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Analytical representations of clinical electron beam central axis depth doses

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Analytical expressions proposed to date to approximate central axis electron beam depth dose distributions are reviewed and their quality of fitting discussed. A recently developed analytical expression based on only four fitting parameters is analyzed. The expression approximates well the measured electron beam data from two commercial linear accelerators in the field size range from $4\times4\,cm^2$ *to* $25\times25\,cm^2$ *and in the energy range from 4 MeV to 22 MeV in all four regions of the depth dose curve: build-up, dose maximum, dose fall-off, and bremsstrahlung contamination.*

Key words: electrons, particle accelerators, depth doses

Introduction

The particular energy loss characteristics of electrons as they penetrate into tissue make electrons suitable for use in treatment of superficial malignant diseases. Advantages of electrons over superficial x-rays and brachytherapy are a better dose homogeneity in the target volume and a lower dose in tissues surrounding the target. The electron beam depth dose distributions consist of four regions: buildup, dose maximum, dose fall-off, and bremsstrahlung contamination. Ever since the first depth dose distributions of clinical electron beams were measured in water, attempts have been made to describe the measured distributions with analytical expressions. In the individual dose regions, it is relatively easy to approximate the dose distributions analytically; however, the distributions are difficult to describe accurately with a single expression covering all four regions simultaneously.

In 1953 Laughlin et al.[1] proposed the first analytical expression to reproduce electron depth doses in water. Since then, attempts to describe analytically depth doses of clinical electron beams have continued with varying degrees of success.[2–12] With each subsequent new proposal, the analytical expressions became more accurate but also, to a certain degree, more complicated, as they depended on an ever-increasing number of empirical parameters involved in the curve fitting process.

In this note, we present a summary of expressions that were proposed to date by various authors to describe electron beam depth dose data analytically. For each expression, we show the fit it provides to a typical measured depth dose distribution. We also provide an analysis of an analytical expression which we developed

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recently for description of electron beams of various energies and field sizes.[12] The expression reproduces electron beam central axis depth doses well in all four regions of the depth dose curve, and it achieves this with a smaller number of empirical parameters than does the most accurate approximation proposed previously.[11]

Analytical expressions for electron beam depth doses

The first analytical representation of clinical electron depth doses was proposed by Laughlin et al.[1] as follows:

$$D(x) = 110 - 10 \exp\left[\mu (x - x_m)\right] , \quad (1)$$

where D(x) is the percentage depth dose at depth x in the medium, μ is an attenuation coefficient, and x_m is the depth of dose maximum. The equation is simple, depends on only two parameters (μ and x_m), but agrees with measured data only for relatively low energy electrons in the dose fall-off region. The equation is valid neither for the dose build-up region nor for the bremsstrahlung region.

To provide a better analytical description of electron depth doses, Bagne[2] proposed a modification to Eq. (1) through the addition of a cubic term in the exponent:

$$D(x) = 110 - 10 \exp\left[\mu_m \left((x-x_m)\,\rho - \lambda\,(x-x_m)^3 \rho^3\right)\right] , \quad (2)$$

where D(x), x and x_m were defined above, μ_m is a mass attenuation coefficient, ϱ the density of the medium, and λ a third adjustable parameter. This equation, although an improvement over Eq. (1), is also only valid for the dose fall-off region.

Pacyniak and Pagnamenta[3] derived the following equation based on physical arguments:

$$D(x) = 100 \left(\frac{R - x}{R - x_m}\right)^A \left(1 + A \frac{x - x_m}{R - x_m}\right) , \quad (3)$$

where R is the practical electron range in the medium and x_m again the depth of maximum dose. The third parameter A is defined as A = μ (R/E), where μ represents an attenuation coefficient and E the average electron energy. In contrast to Equations (1) and (2), Eq. (3) provides a fairly good approximation to the measured depth doses in the build-up region, dose maximum region as well as in the dose fall-off region. At depths beyond the range R, however, the equation generates high negative percent depth doses and therefore cannot account for the bremsstrahlung contamination of the electron beam.

Further attempts to parametrize electron depth dose data were based on different types od mathematical functions which can mimic the dependence of measured depth doses on the depth in phantom. One group the these proposed approximations[4–7] uses polynomial functions as follows:

$$D(x) = 100 + (x-x_m)^2 \left[a_2 + a_3 x + a_4 x^2 + a_5 x^3\right] , \quad (4)$$

$$D(z) = 100 + \sum_{m=2}^{m=5} a_m z^m \text{ with } z = \frac{x - x_m}{E} \quad (5)$$

$$D(x) = 100\left[a_0 + a_1 \frac{x}{x_m} + a_2 \left(\frac{x}{x_m}\right)^2\right]^{\left(\frac{x}{x_m} - 1\right)} , \text{ and} \quad (6)$$

$$D(x) = a_0 \left(a_1 - 2 a_2 s + a_3 a_4 s^{(a_4 - 1)} + a_5 a_6 s(a_6 - 1)\right)$$
$$\exp\left[-(a_1 s + a_2 s^2 + a_3 s (a_4) + a_5 s (a_6)\right] + B \quad (7)$$

where x_m again is the depth of the dose maximum, a_i are adjustable parameters, and parameter B in Eq.(7) is a bremsstrahlung dose-related function. Variables z and s in Equations (5) and (7), respectively, are normalized depths defined as $z = (x - x_m)/E$ where E is the electron beam energy, and $s = x/R_p$ where R_p is the measured practical range of electrons.

It was shown[4–6] that Equations (4), (5) and (6) fit the measured electron depth doses relatively well in the build-up region, dose maximum region, and in the sharp dose fall-off region for electron energies from 5 MeV to 20 MeV and for field sizes from $5 \times 5\,cm^2$ to $25 \times 25\,cm^2$. However, they fail, similarly to Eq. (3), to provide an acceptable approximation in the bremsstrahlung background region in which they generate high negative depth dose values, a property typical of all proposed polynomial representations of electron depth doses. Equation (7) approximates all four electron depth dose regions but is obviously quite complicated as it depends on a large number of fitting parameters.

Another group of mathematical approximations to electron beam depth doses is based on a modified Fermi-Dirac distribution function:[8-10]

$$D(x) = \frac{100}{1 + \exp\left[\frac{(x - x_{50})}{a}\right]} \, , \tag{8}$$

$$D(x) = \frac{108}{1 + a\exp[\mu\,(x - x_s)]} \, , \text{ and} \tag{9}$$

$$D(x) = \frac{a_1 x^2 + a_2 x + a_3}{1 + \exp[a_4(x - a_5)]} \, , \tag{10}$$

where x_{50} in Eq.(8) is the depth in the dose fall-off region at which the dose reaches 50 % of the maximum value and x_s in Eq. (9) is the depth in the fall-off region at which the dose is equal to the surface dose. Parameter a in Equations (8) and (9) and parameters a_i in Eq. (10) are adjustable parameters. Equations (8) and (9) give much better approximations to electron depth doses in the build-up region than Equations (1) and (2), but they are considerably less efficient than the polynomial Equations (3 through 7) in simultaneously approximating both the build-up region and the dose fall-off region. Moreover, they still tend to generate negative percent depth doses in the bremsstrahlung contamination region.

An excellent quality of curve fitting was achieved with Eq. (10), which has five fitting parameters and gives a very good approximation for electron depth doses in the build-up region, dose maximum region, and the dose fall-off region. However, at large depths Eq. (10) yields either zero or negative values and thus cannot reproduce the dose behaviour in the bremsstrahlung region.

A more recent and very successful analytical representation of electron depth doses as a function of the depth in phantom was proposed by Strydom[11] as follows:

$$D(x) = \frac{(100-B)}{1 - a_1(x - x_m)} \exp\{-(x - x_m)^2[a_2 + a_3(x - x_m) + a_4(x - x_m)^2]\} + B \tag{11}$$

where x_m again is the depth of the dose maximum, B represents the bremsstrahlung dose bakcground, and a_i are adjustable parameters. This equation has in effect six varying parameters: (four fitted parameters: a_1, a_2, a_3, a_4 and two measured parameters: x_m and B), and describes very well the electron depth doses in all dose regions, including the bremsstrahlung dose background region.

The various approaches to analytical descriptions of electron beam depth dose distributions discussed above and given by Equations (1) though (11) are illustrated in Figure 1. A typical elecron beam depth dose curve measured in water (9 MeV, field size: $10 \times 10\,cm^2$), and shown as data points, is approximated by various expressions (solid curves) proposed to date, starting (a) with the rudimentary initial proposal of Laughlin et al.[1] with two parameters and ending with (j) the excellent fitting based on six parameters proposed by Strydom.

The varying quality of the curve fitting results for each to the approaches proposed to date is clearly evident from Figure 1, which also gives for each of the approaches the number of parameters required for the curve fitting procedure and the year of the proposal. It is evident that the curve fitting proposals improved with time but they also became considerably more complicated as they depended on ever-increasing numbers of fitted and measured parameters.

As shown in Figure 1 (j), Strydom's equation based on six parameters provides an excellent approximation to measured electron depth doses with four fitted parameters in addition to two measured parameters: the depth of dose maximum and the bremsstrahlung contamination. Thus, the objective of an accurate approximation of the whole electron beam central axis dose distribution has been met successfully with six parameters in Eq. (11). It is clear that new approaches will not be able to improve the quality of fitting; however, they might simplify the fitting procedure by using equations which

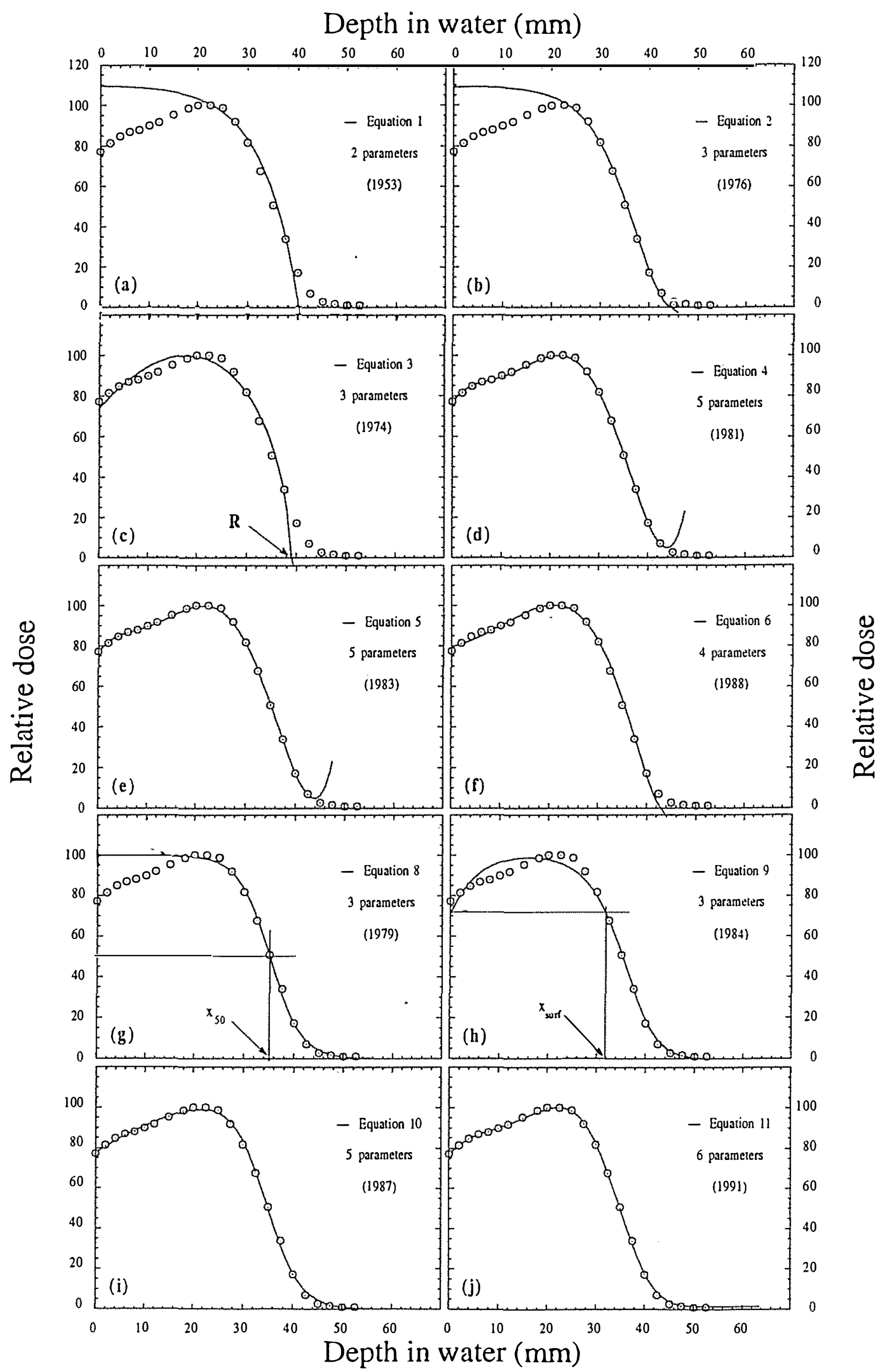

Figure 1. Central axis electron beam depth doses calculated from analytical expressions given by Equations (1) through (6) and (8) through (11), compared to data measured for a 9 MeV electron beam with a field size of $10 \times 10\,\text{cm}^2$. Calculated data are shown with solid curves, measured data as points. For each analytical expression, the number of required parameters and the year the expression was developed are also given. Equations (1) through (6) correspond to parts (a) through (f), respectively. Equations (8) through (11) correspond to parts (g) through (f), respectively.

achieve a similar quality fit with a lower number of parameters.

We have recently proposed[12] a new analytical equation which contains only four parameters and is able to fit the measured electron depth doses in all four regions for various nominal energies of the electron beam as well as for various field sizes. The equation is given as follows:

$$D(x) = \frac{(100 - B)}{1 + a(x + c)(x - c)^2 \exp[b\, x(x + c)(x - c)^2]} - 5 \exp\left(- \frac{3x}{c}\right) + B \quad , \tag{12}$$

where a, b, and c are the fitted parameters, B is a measured parameter representing the bremsstrahlung contamination, and x is the depth in medium.

Materials and methods

The validity of the approximation given by Eq. (12) was verified with electron beam depth dose data which were measured in water at a source-surface distance (SSD) of 100 cm using a 3-D isodose plotter with a p-type semi-conductor detector. In the build-up region the percentage depth doses were measured in polystyrene with a parallel-plate ionization chamber (Markus-type PTW, model 329). Two linear accelerators (Philips SL-25 and Varian Clinac 2300 C/D) were used as sources of electron beams with square field sizes in the range from $4 \times 4\,cm^2$ to $25 \times 25\,cm^2$ and beam energies in the range from 4 MeV to 22 MeV.

Nonlinear curve-fitting was performed with a commercially available graphics software package (KaleidaGraph by Abelbeck Software) on a MacIntosh computer. The general curve fitting space is missing program, based on a Marquardt algorithm,[13] is both powerful and efficient, able to fit any arbitrary single variable function containing up to nine fitted parameters. Moreover, the program allows fitting of weighted data as well as the use of partial derivatives. In our curve fitting procedure we used wqually-weighted data points.

Results and discussion

The fitting of Eq. (12) to measured electron beam percent depth doses resulted in a set of optimized numerical values for parameters a, b and c as a function of field size and nominal electron beam energy. The numerical values of parameters a, b, c and B depend on the field size as well as on the nominal energy of the beam. The second term in the right-hand side of Eq. (12) adjusts the shape of the function to the dose measured at the phantom surface and in its vicinity.

For curve fitting purposes, the initial values of the four parameters are set as follows: the initial value of c is set equal to x_m, the measured depth of dose maximum; the initial value of a is obtained from Eq. (12) for the phantom surface, i.e., $x = 0$ and $D(0) = D_s$ as:

$$a = \frac{100 - (D_s + 5)}{c^3 (D_s + 5 - B)} \tag{13}$$

The initial value of b is set a few (typically six) times smaller than the initial value of a; and B is set equal to the measured bremsstrahlung contamination which is assumed constant for a given electron beam energy. The non-linear curve fitting starts with the initial values for a, b, and c and reaches the optimal values for a, b, and c through an iterative process. Note that Eq. (13) is used only for estimation of the initial value of parameter a using measured values for the surface dose D_s and bremsstrahlung contamination B, and the initial value for c as equal to x_m. The final optimal values for parameters a and c are generally not related through Eq. (13).

An example of Eq. (12) used in fitting experimental electron depth doses is shown in Figure 2 for a field size of $10 \times 10\,cm^2$ and electron beams produced by two of our high energy linear accelerators. Measured data are shown as data points and the corresponding calculated depth doses by solid curves. Parts (b) and (d) of Figure 2 show on an expanded scale the build-up regions of parts (a) and (c), respectively. The agreement between the measured data and the fitted data in all four regions of the electron depth dose curves is excellent, proving that Eq. (12) with four parameters offers an excellent analytical approximation to measured data. Optimized parameters a, b, and c as well as the two measured parameters B and D_s for beams of Figure 2 are given in Table 1.

The quality of fitting Eq. (12) to the electron beam data set of Table 1 was evaluated for percentage depth doses above 20%. The results of a statistical comparison between calculated and measured data representing 655 analyzed points ($10 \times 10\,cm^2$ field size, six beam energies for each of the two linear accelerators) are as follows: at the phantom surface, in the build-up region, and in the dose fall-off region, 61% of calculated points matched the measured data within 1%, 92% within 2%, and 98% within 3%. In the fall-off region, the difference between the measured and calculated depths corresponding to the 50% depth dose was within 0.4 mm for all electron energies produced by the two linear accelerators.

Fitting of Eq. (12) to other sets of measured electron beam data gave results similar to those shown in Figure 2. for the $10 \times 10\,cm^2$ field size. Thus a conclusion can be made that the quality of fitting is independent of electron beam machine, field size, or electron beam energy, and all four regions of the electron depth dose curves are approximated well with

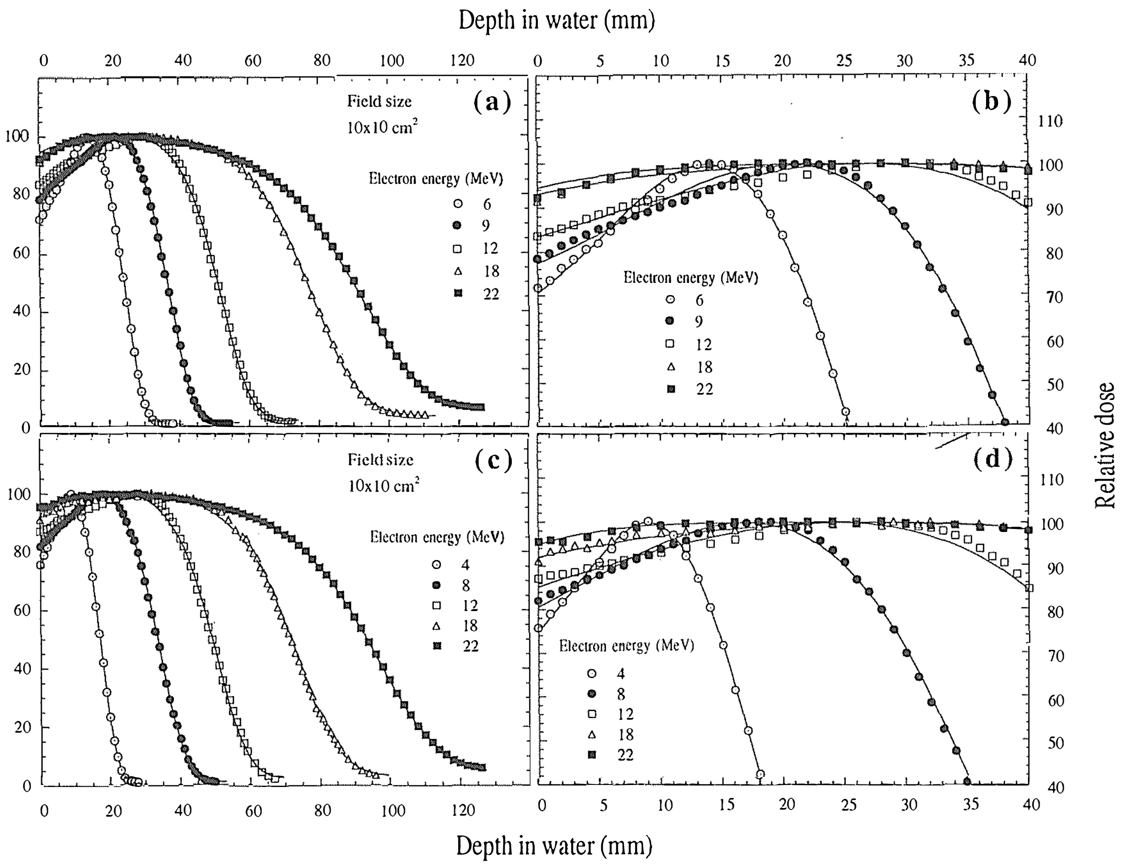

Figure 2. Central axis percentage depth doses for a field size of $10 \times 10\,cm^2$ and various clinical electron beams produced by two linear accelerators: (a) and (b) Varian Clinac 2300C/D; and (c) and (d) Philips SL-25. Measured data are shown as data points and corresponding dose distributions calculated with Eq. (12) are represented by solid curves. Parts (b) and (d) show in greater detail the build-up and dose maximum regions of parts (a) and (c), respectively.

Table 1. Optimized values of fitting parameters a, b, and c and measured parameters B and D_s for $10 \times cm^2$ electron beams with various nominal energies for two commercial linear accelerators: a Varian Clinac 2300C/D and a Philips SL-25.

	Nominal electron energy (MeV)	Parameter $a \times 10^7$ (mm^{-3})	$b \times 10^8$ (mm^{-4})	c (mm)	B	D_s
VARIAN	6	1283.3	595.82	13.65	0.65	70.6
CLINAC	9	278.29	146.30	19.89	1.31	77.3
2300 C/D	12	76.647	42.505	25.96	2.14	83.4
	15	24.050	15.052	27.71	3.44	90.3
	18	11.810	6.1579	25.88	4.13	93.1
	22	7.1493	2.2432	20.44	5.14	94.4
PHILIPS	4	3652.7	1295.1	8.86	0.036	74.7
SL-25	6	1104.0	406.77	12.5	0.54	77.3
	8	369.59	135.71	16.74	1.08	80.5
	10	178.34	78.031	20.54	1.47	81.8
	12	85.550	39.118	23.54	2.49	85.2
	15	33.882	16.287	24.52	3.12	90.4
	18	15.922	8.5102	27.14	3.31	92.0
	20	8.3512	4.1550	23.33	3.68	94.0
	22	4.7616	1.9890	15.79	4.06	94.8

the fitting procedure. Equation (12) with four fitting parameters thus provides a relatively simple yet precise means for expressing clinical electron beams analytically.

Conclusions

Ever since electron beams have been used clinically, attempts have been made to describe analytically the measured central axis depth dose distributions. These distributions consist of four regions: dose buildup, dose maximum, dose falloff and bremsstrahlung contamination. Numerous analytical expressions to approximate electron depth doses in all four regions have been proposed to date. The quality of fitting generally improved with each new proposal but the curve fitting equations were becoming increasingly more complex as they depended on larger and larger numbers of fitting parameters.

Analytical expressions developed in recent years for descriptions for electron beam depth doses provide an excellent fit to measured data. Improvements in this area can in the future only be achieved in developing simpler expressions which rely on a smaller number of parameters. We have recently developed a relatively simple expression based on only four parameters. We show in this paper that the expression represents, with a high degree of precision, measured electron beam depth doses for various beam energies from two commercial linear accelerators. The conclusion can be made that the expression may be applied to describe the electron beam depth doses generally for any linear accelerator, any field size, and any beam energy.

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