

On flag-transitive automorphism groups of symmetric designs*

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Abstract

In this article, we study flag-transitive automorphism groups of non-trivial symmetric (v, k, λ) designs, where λ divides k and $k \geq \lambda^2$. We show that such an automorphism group is either point-primitive of affine or almost simple type, or point-imprimitive with parameters $v = \lambda^2(\lambda + 2)$ and $k = \lambda(\lambda + 1)$, for some positive integer λ . We also provide some examples in both possibilities.

Keywords: Symmetric design, flag-transitive, point-primitive, point-imprimitive, automorphism group.

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1 Introduction

A t -design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ with parameters (v, k, λ) is an incidence structure consisting of a set \mathcal{P} of v points, and a set \mathcal{B} of k -element subsets of \mathcal{P} , called *blocks*, such that every t -element subset of points lies in exactly λ blocks. The design \mathcal{D} is *non-trivial* if $t < k < v - t$, and is *symmetric* if $|\mathcal{B}| = v$. By [7, Theorem 1.1], if \mathcal{D} is symmetric and non-trivial, then $t \leq 2$, see also [12, Theorem 1.27]. Thus we study non-trivial symmetric 2-designs with parameters (v, k, λ) which we simply call non-trivial *symmetric* (v, k, λ) designs. A *flag* of \mathcal{D} is an incident pair (α, B) , where α and B are a point and a block of \mathcal{D} , respectively. An *automorphism* of a symmetric design \mathcal{D} is a permutation of the points permuting the blocks and preserving the incidence relation. An automorphism group G of \mathcal{D} is called *flag-transitive* if it is transitive on the set of flags of \mathcal{D} . If G leaves invariant a non-trivial partition of \mathcal{P} , then G is said to be *point-imprimitive*; otherwise G is called

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point-primitive. We here adopt the standard notation as in [8, 23] for finite simple groups of Lie type. For example, we use $\text{PSL}_n(q)$, $\text{PSp}_n(q)$, $\text{PSU}_n(q)$, $\text{P}\Omega_{2n+1}(q)$ and $\text{P}\Omega_{2n}^\pm(q)$ to denote the finite classical simple groups. Symmetric and alternating groups on n letters are denoted by S_n and A_n , respectively. Further notation and definitions in both design theory and group theory are standard and can be found, for example in [10, 12, 14]. We also use the software GAP [21] for computational arguments.

Flag-transitive incidence structures have been of most interest. In 1961, Higman and McLaughlin [11] proved that a flag-transitive automorphism group of a linear space must act primitively on its points set, and then Buekenhout, Delandtsheer and Doyen [5] studied this action in details and proved that a linear space admitting a flag-transitive automorphism group (which is in fact point-primitive) is either of affine, or almost simple type. Thereafter, a deep result [6], namely the classification of flag-transitive finite linear spaces relying on the Classification of Finite Simple Groups (CFSG) was announced. Although, flag-transitive symmetric designs are not necessarily point-primitive, Regueiro [18] proved that a flag-transitive and point-primitive automorphism group of such designs for $\lambda \leq 4$ is of affine or almost simple type, and so using CFSG, she determined all flag-transitive and point-primitive biplanes ($\lambda = 2$). In conclusion, she gave a classification of flag-transitive biplanes except for the 1-dimensional affine case [17]. Tian and Zhou [22] proved that a flag-transitive and point-primitive automorphism group of a symmetric design with $\lambda \leq 100$ must be of affine or almost simple type. Generally, Zieschang [25] proved in 1988 that a flag-transitive automorphism group of a 2-design with $\gcd(r, \lambda) = 1$ is (point-primitive) of affine or almost simple type, and this result has been generalised by Zhuo and Zhan [24] for $\lambda \geq \gcd(r, \lambda)^2$.

1.1 Main result

In this paper, we study flag-transitive automorphism groups of symmetric (v, k, λ) designs, where λ divides k and $k \geq \lambda^2$, and we show that such an automorphism group is not necessarily point-primitive:

Theorem 1.1. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a non-trivial symmetric (v, k, λ) design with $\lambda \geq 1$, and let G be a flag-transitive automorphism group of \mathcal{D} . If λ divides k and $k \geq \lambda^2$, then one of the following holds:*

- (a) G is point-primitive of affine or almost simple type;
- (b) G is point-imprimitive and $v = \lambda^2(\lambda + 2)$ and $k = \lambda(\lambda + 1)$, for some positive integer λ . In particular, if G has d classes of imprimitivity of size c , then there is a constant l such that, for each block B and each class Δ , the size $|B \cap \Delta|$ is either 0, or l , and $(c, d, l) = (\lambda^2, \lambda + 2, \lambda)$ or $(\lambda + 2, \lambda^2, 2)$.

We highlight here that if λ divides k , then $\gcd(k, \lambda)^2 = \lambda^2 > \lambda$ which does not satisfy the conditions which have been studied in [24, 25]. Moreover, in Section 1.2, we provide some examples to show that both possibilities in Theorem 1.1 can actually occur.

In order to prove Theorem 1.1(a), we apply O’Nan-Scott Theorem [15] and discuss possible types of primitive groups in Section 3. We further note that our proof for part (a) relies on CFSG. To prove part (b), we use an important result by Praeger and Zhou [20, Theorem 1.1] on characterisation of imprimitive flag-transitive symmetric designs.

1.2 Examples and comments on Theorem 1.1

Here, we give some examples of symmetric (v, k, λ) designs admitting flag-transitive automorphism groups, where λ divides k and $k \geq \lambda^2$. In Table 1, we list some small examples of such designs with $\lambda \leq 3$. To our knowledge the design in Line 2 is the only point-primitive example of symmetric designs with $v \leq 2500$ satisfying the conditions of Theorem 1.1 and this motivates the authors to investigate symmetric designs admitting symplectic automorphism groups [3]. More examples of symmetric designs admitting flag-transitive and point-imprimitive automorphism groups can be found in [20] and references therein.

Line 1. Hussain [13] showed that there are exactly three symmetric $(16, 6, 2)$ designs, and Regueiro proved that exactly two of such designs are flag-transitive and point-imprimitive [18, p. 139].

Line 2. The symmetric design in this line arises from the study of primitive permutation groups with small degrees. This design belongs to a class of symmetric designs with parameters $(3^m(3^m + 1)/2, 3^{m-1}(3^m - 1)/2, 3^{m-1}(3^{m-1} - 1)/2)$, for some positive integer $m > 1$, see [4, 9]. If $m = 2$, then we obtain the symmetric $(45, 12, 3)$ design admitting $\text{PSP}_4(3)$ or $\text{PSp}_4(3) : 2$ as flag-transitive automorphism group of rank 3, see [4].

Lines 3–4. Mathon and Spence [16] constructed 2616 pairwise non-isomorphic symmetric $(45, 12, 3)$ designs with non-trivial automorphism groups. Praeger [19] proved that there are exactly two flag-transitive symmetric $(45, 12, 3)$ designs, exactly one of which admits a point-imprimitive group, and this example satisfies Line 4, but not Line 3.

Table 1: Some symmetric designs satisfying the conditions in Theorem 1.1.

Line	v	k	λ	c	d	l	Case	Examples	Reference	Comments
1	16	6	2	4	4	2	(b)	2	[13], [18]	imprimitive
2	45	12	3	–	–	–	(a)	1	[4]	primitive
3	45	12	3	5	9	2	(b)	None	[19]	imprimitive
4	45	12	3	9	5	3	(b)	1	[19]	imprimitive

2 Preliminaries

In this section, we state some useful facts in both design theory and group theory.

Lemma 2.1 ([1, Lemma 2.1]). *Let \mathcal{D} be a symmetric (v, k, λ) design, and let G be a flag-transitive automorphism group of \mathcal{D} . If α is a point in \mathcal{P} and $H := G_\alpha$, then*

- (a) $k(k - 1) = \lambda(v - 1)$;
- (b) k divides $|H|$ and $\lambda v < k^2$.

Lemma 2.2 ([2, Corollary 4.3]). *Let T be a finite simple classical group of dimension n over a finite field \mathbb{F}_q of size q . Then*

- (a) If $T = \text{PSL}_n(q)$ with $n \geq 2$, then $|T| > q^{n^2-2}$;
- (b) If $T = \text{PSU}_n(q)$ with $n \geq 3$, then $|T| > (1 - q^{-1})q^{n^2-2}$;
- (c) If $T = \text{PSp}_n(q)$ with $n \geq 4$, then $|T| > q^{\frac{1}{2}n(n+1)}/(2\alpha)$, where $\alpha = \gcd(2, q - 1)$;
- (d) If $T = \text{P}\Omega_n^\epsilon(q)$ with $n \geq 7$, then $|T| > q^{\frac{1}{2}n(n-1)}/(4\beta)$, where $\beta = \gcd(2, n)$.

Lemma 2.3. *Let T be a non-abelian finite simple group satisfying*

$$|T| < 8 \cdot |\text{Out}(T)|^3. \quad (2.1)$$

Then T is isomorphic to A_5 or A_6 .

Proof. If T is a sporadic simple group or an alternating group A_n with $n \geq 7$, then $|\text{Out}(T)| \in \{1, 2\}$, and so by (2.1), we must have $|T| < 64$, which is a contradiction. Note that the alternating groups A_5 and A_6 satisfy (2.1) as claimed. Therefore, we only need to consider the case where T is a finite simple group of Lie type. In what follows, we discuss each case separately.

Let $T = \text{PSL}_n(q)$ with $q = p^a$ and $n \geq 2$. If $n = 2$, then $q \geq 4$ and $|\text{Out}(T)| = a \cdot \gcd(2, q - 1)$, and so by Lemma 2.2(a) and (2.1), we have that $q^2 < |\text{PSL}_2(q)| < 8a^3 \cdot \gcd(2, q - 1)^3 \leq 64a^3$. Thus, $q^2 < 64a^3$. This inequality holds only for $(p, a) \in \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (3, 1), (3, 2), (3, 3), (5, 1), (7, 1)\}$. Note in this case that $q \geq 4$, and hence by (2.1), we conclude that T is either

$$\text{PSL}_2(4) \cong \text{PSL}_2(5) \cong A_5, \quad \text{or} \quad \text{PSL}_2(9) \cong A_6,$$

as claimed. If $n = 3$, then by Lemma 2.2(a), we have that $q^7 < 64a^3 \cdot \gcd(3, q - 1)^3 < 64a^3q^3$, and so $q^4 < 64a^3$. If q would be odd, then we would have $3^{4a} < 64a^3$, which is impossible. If $q = 2^a$, then $2^{2a} < 64a^3$ would hold only for $a = 1, 2$. Therefore, T is isomorphic to $\text{PSL}_3(2)$ or $\text{PSL}_3(4)$. These simple groups do not satisfy (2.1). If $n \geq 4$, then (2.1) implies that $q^{11} < 64a^3$, but this inequality has no possible solution.

Let $T = \text{PSU}_n(q)$ with $q = p^a$ and $n \geq 3$. By Lemma 2.2(b), we have that $|T| > (1 - q^{-1})q^{n^2-2}$, and so (2.1) implies that $(1 - q^{-1})q^{n^2-2} < 64a^3 \cdot \gcd(n, q + 1)^3$. If $n = 3$, then $(1 - q^{-1})q^7 < 64a^3 \cdot \gcd(n, q + 1)^3$, and so $q^6 < 27 \cdot 64a^3$. This inequality holds only for $(p, a) \in \{(2, 1), (2, 2), (3, 1)\}$. Note that $\text{PSU}_3(2)$ is not simple. Therefore, T is isomorphic to $\text{PSU}_3(3)$ or $\text{PSU}_3(4)$. These simple groups do not satisfy (2.1). If $n \geq 4$, then since $(q + 1)^3 < 4 \cdot q^3(q - 1)$, we would have $q^{n^2-3} < 64a^3 \cdot \gcd(n, q + 1)^3 / (q - 1) < 4 \cdot 64a^3(q + 1)^3 / 4(q - 1) < 4 \cdot 64a^3q^3$, and so $q^{n^2-6} < 4 \cdot 64a^3$, and hence $q^{10} < 4 \cdot 64a^3$, which is impossible.

Let $T = \text{PSp}_n(q)$ with $q = p^a$ and $n \geq 4$. By Lemma 2.2(c), we observe that $|T| > q^{\frac{1}{2}n(n+1)}/2 \gcd(2, q - 1) \geq q^{\frac{1}{2}n(n+1)}/4$. By (2.1), we have that $q^{10} \leq q^{\frac{1}{2}n(n+1)} < 4 \cdot 64a^3$, and so $q^{10} < 4 \cdot 64a^3$, which is impossible.

Let $T = \text{P}\Omega_n^\epsilon(q)$ with $q = p^a$ odd and $n \geq 7$. Then we conclude by Lemma 2.2(d) that $|T| > q^{\frac{1}{2}n(n-1)}/8$. Since $|\text{Out}(T)| = 2a$ and $n \geq 7$, it follows from (2.1) that $q^{21} < 8^3a^3$, which is impossible.

Let $T = \text{P}\Omega_n^\epsilon(q)$ with $q = p^a$ and $n \geq 8$ and $\epsilon = \pm$. It follows from Lemma 2.2(d) that $|T| > q^{\frac{1}{2}n(n-1)}/8$. Note that $|\text{Out}(T)| \leq 6a \cdot \gcd(4, q^{\frac{n}{2}} - \epsilon) \leq 24a$. Then (2.1) implies that $q^{28} < 8^2 \cdot 24^3a^3$, which is impossible.

Let T be one of the finite exceptional groups

$$F_4(q), E_6(q), E_7(q), E_8(q), {}^2F_4(q) (q = 2^{2m+1}), {}^3D_4(q) \text{ and } {}^2E_6(q).$$

Then $|T| > q^{20}$, and so (2.1) implies that $q^{20} < 8 \cdot 2^3 \cdot 3^3 a^3$, which is impossible. If $T = G_2(q)$ with $q = p^a \neq 2$. Then by (2.1), we have that $q^{12} < q^6(q^2 - 1)(q^6 - 1) < 8 \cdot 2^3 a^3$, and so $q^{12} < 8 \cdot 2^3 a^3$, which is impossible. Similarly, if T is one of the groups ${}^2B_2(q)$ with $q = 2^{2m+1}$ and ${}^2G_2(q)$ with $q = 3^{2m+1}$, then $|T| > q^4$, and so (2.1) implies that $q^4 < 8a^3$, which is impossible. \square

3 Point-primitive designs

In what follows, we assume that $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a non-trivial symmetric (v, k, λ) design admitting a flag-transitive and point-primitive automorphism group G . Let also λ divide k and $k \geq \lambda^2$ and set $t := k/\lambda$. Notice that $\lambda < k$, and so $t \geq 2$. We moreover observe by Lemma 2.1(a) that

$$k = \frac{v + t - 1}{t}; \tag{3.1}$$

$$\lambda = \frac{v + t - 1}{t^2}. \tag{3.2}$$

Since also G is a primitive permutation group on \mathcal{P} , by O’Nan-Scott Theorem [15], G is of one of the following types:

- (a) Affine;
- (b) Almost simple;
- (c) Simple diagonal;
- (d) Product;
- (e) Twisted wreath product.

3.1 Product and twisted wreath product type

In this section, we assume that G is a primitive group of product type on \mathcal{P} , that is to say, $G \leq H \wr S_\ell$, where H is of almost simple or diagonal type on the set Γ of size $m := |\Gamma| \geq 5$ and $\ell \geq 2$. In this case, $\mathcal{P} = \Gamma^\ell$.

Lemma 3.1. *Let G be a flag-transitive point-primitive automorphism group of product type. Then k divides $\lambda\ell(m - 1)$.*

Proof. See the proof of Lemma 4 in [18]. \square

Proposition 3.2. *If $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a non-trivial symmetric (v, k, λ) design admitting a flag-transitive and point-primitive automorphism group G , where λ divides k and $k \geq \lambda^2$, then G is not of product type.*

Proof. Assume the contrary. Suppose that G is of product type. Then $v = m^\ell$. Note by Lemma 3.1 that k divides $\lambda\ell(m - 1)$, and so $t = k/\lambda$ divides $\ell(m - 1)$. We also note by

Lemma 2.1(b) that $\lambda v < k^2$. Then $v < \lambda t^2$, and since $\lambda \leq t$, we have that $v < t^3$. Recall that t divides $\ell(m - 1)$. Hence

$$m^\ell < \ell^3(m - 1)^3. \tag{3.3}$$

Then $m^\ell < \ell^3 m^3$, or equivalently, $m^{\ell-3} < \ell^3$. Since $m \geq 5$, it follows that $5^{\ell-3} < \ell^3$, and this is true for $2 \leq \ell \leq 6$. If $\ell = 6$, then since $m^{6-3} < 6^3$, we conclude that $m = 5$, but $(m, \ell) = (5, 6)$ does not satisfy (3.3). Therefore, $2 \leq \ell \leq 5$.

Suppose first that $\ell = 5$. Then by (3.3), we have that $m^5 < 5^3(m - 1)^3$, and so $5 \leq m \leq 9$. It follows from (3.1) that t divides $m^5 - 1$. For each $5 \leq m \leq 9$, we can obtain divisors t of $m^5 - 1$. Note by (3.2) that t^2 must divide $m^5 - t + 1$. This is true only for $m = 7$ when $t = 2$ or 6 for which

$$(v, k, \lambda) = (16807, 8404, 4202) \text{ or } (16807, 2802, 467),$$

respectively. Since $\lambda^2 \leq k$, these parameters can be ruled out.

Suppose that $\ell = 4$. Then by (3.3), we have that $m^5 < 4^3(m - 1)^3$, and so $5 \leq m \leq 9$. By the same argument as in the case where $\ell = 5$, by (3.1) and (3.2), we obtain possible parameters (m, t, v, k, λ) as in Table 2. Note by Lemma 3.1 that k must divide $4\lambda(m - 1)$, and this is not true, for all parameters in Table 2.

Table 2: Possible values for (m, t, v, k, λ) when $\ell = 4$.

m	t	v	k	λ
13	51	28561	561	11
31	555	923521	1665	3
47	345	4879681	14145	41
57	416	10556001	25376	61

Suppose now that $\ell = 3$. We again apply Lemma 3.1 and conclude that t divides $3(m - 1)$. Then there exists a positive integer x such that $3(m - 1) = tx$, and so $m = (tx + 3)/3$. By (3.2), we have that

$$\lambda = \frac{m^2 + t - 1}{t^2} = \frac{t^2x^3 + 9tx^2 + 27x + 27}{27t}.$$

Then $27\lambda t = t^2x^3 + 9tx^2 + 27x + 27$. Therefore, t must divide $27x + 27$, and so $ty = 27x + 27$, for some positive integer y . Thus,

$$\lambda = \frac{t(ty - 27)^3 + 9 \cdot 27(ty - 27)^2 + 27^3y}{27^4}, \tag{3.4}$$

for some positive integers t and y . Since $\lambda^2 \leq k$, we have that $\lambda \leq t$, and so

$$t(ty - 27)^3 + 9 \cdot 27(ty - 27)^2 + 27^3y \leq 27^4t. \tag{3.5}$$

If $y \geq 32$, then

$$\begin{aligned} & t(ty - 27)^3 + 9 \cdot 27(ty - 27)^2 + 27^3y \\ & \geq t(32t - 27)^3 + 9 \cdot 27(32t - 27)^2 + 32 \cdot 27^3 > 27^4t, \end{aligned}$$

for $t \geq 2$. Thus $1 \leq y \leq 31$, and so by (3.5), we conclude that $2 \leq t \leq 107$. For each such y and t , by straightforward calculation, we observe that λ as in (3.4) is not a positive integer.

Suppose finally that $\ell = 2$. Recall by Lemma 3.1 that t divides $2(m - 1)$. Then $2(m - 1) = tx$ for some positive integer x , and so $m = (tx + 2)/2$. It follows from (3.2) that $\lambda = (tx^2 + 4x + 4)/4t$, or equivalently, $4t\lambda = tx^2 + 4x + 4$. This shows that t divides $4x + 4$, and so $ty = 4x + 4$, for some positive integer y . Therefore, $4^3\lambda = (ty - 4)^2 + 16y$. Since $\lambda^2 \leq k$, we have that $\lambda \leq t$, and so $(ty - 4)^2 + 16y \leq 4^3t$. If $y \geq 6$, then $(6t - 4)^2 + 6 \cdot 16 \leq 4^3t$, which has no possible solution for t . Thus $1 \leq y \leq 5$. Since also $(t - 4)^2 + 16 \leq 4^3t$, we conclude that $2 \leq t \leq 71$, and so (3.1) and (3.2) imply that

$$k = \frac{t(t^2y^2 - 8ty + 16y + 16)}{64} \quad \text{and} \quad \lambda = \frac{(ty - 4)^2 + 16y}{64},$$

where $2 \leq t \leq 71$ and $1 \leq y \leq 5$. For these values of t and y , considering the fact that $m \geq 5$, $k \geq \lambda^2$ and λ divides k , we obtain $(v, k, \lambda) = (121, 25, 5)$ or $(441, 56, 7)$ respectively when $(t, y) = (5, 4)$ or $(8, 3)$. These possibilities can be ruled out by [4] or [22, Theorem 1.1]. \square

Proposition 3.3. *If $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a non-trivial symmetric (v, k, λ) design admitting a flag-transitive and point-primitive automorphism group G , where λ divides k and $k \geq \lambda^2$, then G is not of twisted wreath product type.*

Proof. If G would be of twisted wreath product type, then by [15, Remark 2(ii)], it would be contained in the wreath product $H \wr S_m$ with $H = T \times T$ of simple diagonal type, and so G would act on \mathcal{P} by product action, and this contradicts Proposition 3.2. \square

3.2 Simple diagonal type

In this section, we suppose that G is a primitive group of diagonal type. Let $M = \text{Soc}(G) = T_1 \times \dots \times T_m$, where $T_i \cong T$ is a non-abelian finite simple group, for $i = 1, \dots, m$. Then G may be viewed as a subgroup of $M \cdot (\text{Out}(T) \times S_m)$. Here, G_α is isomorphic to a subgroup of $\text{Aut}(T) \times S_m$ and $M_\alpha \cong T$ is a diagonal subgroup of M , and so $|\mathcal{P}| = |T|^{m-1}$.

Lemma 3.4. *Let G be a flag-transitive point-primitive automorphism group of simple diagonal type with socle T^m . Then k divides $\lambda m_1 h$, where $m_1 \leq m$ and h divides $|T|$.*

Proof. See the proof of Proposition 3.1 in [22]. \square

Proposition 3.5. *If $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a non-trivial symmetric (v, k, λ) design admitting a flag-transitive and point-primitive automorphism group G , where λ divides k and $k \geq \lambda^2$, then G is not of simple diagonal type.*

Proof. Suppose by contradiction that G is a primitive group of simple diagonal type. Then $v = |T|^{m-1}$, and so by Lemma 2.1(b), $\lambda v < k^2$. This implies that $\lambda|T|^{m-1} < k^2 = \lambda^2 t^2$. Since $\lambda^2 \leq k$, we must have $\lambda \leq t$, and hence

$$|T|^{m-1} < t^3. \tag{3.6}$$

Note by Lemma 3.4 that k divides $\lambda m_1 h$ and $m_1 h \leq m|T|$. Then t divides $m_1 h$, and so $t \leq m|T|$. We now apply (3.6) and conclude that $|T|^{m-1} < m^3|T|^3$. Therefore,

$|T|^{m-4} < m^3$. Since $|T| \geq 60$, we must have $m < 6$. If $m = 5$, then $|T| < 5^3$, and it follows that $T \cong A_5$. Note that k divides $\lambda(v - 1) = \lambda(|T|^{m-1} - 1)$. Then t divides $|T|^{m-1} - 1 = 60^4 - 1 = 13 \cdot 59 \cdot 61 \cdot 277$. Since $t \leq m|T| = 300$ and $t \geq 2$, it follows that $t \in \{13, 59, 61, 277\}$. For each such t , we have that $\lambda \leq t$ and $k = t\lambda$, and so we easily observe that these parameters does not satisfy Lemma 2.1(a). Therefore $m \in \{2, 3, 4\}$. Note that G_α is isomorphic to a subgroup of $\text{Aut}(T) \times S_m$. Then by Lemma 2.1(b), the parameter k divides $|G_\alpha|$, and so k divides $(m!) \cdot |T| \cdot |\text{Out}(T)|$. On the other hand, Lemma 2.1(a) implies that k divides $\lambda(|T|^{m-1} - 1)$, and so t divides $|T|^{m-1} - 1$ implying that $\gcd(t, |T|) = 1$. Since k divides $(m!) \cdot |T| \cdot |\text{Out}(T)|$ and t is a divisor of k , we conclude that t divides $(m!) \cdot |\text{Out}(T)|$. Recall by (3.6) that $|T|^{m-1} < t^3$. Therefore,

$$|T|^{m-1} < (m!)^3 \cdot |\text{Out}(T)|^3, \tag{3.7}$$

where $m \in \{2, 3, 4\}$.

If $m = 2$, then $|T| < 8 \cdot |\text{Out}(T)|^3$. If $m = 3$, then $|T|^2 < 6^3 |\text{Out}(T)|^3$, and so $|T| < 6^{\frac{3}{2}} |\text{Out}(T)|$. If $m = 4$, then $|T|^3 < 24^3 |\text{Out}(T)|^3$, and $|T| < 24 |\text{Out}(T)|$. Thus for $m \leq 4$, we always have

$$|T| < 8 \cdot |\text{Out}(T)|^3,$$

where T is a non-abelian finite simple group. We now apply Lemma 2.3 and conclude that T is isomorphic to A_5 or A_6 . If $m = 2$, then since t divides $|T|^{m-1} - 1 = |T| - 1$, we have that t divides 59 or 359 when T is isomorphic to A_5 or A_6 , respectively. Thus $(v, k, \lambda) = (60, 59\lambda, \lambda)$ or $(v, k, \lambda) = (360, 359\lambda, \lambda)$. Since $\lambda > 1$, in each case, we conclude that $k > v$, which is a contradiction. For $m = 3, 4$, since $|\text{Out}(A_5)| = 2$ and $|\text{Out}(A_6)| = 4$, it follows from (3.7) that $|T| < 48$ or $|T| < 96$ when T is isomorphic to A_5 or A_6 , respectively, which is a contradiction. \square

4 Proof of the main result

In this section, we prove Theorem 1.1. Suppose that $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a non-trivial symmetric (v, k, λ) design with λ divides k and $k \geq \lambda^2$. Suppose also that G is a flag-transitive automorphism group of \mathcal{D} .

Proof of Theorem 1.1. If G is point-primitive, then by O’Nan-Scott Theorem [15] and Propositions 3.2, 3.3 and 3.5, we conclude that G is of affine or almost simple type. Suppose now that G is point-imprimitive. Then G leaves invariant a non-trivial partition \mathcal{C} of \mathcal{P} with d classes of size c . By [20, Theorem 1.1], there is a constant l such that, for each $B \in \mathcal{B}$ and $\Delta \in \mathcal{C}$, $|B \cap \Delta| \in \{0, l\}$ and one of the following holds:

- (a) $k \leq \lambda(\lambda - 3)/2$;
- (b) $(v, k, \lambda) = (\lambda^2(\lambda + 2), \lambda(\lambda + 1), \lambda)$ with $(c, d, l) = (\lambda^2, \lambda + 2, \lambda)$ or $(\lambda + 2, \lambda^2, 2)$;
- (c)

$$(v, k, \lambda, c, d, l) = \left(\frac{(\lambda + 2)(\lambda^2 - 2\lambda + 2)}{4}, \frac{\lambda^2}{2}, \lambda, \frac{\lambda + 2}{2}, \frac{\lambda^2 - 2\lambda + 2}{2}, 2 \right),$$

and either $\lambda \equiv 0 \pmod{4}$, or $\lambda = 2u^2$, where u is odd, $u \geq 3$, and $2(u^2 - 1)$ is a square;

(d)

$$(v, k, \lambda, c, d, l) = \left(\frac{(\lambda + 6)(\lambda^2 + 4\lambda - 1)}{4}, \frac{\lambda(\lambda + 5)}{2}, \lambda, \lambda + 6, \frac{\lambda^2 + 4\lambda - 1}{4}, 3 \right),$$

where $\lambda \equiv 1$ or $3 \pmod{6}$.

We easily observe that the cases (a) and (c) can be ruled out as $k \geq \lambda^2$. If case (d) occurs, then $\lambda(\lambda + 5)/2 = k \geq \lambda^2$ implying that $\lambda \leq 5$. Since $\lambda \equiv 1$ or $3 \pmod{6}$, it follows that $\lambda = 3$ for which $(v, k, \lambda, c, d, l) = (45, 12, 3, 9, 5, 3)$ which satisfies the condition in Theorem 1.1(b). Therefore, the case (b) can occur as claimed. \square

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