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Arc-transitive graphs of valency twice a prime admit a semiregular automorphism*

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Abstract

We prove that every finite arc-transitive graph of valency twice a prime admits a nontrivial semiregular automorphism, that is, a non-identity automorphism whose cycles all have the same length. This is a special case of the Polycirculant Conjecture, which states that all finite vertex-transitive digraphs admit such automorphisms.

Keywords: Arc-transitive graphs, polycirculant conjecture, semiregular automorphism.

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1 Introduction

All graphs in this paper are finite. In 1981, Marušič asked if every vertex-transitive digraph admits a nontrivial semiregular automorphism [13], that is, an automorphism whose cycles all have the same length. This question has attracted considerable interest and a generalisation of the affirmative answer is now referred to as the Polycirculant Conjecture [4]. See [1] for a recent survey on this problem.

One line of investigation of this question has been according to the valency of the graph or digraph. Every vertex-transitive graph of valency at most four admits such an automorphism [7, 14], and so does every vertex-transitive digraph of out-valency at most three [9].

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Every arc-transitive graph of prime valency has a nontrivial semiregular automorphism [10] and so does every arc-transitive graph of valency 8 [17]. Partial results were also obtained for arc-transitive graphs of valency a product of two primes [18]. We continue this theme by proving the following theorem.

Theorem 1.1. *Arc-transitive graphs of valency twice a prime admit a nontrivial semiregular automorphism.*

2 Preliminaries

If G is a group of automorphisms of a graph Γ and v is a vertex of Γ, we denote by G_v the stabiliser in G of v, by $\Gamma(v)$ the neighbourhood of v, and by $G_v^{\Gamma(v)}$ the permutation group induced by G_v on $\Gamma(v)$. We will need the following well-known results.

Lemma 2.1. *Let Γ be a connected graph and $G \leqslant \mathrm{Aut}(\Gamma)$. If a prime p divides $|G_v|$ for some $v \in \mathrm{V}(\Gamma)$, then there exists $u \in \mathrm{V}(\Gamma)$ such that p divides $|G_u^{\Gamma(u)}|$.*

Proof. Since p divides $|G_v|$, there exists an element g of order p in G_v. As $g \neq 1$, there are vertices not fixed by g. Among these vertices, let w be one at minimal distance from v. Let P be a path of minimal length from v to w and let u be the vertex preceding w on P. By the definition of w, we have that u is fixed by g, so $g \in G_u$. On the other hand, g does not fix the neighbour w of u, so $g^{\Gamma(u)} \neq 1$ hence $|g^{\Gamma(u)}| = p$ and the result follows. □

Lemma 2.2. *Let G be a permutation group and let K be a normal subgroup of G such that G/K acts faithfully on the set of K-orbits. If G/K has a semiregular element Kg of order r coprime to $|K|$, then G has a semiregular element of order r.*

Proof. See for example [17, Lemma 2.3]. □

Lemma 2.3. *A transitive group of degree a power of a prime p contains a semiregular element of order p.*

Proof. In a transitive group of degree a power of a prime p, every Sylow p-subgroup is transitive. A non-trivial element of the center of this subgroup must be semiregular. □

Recall that a permutation group is *quasiprimitive* if every non-trivial normal subgroup is transitive, and *biquasiprimitive* if it is transitive but not quasiprimitive and every non-trivial normal subgroup has at most two orbits.

3 Arc-transitive graphs of prime valency

In the most difficult part of our proof, the arc-transitive graph of valency twice a prime will have a normal quotient with prime valency. We will thus need a lot of information about arc-transitive graphs of prime valency, which we collect in this section. We start with the following result, which is [3, Theorem 5]:

Theorem 3.1. *Let Γ be a connected G-arc-transitive graph of prime valency p such that the action of G on $\mathrm{V}(\Gamma)$ is either quasiprimitive or biquasiprimitive. Then one of the following holds:*

(1) G contains a semiregular element of odd prime order;

(2) $|\mathrm{V}(\Gamma)|$ *is a power of* 2*;*

(3) $\Gamma = \mathrm{K}_{12}$, $G = \mathrm{M}_{11}$ *and* $p = 11$*;*

(4) $|\mathrm{V}(\Gamma)| = (p^2 - 1)/2s$ *and* $G = \mathrm{PSL}_2(p)$ *or* $\mathrm{PGL}_2(p)$*, where* p *is a Mersenne prime and* s *is a proper divisor of* $(p-1)/2$ *but also a multiple of the product of the distinct prime divisors of* $(p-1)/2$*;*

(5) $|\mathrm{V}(\Gamma)| = (p^2 - 1)/s$ *and* $G = \mathrm{PGL}_2(p)$*, where* p *and* s *are as in part (4), and* Γ *is the canonical double cover of the graph given in (4).*

(Recall that the *canonical double cover* of a graph Γ is $\Gamma \times \mathrm{K}_2$.) We note that in cases (4) and (5), we must have $p \geqslant 127$, since this is the smallest Mersenne prime p such that $(p-1)/2$ is not squarefree. This fact will be used at the end of Section 4.

Corollary 3.2. *Let* Γ *be a connected* G*-arc-transitive graph of prime valency. Then one of the following holds:*

(1) G *contains a semiregular element of odd prime order;*

(2) $|\mathrm{V}(\Gamma)|$ *is a power of* 2*;*

(3) G *contains a normal* 2*-subgroup* P *such that* $(\Gamma/P, G/P)$ *is one of the graph-group pairs in (3–5) of Theorem 3.1.*

Proof. Suppose that $|\mathrm{V}(\Gamma)|$ is not a power of 2. If G is quasiprimitive or biquasiprimitive on $\mathrm{V}(\Gamma)$, then the result follows immediately from Theorem 3.1 (with $P = 1$ in case (3)). We thus assume that this is not the case and let P be a normal subgroup of G that is maximal subject to having at least three orbits on $\mathrm{V}(\Gamma)$. In particular, P is the kernel of the action of G on the set of P-orbits. Hence G/P acts faithfully, and quasiprimitively or biquasiprimitively on $\mathrm{V}(\Gamma/P)$. Since Γ has prime valency, is connected and G-arc-transitive, [12, Theorem 9] implies that P is semiregular. We may thus assume that P is a 2-group. (Otherwise P contains a semiregular element of odd prime order.) If G/P contains a semiregular element of odd prime order, then Lemma 2.2 implies that so does G. We may assume that this is not the case. Similarly, we may assume that $|\mathrm{V}(\Gamma/P)|$ is not a power of 2. (Otherwise, $|\mathrm{V}(\Gamma)|$ is a power of 2.) It follows that Γ/P and G/P are as is (3–5) of Theorem 3.1. □

We will now prove some more results about the graphs that appear in (3–5) of Theorem 3.1. Let us first recall the notion of *coset graphs*. Let G be a group with a subgroup H and let $g \in G$ such that $g^2 \in H$ but $g \notin \mathrm{N}_G(H)$. The graph $\mathrm{Cos}(G, H, HgH)$ has vertices the right cosets of H in G, with two cosets Hx and Hy adjacent if and only if $xy^{-1} \in HgH$. Observe that the action of G on the set of vertices by right multiplication induces an arc-transitive group of automorphisms such that H is the stabiliser of a vertex. Moreover, every arc-transitive graph can be constructed in this way [16, Theorem 2].

Lemma 3.3. *The graphs in (3) and (4) of Theorem 3.1 have a* 3*-cycle.*

Proof. Clearly K_{12} has a 3-cycle so suppose that Γ is one of the graphs given in (4). Let G be as in Theorem 3.1 and let $v \in \mathrm{V}(\Gamma)$. Then G is one of $\mathrm{PSL}_2(p)$ or $\mathrm{PGL}_2(p)$ and acts arc-transitively on Γ. In both cases, let $X = \mathrm{PSL}_2(p)$, so $G = X$ or $|G : X| = 2$. By [3, Lemma 5.3], we have that $G_v \cong \mathrm{C}_p \rtimes \mathrm{C}_s$ if $G = \mathrm{PSL}_2(p)$, and $\mathrm{C}_p \rtimes \mathrm{C}_{2s}$ if

$G = \mathrm{PGL}_2(p)$. By [5, pp. 285–286], $\mathrm{PGL}_2(p)$ has a unique conjugacy class of subgroups of order p and the normaliser of such a subgroup is isomorphic to $\mathrm{C}_p \rtimes \mathrm{C}_{p-1}$, which is the stabiliser in $\mathrm{PGL}_2(p)$ of a 1-dimensional subspace of the natural 2-dimensional vector space. The intersection of this subgroup with X is isomorphic to $\mathrm{C}_p \rtimes \mathrm{C}_{(p-1)/2}$, which has odd order. It follows that if $|G : X| = 2$, then G_v is not contained in X. Thus, in both cases, X is transitive on $\mathrm{V}(\Gamma)$. Since X is normal in G, $X_v \neq 1$ and Γ has prime valency, it follows that X is arc-transitive on Γ and so $\Gamma = \mathrm{Cos}(X, H, HgH)$ where $H \cong \mathrm{C}_p \rtimes \mathrm{C}_s$ and $g \in X \backslash \mathrm{N}_X(H)$ such that HgH is a union of p distinct right cosets of H.

Since H has a characteristic subgroup Y of order p, it follows that $\mathrm{N}_X(H)$ normalises Y and so $\mathrm{N}_X(H) \leqslant \mathrm{N}_X(Y) \cong \mathrm{C}_p \rtimes \mathrm{C}_{(p-1)/2}$. Since $\mathrm{N}_X(Y)/Y$ is cyclic it follows that $\mathrm{N}_X(H) = \mathrm{N}_X(Y)$. Also note that $\mathrm{N}_X(H)$ is the stabiliser in X of a 1-dimensional subspace and so the action of X on the set of right cosets of $\mathrm{N}_X(H)$ is 2-transitive and the stabiliser of any two 1-dimensional subspaces is isomorphic to $\mathrm{C}_{(p-1)/2}$. Let $x \in X$. The stabiliser in X of the coset Hx is H^x and so the orbit of Hx under H has length $|H : H \cap H^x|$. In particular, H fixes the coset Hx if and only if $x \in \mathrm{N}_X(H)$, and so H fixes $|\mathrm{N}_X(H) : H| = (p(p-1)/2)/ps = (p-1)/2s$ points of $\mathrm{V}(\Gamma)$. Moreover, since the stabiliser of two 1-dimensional subspaces is isomorphic to $\mathrm{C}_{(p-1)/2}$ it follows that if $x \notin \mathrm{N}_X(H)$ then $H \cap H^x \cong \mathrm{C}_{(p-1)/2}$ and so $|H : H \cap H^x| = p$. Thus the points of $\mathrm{V}(\Gamma)$ that are not fixed by H are permuted by H in orbits of size p and so for any $g \notin \mathrm{N}_X(H)$ we have that HgH is a union of p distinct right cosets of H.

For each $x \in \mathrm{N}_X(H)$ define the bijection λ_x of $\mathrm{V}(\Gamma)$ by $Hy \mapsto x^{-1}Hy = Hx^{-1}y$. Since X acts on $\mathrm{V}(\Gamma)$ by right multiplication, we see that λ_x commutes with each element of X. Moreover, λ_x is nontrivial if and only if $x \notin H$. Let $C = \{\lambda_x \mid x \in \mathrm{N}_G(H)\} \leqslant \mathrm{Sym}(\mathrm{V}(\Gamma))$. Since X acts transitively on $\mathrm{V}(\Gamma)$ and C centralises X, it follows from [6, Theorem 4.2A] that C acts semiregularly on $\mathrm{V}(\Gamma)$. Now $|C| = |\mathrm{N}_X(H) : H| = (p-1)/2s$ and $X \times C \leqslant \mathrm{Sym}(\mathrm{V}(\Gamma))$. Since $C \trianglelefteq X \times C$, the set of orbits of C forms a system of imprimitivity for $X \times C$ and hence for X. Moreover, since C is semiregular, comparing orders yields that C has $|\mathrm{V}(\Gamma)|/|C| = p + 1$ orbits. One of these orbits is the set of fixed points of H and H transitively permutes the remaining p orbits of C. In particular, it follows that C transitively permutes the nontrivial orbits of H and so the isomorphism type of Γ does not depend on the choice of the double coset HgH.

Let Z be the subgroup of scalar matrices in $\mathrm{SL}_2(p)$ and let $\hat{H}$ be the subgroup of the stabiliser in $\mathrm{SL}_2(p)$ of the 1-dimensional subspace $\langle(1, 0)\rangle$ such that $H = \hat{H}/Z$. Note that $\hat{h} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \hat{H}$ and let $\hat{g} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In particular, letting $h = \hat{h}Z$ and $g = \hat{g}Z$ we have that $g \notin \mathrm{N}_X(H)$ and so we may assume that $\Gamma = \mathrm{Cos}(X, H, HgH)$. Now Hg and Hgh are both adjacent to H and one can check that $g(gh)^{-1} = hgh \in HgH$ and so $\{H, Hg, Hgh\}$ is a 3-cycle in Γ. □

Definition 3.4. Let Γ be a graph and let S_0 be a subset of $\mathrm{V}(\Gamma)$. Let $S = S_0$. If a vertex u outside S has at least two neighbours in S, add u to S. Repeat this procedure until no more vertices outside S have this property. If at the end of the procedure, we have $S = \mathrm{V}(\Gamma)$, then we say that Γ is *dense with respect to* S_0.

It is an easy exercise to check that denseness is well-defined.

Corollary 3.5. *Let Γ be a graph in (3) or (4) of Theorem 3.1 and let $S_0 = \{u, v\}$ be an edge of Γ. Then Γ is dense with respect to S_0.*

Proof. Since Γ is arc-transitive of prime valency p, the local graph at v (that is, the subgraph induced on $\Gamma(v)$) is a vertex-transitive graph of order p and thus vertex-primitive. By Lemma 3.3, Γ has a 3-cycle so the local graph has at least one edge and thus must be connected. It follows that, running the process described in Definition 3.4 starting at $S = S_0$, eventually S will contain all neighbours of v. Repeating this argument and using connectedness of Γ yields the desired conclusion. □

The following is immediate from Definition 3.4.

Lemma 3.6. *Let Γ be a graph and S_0 be a set of vertices such that Γ is dense with respect to S_0. Then the canonical double cover of Γ, with vertex-set $\mathrm{V}(\Gamma) \times \{0, 1\}$, is dense with respect to $S_0 \times \{0, 1\}$.*

Proof. Let S_i be the sequence of subsets of $\mathrm{V}(\Gamma)$ obtained when running the procedure from Definition 3.4 starting with S_0 and ending with S_n for some n. Since Γ is dense with respect to S_0, we have $S_n = \mathrm{V}(\Gamma)$. For $i \in \{1, \dots, n\}$, let $v_i = S_i \setminus S_{i-1}$. (In other words, v_1 is the first vertex added to S_0 to get S_1, then v_2 is added to S_1 to get S_2, etc.)

Now, let $\Gamma' = \Gamma \times \mathrm{K}_2$ be the canonical double cover of Γ and let $S_0' = S_0 \times \{0, 1\} \subseteq \mathrm{V}(\Gamma')$. We now run the procedure from Definition 3.4 starting at S_0'. At the first step, we note that, since v_1 was added to S_0, it must have at least two neighbours in S_0, say u_1 and w_1. It follows that both $(v_1, 0)$ and $(v_1, 1)$ also have at least two neighbours in S_0' (for example, $(u_1, 1)$ and $(w_1, 1)$, and $(u_1, 0)$ and $(w_1, 0)$, respectively). We thus add $(v_1, 0)$ and $(v_1, 1)$ to S_0' to get $S_1' = S_0' \cup \{(v_1, 0), (v_1, 1)\}$. Note that $S_1' = S_1 \times \{0, 1\}$. We then repeat this procedure, preserving the condition $S_i' = S_i \times \{0, 1\}$ at each iteration. At the end of this process, we have $S_n' = S_n \times \{0, 1\} = \mathrm{V}(\Gamma) \times \{0, 1\} = \mathrm{V}(\Gamma')$ and so Γ' is dense with respect to $S_0 \times \{0, 1\}$. □

4 Proof of Theorem 1.1

Let p be a prime, let Γ be an arc-transitive graph of valency $2p$ and let $G = \mathrm{Aut}(\Gamma)$. We may assume that Γ is connected. If G is quasiprimitive or bi-quasiprimitive, then G contains a nontrivial semiregular element, by [8, Theorem 1.2] and [10, Theorem 1.4]. We may thus assume that G has a minimal normal subgroup N such that N has at least three orbits. In particular, Γ/N has valency at least 2.

If N is nonabelian, then G has a nontrivial semiregular element by [18, Theorem 1.1]. We may therefore assume that N is abelian and, in particular, N is an elementary abelian q-group for some prime q.

We may also assume that N is not semiregular that is, $N_v \neq 1$ for some vertex v. It follows from Lemma 2.1 that $1 \neq N_v^{\Gamma(v)} \trianglelefteq G_v^{\Gamma(v)}$. As Γ is G-arc-transitive, we have that $G_v^{\Gamma(v)}$ is transitive and so the orbits of $N_v^{\Gamma(v)}$ all have the same size, either 2 or p. Since N is a q-group, this size is equal to q. Writing d for the valency of Γ/N, we have that either $(d, q) = (2, p)$ or $(d, q) = (p, 2)$.

If $d = 2$ and $q = p$, then it follows from [15, Theorem 1] that Γ is isomorphic to the graph denoted by $\mathrm{C}(p, r, s)$ in [15]. By [15, Theorem 2.13], $\mathrm{Aut}(\mathrm{C}(p, r, s))$ contains the nontrivial semiregular automorphism ϱ defined in [15, Lemma 2.5].

We may thus assume that $d = p$ and $q = 2$. In this case, if u is adjacent to v, then u has exactly $2 = 2p/d$ neighbours in v^N. Let K be the kernel of the action of G on N-orbits. By the previous observation, the orbits of $K_v^{\Gamma(v)}$ have size 2 and thus it is a 2-group. It follows from Lemma 2.1 that K_v is a 2-group and thus so is $K = NK_v$.

Now, G/K is an arc-transitive group of automorphisms of Γ/N, so we may apply Corollary 3.2. If G/K has a semiregular element of odd prime order, then so does G, by Lemma 2.2. If $|\mathrm{V}(\Gamma/N)|$ is a power of 2, then so is $|\mathrm{V}(\Gamma)|$ and, in this case, G contains a semiregular involution by Lemma 2.3. We may thus assume that we are in case (3) of Corollary 3.2, that is, G/K contains a normal 2-subgroup P/K such that $(\Gamma/P, G/P)$ is one of the graph-group pairs in (3–5) of Theorem 3.1. Note that P is a 2-group. Let M be a minimal normal subgroup of G contained in the centre of P. Note that M is an elementary abelian 2-group. We may assume that M is not semiregular hence $M_v \neq 1$ and so by Lemma 2.1, $M_v^{\Gamma(v)} \neq 1$. Moreover, $|M| \neq 2$ as otherwise $M_v = M$ and we would deduce that M fixes each element of $\mathrm{V}(\Gamma)$, a contradiction. Since M is central in P, M_v fixes every vertex in v^P.

Note that the G-conjugates of M_v must cover M, otherwise M contains a nontrivial semiregular element. By the previous paragraph, the number of conjugates of M_v is bounded above by the number of P-orbits, that is $|\mathrm{V}(\Gamma/P)|$, so we have

$$|M| \leqslant |M_v||\mathrm{V}(\Gamma/P)|.$$

Since Γ is connected and G-arc-transitive, there are no edges within P-orbits. As $M_v^{\Gamma(v)} \neq 1$, there exists $g \in M_v$ such that w and w^g are distinct neighbours of v. Let u be the other neighbour of w in v^P. Since M_v fixes every element of v^P it follows that u is also a neighbour of w and w^g and so $\{v, w, u, w^g\}$ is a 4-cycle in Γ. Thus the graph induced between adjacent P-orbits is a union of C_4's.

If x is a vertex and y^P is a P-orbit adjacent to x, then there is a unique C_4 containing x between x^P and y^P, and thus a unique vertex z antipodal to x in this C_4. We say that z is the *buddy of x with respect to y^P*. The set of buddies of v is equal to $\Gamma_2(v) \cap v^P$, which is clearly fixed setwise by G_v. Moreover, each vertex has the same number of buddies. Furthermore, since G_v transitively permutes the set of p P-orbits adjacent to v^P, either v has a unique buddy or it has exactly p buddies.

If v has a unique buddy z, then $\Gamma(v) = \Gamma(z)$, and so swapping every vertex with its unique buddy is a nontrivial semiregular automorphism. Thus it remains to consider the case where v has p buddies. We first prove the following.

Claim. *If X is a subgroup of M that fixes pointwise both a^P and b^P, and c^P is a P-orbit adjacent to both a^P and b^P, then X fixes c^P pointwise.*

Proof. Suppose that some $x \in X$ does not fix c. Now x fixes a^P pointwise, so c^x must be the buddy of c with respect to a^P. Similarly, c^x must be the buddy of c with respect to b^P. These are distinct, which is a contradiction. It follows that X fixes c and, since $X \leqslant M$, also c^P. □

Let $s \geqslant 1$, let $\alpha = (v_0, \ldots, v_s)$ be an s-arc of Γ and let $\alpha' = (v_0, \ldots, v_{s-1})$. Now $|{v_s}^{M_{v_{s-1}}}| = 2$, so $|M_{v_{s-1}} : M_{v_{s-1}v_s}| = 2$ and $|M_{\alpha'} : M_\alpha| \leqslant 2$. Applying induction yields that

$$|M_{v_0} : M_\alpha| \leqslant 2^s. \tag{4.1}$$

We first assume that Γ/P and G/P are as in (3) or (4) of Theorem 3.1. Let $\{u, v\}$ be an edge of Γ. By the previous paragraph, we have $|M_v : M_{uv}| \leqslant 2$. Recall that M_v fixes all vertices in v^P, so M_{uv} fixes all vertices in $v^P \cup u^P$. Combining the claim with Corollary 3.5

yields that $M_{uv} = 1$ and thus $|M_v| = 2$. It follows that $|M| \leqslant |M_v||\mathrm{V}(\Gamma/P)|$ so $|M| \leqslant 2|\mathrm{V}(\Gamma/P)|$. Since M is minimal normal in G, it is an irreducible G-module over $\mathrm{GF}(2)$, of dimension at least two. In fact, since M is central in P, it is also an irreducible (G/P)-module. Since G/P is nonabelian simple or has a nonabelian simple group as an index two subgroup, this implies that M is a faithful irreducible (G/P)-module over $\mathrm{GF}(2)$. If $G/P = \mathrm{M}_{11}$, then $|M| \geqslant 2^{10}$ [11, Theorem 8.1], contradicting $|M| \leqslant 2 \cdot 12 = 24$. If $G/P = \mathrm{PSL}_2(p)$ or $\mathrm{PGL}_2(p)$ then by [2, Section VIII], $|M| \geqslant 2^{(p-1)/2}$. Recall that $p \geqslant 127$ and so this contradicts $|M| \leqslant 2(p^2-1)/2s < p^2 - 1$.

Finally, we assume that Γ/P is in (5) of Theorem 3.1, that is, Γ/P is the canonical double cover of a graph Γ' which appears in (4) of Theorem 3.1. In particular, $\mathrm{V}(\Gamma/P) = \mathrm{V}(\Gamma') \times \{0,1\}$. By Lemma 3.3, Γ' has a 3-cycle, say (u,v,w). By Corollary 3.5 and Lemma 3.6, Γ/P is dense with respect to $\{u,v\} \times \{0,1\}$. Now, let

$$\overline{\alpha} = ((u,0),(v,1),(w,0),(u,1),(v,0)).$$

Since $\overline{\alpha}$ contains $\{u,v\} \times \{0,1\}$, Γ/P is dense with respect to $\overline{\alpha}$. Note that $\overline{\alpha}$ is a 4-arc of Γ/P. Let α be a 4-arc of Γ that projects to $\overline{\alpha}$. Since Γ/P is dense with respect to $\overline{\alpha}$, arguing as in the last paragraph yields $M_\alpha = 1$. On the other hand, if $v \in \mathrm{V}(\Gamma)$ is the the initial vertex of α, then by (4.1), we have $|M_v : M_\alpha| \leqslant 2^4$ and thus $|M_v| \leqslant 2^4$. Since $|M| \leqslant |M_v||\mathrm{V}(\Gamma/P)|$ it follows that $|M| \leqslant 2^4(p^2-1)/s$. As above, M is a faithful irreducible (G/P)-module over $\mathrm{GF}(2)$ of dimension at least two. Since $G/P = \mathrm{PGL}_2(p)$ we have from [2] that $|M| \geqslant 2^{(p-1)/2}$, which again contradicts $|M| \leqslant 2^4(p^2-1)/s < 2^4(p^2-1)$.

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References

[1] M. Arezoomand, A. Abdollahi and P. Spiga, On problems concerning fixed-point-free permutations and on the polycirculant conjecture—a survey, *Trans. Comb.* **8** (2019), 15–40, doi:10.22108/toc.2018.112665.1585.

[2] R. Burkhardt, Die Zerlegungsmatrizen der Gruppen $\mathrm{PSL}(2,p^f)$, *J. Algebra* **40** (1976), 75–96, doi:10.1016/0021-8693(76)90088-0.

[3] T. C. Burness and M. Giudici, Permutation groups and derangements of odd prime order, *J. Comb. Theory Ser. A* **151** (2017), 102–130, doi:10.1016/j.jcta.2017.04.007.

[4] P. J. Cameron, M. Giudici, G. A. Jones, W. M. Kantor, M. H. Klin, D. Marušič and L. A. Nowitz, Transitive permutation groups without semiregular subgroups, *J. London Math. Soc. (2)* **66** (2002), 325–333, doi:10.1112/s0024610702003484.

[5] L. E. Dickson, *Linear Groups: With an Exposition of the Galois Field Theory*, Dover Publications, New York, 1958.

[6] J. D. Dixon and B. Mortimer, *Permutation Groups*, volume 163 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1996, doi:10.1007/978-1-4612-0731-3.

[7] E. Dobson, A. Malnič, D. Marušič and L. A. Nowitz, Semiregular automorphisms of vertex-transitive graphs of certain valencies, *J. Comb. Theory Ser. B* **97** (2007), 371–380, doi:10.1016/j.jctb.2006.06.004.

[8] M. Giudici, Quasiprimitive groups with no fixed point free elements of prime order, *J. London Math. Soc. (2)* **67** (2003), 73–84, doi:10.1112/s0024610702003812.

[9] M. Giudici, L. Morgan, P. Potočnik and G. Verret, Elusive groups of automorphisms of digraphs of small valency, *European J. Combin.* **46** (2015), 1–9, doi:10.1016/j.ejc.2014.11.004.

[10] M. Giudici and J. Xu, All vertex-transitive locally-quasiprimitive graphs have a semiregular automorphism, *J. Algebraic Combin.* **25** (2007), 217–232, doi:10.1007/s10801-006-0032-5.

[11] G. D. James, The modular characters of the Mathieu groups, *J. Algebra* **27** (1973), 57–111, doi:10.1016/0021-8693(73)90165-8.

[12] P. Lorimer, Vertex-transitive graphs: symmetric graphs of prime valency, *J. Graph Theory* **8** (1984), 55–68, doi:10.1002/jgt.3190080107.

[13] D. Marušič, On vertex symmetric digraphs, *Discrete Math.* **36** (1981), 69–81, doi:10.1016/0012-365x(81)90174-6.

[14] D. Marušič and R. Scapellato, Permutation groups, vertex-transitive digraphs and semiregular automorphisms, *European J. Combin.* **19** (1998), 707–712, doi:10.1006/eujc.1997.0192.

[15] C. E. Praeger and M. Y. Xu, A characterization of a class of symmetric graphs of twice prime valency, *European J. Combin.* **10** (1989), 91–102, doi:10.1016/s0195-6698(89)80037-x.

[16] G. Sabidussi, Vertex-transitive graphs, *Monatsh. Math.* **68** (1964), 426–438, doi:10.1007/bf01304186.

[17] G. Verret, Arc-transitive graphs of valency 8 have a semiregular automorphism, *Ars Math. Contemp.* **8** (2015), 29–34, doi:10.26493/1855-3974.492.37d.

[18] J. Xu, Semiregular automorphisms of arc-transitive graphs with valency pq, *European J. Combin.* **29** (2008), 622–629, doi:10.1016/j.ejc.2007.04.008.