HEIGHTS, POTENTIALS AND GEOPOTENTIAL NUMBERS

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Received for publication: 21 January 1998
Prepared for publication: 20 March 1998

Abstract
The paper presents the connection between measured height difference and the acceleration of gravity which defines the difference in potentials and geopotential height numbers which serve as a basis for the calculation of point heights in various height systems.
Keywords: *acceleration of gravity, difference in potential, geopotential numbers, height difference, level surface, potential*

1 INTRODUCTION

Heights and height differences have both a geometrical and a physical meaning. One usually envisualises height to be the vertical distance of a certain point, located above a certain reference surface, from that surface. In this case, height is defined as a geometrical quantity. Practical experience has also taught us that two points have the same height when water between them does not move, which means that they lie on the same level surface and that the height difference between them is proportional to the difference in their potentials. This example illustrates the physical meaning of heights and height differences. Which of the two meanings of height or height difference is more important, the geometrical or the physical one, depends above all on the purpose of use of the heights or height differences. The physical interpretation of heights and height differences proves to be more suitable for the majority of natural and artificial dynamic processes which take place on Earth (the movement of water and vehicles and the dynamics of constructed buildings). For geodesy and determination of the position of points in three-dimensional space, on the other hand, increasingly greater significance is ascribed to the geometrical interpretation of heights and height differences. Geometrical interpretation has gained a special significance with the introduction of GPS technology into geodetic surveys.

2 POTENTIAL AND DIFFERENCE IN POTENTIAL

The vector of gravity acts on each point on the Earth's surface and is perpendicular to the level surface on which this point lies. All points therefore have a specific potential. All points with the same potential lie on the same level surface. Geodesy is concerned with the connection between the acceleration of gravity g and measured height difference, dh. For easier analysis, a scalar field is assigned to the vector field W(x, y, z) of the acceleration of gravity, such that

$$\bar{g} = \text{grad } W.$$

The magnitude of the vector of gravity is the acceleration of gravity g

$$g = \frac{dW}{dh}. \qquad 2.1$$

The height difference between two points is determined by the length of the vertical line between their level surfaces, for which W(x, y, z) = constant (see Figure 2.2). The negative sign in the equation means that g and the height difference (change in height) dh are inversely proportional. There is a constant potential difference $(W_{P_1} - W_{P_2})$. W_{P_1} and W_{P_2} between the two level surfaces which run through points P_1 and P_2 with constant potentials and . These two level surfaces are at a distance of dh. Due to the irregular distribution of mass in the Earth's interior, the acceleration of gravity g on level surfaces is a variable. Changes in the acceleration of gravity (Δg) can be measured with great accuracy with the use of gravimeters. The unit for Δg used in geodesy is 1 gal = 10^{-2} m s^{-2} and is not included in the international system of units, SI. If equation 2.1 is written in the following form

$$dW = -\, g\, dh = \text{constant} \qquad 2.2$$

and taking into account the variation of g, it is clear that the difference between the level surfaces dh also varies in inverse proportion. It follows from this that the neighbouring level surfaces are not parallel. At higher levels of g, the distances between level surfaces dh is smaller (Bretterbauer, 1986).

If one desires to determine unambiguous heights, independent of the levelling route, the definition of heights must be bound to potentials. We are interested in the potential difference, and this is independent of the route. A question arises of whether it is possible to determine the potential difference from data on the measured acceleration of gravity and levelled height difference. The first person to give an answer to this question was Helmert in 1884. The problem of determining the potential difference from measured height differences and the acceleration of gravity was first analysed for a single instrument station point.

The measured height difference between points P_z and P_s is obtained as a difference between readings (Z, S) on a vertically placed levelling staff at points P_z and P_s, with a horizontal line of sight. In this case, the line of sight is a tangent to the level surface which runs through the optical centre of the objective. The potential difference between points P_z and P_s is (Leismann et al., 1992):

$$W_S - W_Z = -\int_{P_Z}^{P_S} g\,dh = -\,(l_Z \overline{g_Z} - l_S \overline{g_S}), \qquad 2.3$$

where:

l_Z, l_S ... length of the vertical line through points P_Z, P_S between level surfaces (W_Z = constant, W_S = constant and W = constant)

$\overline{g_Z}, \overline{g_S}$ ... corresponding mean value of the acceleration of gravity in the corresponding part of the vertical line.

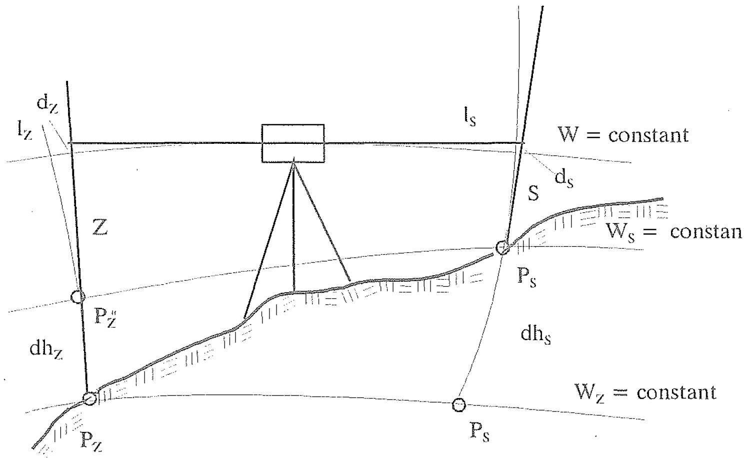

Figure 2.1

It can be seen in Figure 2.1 that the slightly curved parts of the vertical line can be replaced with the readings on the levelling staff Z and S, which are reduced by the values d_Z and d_S. According to Helmert, the error in the levelling line is negligible. The values (Leismann et al., 1992)

$$\overline{g_Z} = \frac{1}{l_Z} \int g dh \quad \text{and} \quad \overline{g_S} = \frac{1}{l_S} \int g dh \tag{2.4}$$

can therefore be approximated with the value obtained on one half of readings on the levelling staff, since at small height differences it can be assumed that the acceleration of gravity falls linearly with height. Equation 2.3 can therefore be written in the following form

$$W_S - W_Z \cong -(Z - d_Z)\,\overline{g_Z} + (S - d_S)\,\overline{g_S}. \tag{2.5}$$

If equation 2.5 is transformed such that individual terms are expressed as a sum or difference of the readings on the levelling staffs (Z and S), the mean values of the acceleration of gravity (g_Z, g_S) and the values d_Z and d_S, the following expression is obtained

$$W_S - W_Z = -\frac{(Z-S)}{2}(\overline{g_Z} + \overline{g_S}) - \frac{(Z+S)}{2}(\overline{g_Z} - \overline{g_S}) + \\ + \frac{(d_Z - d_S)}{2}(\overline{g_Z} + \overline{g_S}) + \frac{(d_Z + d_S)}{2}(\overline{g_Z} - \overline{g_S}). \tag{2.6}$$

If levelling is performed from the middle, as prescribed for precise levelling, the difference (d_Z - d_S) is negligible because the curvatures of the level surface between the instrument station point and the station points of the levelling staffs (at the front, at the back) are almost equal. A similar consideration applies to the levelling line, therefore the third term in equation 2.6 can be omitted. It was established on the basis of research performed by Baeschlin, Ramsayer and Zeger that the second and fourth terms in the above-mentioned equation are also negligible, and therefore also the value of $(\overline{g_Z - g_S})$. The following expression is obtained

$$W_S - W_Z = -\frac{(Z-S)}{2}(\overline{g_Z} + \overline{g_S}). \qquad 2.7$$

In addition, the following relation applies to the potential difference W_S - W_Z

$$W_S - W_Z = dh_Z\, g''_Z = dh_S\, g''_S, \qquad 2.8$$

where

dh_Z, dh_S ... distance between level surfaces W_Z = constant and W_S = constant in point P_Z or P_S

g''_Z, g''_S ... mean value of the acceleration of gravity in the intervals $P_Z - P''_Z$ and $P_S - P''_S$ (see Figure 2.1).

If g is taken to be

$$\overline{g} = \frac{1}{2}(\overline{g_Z} + \overline{g_S}) \qquad 2.9$$

equations 2.7 and 2.8 are equalised, and the equation showing the difference in readings on the levelling staff is solved, the following equations are obtained

$$-(Z-S) \cong dh_Z \frac{g''_Z}{g} \qquad \text{and} \qquad -(Z-S) \cong dh_S \frac{g''_S}{g}. \qquad 2.10$$

The scale factors, $\frac{g''_Z}{g}$ and $\frac{g''_S}{g}$, need not be taken into account because their influence is smaller than $1x10^{-8}$. Equations 2.10 can therefore be written with sufficient accuracy as follows

$$dh_Z \cong -(Z-S) \cong dh_S. \qquad 2.11$$

This means that the difference in readings on levelling staffs (Z - S), at one station point of the instrument can be approximated with sufficient accuracy using the distance between level surfaces which run through the station points of the levelling staffs. The difference between the station points of levelling staffs is obtained by multiplying the measured height difference with the acceleration of gravity at the station point (see equation 2.9). For each individual station point, this value equals the acceleration of gravity at the height of $\frac{Z+S}{4}$ above the station point of the instrument (Leismann et al., 1992). Since the measurement of the acceleration of gravity at this point is impractical, the value of the acceleration of gravity in equation 2.9 can according to Helmert be taken to be the arithmetic mean of the accelerations of gravity which were measured on the station points of the levelling staffs. On the basis of the above-mentioned equations it can be seen that the potential difference between points P_Z and P_S can be determined on the basis of data on the measured acceleration of gravity and the levelled height difference.

The potential difference in the levelling line between points P_1 and P_2 can be determined in a similar manner. If the height difference on one station point of the instrument (reading at the back of the staff - reading at the front of the staff) is designated as δh_i, then the measured height difference between points P_1 and P_2 equals

$$dh_{P_1}^{P_2} = \sum_{i=P_1}^{P_2} \delta h_i .$$

Since level surfaces of the gravitational field are not parallel to each other, and because the calibration of the level vial and the position of the spirit level compensator are closely connected with the gravitational field, point heights cannot be determined independently of the levelling route. It can be seen in Figure 2.2 that the levelled height difference depends on the route. If levelling is performed from P_1 through P_2'' to P_2 or from P_1 through P_1'' to P_2, different results are obtained, because the levelling value along the level surfaces P_1P_2'' and $P_1''P_2$ equals zero. Only the potential difference $(W_{P_2} - W_{P_1})$, which is obtained by integrating equation 2.2, is independent of the route. In practice, the integral is approximated using a sum and the following expression is obtained (Bretterbauer, 1986):

$$W_{P_2} - W_{P_1} = -\int_{P_1}^{P_2} g \, dh_{P_1}^{P_2} \cong \sum_{i=P_1}^{P_2} g_i \delta h_i . \qquad 2.12$$

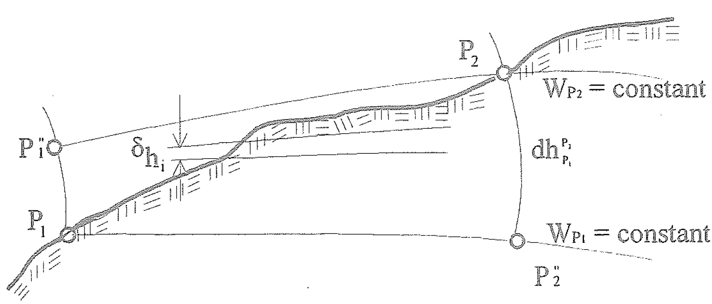

Figure 2.2

Figure 2.2 shows:

δh_i . . . height difference between the two station points of the levelling staffs (difference in readings at the back and at the front of the staff)

$dh_{P_1}^{P}$ . . . height difference between points P_1 and P_2.

3 GEOPOTENTIAL NUMBERS

It was established in the previous section that the potential difference between points P_1 and P_2 can be determined on the basis of data on the measured values of acceleration of gravity and levelled height differences. This type of levelling can be named geopotential levelling. It is defined as levelling which connects direct levelling and the measured acceleration of gravity. Potential differences at individual points with regard to the reference level surface, i.e. the geoid, were named geopotential numbers (C) by a French geodesist P. Tardi. The following expression applies to point P_i:

$$C_{P_i} = W_{P_i^0} - W_{P_i} = \int_{P_i^0}^{P_i} g_i \, dh_{P_i^0}^{P_i} , \qquad 3.1$$

where

$W_{P_i^0}$ ... potential of the reference level surface, the geoid

W_{P_i} ... potential of the level surface through point P_i

P_i^0 ... point on the reference level surface - the geoid, assigned to point P_i.

In practice, the integral is approximated by a sum to yield

$$C_{P_i} \cong \sum_{i=P_i^0}^{P_i} g_i \delta h_i , \qquad 3.2$$

where

δh_i ... height difference on the i-th station point of the instrument

g_i ... mean value of the acceleration of gravity between station points i and i-1, therefore

$$g_i = \frac{g_{i-1}^{izm} + g_i^{izm}}{2}.$$

If the height of the reference level surface or the geoid is taken to be 0, then the potential difference is the natural measure of the heights of points on the Earth's surface. The unit of geopotential numbers is Nm/kg, i.e. work per mass unit. At their conference in Rome in 1954, the International Association of Geodesists adopted a geopotential number unit of 1 kgal m = 1 gpu (geopotential unit) = 10 Nm/kg = 10 m^2/s^2. The differences in geopotential numbers between benchmarks P_1 and P_2 can be calculated as follows (Bilajbegović et al., 1989):

$$\Delta C_{P_1}^{P_2} = \overline{g_{P_1}^{P_2}} dh_{P_1}^{P_2} ,$$

$\overline{g_{P_1}^{P_2}}$ is calculated using the following equation

$$\overline{g_{P_1}^{P_2}} = \frac{g_{P_1} + g_{P_2}}{2},$$

where

g_{P_1} ... acceleration of gravity on the P_1 benchmark

g_{P_2} ... acceleration of gravity on the P_2 benchmark

$dh_{P_1}^{P_2}$ ... measured height difference between benchmarks P_1 and P_2.

4 CONCLUSION

Even though geopotential numbers unambiguously define point heights and point heights are determined independently of levelling route, they are unsuitable for the majority of users because they represent point heights defined entirely physically. The main shortcoming of geopotential numbers is that they cannot be interpreted geometrically and are not expressed in metres, which is vital for many users. These two main shortcomings of geopotential numbers can be removed by dividing geopotential numbers with the acceleration of gravity at a certain point. Geopotential numbers thus represent the basis for the determination of point heights in different elevation systems (orthometric heights and normal heights), naturally with the exception of the determination of the ellipsoid heights of points, which are point heights defined entirely geometrically.

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