the Gray code numbers of k-trees. We use the definition of k-tree given in [9], that is, a *k-tree* is constructed recursively by starting with a complete graph on $k+1$ vertices and connecting each new vertex to an existing clique on k vertices (hence a 1-tree is simply a connected acyclic graph). A vertex of degree k in a k-tree is called a *leaf*. Let H be a k-tree. Then it is clear that the chromatic number of H is $\chi(H) = k+1$, and that $G_{k+1}(H)$ is disconnected. Since every induced subgraph of H has a leaf, $\mathrm{col}(H) = k + 1$. Thus it follows from [6, Theorem 3.4] that $k_0(H) \leq k + 3$, and therefore $k + 2 \leq k_0(H) \leq k + 3$. The problem is therefore reduced to classifying k-trees into those with Gray code number $k+2$ and those with Gray code number $k+3$. The answer appears to be far from trivial. In the current paper, we provide a complete solution for 2-trees; the characterization can be stated in fairly non-technical language, but the proof involves numerous cases and generalizations of the techniques used in [3, 6]. The *join* of graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is obtained from the disjoint union of G_1 and G_2 by adding all edges between vertices of G_1 and vertices of G_2.

Theorem 1.1. *If H is a 2-tree then $k_0(H) = 4$, unless $H \cong T \vee \{u\}$ for some tree T and vertex u, where T is a star on at least three vertices or the bipartition of $V(T)$ has two even parts; in these cases, $k_0(H) = 5$.*

The remainder of the paper is devoted to proving this theorem. We first characterize 2-trees of diameter two (Lemma 3.2). We then determine the 2-trees, H, of diameter two for which $G_4(H)$ has a Hamilton cycle (Lemmas 3.3 and 3.5). This is done by considering the structure of $G_3(T)$, where T is a tree (Lemmas A.2 and 3.6). We show that if H is a 2-tree with diameter at least three, then $G_4(H)$ has a Hamilton cycle (Lemmas 6.3 and 6.4). To do so we describe a specific recursive procedure for constructing 2-trees of diameter at least three (Theorem 4.3).

2 Preliminaries

Definition 2.1. Let H be a graph, and let X and Y be disjoint subsets of $V(H)$. We denote by $[X, Y]$ the set of edges of H that have one end in X and the other end in Y.

Definition 2.2. Let H be a graph and L a function that assigns to each vertex $v \in V(H)$ a set of positive integers, $L(v)$, called the *list* of v. An L*-colouring* of H (also called a *list colouring* of H with respect to L) is a (proper) colouring $c\colon V(H) \to \mathbb{N}$ such that $c(v) \in L(v)$ for each $v \in V(H)$. We define the L*-colouring graph of* H, denoted $G_L(H)$, to be the graph whose vertex set consists of all L-colourings of H; two L-colourings are joined by an edge of $G_L(H)$ if they differ in colour on just one vertex of H.

Remark 2.3. If $k \geq \chi(H)$ and $L(v) := \{1, 2, \ldots, k\}$ for each $v \in V(H)$, then $G_L(H) \cong G_k(H)$, the k-colouring graph of H.

We use $G \,\Box\, H$ to denote the Cartesian product of graphs G and H, and Q_n to denote the n-dimensional hypercube, defined as the Cartesian product of n copies of K_2.

Remark 2.4. Let H_1 and H_2 be vertex disjoint graphs and let L be an assignment of lists to the vertices of $H_1 \cup H_2$. Then

$$G_L(H_1 \cup H_2) = G_L(H_1) \,\Box\, G_L(H_2).$$

Lemma 2.5. *Let H be a 2-tree with clique $X = \{x_1, x_2, \ldots, x_\ell\}$ where $\ell \leq 3$, and let L be an assignment of lists to the vertices of H so that*

1. $|L(x_i)| = 1$ *and* $L(x_i) \subseteq \{1, 2, 3, 4\}$, $1 \leq i \leq \ell$;

2. $L(x_i) \neq L(x_j)$ *for all* $1 \leq i \neq j \leq \ell$;

3. $L(v) = \{1, 2, 3, 4\}$ *for each* $v \in V(H) \setminus X$.

Then $G_L(H)$ has a spanning tree T with $\Delta(T) \leq 4$.

Proof. The proof is by induction on $|V(H)|$. For the basis, $H \cong K_3$ with $V(H) := \{v_1, v_2, v_3\}$. When $\ell = 0$, $G_L(H) = G_4(K_3)$, which has a Hamilton cycle [6, Theorem 4.1], so take T to be a Hamilton path in $G_4(K_3)$. When $\ell = 1$, we may assume, without loss of generality, that $L(v_1) := \{1\}$ and $L(v_2) = L(v_3) := \{1, 2, 3, 4\}$. Then $G_L(H) \cong G_3(K_2)$, which has a Hamilton cycle [6, Theorem 4.1], so take T to be a Hamilton path in $G_3(K_2)$. When $\ell = 2$, we may may assume, without loss of generality, that $L(v_1) := \{1\}$, $L(v_2) := \{2\}$, and $L(v_3) := \{1, 2, 3, 4\}$. Then $G_L(H) \cong K_2$, and the result holds. Finally, when $\ell = 3$, we may assume, without loss of generality, that $L(v_1) := \{1\}$, $L(v_2) := \{2\}$, and $L(v_3) := \{3\}$. Then $G_L(H) \cong K_1$, and the result holds.

Now suppose H is a 2-tree with $|V(H)| > 3$ and clique $X = \{x_1, \ldots, x_\ell\}$. Since $\ell \leq 3$, there is a leaf $v \in V(H)$ with $v \notin X$, and thus $H - v$ is a 2-tree containing the clique X. It follows by induction that $G_L(H - v)$ has a spanning tree $\mathcal{T}$ with $\Delta(\mathcal{T}) \leq 4$. Let $V(G_L(H - v)) := \{f_0, f_1, \ldots, f_{N-1}\}$, and for each $f_j \in V(G_L(H - v))$, let $F_j \subseteq V(G_L(H))$ be the set of L-colourings of H that agree with f_j on $V(H - v)$. Then $\{F_0, F_1, \ldots, F_{N-1}\}$ is a partition of the vertices of $G_L(H)$. Since v is a leaf of H, there are two ways to extend an L-colouring of $H - v$ to an L-colouring of H, and hence for each j, $0 \leq j \leq N - 1$, $F_j = \{a_j, b_j\}$ is a clique.

For each edge $f_i f_j \in E(\mathcal{T})$, there is a unique vertex $w \in V(H - v)$ for which $f_i(w) \neq f_j(w)$. If $wv \notin E(H)$, then $[F_i, F_j]$ consists of two disjoint edges, and the subgraph of $G_L(H)$ induced by $F_i \cup F_j$ is a 4-cycle. Otherwise, $wv \in E(H)$, so $[F_i, F_j]$ has only one edge, and the subgraph of $G_L(H)$ induced by $F_i \cup F_j$ is a path of length three. In both cases, label the edge $f_i f_j$ in $\mathcal{T}$ with $|[F_i, F_j]|$.

Let S denote the spanning subgraph of $G_L(H)$ corresponding to the spanning tree $\mathcal{T}$ of $G_L(H - v)$ as described above, that is, S has edge set

$$\left(\bigcup_{f_i f_j \in E(\mathcal{T})} [F_i, F_j] \right) \cup \{a_j b_j \mid 0 \leq j \leq N - 1\}.$$

Since $[F_i, F_j]$ is nonempty for each $f_i f_j \in E(\mathcal{T})$, S is connected. Also, since $\Delta(\mathcal{T}) \leq 4$ and $[F_i, F_j]$ contains either one edge or two disjoint edges, $\Delta(S) \leq 5$. Furthermore, since there are only two vertices adjacent to v in H, at most two edges incident to f_i in $G_L(H-v)$ have label '1'.

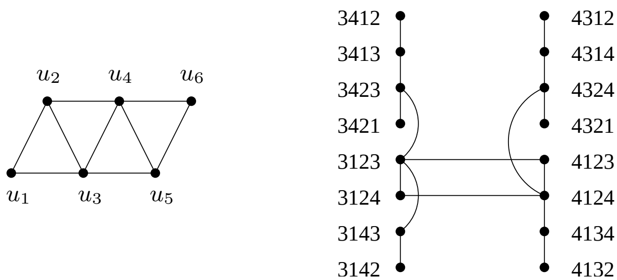
Let S' be the graph obtained from S by deleting the edges $a_j b_j$ for each $f_j \in V(\mathcal{T})$ with $d_{\mathcal{T}}(f_j) = 4$. Then $\Delta(S') \leq 4$, since if $a_j \in V(S)$ has $d_S(a_j) = 5$, then $d_{\mathcal{T}}(f_j) = 4$, and thus $d_{S'}(a_j) = 4$. We also claim that S' is connected. To prove this, it suffices to show that there is a path in S' from a_p to b_p for each $f_p \in V(\mathcal{T})$ with $d_{\mathcal{T}}(f_p) = 4$. Suppose $f_p \in V(\mathcal{T})$ has $d_{\mathcal{T}}(f_p) = 4$. Construct a path, P, in $\mathcal{T}$ starting at f_p, using edges labelled '2', and whose final vertex f_q has $d_{\mathcal{T}}(f_q) < 4$. Such a path exists since $\mathcal{T}$ has no cycles, and each degree four vertex in $\mathcal{T}$ is incident to at least two edges labelled '2'. The union

$$\left(\bigcup_{f_i f_j \in E(P)} [F_i, F_j] \right) \cup \{a_q b_q\}$$

gives us a path in S' from a_p to b_p.

Therefore, S' is a connected spanning subgraph of $G_L(H)$, and thus S' has a spanning tree T that is also a spanning tree of $G_L(H)$. Since $\Delta(S') \leq 4$, $\Delta(T) \leq 4$, thus completing the proof of the lemma. □

The result in Lemma 2.5 is best possible in that $\Delta(T)$ cannot be reduced from four to three, as illustrated in the following example. Let D denote the unique 2-tree of diameter three on six vertices, with vertices labelled as shown in Figure 1(a).



(a) D (b) $G_L(D)$

Figure 1: The 2-tree D and $G_L(D)$.

Let $L\colon V(D) \to \{1, 2, 3, 4\}$ be defined as follows.

$$\begin{aligned} L(u_1) &:= \{1\}, \\ L(u_2) &:= \{2\}, \text{ and} \\ L(u_i) &:= \{1, 2, 3, 4\} \text{ for } 3 \leq i \leq 6. \end{aligned}$$

If f is an L-colouring of D, then $f(u_1) = 1$ and $f(u_2) = 2$, and thus we may denote the vertices of $G_L(D)$ by strings $ijk\ell$ where $f(u_3) = i$, $f(u_4) = j$, $f(u_5) = k$ and $f(u_6) = \ell$. Using this convention, $G_L(D)$ is depicted in Figure 1(b). Notice that $G_L(D)$ is unicyclic, and has exactly two nonadjacent vertices of degree four, '3123' and '4124'. Thus every spanning tree of $G_L(D)$ has a vertex of degree four.

As part of their proof [6, Theorem 5.5], Choo and MacGillivray prove the following.

Remark 2.6. Let G be a graph with vertex partition $\{F_0, F_1, \ldots, F_{N-1}\}$, such that

(i) $G[F_i]$ is a 4-cycle for each i, $0 \leq i \leq N-1$;

(ii) $G[F_{i-1} \cup F_i]$ is isomorphic to either $P_4 \,\square\, K_2$ or Q_3 for each i, $1 \leq i \leq N-1$;

(iii) if $G[F_{i-1} \cup F_i]$ and $G[F_i \cup F_{i+1}]$ are both isomorphic to $P_4 \,\square\, K_2$, then $G[F_{i-1} \cup F_i \cup F_{i+1}]$ is not isomorphic to the graph in Figure 2.

Then G has a Hamilton cycle.

The conditions imply that $[F_{i-1}, F_i] \neq \emptyset$, $1 \leq i \leq N-1$, and hence there is a function, h, from a spanning subgraph of G to a path $f_0 f_1 \cdots f_{N-1}$ of length $N-1$ defined by $h(u) = f_i$ for all $u \in F_i$, $0 \leq i \leq N-1$.

In our next lemma, we adapt the result of Choo and MacGillivray to a more general scenario. Suppose G is a graph with vertex partition $\{F_0, F_1, \ldots, F_{N-1}\}$ where $G[F_i]$

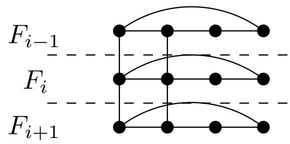


Figure 2: Forbidden subgraph.

contains a spanning cycle for each i, $0 \leq i \leq N-1$ (these cycles form a 2-factor of G). Further, assume that there is a function, h, from a spanning subgraph of G to a tree with vertex set $\{f_0, f_1, \ldots, f_{n-1}\}$ such that $h(u) = f_i$ for all $u \in F_i$, $0 \leq i \leq N-1$. The general idea is to choose, for each edge $f_i f_j$ of the spanning tree, appropriate edges from the set $[F_i, F_j]$ of G so that we are able to construct a Hamilton cycle from among these edges and edges of the 2-factor. See Figure 3 for an illustration of this result.

Lemma 2.7. *Let G be a graph with vertex partition $\{F_0, F_1, \ldots, F_{N-1}\}$, and let T be a tree with $V(T) := \{f_0, f_1, \ldots, f_{N-1}\}$. Suppose there is a function, h, from a spanning subgraph of G to T such that $h(u) = f_i$ for all $u \in F_i$, $0 \leq i \leq N-1$. Furthermore, suppose that for each $f_i f_j \in E(T)$, $0 \leq i, j \leq N-1$, there exist edges $e_{i,j}$ in $G[F_i]$ and $e_{j,i}$ in $G[F_j]$ such that*

(i) if $j \neq k$ and $f_i f_j, f_i f_k \in E(T)$, then $e_{i,j} \neq e_{i,k}$;

(ii) if $e_{i,j} = ac$ and $e_{j,i} = bd$, then $G[\{a, b, c, d\}]$ contains a 4-cycle;

(iii) $G[F_i]$ has a Hamilton cycle C_i such that

$$M_i := \{e_{i,j} \mid f_i f_j \in E(T)\} \subseteq E(C_i).$$

Then G has a Hamilton cycle C such that

$$\bigcup_{i=0}^{N-1} (E(C_i) \setminus M_i) \subseteq E(C).$$

Proof. The proof is by induction on N. The result is trivial when $N = 1$.

Let $N > 1$. For each $f_i f_j \in E(T)$, $0 \leq i \leq N-1$, suppose $e_{i,j} \in E(G[F_i])$, $e_{j,i} \in E(G[F_j])$, and C_i (a Hamilton cycle in $G[F_i]$) satisfy conditions (i), (ii) and (iii). Without loss of generality, we may assume that f_{N-1} is a leaf of T, and that $f_{N-1} f_{N-2} \in E(T)$. Let $G' := G - F_{N-1}$ and $T' := T - f_{N-1}$. Using $e_{i,j}$, $e_{j,i}$ and C_i as previously defined, $0 \leq i \leq N-2$, and M_i as defined in (iii) except with M_{N-2} replaced by $M'_{N-2} := M_{N-2} \setminus \{e_{N-2,N-1}\}$, we apply the induction hypothesis to G'. The result is a Hamilton cycle C' in G' such that

$$\left(\bigcup_{i=0}^{N-3} (E(C_i) \setminus M_i) \right) \cup (E(C_{N-2}) \setminus M'_{N-2}) \subseteq E(C'),$$

and $e_{N-2,N-1} \in E(C')$. Let $e_{N-2,N-1} := ac$, $e_{N-1,N-2} := bd$; without loss of generality, $abdca$ is a 4-cycle in $G[\{a, b, c, d\}]$, and hence

$$C := (C' - e_{N-2,N-1}) \cup (C_{N-1} - e_{N-1,N-2}) + \{ab, cd\}$$

is a Hamilton cycle in G with the required property. □

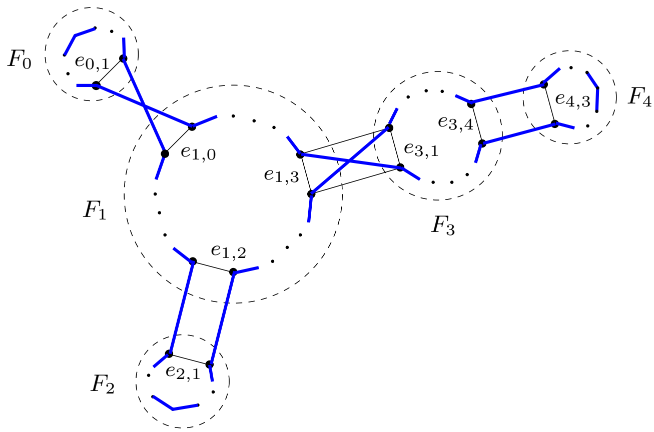


Figure 3: Illustration of Lemma 2.7.

We remark that if $d_T(f_i) = 1$ for some i, then the Hamilton cycle C constructed in Lemma 2.7 contains all except one edge of $E(C_i)$.

3 2-trees of diameter two

In this section we characterize 2-trees H of diameter two in which $G_4(H)$ has a Hamilton cycle. We begin by defining a class of 2-trees that we denote by $T(p, q, r)$ (see Figure 4).

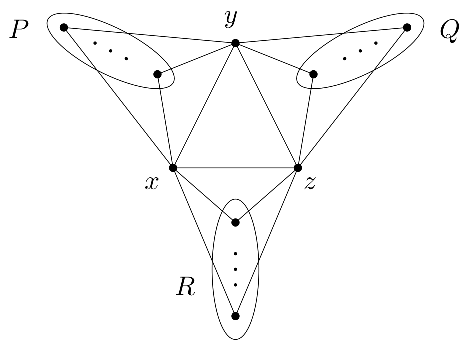


Figure 4: The 2-tree $T(p, q, r)$.

Definition 3.1. Let P, Q, and R be pairwise disjoint independent sets of vertices with $|P| := p$, $|Q| := q$, and $|R| := r$. The graph $T(p, q, r)$ is the graph on $p+q+r+3$ vertices defined on vertex set $\{x, y, z\} \cup P \cup Q \cup R$ where the subgraph induced by $P \cup Q \cup R$ is an independent set, and

- the subgraph induced by $\{x, y\} \cup P$ is isomorphic to $K_{1,1,p}$;
- the subgraph induced by $\{y, z\} \cup Q$ is isomorphic to $K_{1,1,q}$;
- the subgraph induced by $\{z, x\} \cup R$ is isomorphic to $K_{1,1,r}$.

A *dominating vertex* in a graph is a vertex adjacent to all other vertices of the graph.

Lemma 3.2. *A graph H is a 2-tree of diameter two if and only if H has a dominating vertex or $H \cong T(p,q,r)$ for $p,q,r > 0$.*

Proof. Any 2-tree with a dominating vertex has diameter two, and one can easily verify that $T(p,q,r)$ is a 2-tree of diameter two for any $p,q,r > 0$.

For the converse, suppose H is a 2-tree of diameter two. We proceed by induction on $n := |V(H)|$. When $n = 4$, H is isomorphic to the graph obtained from K_4 by deleting one edge, and has a dominating vertex. Now suppose that $n \geq 5$, and let $u \in V(H)$ be a leaf of H, i.e., $d_H(u) = 2$. By the induction hypothesis, $H - u$ has a dominating vertex, or $H - u \cong T(p',q',r')$ for some $p',q',r' > 0$. If $H - u \cong T(p',q',r')$ for some $p',q',r' > 0$, then let $V(H-u) := \{x,y,z\} \cup P' \cup Q' \cup R'$, where $|P'| := p'$, $|Q'| := q'$, and $|R'| := r'$. Since H has diameter two, u must be adjacent to at least two vertices from $\{x,y,z\}$. However $d_H(u) = 2$, and thus $N_H(u) = \{x,y\}$, $N_H(u) = \{y,z\}$, or $N_H(u) = \{z,x\}$. It follows that $H \cong T(p'+1,q',r')$, $H \cong T(p',q'+1,r')$, or $H \cong T(p',q',r'+1)$, respectively.

Now suppose $H - u$ has a dominating vertex, x. If $ux \in E(H)$, then x is a dominating vertex in H. Otherwise, let y and z denote the neighbours of u in H, and note that $yz \in E(H)$. Since H has diameter two, every vertex in $V(H-u)$ is adjacent to x and at least one of y or z. Let P' be the set of vertices in $H - u$ adjacent to both x and y, R' be the set of vertices in $H - u$ adjacent to both x and z, and suppose $|P'| := p'$ and $|R'| := r'$. Since $H - u$ is a 2-tree, $P' \cup R'$ is an independent set. Therefore, $P' \cup R' \cup \{u\}$ is an independent set in H, and thus $H \cong T(p',1,r')$. □

In what follows, we first prove that if $p,q,r > 0$, then $G_4(T(p,q,r))$ has a Hamilton cycle. Let G be a graph with vertex partition $\{F_0, F_1, \dots, F_{N-1}\}$. For each i, $1 \leq i \leq N-1$, let $S_{i-1} \subseteq F_{i-1}$ and $S'_i \subseteq F_i$ denote the vertices incident to the edges of $[F_{i-1}, F_i]$.

Lemma 3.3. *If $p,q,r > 0$, then $G_4(T(p,q,r))$ has a Hamilton cycle.*

Proof. Let $V(K_3) := \{x,y,z\}$. Suppose $f\colon V(K_3) \to \{1,2,3,4\}$ is a 4-colouring of K_3 and $V(G_4(K_3)) := \{f_0, f_1, \dots, f_{N-1}\}$. Since $G_4(K_3)$ has a Hamilton cycle by [6, Theorem 4.1], we may assume that $f_0 f_1 \cdots f_{N-1}$ is a Hamilton path in $G_4(K_3)$.

For $0 \leq i \leq N-1$, let F_i be the set of 4-colourings of $T(p,q,r)$ that agree with f_i on $\{x,y,z\}$. In order to simplify notation, we define $G := G_4(T(p,q,r))$. Then $\{F_0, F_1, \dots, F_{N-1}\}$ is a partition of the vertices of G, and $G[F_i] \cong Q_{p+q+r}$, $0 \leq i \leq N-1$.

Let $st \in [F_{i-1}, F_i]$, where $s \in S_{i-1}$ and $t \in S'_i$ for some $1 \leq i \leq N-1$. Then $s(u) = t(u)$ for all $u \in V(G) \setminus \{v\}$, where v is one of $\{x,y,z\}$, and $s(v) \neq t(v)$. Thus, $[F_{i-1}, F_i]$ is a set of independent edges. If $v = x$, then $s(u) = s(w)$ for all $u,w \in P$ and for all $u,w \in R$, and $t(u) = t(w)$ for all $u,w \in P$ and for all $u,w \in R$. Thus $G[S_{i-1}] \cong Q_q \cong G[S'_i]$, and $G[S_{i-1} \cup S'_i] \cong Q_{q+1}$. Analogously, if $v = y$, $G[S_{i-1}] \cong Q_r \cong G[S'_i]$ and $G[S_{i-1} \cup S'_i] \cong Q_{r+1}$; and if $v = z$, $G[S_{i-1}] \cong Q_p \cong G[S'_i]$, $G[S_{i-1} \cup S'_i] \cong Q_{p+1}$.

Consider the path $f_{i-1} f_i f_{i+1}$ in $G_4(K_3)$, $1 \leq i \leq N-2$. Note that

$$f_{i-1}(x), f_i(x), f_{i+1}(x)$$

use at most two different colours; otherwise, there would be only one colour available for y and z, which is impossible since y and z always receive different colours. Analogously, f_{i-1}, f_i, f_{i+1} assign at most two different colours to each of y and z. It follows that if $|[F_{i-1}, F_i]| = |[F_i, F_{i+1}]| = 2$, then $[F_{i-1}, F_i] \cup [F_i, F_{i+1}] \ncong 2P_3$.

For each edge $f_{i-1}f_i$ with $1 \leq i \leq N-1$, we choose a 4-cycle, B_{i-1}, containing exactly two edges of $[F_{i-1}, F_i]$ as follows.

- For each $|[F_{i-1}, F_i]| = 2$, $G[S_{i-1} \cup S_i']$ is a 2-cube, so we choose $B_{i-1} := G[S_{i-1} \cup S_i'] = a_{i-1}c_{i-1}d_ib_ia_{i-1}$, where $a_{i-1}, c_{i-1} \in F_{i-1}$ and $b_i, d_i \in F_i$.
- For $i = 1, 2, \ldots, N-1$, and $|[F_{i-1}, F_i]| > 2$, first note that $|[F_{i-1}, F_i]| \geq 4$ since $G[S_{i-1} \cup S_i'] \cong Q_n$ for some $n \geq 3$. Thus it is possible to choose edges $a_{i-1}b_i$ and $c_{i-1}d_i$ of $[F_{i-1}, F_i]$ so that $B_{i-1} := a_{i-1}c_{i-1}d_ib_ia_{i-1}$ is edge disjoint from B_{i-2}, and also edge disjoint from B_i if $|[F_i, F_{i+1}]| = 2$ and $i < N-1$.

Let $e_{i,i+1} := G[S_i] \cap B_i$ and $e_{i+1,i} := G[S_{i+1}'] \cap B_i$, $0 \leq i \leq N-2$. Observe that G and the path $f_0f_1 \cdots f_{N-1}$ satisfy conditions (i) and (ii) of Lemma 2.7. Recall that $G[F_i] \cong Q_{p+q+r}$, $0 \leq i \leq N-1$, and $p+q+r \geq 3$. Since any pair of edges of Q_n, $n \geq 2$, is contained in a Hamilton cycle (see [7, Theorem 4.1]), there exist Hamilton cycles C_0 in $G[F_0]$ containing $e_{0,1}$, C_{N-1} in $G[F_{N-1}]$ containing $e_{N-1,N-2}$, and, for $1 \leq i \leq N-2$, C_i in $G[F_i]$ containing $e_{i,i-1}$ and $e_{i,i+1}$. Therefore, by Lemma 2.7, G has a Hamilton cycle. □

In the case where $p > 0$ and $q = r = 0$, $T(p,q,r)$ has a dominating vertex, and is isomorphic to $K_2 \vee \overline{K}_n$.

Lemma 3.4. *For $n \geq 2$, $G_4(K_2 \vee \overline{K}_n)$ has no Hamilton cycle.*

Proof. Let $H := K_2 \vee \overline{K}_n$ for $n \geq 2$, let $\mathcal{H} := G_4(H)$, and let u, v be the two vertices of H of degree $n+1$. For each $1 \leq i \neq j \leq 4$ let

$$V_{ij} := \{c \in V(\mathcal{H}) \mid c(u) = i \text{ and } c(v) = j\}.$$

Then

$$\{V_{12}, V_{13}, V_{14}, V_{21}, V_{23}, V_{24}, V_{31}, V_{32}, V_{34}, V_{41}, V_{42}, V_{43}\}$$

is a partition of $V(\mathcal{H})$. Note that $[V_{\alpha\beta}, V_{\gamma\delta}] \neq \emptyset$ if and only if $\alpha = \gamma$ or $\beta = \delta$. For $1 \leq i \neq j \leq 4$, let L_{ij} be an assignment of lists to the vertices of H such that $L_{ij}(u) := \{i\}$, $L_{ij}(v) := \{j\}$ and $L_{ij}(x) := \{1,2,3,4\}$ for $x \in V(H - \{u,v\})$. Note that $G_{L_{ij}}(H) \cong \mathcal{H}[V_{ij}] \cong Q_n$, for each $1 \leq i \neq j \leq 4$. Define the three-coloured vertices of $\mathcal{H}$ (that is, the colourings of H with three colours) by $c_{ijk} \in V_{ij}$ such that

$$c_{ijk}(x) := \begin{cases} i, & \text{if } x = u, \\ j, & \text{if } x = v, \\ k, & \text{otherwise.} \end{cases}$$

Each V_{ij} has two such vertices, c_{ijk_1} and c_{ijk_2}, where $k_1, k_2 \in \{1,2,3,4\} \setminus \{i,j\}$. Furthermore, $\mathcal{H} - \{c_{ijk_1}, c_{ijk_2}\}$ is disconnected so that any Hamilton cycle in $\mathcal{H}$ must contain the edges of a Hamilton path of $\mathcal{H}[V_{ij}]$ with endpoints c_{ijk_1} and c_{ijk_2}, for each $1 \leq i \neq j \leq 4$.

By [6, Lemma 2.1] there is no Hamilton path in the n-dimensional cube from $00 \cdots 0$ to $11 \cdots 1$ whenever n is even. Thus, for n even, there is no Hamilton path in $\mathcal{H}[V_{ij}]$ with endpoints c_{ijk_1} and c_{ijk_2}. Therefore, $\mathcal{H}$ has no Hamilton cycle when n is even.

For n odd, such Hamilton paths exist and must be used in any Hamilton cycle of $\mathcal{H}$, if one exists. We construct an auxiliary graph A (see Figure 5(a)) where the vertex (i,j)

represents a Hamilton path in $\mathcal{H}[V_{ij}]$ from c_{ijk_1} to c_{ijk_2}. There is an edge in A between (i_1, j_1) and (i_2, j_2) whenever $c_{i_1j_1k}$ is adjacent to $c_{i_2j_2k}$ in $\mathcal{H}$. We label the edge between (i_1, j_1) and (i_2, j_2) by the unique element of $\{1, 2, 3, 4\} \setminus (\{i_1, j_1\} \cup \{i_2, j_2\})$ (see Figure 5(b)). Notice that the edges labelled i, $1 \leq i \leq 4$, induce a 6-cycle, and the edges of these four 6-cycles partition $E(A)$.

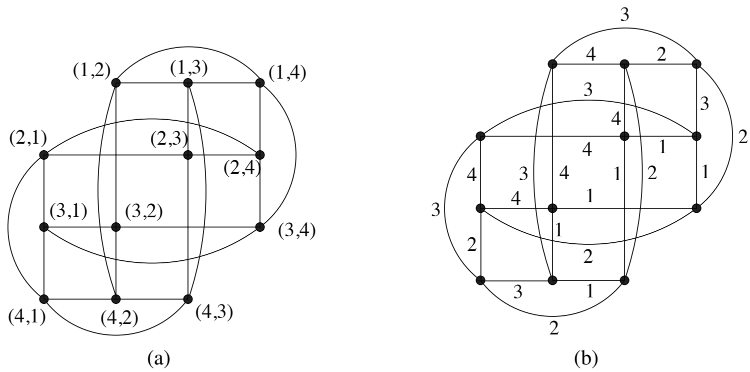


Figure 5: The auxiliary graph A.

A Hamilton cycle in $\mathcal{H}$ corresponds to a Hamilton cycle, $\mathcal{C}$, in A in which any two consecutive edges of $\mathcal{C}$ have different labels. Such a cycle $\mathcal{C}$ uses a matching of size three from each labelled 6-cycle in A. Without loss of generality, we may assume $\mathcal{C}$ contains horizontal edges of the 6-cycle labelled 1. Now, $\mathcal{C}$ uses horizontal edges from one of the remaining labelled 6-cycles, otherwise, $\mathcal{C}$ contains a K_3. Regardless of whether $\mathcal{C}$ uses horizontal edges of the 6-cycle labelled by 2, 3 or 4, using vertical edges of the two remaining 6-cycles gives $\mathcal{C} \cong C_4 \cup C_8$, a contradiction. Therefore $\mathcal{H}$ has no Hamilton cycle. □

Observe that if H is a 2-tree of diameter two having a dominating vertex u, then $H \cong T \vee \{u\}$ for some tree T.

Lemma 3.5. *Let T be a tree on at least two vertices. Then $G_4(T \vee \{u\})$ has a Hamilton cycle unless T is a star on at least three vertices or the bipartition of $V(T)$ has two even parts.*

The proof of this lemma requires a result that we state here, but whose proof is technical and is postponed to the Appendix A.

Lemma 3.6. *Let T be a tree with bipartition (A, B), where $|A| := \ell$ and $|B| := r$, and let $G_3(T)$ be the 3-colouring graph of T with colours $C = \{1, 2, 3\}$. Define c_{ij} to be the vertex of $G_3(T)$ with $c_{ij}(a) = i$ for all $a \in A$ and $c_{ij}(b) = j$ for all $b \in B$.*

(1) If $\ell, r > 0$ are both even, then $G_3(T)$ has no spanning subgraph consisting only of paths whose ends are in $\{c_{12}, c_{13}, c_{21}, c_{23}, c_{31}, c_{32}\}$.

(2) If $\ell > 1$ is odd and $r > 0$ is even, then $G_3(T)$ has a Hamilton path from c_{12} to c_{23}.

(3) If $\ell > 1$ and $r > 1$ are both odd, then $G_3(T)$ has a Hamilton path from c_{12} to c_{13}.

Proof of Lemma 3.5. Let T be a tree with $|V(T)| \geq 2$ and bipartition (A, B), where $A := \{x_1, x_2, \dots, x_\ell\}$ and $B := \{y_1, y_2, \dots, y_r\}$. Let $H := T \vee \{u\}$, and let $\mathcal{H} := G_4(H)$ be the 4-colouring graph of H with colours $\{1, 2, 3, 4\}$. For each $1 \leq k \leq 4$, define

$$V_k := \{c \in V(\mathcal{H}) \mid c(u) = k\}.$$

Then $\{V_1, V_2, V_3, V_4\}$ is a partition of $V(\mathcal{H})$. Define L_k to be an assignment of lists with $L_k(u) := \{k\}$ and $L_k(w) := \{1, 2, 3, 4\}$ for $w \in V(T)$. Note that $G_{L_k}(H) \cong \mathcal{H}[V_k] \cong G_3(T)$. Define the three-coloured vertices of $\mathcal{H}$ (that is, the colourings of H with three colours) by $c_{ijk} \in V_k$ so that $c_{ijk}(x) := i$ for all $x \in A$, $c_{ijk}(y) := j$ for all $y \in B$.

Observe

$$[V_1, V_2] = \{c_{341}c_{342},\ c_{431}c_{432}\}, \qquad [V_1, V_3] = \{c_{241}c_{243},\ c_{421}c_{423}\},$$
$$[V_1, V_4] = \{c_{231}c_{234},\ c_{321}c_{324}\}, \qquad [V_2, V_3] = \{c_{142}c_{143},\ c_{412}c_{413}\},$$
$$[V_2, V_4] = \{c_{132}c_{134},\ c_{312}c_{314}\}, \qquad [V_3, V_4] = \{c_{123}c_{124},\ c_{213}c_{214}\}.$$

Case 1. Suppose $|V(T)| = 2$. Then $T \vee \{u\} \cong K_3$ and $G_4(K_3)$ has a Hamilton cycle by [6, Theorem 4.1].

Case 2. Suppose T is a star with $|V(T)| \geq 3$. Then $G_4(T \vee \{u\}) \cong K_2 \vee \overline{K}_n$ for some $n \geq 2$. By Lemma 3.4, $G_4(T \vee \{u\})$ has no Hamilton cycle.

Case 3. Suppose both $|A|$ and $|B|$ are odd. By Lemma 3.6, there is a Hamilton path in $\mathcal{H}[V_1]$ from c_{421} to c_{431}, in $\mathcal{H}[V_2]$ from c_{432} to c_{132}, in $\mathcal{H}[V_3]$ from c_{134} to c_{124} and in $\mathcal{H}[V_4]$ from c_{123} to c_{423}. The union of these paths with edges $\{c_{431}c_{432},\ c_{132}c_{134},\ c_{124}c_{123},\ c_{423}c_{421}\}$ gives a Hamilton cycle in $\mathcal{H}$.

Case 4. Suppose one of $|A|$ and $|B|$ is even and the other is odd. Without loss of generality, $|A|$ is odd and $|B|$ is even. By Lemma 3.6, there is a Hamilton path in $\mathcal{H}[V_1]$ from c_{231} to c_{341}, in $\mathcal{H}[V_2]$ from c_{342} to c_{412}, in $\mathcal{H}[V_3]$ from c_{413} to c_{123} and in $\mathcal{H}[V_4]$ from c_{124} to c_{234}. The union of these paths with edges $\{c_{341}c_{342},\ c_{412}c_{413},\ c_{123}c_{124},\ c_{234}c_{231}\}$ gives a Hamilton cycle in $\mathcal{H}$.

Case 5. Suppose both $|A|$ and $|B|$ are even, and suppose $\mathcal{H}$ contains a Hamilton cycle $\mathcal{C}$. Then $\mathcal{C}[V_1]$ is a spanning subgraph of $\mathcal{H}[V_1] \cong G_3(T)$ consisting of a union of paths whose endpoints are three-coloured vertices of V_1, contradicting Lemma 3.6. Thus in this case, $\mathcal{H}$ has no Hamilton cycle. □

4 Constructing 2-trees of diameter at least three

To complete the proof of our main result, we must show that if H' is a 2-tree with diameter at least three, then $k_0(H') = 4$. Naively deleting two leaves with the intention of applying Remark 2.6 may be problematic. For example, let H' be a 2-tree with diameter at least three having leaves x and y of distance at least three. Let $H = H' - \{x, y\}$, $N_{H'}(x) = \{x_1, x_2\}$, and suppose $G_4(H)$ has a Hamilton cycle $f_0 f_1 \cdots f_{N-1} f_0$. Let F_i be the set of 4-colourings of H' that agree with f_i on $V(H)$, $0 \leq i \leq N-1$. In the case that $f_{i-1}f_i$ arises from a colour change on x_1 and $f_i f_{i+1}$ arises from a colour change on x_2, the subgraph $G[F_{i-1} \cup F_i \cup F_{i+1}]$ is isomorphic to the forbidden subgraph in Figure 2. Because of this we take a more general approach. Suppose a 2-tree H' is obtained from a 2-tree H by repeatedly adding vertices of degree two. Instead of a Hamilton cycle in $G_4(H)$ we take a spanning tree satisfying certain properties providing the flexibility needed to construct a Hamilton cycle in $G_4(H')$. Our approach does not require $G_4(H)$ to have a Hamilton cycle. To facilitate this procedure, we define nine operations (see Figure 6).

Definition 4.1. Let H be a 2-tree. Then H' is obtained from H by

Operation I if $\{\alpha,\beta,\gamma\} := V(H') \setminus V(H)$, $\{a_1a_2, b_1b_2, c_1c_2\} \subseteq E(H)$, and $N_{H'}(\alpha) = \{a_1,a_2\}$, $N_{H'}(\beta) = \{b_1,b_2\}$, $N_{H'}(\gamma) = \{c_1,c_2\}$.

Operation II if $\{\alpha,\beta,\gamma\} := V(H') \setminus V(H)$, $\{xa, xb, c_1c_2\} \subseteq E(H)$, and $N_{H'}(\alpha) = \{x,a\}$, $N_{H'}(\beta) = \{x,b\}$, $N_{H'}(\gamma) = \{c_1,c_2\}$.

Operation III if $\{\alpha,\beta,\gamma\} := V(H') \setminus V(H)$, $\{ax, xy, yc\} \subseteq E(H)$, and $N_{H'}(\alpha) = \{a,x\}$, $N_{H'}(\beta) = \{x,y\}$, $N_{H'}(\gamma) = \{c,y\}$.

Operation IV if $\{\alpha,\beta,\gamma,\delta\} := V(H') \setminus V(H)$, $\{xy, zw\} \subseteq E(H)$, and $N_{H'}(\alpha) = N_{H'}(\beta) = \{x,y\}$, $N_{H'}(\gamma) = N_{H'}(\delta) = \{w,z\}$.

Operation V if $\{\alpha,\beta,\gamma,\delta\} := V(H') \setminus V(H)$, $\{xy, zw\} \subseteq E(H)$, and $N_{H'}(\alpha) = \{\beta,x,y\}$, $N_{H'}(\beta) = \{\alpha,y\}$, $N_{H'}(\gamma) = N_{H'}(\delta) = \{w,z\}$.

Operation VI if $\{\alpha,\beta,\gamma,\delta\} := V(H') \setminus V(H)$, $\{xy, zw\} \subseteq E(H)$, and $N_{H'}(\alpha) = \{\beta,x,y\}$, $N_{H'}(\beta) = \{\alpha,y\}$, $N_{H'}(\gamma) = \{\delta,w,z\}$, $N_{H'}(\delta) = \{\gamma,z\}$.

Operation VII if $\{\alpha,\beta,\gamma,\delta\} := V(H') \setminus V(H)$, $\{xy, xz\} \subseteq E(H)$, and $N_{H'}(\alpha) = \{\beta,x,y\}$, $N_{H'}(\beta) = \{\alpha,y\}$, $N_{H'}(\gamma) = \{\delta,x,z\}$, $N_{H'}(\delta) = \{\gamma,z\}$.

Operation VIII if $\{\alpha,\beta,\gamma,\delta\} := V(H') \setminus V(H)$, $\{xy, xz\} \subseteq E(H)$, and $N_{H'}(\alpha) = \{\beta,x,y\}$, $N_{H'}(\beta) = \{x,\alpha\}$, $N_{H'}(\gamma) = \{\delta,x,z\}$, $N_{H'}(\delta) = \{\gamma,z\}$.

Operation IX if $\{\alpha,\beta,\gamma,\delta\} := V(H') \setminus V(H)$, $\{xy, xz\} \subseteq E(H)$, and $N_{H'}(\alpha) = \{\beta,x,y\}$, $N_{H'}(\beta) = \{\alpha,y\}$, $N_{H'}(\gamma) = N_{H'}(\delta) = \{x,z\}$.

Remark 4.2. Since each of Operations I through IX can be performed by sequentially adding simplicial vertices of degree two to H, H' is a 2-tree.

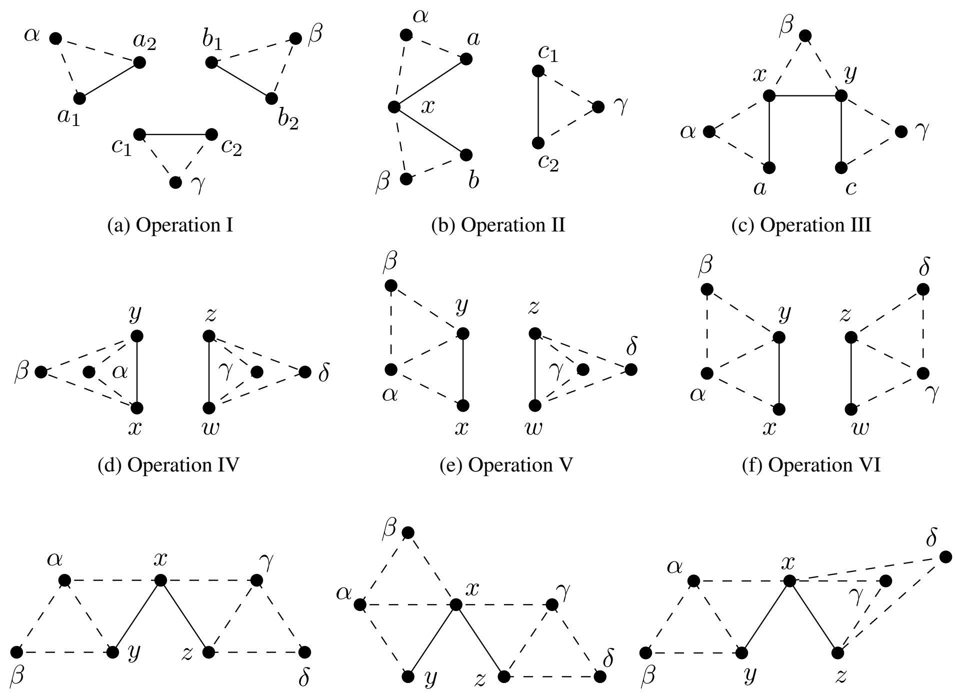
Recall that D denotes the unique 2-tree of diameter three on six vertices (Figure 1(a)).

Theorem 4.3. *A graph H' is a 2-tree of diameter at least three if and only if $H' \cong D$ or H' can be obtained from a 2-tree H by applying one of Operations I through IX.*

Proof. ($\Leftarrow$): If $H' \cong D$, the result is trivially true. Otherwise, it follows from Remark 4.2 that H' is a 2-tree. Since each operation produces two leaves that are distance at least three apart, H' has diameter at least three.

($\Rightarrow$): As already observed, D is the unique 2-tree of diameter three on six vertices. The 2-trees on three, four and five vertices all have diameter less than three. Thus, we may assume that H' is a 2-tree of diameter three and $|V(H')| \geq 7$. Since H' has diameter at least three, there are at least two leaves whose neighbourhoods are vertex disjoint. We consider cases based on the number of edges induced by the neighbourhoods of the leaves of H', and the number of leaves of H'.

Case 1. First assume that the neighbourhoods of the leaves of H' induce at least three distinct edges.



(a) Operation I (b) Operation II (c) Operation III (d) Operation IV (e) Operation V (f) Operation VI (g) Operation VII (h) Operation VIII (i) Operation IX

Figure 6: Operations I through IX applied to a graph H by the addition of vertices α, β, γ, and δ where applicable. The solid edges belong to H and the dotted edges are added to construct H'.

If H' has three leaves whose neighbourhoods are pairwise vertex disjoint, then H' contains vertex disjoint edges a_1a_2, b_1b_2, c_1c_2, and leaves α, β, γ with $N(\alpha) = \{a_1, a_2\}$, $N(\beta) = \{b_1, b_2\}$, $N(\gamma) = \{c_1, c_2\}$. Letting $H := H' - \{\alpha, \beta, \gamma\}$ results in a 2-tree, and applying Operation I to H produces H'.

We may now assume that no three leaves of H' have neighbourhoods that are pairwise vertex disjoint. Choose two leaves α and γ whose neighbourhoods are vertex disjoint, and let $\beta \notin \{\alpha, \gamma\}$ be a leaf such that $N(\beta) \notin \{N(\alpha), N(\gamma)\}$. If $N(\beta)$ intersects exactly one of $N(\alpha)$ and $N(\gamma)$, then we may assume without loss of generality that $|N(\beta) \cap N(\alpha)| = 1$ and $N(\beta) \cap N(\gamma) = \emptyset$. Then H' contains a path of length two, axb and an edge c_1c_2 that is vertex disjoint from axb, such that $N(\alpha) = \{a, x\}$, $N(\beta) = \{x, b\}$, and $N(\gamma) = \{c_1, c_2\}$. Letting $H := H' - \{\alpha, \beta, \gamma\}$ results in a 2-tree, and applying Operation II to H produces H'. Otherwise, $|N(\beta) \cap N(\alpha)| = |N(\beta) \cap N(\gamma)| = 1$, and H' contains a path of length three, c_1xyc_2 such that $N(\alpha) = \{c_1, x\}$, $N(\beta) = \{x, y\}$ and $N(\gamma) = \{y, c_2\}$. Letting $H := H' - \{\alpha, \beta, \gamma\}$ results in a 2-tree, and applying Operation III to H produces H'.

Case 2. We may now assume that the neighbourhoods of the leaves of H' induce exactly two edges.

Case 2(a). Suppose that H' has at least three leaves. Let leaves β and γ have neighbourhoods that are vertex disjoint, and let $\delta \notin \{\beta, \gamma\}$ be a leaf. Without loss of generality, $N(\delta) = N(\gamma)$.

If there exists a leaf α with $N(\alpha) = N(\beta)$, then $H := H' - \{\alpha, \beta, \gamma, \delta\}$ is a 2-tree, and applying Operation IV to H produces H'. Otherwise, no other leaf of H' has the same neighbourhood as β. If we let $N(\beta) := \{\alpha, y\}$, then at least one of $\{\alpha, y\}$ is a leaf of $H' - \beta$; without loss of generality, α is a leaf of $H' - \beta$, and so $d(\alpha) = 3$. It follows that $N(\alpha) = \{\beta, y, x\}$ for some $x \notin \{\alpha, \beta, \gamma, \delta, y\}$, and that $xy \in E(H')$.

Let $N(\gamma) = N(\delta) = \{w, z\}$. We consider two cases depending on $|\{x\} \cap \{w, z\}|$. If $|\{x\} \cap \{w, z\}| = 0$, then $H := H' - \{\alpha, \beta, \gamma, \delta\}$ is a 2-tree, and applying Operation V to H produces H'. If $|\{x\} \cap \{w, z\}| = 1$, then $y = w$ or $y = z$, and the two cases are analogous. The graph $H := H' - \{\alpha, \beta, \gamma, \delta\}$ is a 2-tree, and applying Operation IX to H produces H'.

Case 2(b). Finally, assume that H' has exactly two leaves, β and δ, with $N(\beta) = \{\alpha, y\}$ and $N(\delta) = \{\gamma, z\}$. Since $d_{H'}(\beta, \delta) \geq 3$, $\{\alpha, y\} \cap \{\gamma, z\} = \emptyset$. We may assume, without loss of generality, that α and γ are leaves in $H' - \{\beta, \delta\}$, and so $d(\alpha) = 3$ and $d(\gamma) = 3$. It follows that $N(\alpha) = \{\beta, y, p\}$ and $N(\gamma) = \{\delta, z, q\}$ for some $p, q \notin \{\beta, \alpha, \delta, \gamma\}$ with $p \neq y$, $q \neq z$ and $py, qz \in E(H')$.

We consider three cases depending on $|\{p, y\} \cap \{q, z\}|$. If $|\{p, y\} \cap \{q, z\}| = 0$, then $H := H' - \{\alpha, \beta, \gamma, \delta\}$ is a 2-tree, and applying Operation VI to H produces H'. If $|\{p, y\} \cap \{q, z\}| = 1$, then either $p = q$, $p = z$, or $q = y$. In the case $p = q$, $H := H' - \{\alpha, \beta, \gamma, \delta\}$ is a 2-tree, and applying Operation VII to H produces H'. In the case $p = z$, $H := H' - \{\alpha, \beta, \gamma, \delta\}$ is a 2-tree, and applying Operation VIII to H produces H'. The case $q = y$ is analogous to the case $p = z$. Finally if $|\{p, y\} \cap \{q, z\}| = 2$, then the fact that $y \neq z$ implies that $p = z$ and $q = y$, and hence $H' \cong D$, contradicting the fact that $|V(H')| \geq 7$. □

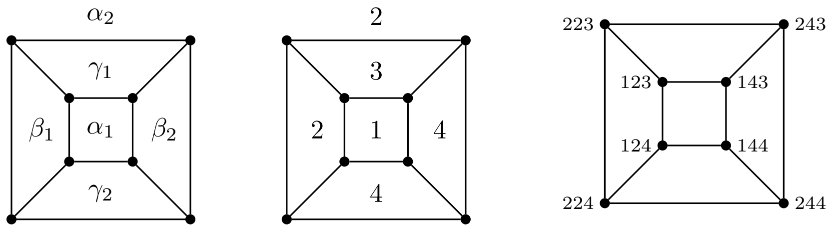
5 Operations and the 4-colouring graph

Let H be a 2-tree, and let H' be the 2-tree obtained from H by applying one of the Operations I through IX. As before, let $V(G_4(H)) := \{f_0, f_1, \ldots, f_{N-1}\}$, and let $F_j \subseteq V(G_4(H'))$ be the set of 4-colourings of H' that agree with f_j on the vertices of H, $0 \leq j \leq N-1$. For each Operation I through IX, what follows is a description of the subgraph of $G := G_4(H')$ induced by F_i, $0 \leq i \leq N-1$, and also a description of the subgraph of G induced by $F_i \cup F_j$ when $f_i f_j \in E(G_4(H))$, $0 \leq i, j \leq N-1$. Each edge $f_i f_j$ of $G_4(H)$ is also assigned a label to indicate the structure of $G[F_i \cup F_j]$. We remark that for a path $f_i f_j f_k$ of length two in $G_4(H)$, if $f_i(u) \neq f_j(u)$ for some $u \in V(H)$, then $f_j(u) = f_k(u)$.

5.1 Operation I

If H' is obtained from H using Operation I, then there are two choices of colour for each of the vertices α, β, and γ, so for each i, $0 \leq i \leq N-1$, $G[F_i] \cong Q_3$. To simplify the labelling of the vertices of $G[F_i]$, we label the faces of a plane drawing of Q_3 as shown in Figure 7(a), where α_1 and α_2 are the possible colours of vertex α, β_1 and β_2 are the possible colours of vertex β, and γ_1 and γ_2 are the possible colours of vertex γ. Without loss of generality, assume these colour choices are as shown in Figure 7(b). A vertex u of

Q_3 is assigned label $\alpha_i\beta_j\gamma_k$ where $i, j, k \in \{1, 2\}$, and u is incident with the faces labelled α_i, β_j and γ_k (see Figure 7(c)).

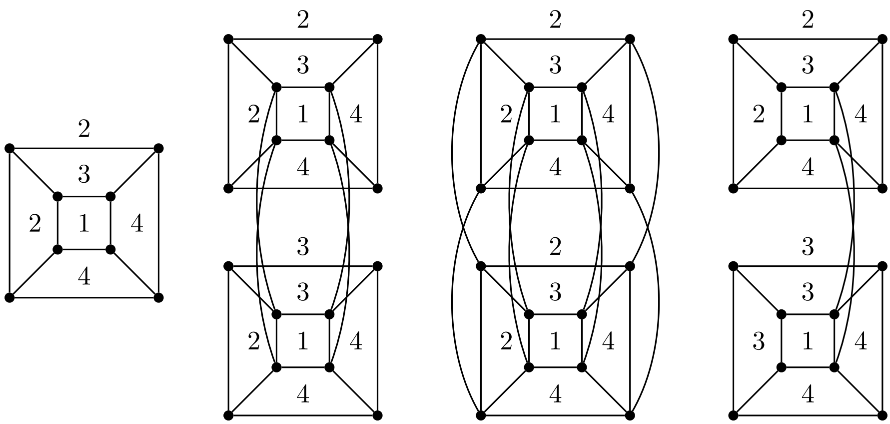


(a) (b) (c)

Figure 7: Labelling $G[F_i]$.

The following arguments can be made with sets of colours $\{f_i(a_1), f_i(a_2)\}$, $\{f_i(b_1), f_i(b_2)\}$ and $\{f_i(c_1), f_i(c_2)\}$ each chosen independently as a subset of $\{1, 2, 3, 4\}$. To ease notation, we assume that $f_i(a_1) = 3$, $f_i(a_2) = 4$, $f_i(b_1) = 1$, $f_i(b_2) = 3$, $f_i(c_1) = 1$, and $f_i(c_2) = 2$. Then the colour choices for α are $\{1, 2\}$, for β are $\{2, 4\}$, and for γ are $\{3, 4\}$. As already noted, $G[F_i] \cong Q_3$; assume that $G[F_i]$ has been drawn in the plane and labelled as in Figure 8(a).

If $f_if_j \in E(G_4(H))$, then f_j is obtained from f_i by changing the colour of a single vertex in $V(H)$. We label each edge $f_if_j \in E(G_4(H))$ according to the structure of $G[F_i \cup F_j]$.



(a) (b) (c) (d)

Figure 8: $G[F_i]$ and $G[F_i \cup F_j]$ for Operation I, II and III.

(i) $f_i(v) \neq f_j(v)$ for $v \in \{a_1, a_2\}$. Without loss of generality, suppose $v = a_1$. Since H is a 2-tree, there is a vertex $a_3 \in V(H)$ such that $H[\{a_1, a_2, a_3\}] \cong K_3$. Observe $f_i(a_3) \in \{1, 2\}$; we may assume $f_i(a_3) = 1$. Then $f_j(a_1) = 2$.[1] Since $f_j(a_2) =$

[1]Since $f_j(a_1)$ is uniquely determined there is no f_ℓ with $\ell \neq j$ such that f_i and f_ℓ differ on a_1.

$f_i(a_2) = 4$, the colours available for α are $\{1, 3\}$; the colours available for β and γ are unchanged. Therefore $[F_i, F_j]$ is a matching of size four between the 4-cycle bounding the α_1 face in $G[F_i]$ and the 4-cycle bounding either the α_1 face or the α_2 face in $G[F_j]$. Thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 8(b). We label the edge $f_i f_j$ by a-sq. It follows from Footnote 1 that there are at most two edges incident to f_i having label a-sq. Furthermore, we remark that if $f_i f_{j_1}, f_i f_{j_2} \in E(G_4(H))$ both have label a-sq, then the four vertices of F_i incident to the edges of $[F_i, F_{j_1}]$ are the same four vertices of F_i that are incident to the edges of $[F_i, F_{j_2}]$, and induce a 4-cycle in $G[F_i]$ that bounds a face with label α_1 or α_2.

(ii) $f_i(v) \neq f_j(v)$ for $v \in \{b_1, b_2\}$. As in (i), we label the edge $f_i f_j$ by b-sq. Note that there are at most two edges incident to f_i having label b-sq. If $f_i f_{j_1}, f_i f_{j_2} \in E(G_4(H))$ both have label b-sq, then the four vertices of F_i incident to the edges of $[F_i, F_{j_1}]$ are the same four vertices of F_i that are incident to the edges of $[F_i, F_{j_2}]$, and induce a 4-cycle in $G[F_i]$ that bounds a face with label β_1 or β_2.

(iii) $f_i(v) \neq f_j(v)$ for $v \in \{c_1, c_2\}$. As in (i) and (ii), we label the edge $f_i f_j$ by c-sq. Note that there are at most two edges incident to f_i having label c-sq. If $f_i f_{j_1}, f_i f_{j_2} \in E(G_4(H))$ both have label c-sq, then the four vertices of F_i incident to the edges of $[F_i, F_{j_1}]$ are the same four vertices of F_i that are incident to the edges of $[F_i, F_{j_2}]$, and induce a 4-cycle in $G[F_i]$ that bounds a face with label γ_1 or γ_2.

(iv) $f_i(u) \neq f_j(u)$ for $u \in V(H) \setminus \{a_1, a_2, b_1, b_2, c_1, c_2\}$. In this case, the vertex labels on $G[F_i]$ and $G[F_j]$ are identical. Thus $[F_i, F_j]$ is a perfect matching, and $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 8(c). We label the edge $f_i f_j$ by pm.

Table 1: Summary of Operation I.

Vertex whose colour is changed	Subgraph induced by $F_i \cup F_j$	Label of $f_i f_j$
a_1, a_2	Figure 8(b)	a-sq
b_1, b_2	Figure 8(b)	b-sq
c_1, c_2	Figure 8(b)	c-sq
$u \in V(H) \setminus \{a_1, a_2, b_1, b_2, c_1, c_2\}$	Figure 8(c)	pm

5.2 Operation II

As with Operation I, $G[F_i] \cong Q_3$ for each i, $0 \leq i \leq N-1$. We may assume that $f_i(a) = 4$, $f_i(x) = 3$, $f_i(b) = 1$, $f_i(c_1) = 1$, and $f_i(c_2) = 2$. Then the colour choices for α are $\{1, 2\}$, for β are $\{2, 4\}$, and for γ are $\{3, 4\}$. Using the same labelling convention as for Operation I, we assume that $G[F_i]$ is drawn in the plane and labelled as in Figure 8(a).

If $f_i f_j \in E(G_4(H))$, then f_j is obtained from f_i by changing the colour of a single vertex in $V(H)$. We label each edge $f_i f_j \in E(G_4(H))$ according to the structure of $G[F_i \cup F_j]$.

(i) $f_i(v) \neq f_j(v)$ for $v \in \{a, b, c_1, c_2\}$. This is analogous to Operation I when the colour of one of $\{a_1, a_2, b_1, b_2, c_1, c_2\}$ is changed, and thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 8(b). We label the edge f_if_j by either a-sq, b-sq or c-sq as in Operation I.

(ii) $f_i(x) \neq f_j(x)$. We may assume that $f_j(x) = 2$. Then the colours available for α are $\{1, 3\}$ and the colours available for β are $\{3, 4\}$; the colours available for γ are unchanged. Therefore, $[F_i, F_j]$ is a matching of size two where $G[F_i]$ and $G[F_j]$ are edges whose endpoint labels are unchanged, and thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 8(d). We label the edge f_if_j by e.

(iii) $f_i(u) \neq f_j(u)$ for $u \in V(H) \setminus \{a, b, x, c_1, c_2\}$. In this case, the vertex labels on $G[F_i]$ and $G[F_j]$ are identical. Thus $[F_i, F_j]$ is a perfect matching, and $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 8(c). We label the edge f_if_j by pm.

Table 2: Summary of Operation II.

Vertex whose colour is changed	Subgraph induced by $F_i \cup F_j$	Label of f_if_j
a	Figure 8(b)	a-sq
b	Figure 8(b)	b-sq
c_1, c_2	Figure 8(b)	c-sq
x	Figure 8(d)	e
$u \in V(H) \setminus \{a, b, x, c_1, c_2\}$	Figure 8(c)	pm

5.3 Operation III

As with Operations I and II, $G[F_i] \cong Q_3$ for each i, $0 \leq i \leq N-1$. We may assume that $f_i(x) = 3$, $f_i(y) = 1$, $f_i(a) = 4$, and $f_i(c) = 2$. Then the colour choices for α are $\{1, 2\}$, for β are $\{2, 4\}$, and for γ are $\{3, 4\}$. Using the same labelling convention as for Operation I, we assume that $G[F_i]$ is drawn in the plane and labelled as in Figure 8(a).

If $f_if_j \in E(G_4(H))$, then f_j is obtained from f_i by changing the colour of a single vertex in $V(H)$. We label each edge $f_if_j \in E(G_4(H))$ according to the structure of $G[F_i \cup F_j]$.

(i) $f_i(v) \neq f_j(v)$ for $v \in \{a, c\}$. This is analogous to Operation I when the colour is changed on one of $\{a_1, a_2, c_1, c_2\}$, and thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 8(b). We label the edge f_if_j by either a-sq or c-sq as in Operation I.

(ii) $f_i(v) \neq f_j(v)$ for $v \in \{x, y\}$. This is analogous to Operation II when the colour of x is changed, and thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 8(d). We label the edge f_if_j by e.

(iii) $f_i(u) \neq f_j(u)$ for vertex $u \in V(H) \setminus \{a, c, x, y\}$. In this case, the vertex labels on $G[F_i]$ and $G[F_j]$ are identical. Thus $[F_i, F_j]$ is a perfect matching, and $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 8(c). We label the edge f_if_j by pm.

Remark 5.1. We note that for Operations I – III, if $f_if_{j_1}, f_if_{j_2} \in E(G_4(H))$ have the same label that is not e, and $F_i^1, F_i^2 \subseteq F_i$ are incident to the edges of $[F_i, F_{j_1}], [F_i, F_{j_2}]$, respectively, then $F_i^1 = F_i^2$.

Table 3: Summary of Operation III.

Vertex whose colour is changed	Subgraph induced by $F_i \cup F_j$	Label of $f_i f_j$
a	Figure 8(b)	a-sq
c	Figure 8(b)	c-sq
x, y	Figure 8(d)	e
$u \in V(H) \setminus \{a, c, x, y\}$	Figure 8(c)	pm

5.4 Operation IV

We may assume that $f_i(x) = 1$, $f_i(y) = 2$, $f_i(w) = 2$ and $f_i(z) = 3$. Then the pairs of colour available for α and β, respectively, are

$$\{(4,3),(3,3),(3,4),(4,4)\},$$

and the pairs of colours available for γ and δ, respectively, are

$$\{(1,4),(1,1),(4,1),(4,4)\}.$$

Thus the subgraph of G induced by F_i is isomorphic to $C_4 \,\square\, C_4$, and we assume that it is drawn as shown in Figure 9(a), with the rows labelled by the pairs of colours available for α and β, respectively, and the columns labelled by the pairs of colours available for γ and δ, respectively.

If $f_i f_j \in E(G_4(H))$, then f_j is obtained from f_i by changing the colour of a single vertex in $V(H)$; there are three cases to consider.

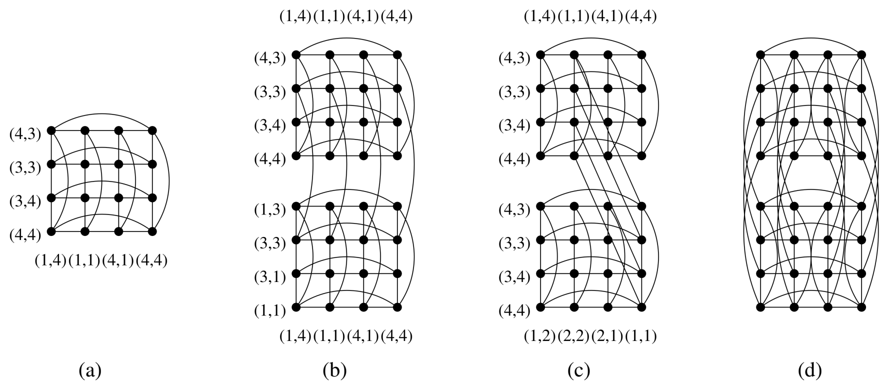


Figure 9: $G[F_i]$ and $G[F_i \cup F_j]$ for Operation IV.

(i) $f_i(v) \neq f_j(v)$ for $v \in \{x, y\}$. We may assume that $f_j(x) = 4$. Then the pairs of colours available for α and β, respectively, are

$$\{(1,3),(3,3),(3,1),(1,1)\},$$

and the pairs of colours available for γ and δ are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 9(b).

(ii) $f_i(v) \neq f_j(v)$ for $v \in \{w, z\}$. We may assume that $f_j(w) = 4$. Then the pairs of colours available for γ and δ, respectively, are

$$\{(2,2),(2,1),(1,1),(1,2)\},$$

and the pairs of colours available for α and β are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 9(c).

(iii) $f_i(u) \neq f_j(u)$ for $u \in V(H) \setminus \{x, y, w, z\}$. In this case, the vertex labels on $G[F_i]$ and $G[F_j]$ are identical. Thus $[F_i, F_j]$ is a perfect matching, and $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 9(d).

Table 4: Summary of Operation IV.

Vertex whose colour is changed	Subgraph induced by $F_i \cup F_j$	Label of $f_i f_j$
x, y	Figure 9(b)	r
z, w	Figure 9(c)	c
$u \in V(H) \setminus \{x, y, z, w\}$	Figure 9(d)	pm

Remark 5.2. We note that for Operation IV, if $f_i f_j \in E(G_4(H))$ has label r and $e \in [F_i, F_j]$, then each colouring corresponding to an end of e assigns the same colour to α and β. Similarly, if $f_i f_j \in E(G_4(H))$ has label c and $e \in [F_i, F_j]$, then each colouring corresponding to an end of e assigns the same colour to γ and δ.

5.5 Operation V

We may assume that $f_i(x) = 1$, $f_i(y) = 2$, $f_i(w) = 2$ and $f_i(z) = 3$. Then the pairs of colour available for α and β, respectively, are

$$\{(3,4),(3,1),(4,1),(4,3)\},$$

and the pairs of colours available for γ and δ, respectively, are

$$\{(4,1),(1,1),(1,4),(4,4)\}.$$

Thus the subgraph of $G_4(H')$ induced by F_i is isomorphic to $P_4 \,\square\, C_4$, and we assume that it is drawn as shown in Figure 10(a), with the rows labelled by the pairs of colours available for α and β, respectively, and the columns labelled by the pairs of colours available for γ and δ, respectively.

If $f_i f_j \in E(G_4(H))$, then f_j is obtained from f_i by changing the colour of a single vertex in $V(H)$; there are five cases to consider. Since H is a 2-tree, there are vertices $a, b \in V(H)$ such that $H[\{x, y, a\}] \cong K_3$ and $H[\{w, z, b\}] \cong K_3$. Observe $f_i(a) \in \{3, 4\}$ and $f_i(b) \in \{1, 4\}$. We may assume that $f_i(a) = 4$ and $f_i(b) = 1$. Even though b (respectively, a) could be equal to x or y (respectively, w or z), this does not affect the argument.

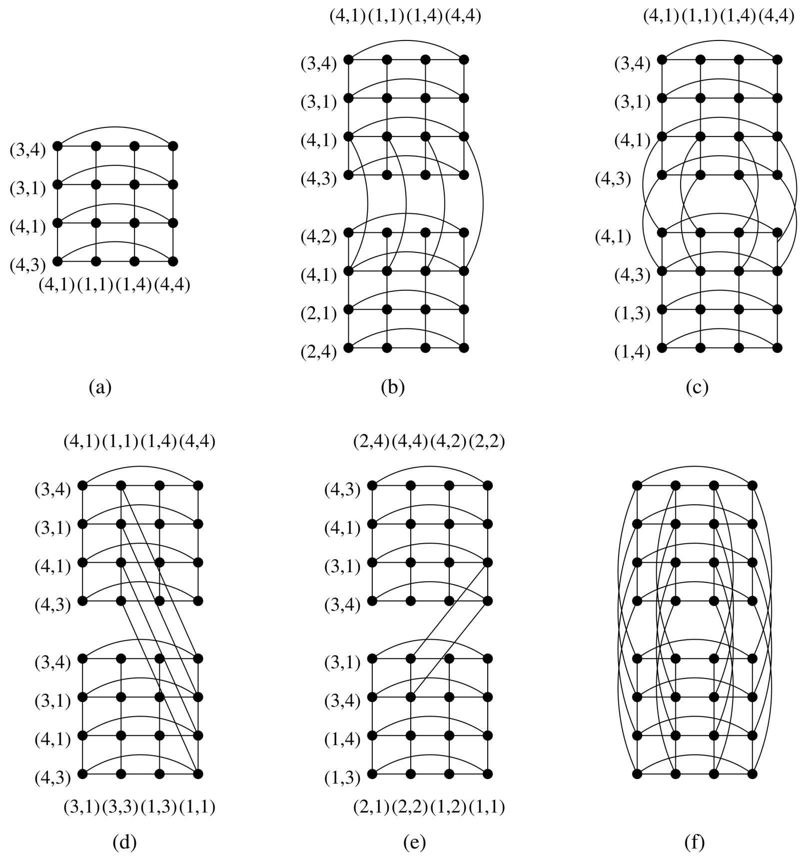


Figure 10: Subgraphs induced by $F_i \cup F_j$ for Operations V and IX.

(i) $f_i(y) \neq f_j(y)$. Then $f_j(y) = 3$ and the pairs of colours available for α and β, respectively, are

$$\{(4,2),(4,1),(2,1),(2,4)\},$$

and the pairs of colours available for γ and δ are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 10(b) with appropriate labels.

(ii) $f_i(x) \neq f_j(x)$. Then $f_j(x) = 3$ and the pairs of colours available for α and β, respectively, are

$$\{(4,1),(4,3),(1,3),(1,4)\},$$

and the pairs of colours available for γ and δ are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 10(c).

(iii) $f_i(z) \neq f_j(z)$. Then $f_j(z) = 4$ and the pairs of colours available for γ and δ, respectively, are

$$\{(3,3),(1,3),(1,1),(3,1)\},$$

and the pairs of colours available for α and β are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 10(d).

(iv) $f_i(w) \neq f_j(w)$. Then $f_j(w) = 4$ and the pairs of colours available for γ and δ, respectively, are

$$\{(3,3),(1,3),(1,1),(3,1)\},$$

and the pairs of colours available for α and β are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 10(d).

(v) $f_i(u) \neq f_j(u)$ $u \in V(H) \setminus \{x, y, w, z\}$. In this case, the vertex labels on $G[F_i]$ and $G[F_j]$ are identical. Thus $[F_i, F_j]$ is a perfect matching, and $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 10(f).

Table 5: Summary of Operation V.

Vertex whose colour is changed	Subgraph induced by $F_i \cup F_j$	Label of $f_i f_j$
y	Figure 10(b)	r
x	Figure 10(c)	rr
z, w	Figure 10(d)	c
$u \in V(H) \setminus \{x, y, z, w\}$	Figure 10(f)	pm

We informally refer to the rows and columns of vertices in $G[F_i]$ according to the drawing in Figure 10(a). For Operations IV and VI through IX we use a similar convention.

Remark 5.3. We note that for Operation V, if $f_i f_j \in E(G_4(H))$ has label r, then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ consists of row two or three[2]. If $f_i f_j \in E(G_4(H))$ has label rr, then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ consists of rows one and two, or rows three and four. If $f_i f_j \in E(G_4(H))$ has label c and $e \in [F_i, F_j]$, then each colouring corresponding to an end of e assigns the same colour to γ and δ.

5.6 Operation VI

We may assume that $f_i(x) = 1$, $f_i(y) = 2$, $f_i(z) = 3$ and $f_i(w) = 1$. Then the pairs of colour available for α and β, respectively, are

$$\{(3,4),(3,1),(4,1),(4,3)\},$$

and the pairs of colours available for γ and δ, respectively, are

$$\{(2,4),(2,1),(4,1),(4,2)\}.$$

Thus $G[F_i]$ is isomorphic to $P_4 \,\square\, P_4$, and we assume that it is drawn in the plane as shown in Figure 11(a), with the rows labelled by the pairs of colours available for α and β, respectively, and the columns labelled by the pairs of colours available for γ and δ, respectively.

[2]Rows are numbered from top to bottom and columns from left to right.

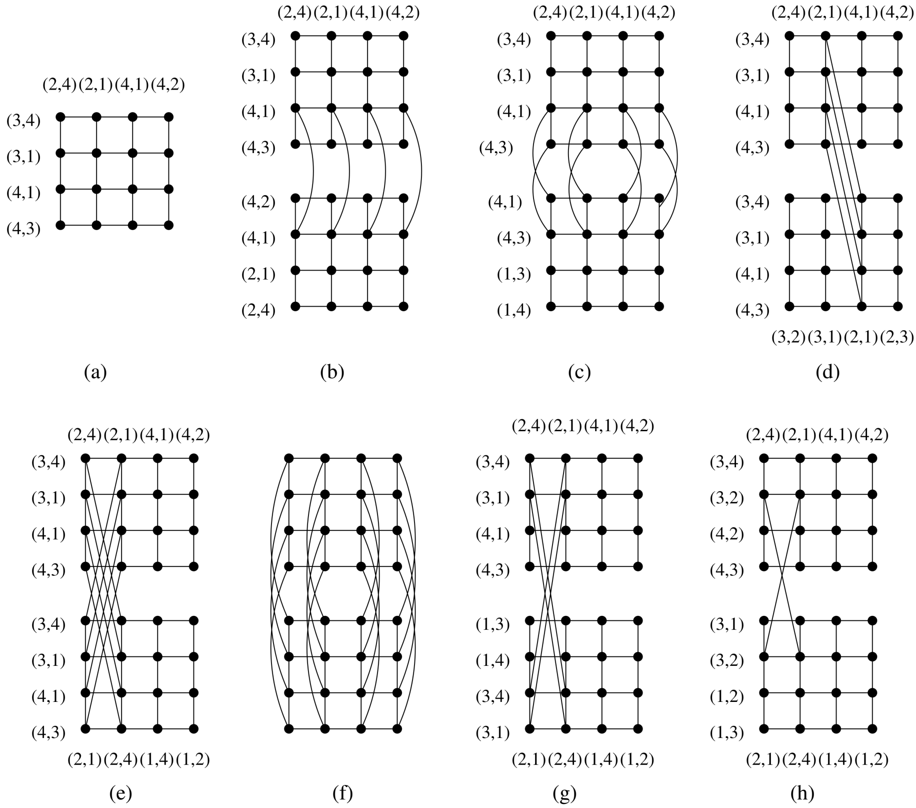


Figure 11: Subgraphs induced by $F_i \cup F_j$ for Operations VI, VII and VIII.

If $f_i f_j \in E(G_4(H))$, then f_j is obtained from f_i by changing the colour of a single vertex in $V(H)$; there are five cases to consider. Since H is a 2-tree, there are vertices $a, b \in V(H)$ such that $H[\{x, y, a\}] \cong K_3$ and $H[\{w, z, b\}] \cong K_3$. Observe $f_i(a) \in \{3, 4\}$ and $f_i(b) \in \{2, 4\}$. We may assume that $f_i(a) = 4$ and $f_i(b) = 2$. Even though b (respectively, a) could be equal to x or y (respectively, w or z), this does not affect the argument.

(i) $f_i(y) \neq f_j(y)$. Then $f_j(y) = 3$, and the pairs of colours available for α and β, respectively, are

$$\{(4, 2), (4, 1), (2, 1), (2, 4)\},$$

and the pairs of colours available for γ and δ are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic the the graph in Figure 11(b).

(ii) $f_i(x) \neq f_j(x)$. Then $f_j(x) = 3$, and the pairs of colours available for α and β, respectively, are

$$\{(4, 1), (4, 3), (1, 3), (1, 4)\},$$

while the pairs of colours available for γ and δ are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(c).

(iii) $f_i(z) \neq f_j(z)$. Then $f_j(z) = 4$ and the pairs of colours available for γ and δ, respectively, are

$$\{(3,2),(3,1),(2,1),(2,3)\},$$

while the pairs of colours available for α and β are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic the the graph in Figure 11(d).

(iv) $f_i(w) \neq f_j(w)$. Then $f_j(w) = 4$ and the pairs of colours available for γ and δ, respectively, are

$$\{(2,1),(2,4),(1,4),(1,2)\},$$

while the pairs of colours available for α and β are unchanged. Hence, $G[F_i \cup F_j]$ is isomorphic the the graph in Figure 11(e).

(v) $f_i(u) \neq f_j(u)$ for some $u \in V(H) \setminus \{x,y,z,w\}$. In this case, the vertex labels on $G[F_i]$ and $G[F_j]$ are identical. Thus $[F_i, F_j]$ is a perfect matching, and $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(f).

Table 6: Summary of Operation VI.

Vertex whose colour is changed	Subgraph induced by $F_i \cup F_j$	Label of f_if_j
y	Figure 11(b)	r
x	Figure 11(c)	rr
z	Figure 11(d)	c
w	Figure 11(e)	cc
$u \in V(H) \setminus \{x,y,z,w\}$	Figure 11(f)	pm

Remark 5.4. We note that for Operation VI, if $f_if_j \in E(G_4(H))$ has label r (respectively, c), then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ consists of row (respectively, column) two or three. If $f_if_j \in E(G_4(H))$ has label rr (respectively, cc), then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ consists of rows (respectively, columns) one and two or rows (respectively, columns) three and four.

5.7 Operation VII

We may assume that $f_i(x) = 1, f_i(y) = 2$ and $f_i(z) = 3$. Then the pairs of colours available for α and β, respectively, are

$$\{(3,4),(3,1),(4,1),(4,3)\},$$

and the pairs of colours available for γ and δ, respectively, are

$$\{(2,4),(2,1),(4,1),(4,2)\}.$$

Thus $G[F_i]$ is isomorphic to $P_4 \,\square\, P_4$, and we assume that it is drawn in the plane as shown in Figure 11(a) with rows labelled by the pairs of colours available for α and β, respectively, and the columns labelled by the pairs of colours available for γ and δ, respectively.

If $f_if_j \in E(G_4(H))$, then f_j is obtained from f_i by changing the colour of a single vertex in $V(H)$. There are four cases to consider.

(i) $f_i(x) \neq f_j(x)$. We may assume that $f_j(x) = 4$. Then the pairs of colours available for α and β, respectively, are

$$\{(3,1),(3,4),(1,4),(1,3)\},$$

and the pairs of colours available for γ and δ, respectively, are

$$\{(2,1),(2,4),(1,4),(1,2)\}.$$

Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(g).

(ii) $f_i(y) \neq f_j(y)$. This is analogous to Operation VI when the colour of y is changed, and thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(b).

(iii) $f_i(z) \neq f_j(z)$. This is analogous to Operation VI when the colour of z is changed, and thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(d).

(iv) $f_i(u) \neq f_j(u)$ for some $u \in V(H) \setminus \{x,y,z\}$. In this case, the vertex labels on $G[F_i]$ and $G[F_j]$ are identical. Thus $[F_i, F_j]$ is a perfect matching, and $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(f).

Table 7: Summary of Operation VII.

Vertex whose colour is changed	Subgraph induced by $F_i \cup F_j$	Label of $f_i f_j$
x	Figure 11(g)	sq
y	Figure 11(b)	r
z	Figure 11(d)	c
$u \in V(H) \setminus \{x,y,z\}$	Figure 11(f)	pm

Remark 5.5. We note that for Operation VII, if $f_i f_j \in E(G_4(H))$ has label r (respectively, c), then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ consists of row (respectively, column) two or three. If $f_i f_j \in E(G_4(H))$ has label sq, then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ induces a four-cycle using a degree two vertex of $G[F_i]$.

5.8 Operation VIII

We may assume that $f_i(x) = 1, f_i(y) = 2$ and $f_i(z) = 3$. Then the pairs of colours available for α and β, respectively, are

$$\{(4,3),(4,2),(3,2),(3,4)\},$$

and the pairs of colours available for γ and δ, respectively, are

$$\{(2,4),(2,1),(4,1),(4,2)\}.$$

Thus $G[F_i]$ is isomorphic to $P_4 \,\square\, P_4$, and we assume that it is drawn in the plane as shown in Figure 11(a) with rows labelled by the pairs of colours available for α and β, respectively, and the columns labelled by the pairs of colours available for γ and δ, respectively (but not the same labels as in the figure).

If $f_i f_j \in E(G_4(H))$, then f_j is obtained from f_i by changing the colour of a single vertex in $V(H)$, and there are four cases.

(i) $f_i(x) \neq f_j(x)$. We may assume that $f_j(x) = 4$. Then the pairs of colours available for α and β, respectively, are

$$\{(1,3),(1,2),(3,2),(3,1)\},$$

and the pairs of colours available for γ and δ, respectively, are

$$\{(2,1),(2,4),(1,4),(1,2)\}.$$

Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(h) with appropriate labels.

(ii) $f_i(y) \neq f_j(y)$. This is analogous to Operation VI when the colour of x is changed, and thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(c) with appropriate labels.

(iii) $f_i(z) \neq f_j(z)$. This is analogous to Operation VI when the colour of z is changed, and thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(d) with appropriate labels.

(iv) $f_i(u) \neq f_j(u)$ for some $u \in V(H) \setminus \{x,y,z\}$. In this case, the vertex labels on $G[F_i]$ and $G[F_j]$ are identical. Thus $[F_i, F_j]$ is a perfect matching, and $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 11(f).

Table 8: Summary of Operation VIII.

Vertex whose colour is changed	Subgraph induced by $F_i \cup F_j$	Label of $f_i f_j$
x	Figure 11(h)	e
y	Figure 11(c)	rr
z	Figure 11(d)	c
$u \in V(H) \setminus \{x,y,z\}$	Figure 11(f)	pm

Remark 5.6. We note that for Operation VIII, if $f_i f_j \in E(G_4(H))$ has label c, then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ consists of column two or three. If $f_i f_j \in E(G_4(H))$ has label rr, then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ consists of rows one and two or rows three and four. If $f_i f_j \in E(G_4(H))$ has label e, then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ induces an edge that is the first or last edge of row two or row three.

5.9 Operation IX

We may assume that $f_i(x) = 1$, $f_i(y) = 2$ and $f_i(z) = 3$. Then the pairs of colours available for α and β, respectively, are

$$\{(4,3),(4,1),(3,1),(3,4)\},$$

and the pairs of colours available for γ and δ, respectively, are

$$\{(2,2),(2,4),(4,4),(4,2)\}.$$

Thus $G[F_i]$ is isomorphic to $P_4 \,\square\, C_4$, and we assume that it is drawn in the plane as shown in Figure 10(a) with rows labelled by the pairs of colours available for α and β, respectively, and the columns labelled by the pairs of colours available for γ and δ, respectively (but not the same labels as in the figure).

If $f_i f_j \in E(G_4(H))$, then f_j is obtained from f_i by changing the colour of a single vertex in $V(H)$; there are four cases.

(i) $f_i(x) \neq f_j(x)$. We may assume that $f_j(x) = 4$. Then the pairs of colours available for α and β, respectively, are

$$\{(3,1),(3,4),(1,4),(1,3)\},$$

and the pairs of colours available for γ and δ, respectively, are

$$\{(2,1),(2,2),(1,2),(1,1)\}.$$

Hence, $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 10(e) with appropriate labels.

(ii) $f_i(y) \neq f_j(y)$. This is analogous to Operation V when the colour of y is changed, and thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 10(b) with appropriate labels.

(iii) $f_i(z) \neq f_j(z)$. This is analogous to Operation V when the colour of z is changed, and thus $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 10(d) with appropriate labels.

(iv) $f_i(u) \neq f_j(u)$ for some $u \in V(H) \setminus \{x,y,z\}$. In this case, the vertex labels on $G[F_i]$ and $G[F_j]$ are identical. Thus $[F_i, F_j]$ is a perfect matching, and $G[F_i \cup F_j]$ is isomorphic to the graph in Figure 10(f).

Table 9: Summary of Operation IX.

Vertex whose colour is changed	Subgraph induced by $F_i \cup F_j$	Label of $f_i f_j$
x	Figure 10(e)	e
y	Figure 10(b)	r
z	Figure 10(d)	c
$u \in V(H) \setminus \{x,y,z\}$	Figure 10(f)	pm

Remark 5.7. We note that for Operation IX, if $f_i f_j \in E(G_4(H))$ has label r and $e \in [F_i, F_j]$, then each colouring corresponding to an end of e assigns the same colour to α and β. Similarly, if $f_i f_j \in E(G_4(H))$ has label c and $e \in [F_i, F_j]$, then each colouring corresponding to an end of e assigns the same colour to γ and δ. If $f_i f_j \in E(G_4(H))$ has label e and $e \in [F_i, F_j]$, then the set of vertices $S_{i,j} \subseteq F_i$ incident to the edges of $[F_i, F_j]$ induces an edge that is either the first or last edge in a column, and each colouring corresponding to an end of e assigns the same colour to γ and δ.

6 4-colouring graphs of 2-trees of diameter at least three

Let H be a 2-tree, and let H' be a 2-tree obtained from H by applying one of the Operations I through IX. We prove $G_4(H')$ has a Hamilton cycle.

6.1 Operations I, II and III

We first prove a result about Hamilton cycles in a cube Q_3 that will later be used to show the existence of edges satisfying Lemma 2.7. In this section, we let each face label in Figure 12(a) denote the 4-cycle bounding that face.

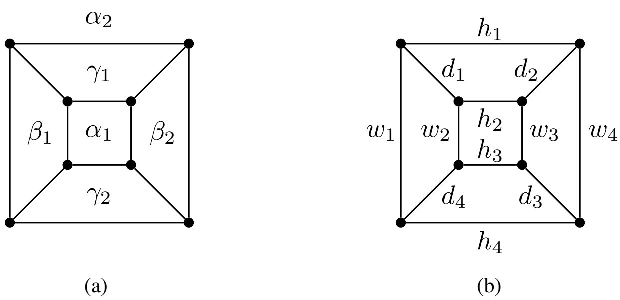


Figure 12: Labelling $G[F_i]$.

We label the six Hamilton cycles of a plane drawing of Q_3 as shown in Figure 13.

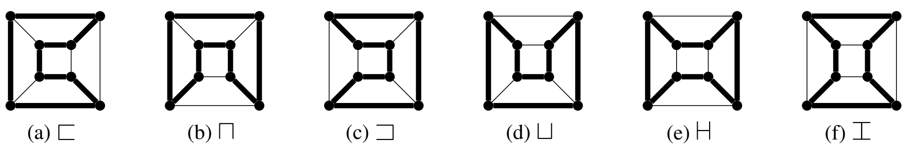


Figure 13: Labels for the Hamilton cycles of Q_3.

To simplify notation for multisets, we write nZ to mean n copies of Z.

Lemma 6.1. *Let $Q \cong Q_3$ be drawn as in Figure 12(a), and let e be an edge of Q. Let $\mathcal{Z} = \{Z_1, Z_2, \ldots, Z_n\}$ be a multiset such that $\mathcal{Z} \subseteq \{2\alpha_1, 2\beta_1, 2\gamma_1, 5Q_3\}$ and $n \leq 5$. That is, each Z_i is an induced subgraph of Q and is either the entire 3-cube or one of the 4-cycles of Q labelled by α_1, β_1, or γ_1. Then there exists a Hamilton cycle in Q containing distinct edges $\{e, e_1, e_2, \ldots, e_n\}$ such that $e_i \in E(Z_i)$, for $1 \leq i \leq n$.*

Proof. It is enough to prove the result when $n = 5$ and $\mathcal{Z} = \{2\alpha_1, 2\beta_1, \gamma_1\}$. Note that the Hamilton cycles $\sqcup$ and $\sqcap$ in Q each contain three edges of α_1, three edges of β_1, and two edges of γ_1. When any single edge is deleted from $\sqcup$ or $\sqcap$, we see that the resulting Hamilton path contains five distinct edges, two from α_1, two from β_1 and one from γ_1. Since at least one of $\sqcup$ and $\sqcap$ contains the edge e, either $\sqcup$ or $\sqcap$ contains distinct edges $\{e, e_1, e_2, \ldots, e_5\}$ such that $e_i \in E(Z_i)$, for $1 \leq i \leq 5$. □

Lemma 6.2. *Let $Q \cong Q_3$ be drawn as in Figure 12(a) with edges labelled as in Figure 12(b). Assume $\{e, e'\} \subseteq E(Q)$ with $e \in \{w_1, w_2, w_3, w_4\}$. Let $\mathcal{Z} = \{Z_1, Z_2, \ldots, Z_n\}$ be a multiset such that $\mathcal{Z} \subseteq \{\alpha_1, \beta_1, 2\gamma_1, 4Q_3\}$ and $n \leq 4$. That is, each Z_i is an induced subgraph of Q and is either the entire 3-cube or one of the 4-cycles of Q labelled by α_1, β_1, or γ_1. Then there exists a Hamilton cycle in Q containing distinct edges $\{e, e', e_1, e_2, \ldots, e_n\}$ such that $e_i \in E(Z_i)$, for $1 \leq i \leq n$.*

Proof. It is enough to prove the result when $n = 4$ and $\mathcal{Z} = \{\alpha_1, \beta_1, 2\gamma_1\}$ with $Z_1 := \alpha_1$, $Z_2 := \beta_1$, $Z_3 := \gamma_1$ and $Z_4 := \gamma_1$. Let $H := \{h_1, h_2, h_3, h_4\}$, $W := \{w_1, w_2, w_3, w_4\}$ and $D := \{d_1, d_2, d_3, d_4\}$. For each Z_i, we designate a set of candidate edges for e_i as follows (see Table 10).

Table 10: Cases in the proof of Lemma 6.2.

e', e	Cycle in Q	Edges e_i assigned to Z_i: Z_1	Z_2	Z_3	Z_4
$e' \in H, e \in \{w_1, w_2\}$	$\sqsubset$	$\{h_2, h_3\} \setminus \{e'\}$	$\{w_1, w_2\} \setminus \{e\}$	d_2	$\{h_1, h_2\} \setminus \{e'\}$
$e' \in H, e \in \{w_3, w_4\}$	$\sqsupset$	$\{h_2, h_3\} \setminus \{e'\}$	d_4	d_1	$\{h_1, h_2\} \setminus \{e'\}$
$\{e, e'\} = \{w_1, w_2\}$	$\sqcap$	w_3	d_4	h_1	h_2
$e, e' \in W, \{e, e'\} \neq \{w_1, w_2\}$	$\sqcup$	h_3	$\{w_1, w_2\} \setminus \{e, e'\}$	d_1	d_2
$e' \in D, e \in \{w_1, w_4\}$	H	h_3	$\{d_1, d_4\} \setminus \{e'\}$	h_1	$\{d_1, d_2\} \setminus \{e'\}$
$e' \in D, e \in \{w_2, w_3\}$	I	$\{w_2, w_3\} \setminus \{e\}$	$\{d_1, d_4\} \setminus \{e'\}$	h_1	$\{d_1, d_2\} \setminus \{e'\}$

As indicated in Table 10, the first case has $e' \in H$ and $e \in \{w_1, w_2\}$. We claim that $\sqsubset$ has the required property. We take $e_3 := d_2$, and as $|\{w_1, w_2\} \setminus \{e\}| = 1$, we take $\{e_2\} := \{w_1, w_2\} \setminus \{e\}$. Observe that the sets $\{h_2, h_3\} \setminus \{e'\}$ and $\{h_1, h_2\} \setminus \{e'\}$ are distinct and non-empty. Thus, we may take $e_1 \in \{h_2, h_3\} \setminus \{e'\}$ and $e_4 \in \{h_1, h_2\} \setminus \{e'\}$ so that $e_1 \neq e_4$. The remaining five cases follow using analogous arguments. □

Lemma 6.3. *Suppose H' is obtained from a 2-tree H by applying one of Operations I, II or III. Then $G_4(H')$ has a Hamilton cycle.*

Proof. **Case 1.** Suppose H' is obtained from H by applying Operation I. By Lemma 2.5, $G_4(H)$ has a spanning tree T with $\Delta(T) \leq 4$. Let $V(T) := \{f_0, f_1, \ldots, f_{N-1}\}$ such that f_0 is a leaf, and root T at f_0, turning T into a branching, $\overrightarrow{T}$, by directing all arcs away from f_0.

Let $G := G_4(H')$, and let F_i be the set of 4-colourings of H' that agree with f_i on $V(G_4(H))$, $0 \leq i \leq N-1$. Label each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with the label of $f_i f_j \in E(G_4(H))$, as described in Section 5, and let $S_{i,j} \subseteq F_i$ denote the vertices incident to the edges of $[F_i, F_j]$.

For each arc $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$, we choose edges $e_{i,j}$ in $G[F_i]$ and $e_{j,i}$ in $G[F_j]$ satisfying conditions (i) and (ii) of Lemma 2.7 as follows. Suppose $\overrightarrow{f_0 f_1} \in A(\overrightarrow{T})$. The fact that $G[F_0 \cup F_1]$ is isomorphic to the graph in Figure 8(b) or 8(c) gives us the flexibility to choose $e_{0,1} \in E(G[F_0])$ and $e_{1,0} \in E(G[F_1])$ satisfying (ii) of Lemma 2.7.

Edge choosing procedure. Now suppose for some i, $e_{i,k}$ has been chosen in $G[F_i]$ but $e_{i,j}$ has not yet been chosen for each j where $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$. For this i, let $J := \{j \mid$

$\overrightarrow{f_if_j} \in A(\overrightarrow{T})\}$. We choose edges $e_{i,j}$ and $e_{j,i}$ for $j \in J$ as follows. Assume that $G[F_i]$ is drawn as in Figure 12(a). By Remark 5.1, $S_{i,j_1} = S_{i,j_2}$ whenever $\overrightarrow{f_if_{j_1}}$ and $\overrightarrow{f_if_{j_2}}$ have the same label. Without loss of generality, we may assume that an arc $\overrightarrow{f_if_j}$ with label a-sq has $G[S_{i,j}]$ isomorphic to the 4-cycle α_1, an arc $\overrightarrow{f_if_j}$ with label b-sq has $G[S_{i,j}]$ isomorphic to the 4-cycle β_1, and an arc $\overrightarrow{f_if_j}$ with label c-sq has $G[S_{i,j}]$ isomorphic to the 4-cycle γ_1. For each $j \in J$, let $Z_j := G[S_{i,j}]$, and define the multiset $\mathcal{Z} := \{Z_j \mid j \in J\}$. Then each Z_j is either a 3-cube or one of the 4-cycles α_1, β_1, or γ_1. Since f_i is incident to at most two edges with label a-sq, at most two edges with label b-sq, and at most two edges with label c-sq, $\mathcal{Z} \subseteq \{2\alpha_1, 2\beta_1, 2\gamma_1, 5Q_3\}$. Observe $|\mathcal{Z}| \leq 4$ since $\Delta(T) \leq 4$. By Lemma 6.1, using $e := e_{i,k}$, there is a Hamilton cycle C_i in $G[F_i]$ and an edge $e_{i,j} \in E(Z_j)$ for each $Z_j \in \mathcal{Z}$ such that $e_{i,j_1} \neq e_{i,j_2}$ whenever $j_1 \neq j_2$. Thus (i) of Lemma 2.7 is satisfied. Furthermore, for each $j \in J$ there is an edge $e_{j,i} \in E(G[F_j])$ such that $e_{i,j}$ and $e_{j,i}$ satisfy (ii) of Lemma 2.7.

Now suppose for every $\overrightarrow{f_if_j} \in A(\overrightarrow{T})$, $e_{i,j}$ and $e_{j,i}$ have been chosen as above. By construction, (i) and (ii) of Lemma 2.7 are satisfied, and for each $G[F_i]$, $0 \leq i \leq N-1$, the Hamilton cycle C_i satisfies condition (iii) of Lemma 2.7. Therefore, G has a Hamilton cycle.

Case 2. Suppose H' is obtained from H by applying Operation II. Let $\mathcal{H} := G_4(H)$ and $V(H) := \{f_0, f_1, \ldots, f_{N-1}\}$. For each $1 \leq i \leq 4$ let $V_i := \{c \in V(\mathcal{H}) \mid c(x) = i\}$. Then $\{V_1, V_2, V_3, V_4\}$ is a partition of $V(\mathcal{H})$. Let L_i be an assignment of lists with $L_i(x) := \{i\}$ and $L_i(w) := \{1, 2, 3, 4\}$ for $w \in V(H) \setminus \{x\}$. Note that $G_{L_i}(H) \cong \mathcal{H}[V_i]$ and that $\mathcal{H}[V_1] \cong \mathcal{H}[V_2] \cong \mathcal{H}[V_3] \cong \mathcal{H}[V_4]$. Thus, $\mathcal{H}$ can be partitioned into four copies isomorphic to $G_{L_1}(H)$ with edges between pairs of copies. Furthermore, each edge in $E(\mathcal{H}[V_i])$, $1 \leq i \leq 4$, has label a-sq, b-sq, c-sq or pm, and each edge with one endpoint in V_i and the other endpoint in V_j, $i \neq j$, has label e.

By Lemma 2.5, $\mathcal{H}[V_i]$, $1 \leq i \leq 4$, has a spanning tree T_i with $\Delta(T_i) \leq 4$. Note that $[V_i, V_j] \neq \emptyset$ for $1 \leq i \neq j \leq 4$. Choose one edge from each of $[V_1, V_2]$, $[V_2, V_3]$, and $[V_3, V_4]$. Without loss of generality, suppose the chosen edges are $f_1f_2 \in [V_1, V_2]$, $f_2'f_3 \in [V_2, V_3]$, and $f_3'f_4 \in [V_3, V_4]$ such that $f_i \in V_i$, $1 \leq i \leq 4$. Since f_1f_2, $f_2'f_3$ and $f_3'f_4$ each have label e in $\mathcal{H}$ and each vertex of $V(\mathcal{H})$ is incident to at most one edge with label e, the vertices $f_1, f_2, f_2', f_3, f_3', f_4$ are distinct. Thus, we may assume that $f_2' = f_0$ and $f_3' = f_5$.

Let T be the spanning tree of $\mathcal{H}$ consisting of the union of T_1, T_2, T_3, T_4, and the edges $\{f_1f_2, f_0f_3, f_5f_4\}$. Then $\Delta(T) \leq 5$ and the only edges of T with label e are f_1f_2, f_0f_3 and f_4f_5. Root T at f_1, turning T into a branching, $\overrightarrow{T}$, by directing all arcs away from f_1. This gives a branching $\overrightarrow{T_i}$ for each T_i, $1 \leq i \leq 4$, and by our choice of labels, f_i is the root of $\overrightarrow{T_i}$.

Let $G := G_4(H')$, and let F_i be the set of 4-colourings of H' that agree with f_i on $V(G_4(H))$, $0 \leq i \leq N-1$. Label each $\overrightarrow{f_if_j} \in A(\overrightarrow{T})$ with the label of $f_if_j \in E(\mathcal{H})$, and let $S_{i,j} \subseteq F_i$ and $S_j' \subseteq F_j$ denote the vertices incident to the edges of $[F_i, F_j]$.

For each arc $\overrightarrow{f_if_j} \in A(\overrightarrow{T})$, we choose edges $e_{i,j}$ in $G[F_i]$ and $e_{j,i}$ in $G[F_j]$ satisfying conditions (i) and (ii) of Lemma 2.7 as follows. For $(i, j) \in \{(1, 2), (0, 3), (5, 4)\}$ (where $\overrightarrow{f_if_j}$ has label e), let $e_{i,j}$ be the unique edge of $G[S_{i,j}] \subseteq G[F_i]$ and $e_{j,i}$ be the unique edge of $G[S_j'] \subseteq G[F_j]$. For $\overrightarrow{T_1}$ and $\overrightarrow{T_4}$ we apply the edge choosing procedure used in Case 1, starting with with $e_{1,2}$ in $G[F_1]$ for $\overrightarrow{T_1}$, and $e_{4,5}$ in $G[F_4]$ for $\overrightarrow{T_4}$. The resulting set of edges

$$\{e_{i,j}, e_{j,i} \mid \overrightarrow{f_i f_j} \in (A(\overrightarrow{T_1} \cup \overrightarrow{T_4}) \cup \{\overrightarrow{f_1 f_2}, \overrightarrow{f_5 f_4}\})\}$$

and set of cycles $\{C_i \mid f_i \in V_1 \cup V_4\}$ satisfy Lemma 2.7. For $\overrightarrow{T_2}$, we let f_M be the parent of f_0 and f_k the parent of f_M. Let $\overrightarrow{T_2}'$ be the subtree of $\overrightarrow{T_2}$ obtained by deleting the descendants of f_M and $\overrightarrow{T_2}''$ be the subtree of $\overrightarrow{T_2}$ rooted at f_M. If $M = 2$ then $k = 1$, so $e_{M,k} = e_{2,1}$. Otherwise apply the apply the edge choosing procedure used in Case 1 to $\overrightarrow{T_2}'$, starting with $e_{2,1}$ in $G[F_2]$. This leads to the designation of an edge $e_{M,k}$ in $G[F_M]$.

Let $J := \{j \mid \overrightarrow{f_M f_j} \in A(\overrightarrow{T_2})\}$. We choose edges $e_{M,j}$ and $e_{j,M}$ for $j \in J$ as follows. Without loss of generality, we may assume that an arc $\overrightarrow{f_M f_j}$ with label a-sq has $G[S_{M,j}]$ isomorphic to the 4-cycle α_1, an arc $\overrightarrow{f_M f_j}$ with label b-sq has $G[S_{M,j}]$ isomorphic to the 4-cycle β_1, an arc $\overrightarrow{f_M f_j}$ with label c-sq has $G[S_{M,j}]$ isomorphic to the 4-cycle γ_1, and an arc with label pm has $G[S_{M,j}]$ isomorphic to Q_3. Let $\mathcal{Z}$ be a multiset consisting of the graphs $Z_j := G[S_{M,j}]$ for $j \in J$.

Suppose there exists $\ell \in (J \setminus \{0\})$ such that both $\overrightarrow{f_M f_0}$ and $f_M f_\ell$ have label c-sq. We apply the edge choosing procedure used in Case 1 to $\overrightarrow{T_2}''$, with chosen edge $e_{M,k}$. The resulting set of edges $\{e_{M,j} \mid j \in J\}$ and cycle C_M satisfy Lemma 2.7. Observe that $e_{M,0}, e_{M,\ell} \in E(\gamma_1)$. Suppose $e_{M,0} = ab$, and let the vertices of $[\{a, b\}, F_0]$ incident to F_0 be $\{c, d\}$. If $e_{0,3} = cd$ then exchange $e_{M,0}$ with $e_{M,\ell}$. It now follows that for each $j \in J$ there is an edge $e_{j,M} \in E(G[F_j])$ such that $e_{M,j}$ and $e_{j,M}$ satisfy (ii) of Lemma 2.7, with $e_{0,M} \neq e_{0,3}$.

Otherwise $\overrightarrow{f_M f_0}$ is the only arc in $\{\overrightarrow{f_M f_j} \mid j \in J\}$ labelled c-sq, or $\overrightarrow{f_M f_0}$ has label a-sq, b-sq, or pm. To ensure (i) of Lemma 2.7 is satisfied for $i = 0$, we duplicate $Z_0 \in \mathcal{Z}$ and apply Lemma 6.1. Let $Z_{0'} := Z_0$ and $F_{0'} := F_0$. Observe $|\mathcal{Z} \cup \{Z_{0'}\}| \leq 5$ since $\Delta(T) \leq 4$. Since f_M is incident to at most one edge with label a-sq, at most one edge with label b-sq, and at most two edges with label c-sq, $(\mathcal{Z} \cup \{Z_{0'}\}) \subseteq \{2\alpha_1, 2\beta_1, 2\gamma_1, 5Q_3\}$. By Lemma 6.1 applied to $\mathcal{Z} \cup \{Z_{0'}\}$, and using $e := e_{M,k}$, there is a Hamilton cycle C_M in $G[F_M]$ and edges $e_{M,j} \in E(Z_j)$ for each $j \in (J \cup \{0'\})$ such that $e_{M,j_1} \neq e_{M,j_2}$ whenever $j_1 \neq j_2$. Thus (i) of Lemma 2.7 is satisfied for $i = M$. Furthermore, for each $j \in (J \cup \{0'\})$ there is an edge $e_{j,M} \in E(G[F_j])$ such that $e_{j,M}$ and $e_{M,j}$ satisfy (ii) of Lemma 2.7. Since $e_{M,0'} \neq e_{M,0}$, we have $e_{0,M} \neq e_{0',M}$, and hence, one of $e_{0,M}$ and $e_{0',M}$ is different from $e_{0,3}$. If $e_{0,M} \neq e_{0,3}$ then we ignore $e_{0',M}$ and $e_{M,0'}$; otherwise, redefine $e_{0,M}$ to be $e_{0',M}$ and $e_{M,0}$ to be $e_{M,0'}$ so that (i) of Lemma 2.7 is satisfied for $i = 0$.

Finally, let $G[F_0] \cong Q_3$ be drawn as in Figure 12(a) with edges labelled as in Figure 12(b). Observe that $e = e_{0,3} \in \{w_1, w_2, w_3, w_4\}$ and let $e' := e_{0,M}$. Let $J := \{j \mid \overrightarrow{f_0 f_j} \in A(\overrightarrow{T_2})\}$. We choose edges $e_{0,j}$ and $e_{j,0}$ for $j \in J$ as follows. Without loss of generality, we may assume that an arc $\overrightarrow{f_0 f_j}$ with label a-sq has $G[S_{0,j}]$ isomorphic to the 4-cycle α_1, an arc $\overrightarrow{f_0 f_j}$ with label b-sq has $G[S_{0,j}]$ isomorphic to the 4-cycle β_1, an arc $\overrightarrow{f_0 f_j}$ with label c-sq has $G[S_{0,j}]$ isomorphic to the 4-cycle γ_1, and an arc with label pm has $G[S_{0,j}]$ isomorphic to Q_3. Let $\mathcal{Z}$ be a multiset consisting of the graphs $Z_j := G[S_{0,j}]$ for $j \in J$. Observe $\mathcal{Z} \subseteq \{\alpha_1, \beta_1, 2\gamma_1, 4Q_3\}$ and $|\mathcal{Z}| \leq 4$. By Lemma 6.2, there exists a Hamilton cycle C_0 in $G[F_0]$ containing distinct edges $\{e, e'\} \cup \{e_{0,j} \mid j \in J\}$ such that

$e_{0,j} \in E(Z_j)$ for $j \in J$. Thus (i) of Lemma 2.7 is satisfied for $i = 0$. Furthermore, for each $j \in J$ there is an edge $e_{j,0} \in E(G[F_j])$ so that $e_{j,0}$ and $e_{0,j}$ satisfy (ii) of Lemma 2.7.

We now apply the edge choosing procedure used in Case 1 to the remaining nodes of $\overrightarrow{T_2}$. This designates edges $e_{i,j}$ in $G[F_i]$ and $e_{j,i}$ in $G[F_j]$, and cycles C_i satisfying Lemma 2.7. For $\overrightarrow{T_3}$ we apply the argument for $\overrightarrow{T_2}$ giving us edges $e_{i,j}$ in $G[F_i]$ and $e_{j,i}$ in $G[F_j]$, and cycles C_i satisfying Lemma 2.7, for each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T_3})$.

Now $e_{i,j}$ and $e_{j,i}$ have been chosen for all $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$. By construction, each such $e_{i,j}$ and $e_{j,i}$ satisfy (ii) of Lemma 2.7, and for each $G[F_i]$, $0 \le i \le N-1$, the Hamilton cycle C_i satisfies condition (iii) of Lemma 2.7. Furthermore, the collection of chosen edges are all distinct. Therefore, G has a Hamilton cycle.

Case 3. Suppose H' is obtained from H by applying Operation III. Let $\mathcal{H} := G_4(H)$ and $V(H) := \{f_0, f_1, \dots, f_{N-1}\}$. For each $1 \le i \ne j \le 4$ let

$$V_{ij} := \{c \in V(\mathcal{H}) \mid c(x) = i \text{ and } c(y) = j\}.$$

Then

$$V := \{V_{12}, V_{13}, V_{14}, V_{21}, V_{23}, V_{24}, V_{31}, V_{32}, V_{34}, V_{41}, V_{42}, V_{43}\}$$

is a partition of $V(\mathcal{H})$. Note that $[V_{\alpha\beta}, V_{\gamma\delta}] \ne \emptyset$ if and only if $\alpha = \gamma$ or $\beta = \delta$. Furthermore, each edge in $E(\mathcal{H}[V_{ij}])$, $1 \le i \ne j \le 4$, has label a-sq, c-sq or pm, and each edge with one endpoint in $V_{i_1 j_1}$ and the other endpoint in $V_{i_2 j_2}$, $(i_1, j_1) \ne (i_2, j_2)$, has label e.

Let $\{i, j, k\} \subset \{1, 2, 3, 4\}$. As H is a 2-tree, H is 3-colourable and for each $1 \le i \ne j \le 4$, there is a unique vertex $c_{ijk} \in V(\mathcal{H})$ with $c_{ijk}(x) = i$, $c_{ijk}(y) = j$ and $c_{ijk}(w) \in \{i, j, k\}$ for $w \in V(H) \setminus \{x, y\}$.

Consider the ordering

$$(V_{14}, V_{12}, V_{32}, V_{34}, V_{31}, V_{21}, V_{24}, V_{23}, V_{13}, V_{43}, V_{42}, V_{41})$$

of V. For each $V_{ij} \in V \setminus \{V_{41}\}$, suppose $V_{\ell m}$ immediately follows V_{ij} in the list. Then $|\{i, j\} \cup \{\ell, m\}| = 3$, and hence there is a unique $k_{ij} \in \{1, 2, 3, 4\} \setminus (\{i, j\} \cup \{\ell, m\})$ such that $c_{ijk_{ij}} \in V_{ij}$. The ordering of V ensures that for each $V_{ij} \in V \setminus \{V_{42}, V_{41}\}$ with $V_{\ell m}$ immediately following V_{ij} in the list, $k_{ij} \ne k_{\ell m}$. Choose the edge $c_{14}c'_{14}$ from $[V_{14}, V_{12}]$ with endpoint $c_{14} := c_{143} \in V_{14}$; note that $c'_{14} \ne c_{124}$. For each $[V_{ij}, V_{\ell m}]$ where $V_{\ell m}$ immediately follows V_{ij} in the list, there is a unique edge $c_{ij}c'_{ij}$ with endpoint $c_{ij} = c_{ijk_{ij}} \in V_{ij}$.

By Lemma 2.5, $\mathcal{H}[V_{ij}]$, $1 \le i \ne j \le 4$, has a spanning tree T_{ij} with $\Delta(T_{ij}) \le 4$. Let T be the spanning tree of $\mathcal{H}$ with $T_{ij} \subset T$, $1 \le i \ne j \le 4$, and $c_{ij}c'_{ij} \in E(T)$, $1 \le i \ne j \le 4$ with $(i, j) \ne (4, 1)$. Then $\Delta(T) \le 5$ and the only edges of T with label e are $c_{ij}c'_{ij}$, $1 \le i \ne j \le 4$ with $(i, j) \ne (4, 1)$. Root T at $c_{143} \in V_{14}$, turning T into a branching, $\overrightarrow{T}$, by directing all arcs away from c_{14}. This gives a branching $\overrightarrow{T_{ij}}$ for each T_{ij}, $1 \le i \ne j \le 4$. Now repeat the argument in Case 2 for each $\overrightarrow{T_{ij}}$, $1 \le i \ne j \le 4$. □

6.2 Operations IV to IX

We introduce some labelled Hamilton cycles of $C_4 \,\Box\, C_4$, $P_4 \,\Box\, C_4$ and $P_4 \,\Box\, P_4$ to be used in Lemma 2.7 to show the existence of a Hamilton cycle in $G_4(H')$, where H' is obtained from a 2-tree H by applying one of Operations IV through IX.

Let ⊓⊔ be the edge labelled Hamilton cycle in $C_4 \,\square\, C_4$ shown in Figure 14, with the edges in rows two and four labelled by r, the edges in columns two and four labelled by c, and the remaining edges left unlabelled. Let ⊏ and ⊏′ be the edge labelled Hamilton cycles in $P_4 \,\square\, C_4$ shown in Figure 14, and let ⌶, ⌶′ and ⌶″ be the edge labelled Hamilton cycles in $P_4 \,\square\, P_4$ shown in Figure 14.

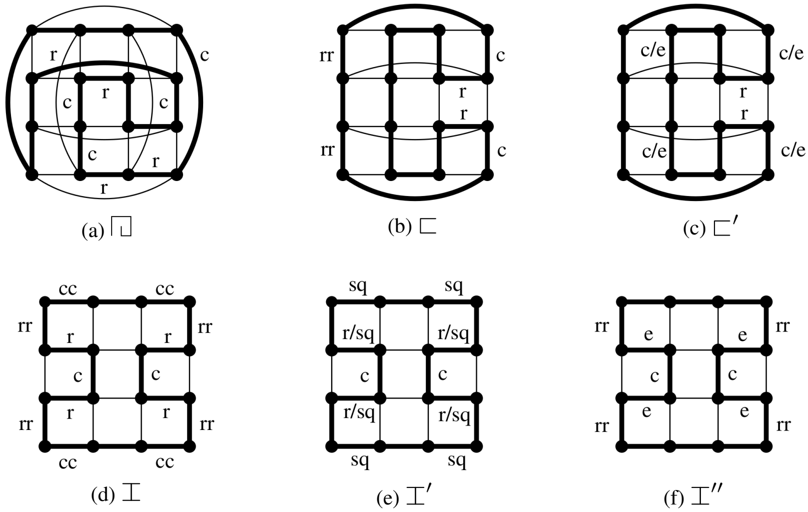


Figure 14: Labelled Hamilton cycles of $C_4 \,\square\, C_4$, $P_4 \,\square\, C_4$ and $P_4 \,\square\, P_4$.

Lemma 6.4. *Suppose H' is obtained from a 2-tree H by applying one of Operations IV through IX. Then $G_4(H')$ has a Hamilton cycle.*

Proof. By Lemma 2.5, $\mathcal{H} := G_4(H)$ has a spanning tree T with $\Delta(T) \leq 4$. Let $V(T) := \{f_0, f_1, \ldots, f_{N-1}\}$ such that f_0 is a leaf, and root T at f_0, turning T into a branching, $\overrightarrow{T}$, by directing all arcs away from f_0. Let $G := G_4(H')$, and let F_i be the set of 4-colourings of H' that agree with f_i on $V(\mathcal{H})$, $0 \leq i \leq N-1$. Label each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with the label of $f_i f_j \in E(\mathcal{H})$, and let $S_{i,j} \subseteq F_i$ and $S'_j \subseteq F_j$ denote the vertices incident to the edges of $[F_i, F_j]$.

We first traverse $\overrightarrow{T}$ using breadth-first search starting at f_0 to construct a drawing of each $G[F_i]$ as shown in Figure 9(a) for Operation IV, Figure 10(a) for Operations V and IX, and Figure 11(a) for Operations VI, VII and VIII, so that each drawing has the following property, denoted $(*)$.

$(*)$ The rows are labelled by the pairs of colours available for α and β, respectively, and the columns are labelled by the pairs of colours available for γ and δ, respectively. In the case of Operations IV, V and IX, we further assume that the second and fourth columns have γ and δ the same colour. Also, in the case of Operation IV, we assume the second and fourth rows have α and β the same colour.

Start with a drawing of $G[F_0]$ satisfying $(*)$. Assume $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ where $G[F_i]$ has been drawn but $G[F_j]$ has not. We draw $G[F_j]$ satisfying $(*)$ as follows. If $\overrightarrow{f_i f_j}$ has label

- **pm**, then draw $G[F_j]$ so that the labels of the rows and columns are in the same order as in $G[F_i]$.

- **r**, then draw $G[F_j]$ so that the column labels of $G[F_j]$ are in the same order as the column labels of $G[F_i]$, and so that the row number of S'_j in $G[F_j]$ is the same as that of $S_{i,j}$ in $G[F_i]$. This can be done for Operation IV due to the cyclic structure of $C_4 \,\square\, C_4$; furthermore, $(*)$ guarantees that $S_{i,j}$ and S'_j have row number two or four. For Operations V, VI, VII and IX, this can be done by Remarks 5.3, 5.4, 5.5 and 5.7. Thus, one of the two possible labellings for the rows of $G[F_j]$ gives a drawing of $G[F_j]$ such that the row number of S'_j in $G[F_j]$ is the same as that of $S_{i,j}$ in $G[F_i]$.

- **c**, then draw $G[F_j]$ so that the row labels of $G[F_j]$ are in the same order as the row labels of $G[F_i]$, and so that the column number of S'_j in $G[F_j]$ is the same as that of $S_{i,j}$ in $G[F_i]$. This can always be done using a similar argument as in the case for $\overrightarrow{f_i f_j}$ having label r. For Operations IV, V and IX, $(*)$ guarantees that $S_{i,j}$ and S'_j have column number two or four. For Operations VI, VII and VIII, by Remarks 5.4, 5.5 and 5.6, $S_{i,j}$ and S'_j have column number two or three.

- **rr**, then draw $G[F_j]$ so that the column labels of $G[F_j]$ are in the same order as the column labels of $G[F_i]$, and so that the set of row numbers of S'_j in $G[F_j]$ is the same as that of $S_{i,j}$ in $G[F_i]$. This can be done for Operations V, VI and VIII, since by Remarks 5.3, 5.4 and 5.6, the set of row numbers of $S_{i,j}$ and S'_j is either $\{1,2\}$ or $\{3,4\}$.

- **cc**, then draw $G[F_j]$ so that the row labels of $G[F_j]$ are in the same order as the row labels of $G[F_i]$, and so that the set of column numbers of S'_j in $G[F_j]$ is the same as that of $S_{i,j}$ in $G[F_i]$. This can be done for Operation VI, since by Remark 5.4, the set of column numbers of $S_{i,j}$ and S'_j is either $\{1,2\}$ or $\{3,4\}$.

- **sq**, then draw $G[F_j]$ satisfying $(*)$.

- **e**, then draw $G[F_j]$ satisfying $(*)$. Note that for Operation IX, $(*)$ guarantees that $S_{i,j}$ and S'_j belong to column number two or four.

For Operation IV (respectively, V, VI, VII, VIII, and IX), for each i, $0 \le i \le N-1$, let C_i be the Hamilton cycle ⊓⊔ (respectively, ⊏, ⌶, ⌶′, ⌶″, and ⊏′) in $G[F_i]$.

We describe how to construct a set of edges

$$\mathcal{E} := \{e_{i,j}, e_{j,i} \mid f_i f_j \in E(T)\}$$

so that for each arc $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$, the edges $e_{i,j}$ in $G[F_i]$ and $e_{j,i}$ in $G[F_j]$ satisfy conditions (i) and (ii) of Lemma 2.7, and so that $e_{i,j} \in E(C_i)$ and $e_{j,i} \in E(C_j)$. Start with $\mathcal{E} := \emptyset$. We consider the arcs of $\overrightarrow{T}$ in the following order according to their labels.

(1) For each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with label e (Operations VIII and IX), $e_{i,j} \in G[F_i]$ and $e_{j,i} \in G[F_j]$ satisfying condition (ii) of Lemma 2.7 are uniquely determined. Note that for Operation VIII, $e_{i,j} \in C_i$ and has label e, and $e_{j,i} \in C_j$ and has label e; this follows by the symmetry of ⌶″ and the drawings $G[F_i]$ and $G[F_j]$. For Operation IX, $e_{i,j} \in C_i$ and has label c/e, and $e_{j,i} \in C_j$ and has label c/e; this follows from the drawings of $G[F_i]$ and $G[F_j]$, and Remark 5.7. Add $e_{i,j}$ and $e_{j,i}$ to $\mathcal{E}$.

(2) For each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with label sq (Operation VII), choose $e_{i,j}$ in C_i with label sq and $e_{j,i}$ in $G[F_j]$ satisfying condition (ii) of Lemma 2.7. Note that $e_{j,i} \in C_j$ and has label sq or label r/sq; this follows from the drawings of $G[F_i]$ and $G[F_j]$, and Remark 5.7. Add $e_{i,j}$ and $e_{j,i}$ to $\mathcal{E}$.

(3) For each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with label rr (Operations V, VI and VIII), choose $e_{i,j}$ in C_i with label rr and $e_{j,i}$ in $G[F_j]$ satisfying condition (ii) of Lemma 2.7. Note that $e_{j,i} \in C_j$ and has label rr; this follows from the drawings of $G[F_i]$ and $G[F_j]$. Add $e_{i,j}$ and $e_{j,i}$ to $\mathcal{E}$.

(4) For each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with label cc (Operation VI), choose $e_{i,j}$ in C_i with label cc and $e_{j,i}$ in $G[F_j]$ satisfying condition (ii) of Lemma 2.7. Note that $e_{j,i} \in C_j$ and has label cc; this follows from the drawings of $G[F_i]$ and $G[F_j]$. Add $e_{i,j}$ and $e_{j,i}$ to $\mathcal{E}$.

(5) For each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with label r (Operations V, VI and IX), choose $e_{i,j}$ in C_i with label r and $e_{j,i}$ in $G[F_j]$ satisfying condition (ii) of Lemma 2.7. Note that $e_{j,i} \in C_j$ and has label r; this follows by the drawings of $G[F_i]$ and $G[F_j]$. Add $e_{i,j}$ and $e_{j,i}$ to $\mathcal{E}$.

For each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with label r (Operation VII), choose $e_{i,j}$ in $E(C_i) \setminus \mathcal{E}$ with label r/sq and $e_{j,i}$ in $G[F_j]$ satisfying condition (ii) of Lemma 2.7. This is possible since ⌶′ has four edges with label r/sq. Note that $e_{j,i} \in C_j$ and has label r/sq; this follows by the drawings of $G[F_i]$ and $G[F_j]$. Add $e_{i,j}$ and $e_{j,i}$ to $\mathcal{E}$.

In the context of Operation IV, let T' be the subgraph of T induced by edges with label r. Each component of T' is a path since each f_i is incident to at most two edges of T with label r. Let P be a component of T', and assume without loss of generality that $P = f_0 f_1 \cdots f_{m-1}$. For each i, $0 \leq i \leq m-2$, starting at $i = 0$, choose $e_{i,i+1}$ in $E(C_i) \setminus \mathcal{E}$ with label r and $e_{i+1,i}$ in $G[F_{i+1}]$ satisfying condition (ii) of Lemma 2.7. This is possible because the Hamilton cycle ⎾⏌ has two edges labelled r in both rows two and four. Note that $e_{i+1,i} \in C_{i+1}$ and has label r; this follows by the drawings of $G[F_i]$ and $G[F_{i+1}]$. Add $e_{i,i+1}$ and $e_{i+1,i}$ to $\mathcal{E}$.

(6) For each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with label c (Operations VI, VII and VIII), choose $e_{i,j}$ in C_i with label c and $e_{j,i}$ in $G[F_j]$ that satisfies condition (ii) of Lemma 2.7. Note that $e_{j,i} \in C_j$ and has label c; this follows by the drawings of $G[F_i]$ and $G[F_j]$. Add $e_{i,j}$ and $e_{j,i}$ to $\mathcal{E}$.

For each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with label c (Operation IX), choose $e_{i,j}$ in $E(C_i) \setminus \mathcal{E}$ with label c/e and $e_{j,i}$ in $G[F_j]$ satisfying condition (ii) of Lemma 2.7. This is possible because the Hamilton cycle $\sqsubset'$ has two edges labelled c/e in columns two and four, and f_i is incident to at most one edge in T with label c and at most one edge in T with label e. Note that $e_{j,i} \in C_j$ and has label c/e; this follows by the drawings of $G[F_i]$ and $G[F_j]$. Add $e_{i,j}$ and $e_{j,i}$ to $\mathcal{E}$.

In the context of Operations IV and V, let T' be the subgraph of T induced by edges with label c. Each component of T' is a path since each f_i is incident to at most two edges of T with label c. Let P be a component of T', and assume without loss of generality that $P = f_0 f_1 \cdots f_{m-1}$. For each i, $0 \leq i \leq m-2$, starting at $i = 0$, choose $e_{i,i+1}$ in $E(C_i) \setminus \mathcal{E}$ with label c and $e_{i+1,i}$ in $G[F_{i+1}]$ satisfying condition

(ii) of Lemma 2.7. This is possible because the Hamilton cycles ⌐⊔ and ⊏ have two edges labelled c in both columns two and four. Note that $e_{i+1,i} \in C_{i+1}$ and has label c; this follows by the drawings of $G[F_i]$ and $G[F_{i+1}]$. Add $e_{i,i+1}$ and $e_{i+1,i}$ to $\mathcal{E}$.

(7) For each $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$ with label pm, choose $e_{i,j}$ in $E(C_i) \setminus \mathcal{E}$ and $e_{j,i}$ in $E(G[F_j]) \setminus \mathcal{E}$ satisfying condition (ii) of Lemma 2.7. Add $e_{i,j}$ and $e_{j,i}$ to $\mathcal{E}$.

Now $e_{i,j}$ and $e_{j,i}$ have been chosen for all $\overrightarrow{f_i f_j} \in A(\overrightarrow{T})$. By construction, (i) and (ii) of Lemma 2.7 are satisfied, and for each $G[F_i]$, $0 \le i \le N-1$, the Hamilton cycle C_i satisfies condition (iii) of Lemma 2.7. Therefore, G has a Hamilton cycle. □

As a consequence of Lemmas 3.3, 3.5, 6.3 and 6.4, we now have our main result.

Theorem 1.1. *If H is a 2-tree then $k_0(H) = 4$, unless $H \cong T \vee \{u\}$ for some tree T and vertex u, where T is a star on at least three vertices or the bipartition of $V(T)$ has two even parts; in these cases, $k_0(H) = 5$.*

As pointed out in Section 1, if H is a k-tree then $k+2 \le k_0(H) \le k+3$. For both 1-trees (i.e., trees) and 2-trees, equality can occur in both the upper and lower bound. By [6, Corollary 5.6] and Theorem 1.1, if H is a tree or 2-tree of diameter at least three, then the lower bound holds. We ask if this extends to k-trees, that is, if H is a k-tree with diameter at least three, is it the case that $k_0(H) = k+2$? On a related note, k-trees are a subclass of chordal graphs. We ask if the techniques presented here can be extended to determine the Gray code numbers of other chordal graphs.

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Appendix A 3-colouring graphs of trees

A 2-tree with a dominating vertex u has the form $T \vee \{u\}$ for some tree T. Lemma 3.5 characterizes when $k_0(T \vee \{u\}) = 5$. Its proof requires a technical result (Lemma 3.6) that we prove in this appendix. To do so, we introduce additional terminology along with structural results on the 3-colouring graphs of trees.

Define L_i to be an assignment of lists to the vertices of $T \vee \{u\}$ with $L_i(u) := \{i\}$ and $L_i(w) := \{1, 2, 3, 4\}$ for $w \in V(T)$. Then $G_{L_i}(T \vee \{u\}) \cong G_3(T)$ and thus, $G_4(T \vee \{u\})$ can be partitioned into four copies of $G_3(T)$ with edges between pairs of copies. We prove that when T has even number of vertices then $G_3(T)$ is bipartite. First, we prove a more general result for $G_L(H)$ where H is a connected graph.

Lemma A.1. *Let H be a connected graph on n vertices and L an assignment of lists to the vertices of H such that $L(v) \subseteq \{1, 2, 3\}$ for each $v \in V(H)$. If H is L-colourable and there is a vertex $w \in V(H)$ with $|L(w)| < 3$, then $G_L(H) \subseteq Q_n$. In particular, $G_L(H)$ is a bipartite graph.*

Proof. The proof is by induction on n. Observe that the result is true for $n = 1$ and $n = 2$.

First suppose that $|L(w)| = 1$ and without loss of generality, $L(w) = \{1\}$. In any list colouring of H, each vertex $v \in N_H(w)$ cannot be coloured '1'. Thus, let $\hat{L}$ be an assignment of lists of allowable colours defined as

$$\hat{L}(v) := \begin{cases} L(v) \setminus \{1\}, & \text{if } v \in N_H(w), \\ L(v), & \text{otherwise.} \end{cases}$$

Denote the components of $H - w$ by $H_1, H_2, \ldots, H_N$. Then by the inductive hypothesis, for each i, $1 \le i \le N$,

$$G_{\hat{L}}(H_i) \subseteq Q_{|V(H_i)|}.$$

Since w must be coloured using the colour '1', we have

$$\begin{aligned} G_L(H) &= G_{\hat{L}}(H - w) \\ &= G_{\hat{L}}\left(\cup_{i=1}^{N} H_i\right) \\ &= \mathop{\square}_{i=1}^{N} G_{\hat{L}}(H_i), && \text{by Remark 2.4} \\ &\subseteq \mathop{\square}_{i=1}^{N} Q_{|V(H_i)|}, && \text{by the inductive hypothesis} \\ &= Q_{n-1}. \end{aligned}$$

Next suppose that $|L(w)| = 2$ and without loss of generality, $L(w) := \{1, 2\}$. For $i = 1, 2$, define L_i as $L_i(w) := \{i\}$ and $L_i(v) := L(v)$ for $v \neq w$ with $v \in V(H)$. Also define $\hat{L}_i$ as $\hat{L}_i(w) := \{i\}$ and $\hat{L}_i(v) := \{1, 2, 3\}$ for $v \neq w$ with $v \in V(H)$. Observe that $G_{L_1}(H) \subseteq G_{\hat{L}_1}(H)$ and $G_{L_2}(H) \subseteq G_{\hat{L}_2}(H)$. Furthermore, $G_{\hat{L}_1}(H) \cong G_{\hat{L}_2}(H)$, and hence, $G_L(H) \subseteq G_{\hat{L}_1}(H) \square K_2$. By the preceding argument, $G_{\hat{L}_1}(H) \subseteq Q_{n-1}$, and therefore

$$G_L(H) \subseteq Q_{n-1} \square K_2 = Q_n. \qquad \square$$

Lemma A.2. *If T is a tree with an even number of vertices then $G_3(T)$ is bipartite.*

Proof. If $T \cong K_2$, then $G_3(T) \cong C_6$. Let T be a tree on an even number of vertices with $|V(T)| \geq 4$ having bipartition (A, B). Fix $u \in A$, $v \in B$ with u adjacent to v in T. Define $\mathcal{H} := G_3(T)$. For each $1 \leq i \neq j \leq 3$ let

$$V_{ij} := \{c \in V(\mathcal{H}) \mid c(u) = i \text{ and } c(v) = j\}.$$

Then $\{V_{12}, V_{13}, V_{23}, V_{21}, V_{31}, V_{32}\}$ is a partition of $V(\mathcal{H})$. Observe each $\mathcal{H}[V_{ij}]$ is connected (this can be shown by induction on $|V(T)|$), and that $[V_{\alpha\beta}, V_{\gamma\delta}] \neq \emptyset$ if and only if $\alpha = \gamma$ or $\beta = \delta$. It follows that $\mathcal{H}_1 = \mathcal{H}[V_{12} \cup V_{13} \cup V_{23} \cup V_{21}]$ is connected, and by Lemma A.1 is bipartite. Similarly, $\mathcal{H}_2 = \mathcal{H}[V_{23} \cup V_{21} \cup V_{31} \cup V_{32}]$ and $\mathcal{H}_3 = \mathcal{H}[V_{31} \cup V_{32} \cup V_{12} \cup V_{13}]$ are connected and bipartite. Denote the two-coloured vertices of $\mathcal{H}$ (that is, the colourings of T with two colours) by $c_{ij} \in V_{ij}$ such that

$$c_{ij}(x) := \begin{cases} i, & \text{if } x \in A, \\ j, & \text{if } x \in B. \end{cases}$$

Suppose the bipartition of each $\mathcal{H}[V_{ij}]$ is (A_{ij}, B_{ij}) where $c_{ij} \in B_{ij}$.

If $|A|$ and $|B|$ are even, then $d_{\mathcal{H}}(c_{ij}, c_{i'j'})$ is even. It follows that

$$(A_{12} \cup A_{13} \cup A_{23} \cup A_{21},\ B_{12} \cup B_{13} \cup B_{23} \cup B_{21})$$

is a bipartition of $\mathcal{H}_1$,

$$(A_{23} \cup A_{21} \cup A_{31} \cup A_{32},\ B_{23} \cup B_{21} \cup B_{31} \cup B_{32})$$

is a bipartition of $\mathcal{H}_2$, and

$$(A_{31} \cup A_{32} \cup A_{12} \cup A_{13},\ B_{31} \cup B_{32} \cup B_{12} \cup B_{13})$$

is a bipartition of $\mathcal{H}_3$. Hence,

$$\mathcal{A} := \bigcup_{1 \leq i \neq j \leq 3} A_{ij} \qquad \text{and} \qquad \mathcal{B} := \bigcup_{1 \leq i \neq j \leq 3} B_{ij}$$

are independent, and thus form a bipartition of $\mathcal{H}$.

If $|A|$ and $|B|$ are both odd, $d_{\mathcal{H}}(c_{ij}, c_{i'j'})$ is even if and only if $i \neq i'$ and $j \neq j'$. It follows that

$$(A_{12} \cup B_{13} \cup A_{23} \cup B_{21},\ B_{12} \cup A_{13} \cup B_{23} \cup A_{21})$$

is a bipartition of $\mathcal{H}_1$,

$$(A_{23} \cup B_{21} \cup A_{31} \cup B_{32},\ B_{23} \cup A_{21} \cup B_{31} \cup A_{32})$$

is a bipartition of $\mathcal{H}_2$, and

$$(A_{31} \cup B_{32} \cup A_{12} \cup B_{13},\ B_{31} \cup A_{32} \cup B_{12} \cup A_{13})$$

is a bipartition of $\mathcal{H}_3$. Hence,

$$\mathcal{A} = \{A_{12} \cup B_{13} \cup A_{23} \cup B_{21} \cup A_{31} \cup B_{32}\}$$

and

$$\mathcal{B} = \{B_{12} \cup A_{13} \cup B_{23} \cup A_{21} \cup B_{31} \cup A_{32}\}$$

are independent, and thus form a bipartition of $\mathcal{H}$. □

Definition A.3. A connected bipartite graph with bipartition (A, B) is *Hamilton laceable* if there is a Hamilton path between any $u \in A$ and $v \in B$.

Remark A.4. The following are Hamilton laceable.

1. $P_{2k_1} \square P_{k_2} \square P_{k_3} \square \cdots \square P_{k_n}$, for $n \geq 2$ and $k_i \geq 1$, $1 \leq i \leq n$ [17].

2. Q_n, for $n \geq 1$ [5].

Definition A.5. A *B-graph with vertex partition* $\{F_0, F_1, \ldots, F_{N-1}\}$ is a bipartite graph G with bipartition $(\mathcal{A}, \mathcal{B})$ together with a partition $\{F_0, F_1, \ldots, F_{N-1}\}$ of $V(G)$ so that, for $i = 0, 1, \ldots, N-1$, $G[F_i]$ is Hamilton laceable.

Lemma A.6. *Let G be a B-graph with vertex partition $\{F_0, F_1, \ldots, F_{N-1}\}$ and bipartition $(\mathcal{A}, \mathcal{B})$. Suppose for each $i = 1, 2, \ldots, N-1$, there is an edge $b_{i-1}a_i$ with $b_{i-1} \in \mathcal{B} \cap F_{i-1}$ and $a_i \in \mathcal{A} \cap F_i$. Then G has a Hamilton path between any vertex in $\mathcal{A} \cap F_0$ and any vertex in $\mathcal{B} \cap F_{N-1}$.*

Proof. Let $a_0 \in \mathcal{A} \cap F_0$ and $b_{N-1} \in \mathcal{B} \cap F_{N-1}$. For each F_i, $i = 0, 1, \ldots, N-1$, choose a Hamilton path P_i in $G[F_i]$ between b_i and a_i. Then

$$\left(\bigcup_{i=0}^{N-2} \{b_i a_{i+1}\}\right) \cup \left(\bigcup_{i=0}^{N-1} E(P_i)\right)$$

are the edges of a Hamilton path in G between a_0 and b_{N-1}. □

Corollary A.7. *Let G be a B-graph with vertex partition $\{F_0, F_1, \ldots, F_{N-1}\}$ and bipartition $(\mathcal{A}, \mathcal{B})$ such that $[F_{i-1}, F_i]$ is a set of independent edges and $|[F_{i-1}, F_i]| \geq 2$, $i = 1, 2, \ldots, N-1$. If for each $i = 1, 2, \ldots, N-1$, the endpoints of any pair of edges in $[F_{i-1}, F_i]$ induces a 4-cycle in G, then G has a Hamilton path between any vertex in $\mathcal{A} \cap F_0$ and any vertex in $\mathcal{B} \cap F_{N-1}$.*

Proof. Let $a_0 \in \mathcal{A} \cap F_0$ and $b_{N-1} \in \mathcal{B} \cap F_{N-1}$. For each $[F_{i-1}, F_i]$, $i = 1, 2, \ldots, N-1$, choose two edges $b_{i-1}a_i$ and $b'_{i-1}a'_i$. Then $G[\{a_i, a'_i, b_{i-1}, b'_{i-1}\}]$ induces a 4-cycle $a_i a'_i b'_{i-1} b_{i-1} a_i$. Note that either $b_{i-1} \in \mathcal{B}$ or $b'_{i-1} \in \mathcal{B}$. Without loss of generality, suppose $b_{i-1} \in \mathcal{B}$. Then $a_i \in \mathcal{A}$. The result follows by Lemma A.6. □

Definition A.8. An *odd flare* is a tree obtained from $K_{1,t}$, $t \geq 3$ and odd, by a single subdivision of one edge.

Lemma 3.6. *Let T be a tree with bipartition (A, B), where $|A| := \ell$ and $|B| := r$, and let $G_3(T)$ be the 3-colouring graph of T with colours $C = \{1, 2, 3\}$. Define c_{ij} to be the vertex of $G_3(T)$ with $c_{ij}(a) = i$ for all $a \in A$ and $c_{ij}(b) = j$ for all $b \in B$.*

(1) If $\ell, r > 0$ are both even, then $G_3(T)$ has no spanning subgraph consisting only of paths whose ends are in $\{c_{12}, c_{13}, c_{21}, c_{23}, c_{31}, c_{32}\}$.

(2) If $\ell > 1$ is odd and $r > 0$ is even, then $G_3(T)$ has a Hamilton path from c_{12} to c_{23}.

(3) If $\ell > 1$ and $r > 1$ are both odd, then $G_3(T)$ has a Hamilton path from c_{12} to c_{13}.

Proof. (1): Suppose $\ell, r > 0$ are both even. By Lemma A.2, $\mathcal{H} := G_3(T)$ is bipartite. Suppose $\mathcal{H}$ has bipartition $(\mathcal{A}, \mathcal{B})$. Without loss of generality, assume $c_{12} \in \mathcal{A}$. Since $d_{\mathcal{H}}(c_{12}, c_{ij})$ is even for each $c_{ij} \in \{c_{12}, c_{13}, c_{21}, c_{23}, c_{31}, c_{32}\}$ and $\mathcal{H}$ is bipartite, we have $\{c_{12}, c_{13}, c_{21}, c_{23}, c_{31}, c_{32}\} \subseteq \mathcal{A}$. By [6, Theorem 5.5] there is a Hamilton cycle in $\mathcal{H}$, and thus $|\mathcal{A}| = |\mathcal{B}|$. It follows that there is no spanning subgraph of $\mathcal{H}$ consisting only of paths whose ends are in $\{c_{12}, c_{13}, c_{21}, c_{23}, c_{31}, c_{32}\}$ (otherwise $|\mathcal{A}| > |\mathcal{B}|$).

(2): Suppose $\ell > 1$ is odd and $r > 0$ is even. Then $\ell + r \geq 5$. We first prove that $G_3(T)$ has a Hamilton path between c_{12} and c_{23} whenever T is P_5 or any odd flare. If $T \cong P_5$ (with $|A| = 3$, $|B| = 2$), then there is a Hamilton path between c_{12} to c_{23} in $G_3(P_5)$, as described in Figure 15. A 3-colouring f of $P_5 = x_1x_2x_3x_4x_5$ is represented by the string $f(x_1)f(x_2)f(x_3)f(x_4)f(x_5)$; for example, $c_{12} = 12121$ and $c_{23} = 23232$.

12121	21313	32321	13131	23231	31212
12321	21323	32121	12131	13231	31232
12323	21321	32123	32131	13232	31231
12313	31321	12123	32132	13212	21231
12312	31323	13123	12132	13213	21232
32312	31313	23123	13132	23213	21212
31312	32313	23121	23132	21213	23212
21312	32323	13121	23131	31213	23232

Figure 15: A Hamilton path in $G_3(P_5)$ from c_{12} to c_{23}.

Let T be an odd flare on n vertices with u denoting the unique vertex of degree two and let $N_T(u) = \{v, v'\}$ where v is the unique vertex of degree $n-2$ (Figure 16).

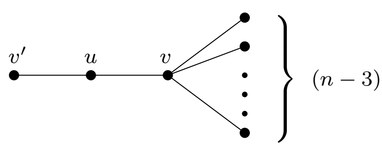

Figure 16: An odd flare T.

Here $|A| = n - 2$ and $|B| = 2$. We partition $\mathcal{H} := G_3(T)$ according to the colours of u and v. For each $1 \leq i \neq j \leq 3$, let $V_{ij} := \{c \in V(\mathcal{H}) \mid c(u) = i \text{ and } c(v) = j\}$ and let L_{ij} be an assignment of lists to the vertices of H such that $L_{ij}(u) := \{i\}$, $L_{ij}(v) := \{j\}$ and $L_{ij}(w) := \{1, 2, 3\}$ for $w \in V(H - \{u, v\})$. Note that $G_{L_{ij}}(H) \cong \mathcal{H}[V_{ij}] \cong Q_{n-2}$, for each $1 \leq i \neq j \leq 3$. Let $\mathcal{H}_1 := \mathcal{H}[V_{32} \cup V_{12}] \cong Q_{n-3} \,\square\, P_4$, $\mathcal{H}_4 := \mathcal{H}[V_{21} \cup V_{31}] \cong Q_{n-3} \,\square\, P_4$, $\mathcal{H}_2 := \mathcal{H}[V_{13}]$, and $\mathcal{H}_3 := \mathcal{H}[V_{23}]$.

For $\{i, j, k\} = \{1, 2, 3\}$, let $d_{ijk} \in V(\mathcal{H})$ denote the vertex with $d_{ijk}(v) = j$, $d_{ijk}(v') = k$, and $d_{ijk}(w) = i$ for all $w \in N_T(v)$. Then $[V_{12}, V_{13}] = \{c_{12}d_{132}, d_{123}c_{13}\}$ and $[V_{23}, V_{21}] = \{c_{21}d_{231}, d_{213}c_{23}\}$.

Claim. For $m \geq 2$, every edge of $Q_m \,\square\, P_4$ is in a Hamilton cycle of $Q_m \,\square\, P_4$.

The claim follows by induction and the fact that any pair of distinct edges of Q_{m-1} ($m \geq 3$) belongs to a Hamilton cycle of Q_{m-1} (for example, see [7, Theorem 4.1]).

In what follows, we define cycles and paths to construct a Hamilton path from c_{12} to c_{23} in $G_3(T)$ when T is an odd flare on n vertices. See Figure 17 for the case $n = 5$; here, the labels of the two columns of V_{ij} represent the colour choices for v' and the labels of the rows of V_{ij} represent the colour choices for the two vertices in $N_T(v) \setminus \{u\}$. Let C_1 be a Hamilton cycle of $\mathcal{H}_1$ containing $c_{12}d_{123}$ and C_4 be a Hamilton cycle of $\mathcal{H}_4$ containing $d_{213}c_{21}$; these exist by the previous claim. Let $c \in N_{\mathcal{H}_3}(c_{23})$ have $c(v') = 3$ and $d \in V_{13}$ be the unique vertex of $\mathcal{H}_2$ adjacent to c. By [7, Theorem 4.1], there is a Hamilton cycle C_3 in $\mathcal{H}_3$ containing the edges cc_{23} and $c_{23}d_{231}$. Observe $d_{\mathcal{H}_2}(d, c_{13})$ is odd. Since $\mathcal{H}_2$ is Hamilton laceable, there is a Hamilton path P between d and c_{13} (see Figure 17).

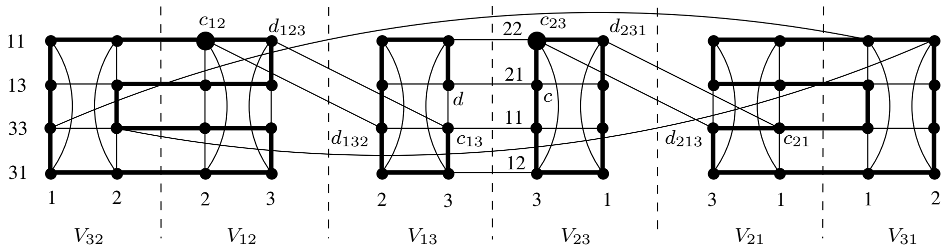

Figure 17: $\mathcal{H} = G_3(T)$, T an odd flare on five vertices, along with C_1, P, C_3, and C_4.

Now,

$$(C_1 - \{c_{12}d_{123}\}) \cup \{d_{123}c_{13}\} \cup P \cup \{dc\} \cup (C_3 - \{c_{23}c, c_{23}d_{231}\}) \\ \cup \{d_{231}c_{21}\} \cup (C_4 - \{c_{21}d_{213}\}) \cup \{d_{213}c_{23}\}$$

is a Hamilton path in $\mathcal{H}$ between c_{12} and c_{23} (see Figure 18 for $n = 5$).

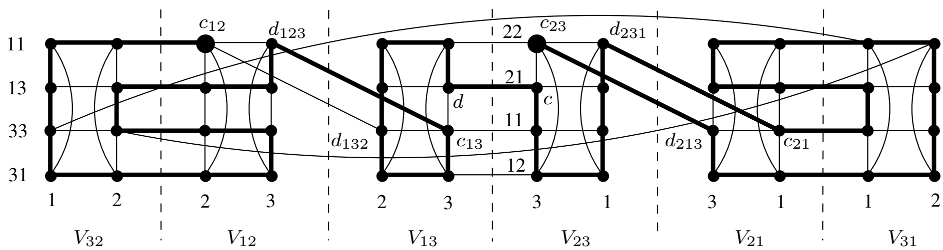

Figure 18: A Hamilton path between c_{12} and c_{23} in $\mathcal{H} = G_3(T)$, T an odd flare on five vertices.

Now suppose T is a tree on $n > 5$ vertices with n odd that is not isomorphic to a star or an odd flare. Then there are leaves $x, y \in V(T)$ with $d_T(x, y) \geq 3$. By choosing leaves x and y so that $d_T(x, y) \geq 3$ is minimum, $T' := T - \{x, y\}$ is not a star. Let $N_T(x) := \{x'\}$ and $N_T(y) := \{y'\}$; since $d_T(x, y) \geq 3$, $x' \neq y'$.

Let (A', B') denote the bipartition of T' with $A' \subseteq A, B' \subseteq B$, and define c'_{ij} to be the vertex of $\mathcal{H}' := G_3(T')$ with $c'_{ij}(a) = i$ for all $a \in A'$ and $c'_{ij}(b) = j$ for all $b \in B'$. By the inductive hypothesis, $\mathcal{H}'$ has a Hamilton path between c'_{12} and c'_{23}. Let $V(\mathcal{H}') := \{f_0, f_1, \ldots, f_{N-1}\}$. Since $\mathcal{H}'$ has a Hamilton path, we may assume that $f_0 f_1 \cdots f_{N-1}$ is a Hamilton path in $\mathcal{H}'$ between $f_0 = c'_{12}$ and $f_{N-1} = c'_{23}$. Since c'_{12} and c'_{23} differ in colour on at least two vertices, f_0 is not adjacent to f_{N-1} in $\mathcal{H}'$.

For $0 \le i \le N-1$, let F_i be the set of 3-colourings of $\mathcal{H}$ that agree with f_i on $V(T')$. Then $\{F_0, F_1, \ldots, F_{N-1}\}$ is a partition of the vertices of $\mathcal{H}$, and $\mathcal{H}[F_i] \cong C_4$, $0 \le i \le N-1$. If f_{i-1} and f_i differ on the colour of a vertex of $V(T') \setminus \{x', y'\}$, then $|[F_{i-1}, F_i]| = 4$ and $\mathcal{H}[F_{i-1} \cup F_i] \cong Q_3$. Otherwise, f_{i-1} and f_i differ on the colour of x' or y', implying that $|[F_{i-1}, F_i]| = 2$, the subgraph of $\mathcal{H}$ induced by the endpoints of the edges of $[F_{i-1}, F_i]$ is a 4-cycle, and $\mathcal{H}[F_{i-1} \cup F_i] \cong P_4 \,\square\, K_2$.

Consider the spanning subgraph G of $\mathcal{H}$ with edge set

$$\left(\bigcup_{i=0}^{N-1} E(\mathcal{H}[F_i])\right) \cup \left(\bigcup_{i=1}^{N-1} [F_{i-1}, F_i]\right).$$

Note that G is a connected B-graph. Let $(\mathcal{A}, \mathcal{B})$ be the bipartition of G and assume $c_{12} \in \mathcal{A}$.

First suppose $c_{23} \in \mathcal{B}$. As $c_{12} \in F_0$ and $c_{23} \in F_{N-1}$, it follows from Corollary A.7 that there is a Hamilton path in G between c_{12} and c_{23}. Now suppose $c_{23} \in \mathcal{A}$. Since $d_G(c_{12}, c_{23})$ is odd, $\mathcal{H}$ must have an edge e with both endpoints in $\mathcal{A}$ or both endpoints in $\mathcal{B}$. Suppose $e \in [F_p, F_q]$, where $0 \le p < q \le N-1$. Since f_0 is not adjacent to f_{N-1} in $\mathcal{H}'$, either $p \ne 0$ or $q \ne N-1$. Without loss of generality we assume $p \ne 0$. Then $f_p f_q \in E(\mathcal{H}')$, and either

(i) $|[F_p, F_q]| = 4$ and $\mathcal{H}[F_p \cup F_q] \cong Q_3$, or

(ii) $|[F_p, F_q]| = 2$, the subgraph of $\mathcal{H}$ induced by the endpoints of the edges of $[F_p, F_q]$ is a 4-cycle, and $\mathcal{H}[F_p \cup F_q] \cong P_4 \,\square\, K_2$.

In either case, there exists another edge $e' \in [F_p, F_q]$ such that $e := uv$, $e' := u'v'$, $u, u' \in F_p$, and $uvv'u'u$ is a 4-cycle. The choice of $e = uv$ ensures that $u, v \in \mathcal{A}$ or $u, v \in \mathcal{B}$.

Consider the spanning tree $\widehat{T}$ of $\mathcal{H}'$ with edge set

$$E(\widehat{T}) := \{f_{i-1} f_i \mid 1 \le i \le q-1\} \cup \{f_{i-1} f_i \mid q+1 \le i \le N-1\} \cup \{f_p f_q\}.$$

We define J to be the spanning subgraph of $\mathcal{H}$ with edge set

$$\left(\bigcup_{i=0}^{N-1} E(\mathcal{H}[F_i])\right) \cup \left(\bigcup_{f_i f_j \in E(\widehat{T})} [F_i, F_j]\right).$$

Then J is a connected B-graph. Let $(\mathcal{K}, \mathcal{L})$ be the bipartition of J; we may assume that $c_{12} \in \mathcal{K}$. Since $f_p f_q \in E(\widehat{T})$, $[F_p, F_q] \subseteq E(J)$; in particular, $e = uv$ and $e' = u'v'$ are edges of J, so u and v are in different parts of the partition $(\mathcal{K}, \mathcal{L})$, and thus $c_{23} \in \mathcal{L}$.

Case 1. If $|[F_p, F_{p+1}]| = |[F_p, F_q]| = 2$, then $f_p f_q, f_p f_{p+1} \in E(\mathcal{H}')$ arise from colour changes on x' and y'. Since there are only three possible vertex colours, there is only one possible colour that x' could change to, and only one possible colour that y' could change to; i.e., one of $f_p f_q, f_p f_{p+1}$ arises from a colour change on x' and the other from a colour change on y'. Assuming that $\mathcal{H}[F_p]$ is the 4-cycle $uu'ww'u$, it follows that u, u' are incident to the edges of $[F_p, F_q]$ and that without loss of generality u', w are incident to the edges of $[F_p, F_{p+1}]$.

Let $[F_p, F_q] = \{uv, u'v'\}$ and $[F_p, F_{p+1}] = \{wz, u'z'\}$. Since $uu' \in E(J)$, exactly one of u, u' is in $\mathcal{L}$.

Case 1(a). First suppose that $u \in \mathcal{L}$. Let J_1 denote the subgraph of J induced by $\cup_{i=0}^{p} F_i$. Then J_1 is a B-graph with vertex partition $\{F_0, F_1, \ldots, F_p\}$ and bipartition $(\mathcal{K}, \mathcal{L})$ satisfying the conditions of Corollary A.7, with $c_{12} \in \mathcal{K} \cap F_0$ and $u \in \mathcal{L} \cap F_p$. Thus J_1 has a Hamilton path R_1 between c_{12} and u. Note that the proof of Corollary A.7 implies that R_1 can be constructed so as to contain the edge $u'w$.

Let J_2 be the subgraph of J induced by $\cup_{i=q}^{N-1} F_i$. Then J_2 is a B-graph with vertex partition $\{F_q, F_{q+1}, \ldots, F_{N-1}\}$ and bipartition $(\mathcal{K}, \mathcal{L})$ satisfying the conditions of Corollary A.7, with $v \in \mathcal{K} \cap F_q$ and $c_{23} \in \mathcal{L} \cap F_{N-1}$. Thus J_2 has a Hamilton path R_2 between v and c_{23}.

Finally, let J_3 be the subgraph of J induced by $\cup_{i=p}^{q-1} F_i$. Then by Lemma 2.7, J_3 has a Hamilton cycle C containing the edges uu', ww' and uw'. Let R_3 be the path between u' and w obtained by deleting u and w' from C.

Now concatenate paths R_1, R_2, R_3 and delete edge $u'w \in R_1$ to form a Hamilton path between c_{12} and c_{23}.

Case 1(b). Now suppose that $u \in \mathcal{K}$. Then $u' \in \mathcal{L}$. Since $p \geq 1$ and $|[F_p, F_{p+1}]| = |[F_p, F_q]| = 2$, we have $|[F_{p-1}, F_p]| = 4$. Let $t \in F_{p-1}$ be such that $tu \in [F_{p-1}, F_p]$. Then $t \in \mathcal{L}$.

Let J_1 denote the subgraph of J induced by $\cup_{i=0}^{p-1} F_i$. Then J_1 is a B-graph with vertex partition $\{F_0, F_1, \ldots, F_{p-1}\}$ and bipartition $(\mathcal{K}, \mathcal{L})$ satisfying the conditions of Corollary A.7, with $c_{12} \in \mathcal{K} \cap F_0$ and $t \in \mathcal{L} \cap F_{p-1}$. Thus J_1 has a Hamilton path R_0 between c_{12} and t. Let R_1 be the concatenation of paths R_0 and $tuw'wu'$.

We define J_2 and R_3 as in Case 1(a). The same argument with v' in place of v gives a Hamilton path R_2 between v' and c_{23}. Now concatenate paths R_1, R_2, R_3 and delete edge $u'w \in R_1$ to form a Hamilton path between c_{12} and c_{23}.

Case 2. Suppose $|[F_p, F_q]| = 4$ and label the 4-cycle of $\mathcal{H}[F_p]$ as $uu'ww'u$. Let J_3 be the subgraph of J induced by $\cup_{i=p}^{q-1} F_i$. Then by Lemma 2.7, J_3 has a Hamilton cycle C. Without loss of generality, suppose C contains the edges uu', ww' and uw'. Let R_3 be the path between u' and w obtained by deleting u and w' from C.

Let J_1 denote the subgraph of J induced by $\cup_{i=0}^{p} F_i$. Then J_1 is a B-graph with vertex partition $\{F_0, F_1, \ldots, F_p\}$ and bipartition $(\mathcal{K}, \mathcal{L})$ satisfying the conditions of Corollary A.7, with $c_{12} \in \mathcal{K} \cap F_0$. Thus J_1 has Hamilton paths R_1' and R_1'' between c_{12} and the two vertices in $\mathcal{L} \cap F_p$. Let R_1 be one of R_1' and R_1'' such that R_1 contains edge $u'w$. Suppose R_1 is between c_{12} and t. Then $t \in \mathcal{L}$. Let $t' \in F_q$ be such that $tt' \in [F_p, F_q]$. Then $t' \in \mathcal{K}$.

We define J_2 as in Case 1(a). The same argument with t' in place of v gives a Hamilton path R_2 between t' and c_{23}. Now concatenate paths R_1, R_2, R_3 and delete edge $u'w \in R_1$ to form a Hamilton path between c_{12} and c_{23}.

Case 3. Suppose $|[F_p, F_{p+1}]| = 4$. Let J_1 denote the subgraph of J induced by $(\cup_{i=0}^{p} F_i) \cup (\cup_{i=q}^{N-1} F_i)$. Then J_1 is a B-graph with vertex partition

$$\{F_0, F_1, \ldots, F_p, F_q, F_{q+1}, \ldots, F_{N-1}\}$$

and bipartition $(\mathcal{K}, \mathcal{L})$ satisfying the conditions of Corollary A.7, with $c_{12} \in \mathcal{K} \cap F_0$ and $c_{23} \in \mathcal{L} \cap F_{N-1}$. Thus J_1 has a Hamilton path R between c_{12} and c_{23}. Note that the proof of Corollary A.7 implies that R can be constructed so as to contain three edges of $\mathcal{H}[F_p]$.

Let J_3 be the subgraph of J induced by $\cup_{i=p+1}^{q-1} F_i$. Then by Lemma 2.7, J_3 has a Hamilton cycle C. Note that C contains three edges of $\mathcal{H}[F_{p+1}]$.

By the Pigeonhole Principle there are $s, t \in F_p$ and $s', t' \in F_{p+1}$ with $st \in R$, $s't' \in C$ such that $ss'tt's$ is a 4-cycle in $\mathcal{H}[F_p \cup F_{p+1}]$. Now $(R \cup C) - \{st, s't'\}$ is a Hamilton path in J between c_{12} and c_{23}.

(3): Finally suppose, $\ell > 1$ and $r > 1$ are both odd. We define $\mathcal{E}^k_{s,t}$ to be the tree obtained from P_k with ends u and v by appending s leaves to u and t leaves to v. The proof is by induction and has the following base cases.

Base Case 1. We first prove that $\mathcal{H} := G_3(T)$ has a Hamilton path between c_{12} and c_{13} when $T \cong \mathcal{E}^{4k+2}_{0,0}$ for $k \geq 0$, that is, when $T \cong P_{4k+2}$. If $T \cong P_2$ then $c_{12}c_{32}c_{31}c_{21}c_{23}c_{13}$ is such a Hamilton path. Suppose $k > 0$ and $T \cong P_{4k+2}$. Let u and v be the leaves of T, $N_T(u) := \{u'\}$, $N_T(v) := \{v'\}$, $N_T(u') := \{u, u''\}$ and $N_T(v') := \{v, v''\}$. Then $T' := T - \{u, u', v, v'\}$ is isomorphic to $P_{4(k-1)+2}$. Let (A', B') denote the bipartition of T' with $A' \subseteq A, B' \subseteq B$, and define c'_{ij} to be the vertex of $\mathcal{H}' := G_3(T')$ with $c'_{ij}(a) = i$ for all $a \in A'$ and $c'_{ij}(b) = j$ for all $b \in B'$. By the inductive hypothesis, $\mathcal{H}'$ has a Hamilton path between c'_{12} and c'_{13}. Let $V(\mathcal{H}') := \{f_0, f_1, \ldots, f_{N-1}\}$. Since $\mathcal{H}'$ has a Hamilton path we may assume that $f_0 f_1 \cdots f_{N-1}$ is a Hamilton path in $\mathcal{H}'$ between $f_0 := c'_{12}$ and $f_{N-1} := c'_{13}$.

For $0 \leq i \leq N-1$, let F_i be the set of 3-colourings of $\mathcal{H}$ that agree with f_i on $V(T')$. Then $\{F_0, F_1, \ldots, F_{N-1}\}$ is a partition of the vertices of $\mathcal{H}$, and $\mathcal{H}[F_i] \cong P_4 \,\square\, P_4$, $0 \leq i \leq N-1$. If f_{i-1} and f_i differ on the colour of a vertex of $V(T') \setminus \{u'', v''\}$, then $|[F_{i-1}, F_i]| = 16$ and $\mathcal{H}[F_{i-1} \cup F_i] \cong (P_4 \,\square\, P_4) \,\square\, K_2$. Otherwise, f_{i-1} and f_i differ on the colour of u'' or v'', implying that $|[F_{i-1}, F_i]| = 8$, and the subgraph of $\mathcal{H}$ induced by the endpoints of the edges of $[F_{i-1}, F_i]$ is $C_4 \,\square\, P_4$.

Consider the spanning subgraph G of $\mathcal{H}$ with edge set

$$\left(\bigcup_{i=0}^{N-1} E(\mathcal{H}[F_i])\right) \cup \left(\bigcup_{i=1}^{N-1} [F_{i-1}, F_i]\right).$$

Note that by Remark A.4, G is a connected B-graph. Let $(\mathcal{A}, \mathcal{B})$ be the bipartition of G and assume $c_{12} \in \mathcal{A}$. Since $\mathcal{H}$ is bipartite by Lemma A.2, and $d_{\mathcal{H}}(c_{12}, c_{13})$ is odd, $c_{13} \in \mathcal{B}$. As $c_{12} \in F_0$ and $c_{13} \in F_{N-1}$, it follows from Corollary A.7 that there is a Hamilton path in G between c_{12} and c_{13}.

Base Case 2. Let $T \cong \mathcal{E}^{4k+2}_{2s,2t}$ or $T \cong \mathcal{E}^{4k}_{2s+1,2t+1}$ with $s, t \geq 1$, $k \geq 0$, and $T' \cong P_{4k+2}$ be obtained from T by deleting $2s$ leaves adjacent to u and $2t$ leaves adjacent to v.

Let (A', B') denote the bipartition of T' with $A' \subseteq A, B' \subseteq B$, and define c'_{ij} to be the vertex of $\mathcal{H}' := G_3(T')$ with $c'_{ij}(a) = i$ for all $a \in A'$ and $c'_{ij}(b) = j$ for all $b \in B'$. By Case 1, $\mathcal{H}'$ has a Hamilton path between c'_{12} and c'_{13}. Let $V(\mathcal{H}') := \{f_0, f_1, \ldots, f_{N-1}\}$. Since $\mathcal{H}'$ has a Hamilton path we may assume that $f_0 f_1 \cdots f_{N-1}$ is a Hamilton path in $\mathcal{H}'$ between $f_0 := c'_{12}$ and $f_{N-1} := c'_{13}$.

For $0 \leq i \leq N-1$, let F_i be the set of 3-colourings of $\mathcal{H} := G_3(T)$ that agree with f_i on $V(T')$. Then $\{F_0, F_1, \ldots, F_{N-1}\}$ is a partition of the vertices of $\mathcal{H}$, and $\mathcal{H}[F_i] \cong Q_{2s+2t}$, $0 \leq i \leq N-1$. If f_{i-1} and f_i differ on the colour of a vertex of $V(T') \setminus \{u, v\}$, then $|[F_{i-1}, F_i]| = 2^{2s+2t}$ and $\mathcal{H}[F_{i-1} \cup F_i] \cong Q_{2s+2t} \,\square\, K_2 \cong Q_{2s+2t+1}$. If f_{i-1} and f_i differ on the colour of u then $|[F_{i-1}, F_i]| = 2^{2t}$, and the subgraph of $\mathcal{H}$ induced by the endpoints of the edges of $[F_{i-1}, F_i]$ is Q_{2t+1}. Otherwise f_{i-1} and f_i differ on the colour of v implying that $|[F_{i-1}, F_i]| = 2^{2s}$, and the subgraph of $\mathcal{H}$ induced by the endpoints of the edges of $[F_{i-1}, F_i]$ is Q_{2s+1}.

Consider the spanning subgraph G of $\mathcal{H}$ with edge set

$$\left(\bigcup_{i=0}^{N-1} E(\mathcal{H}[F_i])\right) \cup \left(\bigcup_{i=1}^{N-1} [F_{i-1}, F_i]\right).$$

Note that by Remark A.4, G is a connected B-graph. Let $(\mathcal{A}, \mathcal{B})$ be the bipartition of G and assume $c_{12} \in \mathcal{A}$. Since $\mathcal{H}$ is bipartite by Lemma A.2, and $d_{\mathcal{H}}(c_{12}, c_{13})$ is odd, $c_{13} \in \mathcal{B}$. As $c_{12} \in F_0$ and $c_{13} \in F_{N-1}$, it follows from Corollary A.7 that there is a Hamilton path in G between c_{12} and c_{13}.

Base Case 3. Let $T \cong \mathcal{E}^{4k}_{2s+1,1}$ with $s, k \geq 1$, and let $T' \cong P_{4k-2}$ be obtained from T by deleting the $2s+2$ leaves and the vertices u and v. Define u', v' as the leaves of T' so that in T, u' is adjacent to u and v' is adjacent to v.

Let (A', B') denote the bipartition of T' with $A' \subseteq A, B' \subseteq B$, and define c'_{ij} to be the vertex of $\mathcal{H}' := G_3(T')$ with $c'_{ij}(a) = i$ for all $a \in A'$ and $c'_{ij}(b) = j$ for all $b \in B'$. By Case 1, $\mathcal{H}'$ has a Hamilton path between c'_{12} and c'_{13}. Let $V(\mathcal{H}') := \{f_0, f_1, \ldots, f_{N-1}\}$. Since $\mathcal{H}'$ has a Hamilton path we may assume that $f_0 f_1 \cdots f_{N-1}$ is a Hamilton path in $\mathcal{H}'$ between $f_0 := c'_{12}$ and $f_{N-1} := c'_{13}$.

For $0 \leq i \leq N-1$, let F_i be the set of 3-colourings of $\mathcal{H} := G_3(T)$ that agree with f_i on $V(T')$. Then $\{F_0, F_1, \ldots, F_{N-1}\}$ is a partition of the vertices of $\mathcal{H}$, and $\mathcal{H}[F_i] \supseteq P_4 \,\square\, P_{2^{2s+2}}$, $0 \leq i \leq N-1$. This follows since $G_L(K_{1,2s+1})$ has a Hamilton path $P_{2^{2s+2}}$ where L is an assignment of lists in which vertices of degree one have lists $\{1, 2, 3\}$ and the remaining vertex has list $\{1, 2\}$. If f_{i-1} and f_i differ on the colour of a vertex of $V(T') \setminus \{u', v'\}$, then $|[F_{i-1}, F_i]| = 4 \cdot 2^{2s+1}$ and $\mathcal{H}[F_{i-1} \cup F_i] \supseteq (P_4 \,\square\, P_{2^{2s+1}}) \,\square\, K_2$. If f_{i-1} and f_i differ on the colour of v' then $|[F_{i-1}, F_i]| = 2 \cdot 2^{2s+1}$, and the subgraph of $\mathcal{H}$ induced by the endpoints of the edges of $[F_{i-1}, F_i]$ contains $(P_2 \,\square\, P_{2^{2s+1}}) \,\square\, K_2$. Otherwise f_{i-1} and f_i differ on the colour of u' implying that $|[F_{i-1}, F_i]| = 4 \cdot 2^{2s}$, and the subgraph of $\mathcal{H}$ induced by the endpoints of the edges of $[F_{i-1}, F_i]$ contains $(P_4 \,\square\, P_{2^{2s}}) \,\square\, K_2$.

Consider the spanning subgraph G of $\mathcal{H}$ with edge set

$$\left(\bigcup_{i=0}^{N-1} E(\mathcal{H}[F_i])\right) \cup \left(\bigcup_{i=1}^{N-1} [F_{i-1}, F_i]\right).$$

Note that by Remark A.4, G is a connected B-graph . Let $(\mathcal{A}, \mathcal{B})$ be the bipartition of G and assume $c_{12} \in \mathcal{A}$. Since $\mathcal{H}$ is bipartite by Lemma A.2, and $d_{\mathcal{H}}(c_{12}, c_{13})$ is odd, $c_{13} \in \mathcal{B}$. As $c_{12} \in F_0$ and $c_{13} \in F_{N-1}$, it follows from Corollary A.7 that there is a Hamilton path in G between c_{12} and c_{13}.

Induction Step. Now suppose T is a tree with bipartition (A, B), where $|A| := \ell > 1$ and $|B| := r > 1$ are both odd and T is not isomorphic to any of the graphs in Base Cases 1 to 3. If every pair of leaves $x, y \in A$ or $x, y \in B$ satisfy $d_T(x, y) \leq 2$, then $T \cong \mathcal{E}^k_{s,t}$; since $\ell, r > 1$ are both odd, T is isomorphic to one of the graphs in Base Cases 1 to 3. Thus, there are leaves $x, y \in A$ (or $x, y \in B$) with $d_T(x, y) \geq 3$.

Case 1. If $T - \{x, y\}$ is a star, then T is the graph obtained from $K_{1,2s+1}$, $s \geq 1$, by subdividing two of its edges. Let $u \in V(T)$ be the vertex of degree $2s+1$ and $v \in V(T)$ a leaf adjacent to u. Define $T' := T[\{u, v\}]$ and observe $T' \cong K_2$.

Let $\mathcal{H}' := G_3(T')$ have Hamilton path $f_0 f_1 f_2 f_3 f_4 f_5 := c'_{12} c'_{32} c'_{31} c'_{21} c'_{23} c'_{13}$, where $c'_{ij}(u) = i$ and $c'_{ij}(v) = j$. For $0 \leq i \leq 5$, let F_i be the set of 3-colourings of $\mathcal{H} := G_3(T)$ that agree with f_i on $V(T')$. Then $\{F_0, F_1, \dots, F_5\}$ is a partition of the vertices of $\mathcal{H}$, and $\mathcal{H}[F_i] \cong P_4 \,\square\, P_4 \,\square\, Q_{2s-2}$, $0 \leq i \leq 5$. If f_{i-1} and f_i differ on the colour of vertex v, then $|[F_{i-1}, F_i]| = 4 \cdot 4 \cdot 2^{2s-2}$ and $\mathcal{H}[F_{i-1} \cup F_i] = (P_4 \,\square\, P_4 \,\square\, Q_{2s-2}) \,\square\, K_2$. If f_{i-1} and f_i differ on the colour of u then $|[F_{i-1}, F_i]| = 4$, and the subgraph of $\mathcal{H}$ induced by the endpoints of the edges of $[F_{i-1}, F_i]$ is C_4.

Consider the spanning subgraph G of $\mathcal{H}$ with edge set

$$\left(\bigcup_{i=0}^{5} E(\mathcal{H}[F_i]) \right) \cup \left(\bigcup_{i=1}^{5} [F_{i-1}, F_i] \right).$$

Note that by Remark A.4, the graph $P_4 \,\square\, P_4 \,\square\, Q_{2s-2}$ is Hamilton laceable since $P_4 \,\square\, P_4 \,\square\, P_{2^{2s-2}}$ is a Hamilton laceable spanning subgraph. Thus, G is a connected B-graph. Let $(\mathcal{A}, \mathcal{B})$ be the bipartition of G and assume $c_{12} \in \mathcal{A}$. Since $\mathcal{H}$ is bipartite by Lemma A.2, and $d_{\mathcal{H}}(c_{12}, c_{13})$ is odd, $c_{13} \in \mathcal{B}$. As $c_{12} \in F_0$ and $c_{13} \in F_5$, it follows from Corollary A.7 that there is a Hamilton path in G between c_{12} and c_{13}.

Case 2. Suppose $N_T(x) := \{x'\}$ and $N_T(y) := \{y'\}$, and that $T' := T - \{x, y\}$ is not a star. Let (A', B') denote the bipartition of T' with $A' \subseteq A, B' \subseteq B$, and define c'_{ij} to be the vertex of $\mathcal{H}' := G_3(T')$ with $c'_{ij}(a) = i$ for all $a \in A'$ and $c'_{ij}(b) = j$ for all $b \in B'$.

By the inductive hypothesis, $\mathcal{H}'$ has a Hamilton path between c'_{12} and c'_{13}. Let $V(\mathcal{H}') := \{f_0, f_1, \dots, f_{N-1}\}$. Since $\mathcal{H}'$ has a Hamilton path we may assume that $f_0 f_1 \cdots f_{N-1}$ is a Hamilton path in $\mathcal{H}'$ between $f_0 := c'_{12}$ and $f_{N-1} := c'_{13}$.

For $0 \leq i \leq N-1$, let F_i be the set of 3-colourings of $\mathcal{H} := G_3(T)$ that agree with f_i on $V(T')$. Then $\{F_0, F_1, \dots, F_{N-1}\}$ is a partition of the vertices of $\mathcal{H}$, and $\mathcal{H}[F_i] \cong C_4$, $0 \leq i \leq N-1$. If f_{i-1} and f_i differ on the colour of a vertex of $V(T') \setminus \{x', y'\}$, then $|[F_{i-1}, F_i]| = 4$ and $\mathcal{H}[F_{i-1} \cup F_i] \cong Q_3$. Otherwise, f_{i-1} and f_i differ on the colour of x' or y', implying that $|[F_{i-1}, F_i]| = 2$, the subgraph of $\mathcal{H}$ induced by the endpoints of the edges of $[F_{i-1}, F_i]$ is a 4-cycle, and $\mathcal{H}[F_{i-1} \cup F_i] \cong P_4 \,\square\, K_2$.

Consider the spanning subgraph G of $\mathcal{H}$ with edge set

$$\left(\bigcup_{i=0}^{N-1} E(\mathcal{H}[F_i]) \right) \cup \left(\bigcup_{i=1}^{N-1} [F_{i-1}, F_i] \right).$$

Note that G is a connected B-graph. Let $(\mathcal{A}, \mathcal{B})$ be the bipartition of G and assume $c_{12} \in \mathcal{A}$. Since $\mathcal{H}$ is bipartite by Lemma A.2 and $d_{\mathcal{H}}(c_{12}, c_{13})$ is odd, $c_{13} \in \mathcal{B}$. As $c_{12} \in F_0$ and $c_{13} \in F_{N-1}$, it follows from Corollary A.7 that there is a Hamilton path in G between c_{12} and c_{13}. □

Author Guidelines

Before submission

Papers should be written in English, prepared in LaTeX, and must be submitted as a PDF file.

The title page of the submissions must contain:

- *Title*. The title must be concise and informative.
- *Author names and affiliations*. For each author add his/her affiliation which should include the full postal address and the country name. If avilable, specify the e-mail address of each author. Clearly indicate who is the corresponding author of the paper.
- *Abstract*. A concise abstract is required. The abstract should state the problem studied and the principal results proven.
- *Keywords*. Please specify 2 to 6 keywords separated by commas.
- *Mathematics Subject Classification*. Include one or more Math. Subj. Class. codes – see http://www.ams.org/msc.

After acceptance

Articles which are accepted for publication must be prepared in LaTeX using class file amcjoucc.cls and the bst file amcjoucc.bst (if you use BibTeX). If you don't use BibTeX, please make sure that all your references are carefully formatted following the examples provided in the sample file.

All files can be found on-line at:

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Abstracts: Be concise. As much as possible, please use plain text in your abstract and avoid complicated formulas. Do not include citations in your abstract. All abstracts will be posted on the website in fairly basic HTML, and HTML can't handle complicated formulas. It can barely handle subscripts and greek letters.

Cross-referencing: All numbering of theorems, sections, figures etc. that are referenced later in the paper should be generated using standard LaTeX \label{. . . } and \ref{. . . } commands. See the sample file for examples.

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